

# Lyapunov Exponents of Hybrid Stochastic Heat Equations

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## Abstract

In this paper, we investigate a class of hybrid stochastic heat equations. By explicit formulae of solutions, we not only reveal the sample Lyapunov exponents but also discuss the  $p$ th moment Lyapunov exponents. Moreover, several examples are established to demonstrate that unstable (deterministic or stochastic) dynamical systems can be stabilized by Markovian switching.

**Keywords:** Stochastic heat equation; Markov chain; Lyapunov exponent; Stabilization; Large deviation.

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## 1 Introduction

Stabilization of (ordinary) stochastic differential equations (SDEs) by noise has been studied extensively in the past few years, e.g., Arnold et al. [1], Has'minskii [6], Mao and Yuan [10], Pardoux and Wihstutz [11, 12], Scheutzow [15]. Recently, there are also many works focusing on such phenomena for stochastic partial differential equations (SPDEs), e.g., Kwiecinska [7] and Kwiecinska [9] discussed by multiplicative noise such problems for heat equations and a class of deterministic evolution equations, respectively; some results on almost sure exponential stabilization of SPDEs were established in Caraballo et al. [3] by a Lyapunov function argument; stabilization by additive noise on solutions to semilinear parabolic SPDEs with quadratic nonlinearities was investigated due to Blömker et al. [2].

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Also, there has been increasing attention to hybrid SDEs (also known as SDEs with Markovian switching and please see, e.g., the monographs [10, 18]), in which continuous dynamics are intertwined with discrete events. One of the distinct features of such systems is that the underlying dynamics are subject to changes with respect to certain configurations while the continuous-time Markov chains are used to delineate many practical systems, where they may experience abrupt changes in their structure and parameters.

It should be pointed out that there are some papers discussing the almost sure exponential stability and the  $p$ th moment exponential stability of stochastic heat equations without Markov switching, e.g., Kwiecinska [7, 8, 9] and Xie [17]. However, hybrid SPDEs, e.g., hybrid stochastic heat equations, have so far little been studied. In this paper, we are mainly interested in finding explicit formulae of solutions for a class of hybrid stochastic heat equations, and compute explicitly the sample Lyapunov exponents as well as the  $p$ th moment Lyapunov exponents. In particular, due to the involvement of Markov chains, it is much more complicated to study the  $p$ th moment Lyapunov exponents of hybrid stochastic heat equations, and our results can not get straightforward from [8, 17]. To cope with the difficulties arising from the Markov switching, large deviation technique has been used in this paper. For large deviation of Markov processes, we refer to, e.g., Donsker and Varadhan [4], Wu [16] and the references therein. On the other hand, we also use our theories to reveal the stabilization of (stochastic) dynamical systems by Markovian switching.

The organization of this paper will be arranged as follows: In Section 2 we recall some notation and notions, and give an existence-and-uniqueness result for hybrid stochastic heat equations. We investigate, in Section 3, the explicit formulae of solutions for hybrid stochastic heat equations, and, in Section 4, the sample Lyapunov exponents, where two examples are constructed to show that Markovian switching can be used to stabilize unstable stochastic dynamical systems. The last section reveals by the large deviation principle the  $p$ th moment Lyapunov exponents.

## 2 Preliminaries

Let  $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\}$  be a complete probability space with a filtration satisfying the usual conditions, and  $B(t)$ ,  $t \geq 0$ , a real-valued Brownian motion defined on the above probability space. For a bounded domain  $\mathcal{O} \subset \mathbb{R}^n$  with  $C^\infty$  boundary  $\partial\mathcal{O}$ , let  $L^2(\mathcal{O})$  denote the family of all real-valued square integrable functions, equipped with the usual inner product  $\langle f, g \rangle := \int_{\mathcal{O}} f(x)g(x)dx$ ,  $f, g \in L^2(\mathcal{O})$ , and the norm  $\|f\| := (\int_{\mathcal{O}} f^2(x)dx)^{\frac{1}{2}}$ ,  $f \in L^2(\mathcal{O})$ . Let  $A := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$  be the Laplace operator, with domain  $\mathcal{D}(A) := H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})$ , where  $H^m(\mathcal{O})$ ,  $m = 1, 2$ , consist of functions of  $L^2(\mathcal{O})$  whose derivatives  $D^\alpha u$  of order  $|\alpha| \leq m$  are in  $L^2(\mathcal{O})$ , and  $H_0^m(\mathcal{O})$  is the subspace of elements of  $H^m(\mathcal{O})$  vanishing on  $\partial\mathcal{O}$ . Furthermore, let  $\{e_n\}_{n \geq 1}$ , forming an orthonormal basis of  $L^2(\mathcal{O})$ , and  $\{\lambda_n\}_{n \geq 1} \uparrow \infty$  be the eigenvector and eigenvalue of  $-A$  respectively, namely

$$-Ae_n = \lambda_n e_n. \quad (2.1)$$

Thus, for any  $f \in L^2(\mathcal{O})$ , we can write  $f = \sum_{n=1}^{\infty} f_n e_n$ , where  $f_n := \langle f, e_n \rangle$ .

Let  $\mathbb{R}_+ := [0, \infty)$  and  $N$  be some positive integer. Let  $\{r(t), t \in \mathbb{R}_+\}$  be a right continuous Markov chain on the probability space  $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\}$  taking values in a finite state space  $\mathbb{S} := \{1, 2, \dots, N\}$ , with generator  $\Gamma := (\gamma_{ij})_{N \times N}$  given by

$$\mathbb{P}(r(t + \Delta) = j | r(t) = i) = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where  $\Delta > 0$  and  $\gamma_{ij} \geq 0$  is the transition rate from  $i$  to  $j$ , if  $i \neq j$ ; while  $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$ . We assume that Markov chain  $r(\cdot)$  is independent of Brownian motion  $B(\cdot)$ . It is known that almost every sample path of  $r(t)$  is a right continuous step function with a finite number of sample jumps in any finite sub-interval of  $\mathbb{R}_+$ . We further assume that the Markov chain  $r(t)$  is irreducible. This is equivalent to the condition that, for any  $i, j \in \mathbb{S}$ , one can find finite numbers  $i_1, i_2, \dots, i_k \in \mathbb{S}$  such that  $\gamma_{i, i_1} \gamma_{i_1, i_2} \dots \gamma_{i_k, j} > 0$ . The algebraic interpretation of irreducibility is  $\text{ran}(\Gamma) = N - 1$ . Under this condition, the Markov chain  $r(t)$  has a unique stationary probability distribution  $\pi := (\pi_1, \pi_2, \dots, \pi_N)$  which can be determined by solving the following linear equation  $\pi\Gamma = 0$  subject to  $\sum_{j=1}^N \pi_j = 1$  and  $\pi_j > 0, \forall j \in \mathbb{S}$ .

In this paper we consider hybrid stochastic heat equation:

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = Au(t, x) + \alpha(r(t))u(t, x) + \beta(r(t))u(t, x)\dot{B}(t), & x \in \mathcal{O}, t > 0; \\ u(t, x) = 0, & x \in \partial\mathcal{O}, t > 0; \\ u(0, x) = u^0(x), & x \in \mathcal{O}. \end{cases} \quad (2.2)$$

Here  $u^0$  is a  $\mathcal{D}(A)$ -valued  $\mathcal{F}_0$ -measurable random variable such that  $\mathbb{E}\|u^0\|^p < \infty$  for any  $p > 0$ , and independent of  $r(\cdot)$  and  $B(\cdot)$ . Moreover,  $\alpha, \beta$  are mappings from  $\mathbb{S} \rightarrow \mathbb{R}$  and we shall write  $\alpha_i := \alpha(i)$ ,  $\beta_i := \beta(i)$  in what follows. In general,  $u(t, x)$  is referred to as the state and  $r(t)$  is regarded as the mode. In its operation, hybrid system will switch from one mode to another according to the law of Markov chain. In this paper, we shall write  $u(t) := u(t, \cdot)$ , and  $u := \{u(t)\}_{t \in \mathbb{R}_+}$ . Hence, for each  $t \geq 0$ ,  $u(t)$  is an  $L^2(\mathcal{O})$ -valued random variable while  $u$  is an  $L^2(\mathcal{O})$ -valued stochastic process. Let us fix an interval  $[0, T]$  for arbitrary  $T > 0$  and recall the definition of solution to Eq. (2.2) from Da Prato et al. [13].

**Definition 2.1.** An  $L^2(\mathcal{O})$ -valued predictable process  $u = \{u(t)\}_{t \in [0, T]}$  is called the solution to Eq. (2.2) if the following conditions are satisfied:

- (i)  $u \in C([0, T]; L^2(\mathcal{O}))$  and, for any  $t \in [0, T]$ ,  $u(t) \in \mathcal{D}(A)$  a.s.;
- (ii) Stochastic integral equation

$$u(t, x) = u^0(x) + \int_0^t [Au(s, x) + \alpha(r(s))u(s, x)]ds + \int_0^t \beta(r(s))u(s, x)dB(s)$$

holds a.s. for any  $t \in [0, T]$  and  $x \in \mathcal{O}$ .

**Theorem 2.1.** Eq. (2.2) admits a unique solution  $u = \{u(t)\}_{t \in \mathbb{R}_+}$  such that  $\mathbb{E}\|u(t)\|^p < \infty, t > 0, p > 0$ .

*Proof.* Recall that almost every sample path of  $r(\cdot)$  is a right-continuous step function with a finite number of sample jumps in any finite time. So there is a sequence  $\{\tau_k\}_{k \geq 0}$  of stopping times such that  $\tau_0 = 0$ ,  $\lim_{k \rightarrow \infty} \tau_k = \infty$  a.s. and  $r(t) = r(\tau_k)$  on  $\tau_k \leq t < \tau_{k+1}$  for  $\forall k \geq 0$ . Let  $T > 0$  be arbitrary. We first consider Eq. (2.2) on  $t \in [0, \tau_1 \wedge T]$  which becomes

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = Au(t,x) + \alpha(r(0))u(t,x) + \beta(r(0))u(t,x)\dot{B}(t), & x \in \mathcal{O}; \\ u(t,x) = 0, & x \in \partial\mathcal{O}; \\ u(0) = u^0(x), & x \in \mathcal{O}. \end{cases}$$

By [7, Proposition 1], Eq. (2.2) has a unique solution  $u \in C([0, \tau_1 \wedge T]; L^2(\mathcal{O}))$  and  $u(t) \in \mathcal{D}(A)$  for  $t \in [0, \tau_1 \wedge T]$  a.s. while  $\mathbb{E}\|u(\tau_1 \wedge T)\|^p < \infty$ . Setting  $u^1(x) := u(\tau_1 \wedge T, x)$ , we next consider Eq. (2.2) on  $t \in [\tau_1 \wedge T, \tau_2 \wedge T]$  which becomes

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = Au(t,x) + \alpha(r(\tau_1))u(t,x) + \beta(r(\tau_1))u(t,x)\dot{B}(t), & x \in \mathcal{O}; \\ u(t,x) = 0, & x \in \partial\mathcal{O}; \\ u(\tau_1 \wedge T, x) = u^1(x), & x \in \mathcal{O}. \end{cases}$$

Again by [7, Proposition 1.], Eq. (2.2) has a unique solution  $u \in C([\tau_1 \wedge T, \tau_2 \wedge T]; L^2(\mathcal{O}))$  and  $u(t) \in \mathcal{D}(A)$  for  $t \in [\tau_1 \wedge T, \tau_2 \wedge T]$  a.s. while  $\mathbb{E}\|u(\tau_2 \wedge T)\|^p < \infty$ . Repeating this procedure, we see that Eq. (2.2) has a unique solution  $u \in C([0, T]; L^2(\mathcal{O}))$ , and, for any  $t \in [0, T]$ ,  $u(t) \in \mathcal{D}(A)$  a.s. while  $\mathbb{E}\|u(T)\|^p < \infty$ . Since  $T$  is arbitrary, the proof is complete.  $\square$

We conclude this section by defining the sample Lyapunov exponent and the  $p$ th moment Lyapunov exponent for the solution of Eq. (2.2)

**Definition 2.2.** The limit

$$\lambda(u^0) := \limsup_{t \rightarrow \infty} \frac{1}{t} \log(\|u(t)\|)$$

is called the sample Lyapunov exponent, while the limit

$$\gamma_p(u^0) := \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}(\|u(t)\|^p)$$

is called the  $p$ th moment Lyapunov exponent.

If  $\lambda(u^0) < 0$  a.s. for any initial data  $u^0$  (obeying the conditions imposed above of course), then any solution of Eq. (2.2) will converge to zero exponentially with probability one. In this case, we say that the solution of Eq. (2.2) is almost surely exponentially stable. Similarly, if  $\gamma_p(u^0) < 0$  for any  $u^0$ , then the solution of Eq. (2.2) is exponentially stable in the  $p$ th moment.

### 3 Explicit Solutions of Hybrid Stochastic Heat Equations

Let us first discuss hybrid stochastic heat equation with external forces and Dirichlet boundary conditions

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = Av(t,x) + \alpha(r(t))v(t,x), & x \in \mathcal{O}, t > 0; \\ v(t,x) = 0, & x \in \partial\mathcal{O}, t > 0; \\ v(0,x) = u^0(x), & x \in \mathcal{O}. \end{cases} \quad (3.1)$$

In the sequel, we shall denote  $u_n^0 := \langle u^0, e_n \rangle$  for  $n \geq 1$ . In general,  $u_n^0$  is a random variable but it becomes a (non-random) number if  $u^0$  is deterministic. When  $u^0$  is deterministic and  $u^0 \neq 0$ , we set  $n_0 := \inf\{n : u_n^0 \neq 0\}$ .

**Theorem 3.1.** The unique solution of Eq. (3.1) has the explicit form

$$v(t,x) = \sum_{n=1}^{\infty} \exp\left\{-\lambda_n t + \int_0^t \alpha(r(s))ds\right\} u_n^0 e_n(x), \quad t \geq 0, x \in \mathcal{O}. \quad (3.2)$$

*Proof.* Clearly, by Theorem 2.1 we can conclude that (3.1) has a unique solution  $v = \{v(t)\}_{t \geq 0}$ . Let  $\{\tau_k\}_{k \geq 0}$  be the same as defined in the proof of Theorem 2.1. For  $t \in [0, \tau_1)$  Eq. (3.1) can be written as

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = Av(t,x) + \alpha(r(0))v(t,x), & x \in \mathcal{O}; \\ v(t,x) = 0, & x \in \partial\mathcal{O}; \\ v(0,x) = u^0(x), & x \in \mathcal{O}. \end{cases}$$

It is well known (see, e.g., [7, 14]) that this heat equation has explicit solution on  $t \in [0, \tau_1]$

$$v(t,x) = \sum_{n=1}^{\infty} \exp\{(-\lambda_n + \alpha(r(0))t)\} u_n^0 e_n(x). \quad (3.3)$$

Set  $v^1(x) := v(\tau_1, x)$  and consider Eq. (3.1) for  $t \in [\tau_1, \tau_2)$ , namely

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = Av(t,x) + \alpha(r(\tau_1))v(t,x), & x \in \mathcal{O}; \\ v(t,x) = 0, & x \in \partial\mathcal{O}; \\ v(\tau_1, x) = v^1(x), & x \in \mathcal{O}. \end{cases}$$

Again this heat equation has explicit solution on  $t \in [\tau_1, \tau_2)$

$$v(t,x) = \sum_{n=1}^{\infty} \exp\{(-\lambda_n + \alpha(r(\tau_1))(t - \tau_1))\} \langle v^1, e_n \rangle e_n(x). \quad (3.4)$$

Letting  $t = \tau_1$  in (3.3) and substituting  $v^1$  into (3.4), we derive that

$$\begin{aligned}
v(t, x) &= \sum_{n=1}^{\infty} \exp\{(-\lambda_n + \alpha(r(\tau_1))(t - \tau_1))\} \\
&\quad \times \left\langle \sum_{j=1}^{\infty} \exp\{(-\lambda_j + \alpha(r(0))\tau_1)\} u_j^0 e_j, e_n \right\rangle e_n(x) \\
&= \sum_{n=1}^{\infty} \exp\{(-\lambda_n + \alpha(r(\tau_1))(t - \tau_1))\} \exp\{(-\lambda_n + \alpha(r(0))\tau_1)\} u_n^0 e_n(x) \\
&= \sum_{n=1}^{\infty} \exp\{-\lambda_n t + \alpha(r(0))\tau_1 + \alpha(r(\tau_1))(t - \tau_1)\} u_n^0 e_n(x) \\
&= \sum_{n=1}^{\infty} \exp\left\{-\lambda_n t + \int_0^t \alpha(r(s)) ds\right\} u_n^0 e_n(x).
\end{aligned}$$

Repeating this procedure, we obtain the required assertion (3.2). The proof is complete.  $\square$

**Theorem 3.2.** The solution of Eq. (3.1) has the following properties:

(1) If  $u^0$  is deterministic and  $u^0 \neq 0$ , then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\|v(t)\|) = -\left(\lambda_{n_0} - \sum_{j=1}^N \pi_j \alpha_j\right) \quad \text{a.s.} \quad (3.5)$$

In particular, the solution of Eq. (3.1) with initial data  $u^0$  will converge exponentially to zero with probability one if and only if

$$\lambda_{n_0} - \sum_{j=1}^N \pi_j \alpha_j > 0.$$

(2) For any initial data  $u^0$ ,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\|v(t)\|) \leq -\left(\lambda_1 - \sum_{j=1}^N \pi_j \alpha_j\right) \quad \text{a.s.} \quad (3.6)$$

In particular, the solution of Eq. (3.1) is almost surely exponentially stable if

$$\lambda_1 - \sum_{j=1}^N \pi_j \alpha_j > 0.$$

**Remark 3.1.** In case (1) above, since  $\lambda_{n_0}$  depends on the initial data  $u^0$ , we only know the asymptotic behavior of the solution with initial data  $u^0$ . However, in case (2), the estimate on the sample Lyapunov exponent holds for any initial data whence the solution of Eq. (3.1) is almost surely exponentially stable if the right-hand-side term of (3.6) is negative.

*Proof.* (1) For any  $t > 0$ , it follows from (3.2) that

$$\begin{aligned} \frac{1}{t} \log(\|v(t)\|) &= \frac{1}{t} \log \left( \sum_{n=n_0}^{\infty} \left| \exp \left\{ -\lambda_{n_0} t + \int_0^t \alpha(r(s)) ds \right\} u_n^0 \right|^2 \right)^{1/2} \\ &\leq \frac{1}{t} \left( -\lambda_{n_0} t + \int_0^t \alpha(r(s)) ds + \log(\|u^0\|) \right), \end{aligned} \quad (3.7)$$

since  $\lambda_n, n \geq 1$ , are increasing. On the other hand, we can also derive that

$$\begin{aligned} \frac{1}{t} \log(\|v(t)\|) &= \frac{1}{t} \log \left( \sum_{n=n_0}^{\infty} \left| \exp \left\{ -\lambda_{n_0} t + \int_0^t \alpha(r(s)) ds \right\} u_n^0 \right|^2 \right)^{1/2} \\ &\geq \frac{1}{t} \left( -\lambda_{n_0} t + \int_0^t \alpha(r(s)) ds + \log(|u_{n_0}^0|) \right). \end{aligned} \quad (3.8)$$

Letting  $t \rightarrow \infty$  in both (3.7) and (3.8) and taking into account the ergodic property of Markov chains, we obtain the first assertion (3.5).

(2) For any initial data  $u^0$ , we observe from (3.7) that

$$\frac{1}{t} \log(\|v(t)\|) \leq \frac{1}{t} \left( -\lambda_1 t + \int_0^t \alpha(r(s)) ds + \log(\|u^0\|) \right).$$

Therefore, the ergodic property of Markov chains yields the other assertion (3.6).  $\square$

**Example 3.1.** Let  $r(t), t \geq 0$ , be a right-continuous Markov chain taking values in  $\mathbb{S} = \{1, 2, 3\}$  with the generator

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

It is straightforward to see that the unique stationary probability distribution of the Markov chain  $r(t)$  is

$$\pi = \left( \frac{7}{15}, \frac{1}{5}, \frac{1}{3} \right).$$

Let us consider hybrid heat equation

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = Av(t,x) + \alpha(r(t))v(t,x), & t > 0, x \in (0, \pi); \\ v(t,0) = v(t,\pi) = 0, & t > 0; \quad v(0,x) = u^0(x), x \in (0, \pi), \end{cases} \quad (3.9)$$

where  $u^0(x) = \sqrt{2/\pi} \sin x$  for  $x \in (0, \pi)$ . Let  $\alpha(1) = 0.1, \alpha(2) = 1.5, \alpha(3) = 0.2$ . Moreover, we recall that  $e_n(x) = \sqrt{2/\pi} \sin nx$ ,  $n = 1, 2, 3, \dots$ , are eigenfunctions of  $-A$ , with positive and increasing eigenvalues  $\lambda_n = n^2$ , and form an orthonormal basis of  $L^2(\mathcal{O})$ .

To see what this example shows us, we regard Eq. (3.9) as the result of the following three equations

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = Av(t,x) + \frac{1}{10}v(t,x), & t > 0, x \in (0, \pi); \\ v(t,0) = v(t,\pi) = 0, & t > 0; \quad v(0,x) = u^0(x), x \in (0, \pi), \end{cases} \quad (3.10)$$

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = Av(t,x) + \frac{3}{2}v(t,x), & t > 0, x \in (0, \pi); \\ v(t,0) = v(t,\pi) = 0, & t > 0; \quad v(0,x) = u^0(x), x \in (0, \pi), \end{cases} \quad (3.11)$$

and

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} = Av(t,x) + \frac{1}{5}v(t,x), & t > 0, x \in (0, \pi); \\ v(t,0) = v(t,\pi) = 0, & t > 0; \quad v(0,x) = u^0(x), x \in (0, \pi), \end{cases} \quad (3.12)$$

switching from one to another according to the movement of the Markov chain  $r(t)$ . We observe that the solutions of Eq. (3.10) and Eq. (3.12) will converge exponentially to zero since the Lyapunov exponents are  $-0.9$  and  $-0.8$ , respectively, while the solution of Eq. (3.11) will explode exponentially since the Lyapunov exponent is  $0.5$ . However, as the result of Markovian switching, the overall behaviour, i.e. the solution of Eq. (3.9) will converge exponentially to zero, since, by Theorem 3.2, the solution of Eq. (3.9) obeys

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\|v(t)\|) = -\frac{8}{15}.$$

To get the explicit expression for the solution of Eq. (2.2), we need one more result.

**Lemma 3.1.** [10, Theorem 5.22, p182] For any positive integer  $n$  and  $t \geq 0$ , hybrid SDE

$$dz_n(t) = (-\lambda_n + \alpha(r(t)))z_n(t)dt + \beta(r(t))z_n(t)dB(t) \quad (3.13)$$

with initial condition  $z_n(0) = u_n^0$  has the explicit solution

$$z_n(t) = u_n^0 \exp \left\{ -\lambda_n t + \int_0^t \left[ \alpha(r(s)) - \frac{1}{2}\beta^2(r(s)) \right] ds + \int_0^t \beta(r(s))dB(s) \right\}. \quad (3.14)$$

We can now state the explicit formula for the solution of Eq. (2.2).

**Theorem 3.3.** The unique solution of Eq. (2.2) has the explicit form

$$u(t,x) = \sum_{n=1}^{\infty} z_n(t)e_n(x) = v(t,x) \exp \left\{ -\frac{1}{2} \int_0^t \beta^2(r(s))ds + \int_0^t \beta(r(s))dB(s) \right\}, \quad (3.15)$$

where  $v(t,x)$  is the solution to (3.1) given explicitly by (3.2).

*Proof.* Set

$$\bar{u}(t,x) := \sum_{n=1}^{\infty} z_n(t)e_n(x). \quad (3.16)$$

and note from (2.1) that

$$A\bar{u}(t,x) = -\sum_{n=1}^{\infty} \lambda_n z_n(t)e_n(x).$$

Substituting (3.13) into (3.16), we then compute

$$\begin{aligned}
\bar{u}(t, x) &= \sum_{n=1}^{\infty} \left\{ u_n^0 e_n(x) + \int_0^t (-\lambda_n + \alpha(r(s))) e_n(x) z_n(s) ds \right. \\
&\quad \left. + \int_0^t \beta(r(s)) e_n(x) z_n(s) dB(s) \right\} \\
&= \sum_{n=1}^{\infty} u_n^0 e_n(x) - \int_0^t \sum_{n=1}^{\infty} \lambda_n e_n(x) z_n(s) ds + \int_0^t \alpha(r(s)) \sum_{n=1}^{\infty} e_n(x) z_n(s) ds \\
&\quad + \int_0^t \beta(r(s)) \sum_{n=1}^{\infty} e_n(x) z_n(s) dB(s) \\
&= u^0(x) + \int_0^t A\bar{u}(s, x) ds + \int_0^t \alpha(r(s)) \bar{u}(s, x) ds + \int_0^t \beta(r(s)) \bar{u}(s, x) dB(s).
\end{aligned}$$

Consequently,  $\bar{u}$  is a solution of Eq. (2.2). While, by the uniqueness of solution of Eq. (2.2), we must have

$$u(t, x) = \sum_{n=1}^{\infty} z_n(t) e_n(x).$$

Taking into account (3.14), we obtain the desired explicit form (3.15).  $\square$

## 4 Sample Lyapunov Exponents

Making use of the explicit solution of Eq. (2.2), we can now discuss its Lyapunov exponent. Let us now begin with the sample Lyapunov exponent.

**Theorem 4.1.** The solution  $\{u(t)\}_{t \geq 0}$  of Eq. (2.2) has the following properties:

(1) If  $u^0$  is deterministic and  $u^0 \neq 0$ , then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\|u(t)\|) = -\left( \lambda_{n_0} - \sum_{j=1}^N \pi_j \left( \alpha_j - \frac{1}{2} \beta_j^2 \right) \right) \quad \text{a.s.}$$

In particular, the solution of Eq. (2.2) with initial data  $u^0$  will converge exponentially to zero with probability one if and only if

$$\lambda_{n_0} > \sum_{j=1}^N \pi_j \left( \alpha_j - \frac{1}{2} \beta_j^2 \right).$$

(2) For any initial data  $u^0$ ,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\|u(t)\|) \leq -\left( \lambda_1 - \sum_{j=1}^N \pi_j \left( \alpha_j - \frac{1}{2} \beta_j^2 \right) \right) \quad \text{a.s.}$$

In particular, the solution of Eq. (2.2) is almost surely exponentially stable if

$$\lambda_1 > \sum_{j=1}^N \pi_j \left( \alpha_j - \frac{1}{2} \beta_j^2 \right).$$

*Proof.* It follows from (3.15) that

$$\frac{1}{t} \log(\|u(t)\|) = \frac{1}{t} \left\{ \log(\|v(t)\|) - \frac{1}{2} \int_0^t \beta^2(r(s)) ds + \int_0^t \beta(r(s)) dB(s) \right\}.$$

By the strong law of large numbers, e.g., [10, Theorem 1.6, p16]

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \beta(r(s)) dB(s) = 0,$$

while by the ergodic property of Markov chains,

$$-\frac{1}{2} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \beta^2(r(s)) ds = -\frac{1}{2} \sum_{j=1}^N \pi_j \beta_j^2.$$

The desired assertions then follow from Theorem 3.2.  $\square$

In the sequel we set an example to demonstrate our theories and reveal that an unstable stochastic system can be stabilized by Markovian switching.

**Example 4.1.** Let  $B(t)$  be a scalar Brownian motion and  $r(t)$  a right-continuous Markov chain taking values in  $\mathbb{S} = \{1, 2\}$  with the generator  $\Gamma = (\gamma_{ij})_{2 \times 2}$ :

$$-\gamma_{11} = \gamma_{12} > 0, \quad -\gamma_{22} = \gamma_{21} > 0.$$

It is straightforward to show that the unique stationary probability distribution of the Markov chain  $r(t)$  is

$$\pi = (\pi_1, \pi_2) = \left( \frac{\gamma_{22}}{\gamma_{11} + \gamma_{22}}, \frac{\gamma_{11}}{\gamma_{11} + \gamma_{22}} \right).$$

Consider stochastic heat equation with Markovian switching:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} &= Au(t,x) + \alpha(r(t))u(t,x) + \beta(r(t))u(t,x)\dot{B}(t), \quad t > 0, x \in (0, \pi); \\ u(t,0) &= u(t,\pi) = 0, t > 0; \quad u(0,x) = \sqrt{2/\pi} \sin x, \quad x \in (0, \pi), \end{cases} \quad (4.1)$$

where  $\alpha(1) = a, \alpha(2) = b, \beta(1) = c, \beta(2) = d$  with  $a, b, c, d \in \mathbb{R}$ . Moreover, we recall that  $e_n(x) = \sqrt{2/\pi} \sin nx, n = 1, 2, 3, \dots$ , are eigenfunctions of  $-A$ , with positive and increasing eigenvalues  $\lambda_n = n^2$ , and  $e_n \in \mathcal{D}(A)$ . Note that initial condition  $u(0,x) = \sqrt{2/\pi} \sin x$  is deterministic and  $u_1^0 = \langle \sqrt{2/\pi} \sin x, \sqrt{2/\pi} \sin x \rangle = 1$ , which implies  $n_0 = 1$ . By Theorem 4.1, the unique solution  $u(t,x)$  converges exponentially to zero if and only if

$$1 - (\pi_1 \alpha_1 + \pi_2 \alpha_2) + \frac{1}{2} (\pi_1 \beta_1^2 + \pi_2 \beta_2^2) > 0.$$

That is,

$$\frac{a\gamma_{22} + b\gamma_{11}}{\gamma_{11} + \gamma_{22}} < 1 + \frac{c^2\gamma_{22} + d^2\gamma_{11}}{2(\gamma_{11} + \gamma_{22})}. \quad (4.2)$$

As a special case, let us set  $a = 2, b = 1, c = 1, d = 1$  and

$$-\gamma_{11} = \gamma_{12} = 4, \quad -\gamma_{22} = \gamma_{21} = \gamma > 0.$$

Then the stochastic system (4.1) can be regarded as the result of two equations

$$\frac{\partial u(t, x)}{\partial t} = Au(t, x) + 2u(t, x) + u(t, x)\dot{B}(t), \quad t \geq 0, x \in (0, \pi) \quad (4.3)$$

and

$$\frac{\partial u(t, x)}{\partial t} = Au(t, x) + u(t, x) + u(t, x)\dot{B}(t), \quad t \geq 0, x \in (0, \pi), \quad (4.4)$$

with the Dirichlet boundary condition and initial condition, switching from one to the other according to the law of Markov chain. By Theorem 4.1, the solution to Eq.(4.3) has the property

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\|u(t)\|) = \frac{1}{2},$$

and the solution to Eq.(4.4) has the property

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\|u(t)\|) = -\frac{1}{2}.$$

That is, the solution of stochastic system (4.3) explodes exponentially, and the solution of stochastic system (4.4) converges exponentially to zero. However, by (4.2), the unique solution  $u(t, x)$  of Eq.(4.1) converges exponentially to zero if and only if  $0 < \gamma < 4$ . This shows once again that Markovian switching plays a key role in the stability of hybrid stochastic heat equations.

## 5 $p$ th Moment Lyapunov Exponents

Let us now turn to the discussion of the  $p$ th moment Lyapunov exponent. The  $p$ th moment exponential stability of stochastic heat equations without Markov switching has been discussed, e.g., in [3, 7, 9, 17]. However, due to Markov switching, it is much more complicated to study the moment Lyapunov exponent of hybrid stochastic heat equations. To cope with the difficulties arising from Markov switching, large deviation techniques will be used in this section.

In what follows, we recall some details with respect to large deviation, see, e.g., Donsker and Varadhan [4]. Let  $(X, \mathcal{B}, \|\cdot\|_X)$  be a Polish space, and  $p(t, x, dy)$  the transition probability of an  $X$ -valued Markov process  $Z(t)$  with  $Z(0) = x$ . Let  $P_t$  be a strongly continuous Markovian semigroup associated with  $Z(t)$ , and define  $P_t f(x) := \int_X f(y)p(t, x, dy)$  for  $f \in C(X)$ , the space of continuous functions on  $X$ . Let  $L$  be the infinitesimal generator of

the semigroup  $P_t$  with domain  $\mathcal{D}(L)$ . Let  $\mathcal{M}$  be the space of all probability measures on  $X$ . For any  $\mu \in \mathcal{M}$ , define the rate function by

$$I(\mu) := - \inf_{u>0, u \in \mathcal{D}} \int_X \left( \frac{Lu}{u}(x) \right) \mu(dx). \quad (5.1)$$

Let  $\Omega_x$  be the space of  $X$ -valued càdlàg functions  $Z(t), 0 \leq t < \infty$ , with  $Z(0) = x$ . For each  $t > 0, \omega \in \Omega_x$ , and Borel set  $A \subset \mathcal{B}$ , the occupation time measure is defined by

$$L_t(\omega, A) := \frac{1}{t} \int_0^t I_A(Z(s)) ds. \quad (5.2)$$

In other words,  $L(\omega, A)$  is the proportion of time up to  $t$  that a particular sample  $\omega = Z(\cdot)$  spends in the set  $A$ . Note that for each  $t > 0$  and each  $\omega$ ,  $L(\omega, \cdot)$  is a probability measure on  $X$ .

**Remark 5.1.** For the Markov chain  $r(t)$ , we remark that

$$X = \mathbb{S} \text{ and } L_t(\omega, i) = \frac{1}{t} \int_0^t I_{\{i\}}(r(s)) ds, \quad i \in \mathbb{S}.$$

**Remark 5.2.** For a continuous time Markov chain with finite state space and  $Q$ -matrices  $(q_{ij})_{N \times N}$ , it is easy to see that the rate function has the following expression (see[5])

$$I(\mu) = - \inf_{u_i > 0} \sum_{i,j=1}^N \frac{\mu_i q_{ij} u_j}{u_i}, \quad (5.3)$$

where  $\mu$  is the probability measure on the state space  $\{1, 2, \dots, N\}$ .

**Lemma 5.1.** [4, Theorem 4] If  $\Phi$  is a real-valued weakly continuous functional on  $\mathcal{M}$ , then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(E\{\exp\{t\Phi(L_t(\omega, \cdot))\}\}) = \sup_{\mu \in \mathcal{M}} [\Phi(\mu) - I(\mu)]$$

where  $L_t(\omega, \cdot)$  and  $I(\mu)$  is defined by (5.2) and (5.3), respectively.

**Theorem 5.1.** Let  $p > 0$ . The solution to Eq. (2.2) has the following properties:

(1) If  $u^0$  is deterministic and  $u^0 \neq 0$ , then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}(\|u(t)\|^p)) = -p\lambda_{n_0} + \sup_{\mu} \left\{ \sum_{i=1}^N g(i)\mu(i) - I(\mu) \right\}, \quad (5.4)$$

where the supremum is taken over probability measures  $\mu$  on  $\mathbb{S}$ ,  $g(i) = p\alpha_i + \frac{p(p-1)}{2}\beta_i^2$ ,  $i = 1, 2, \dots, N$  and  $I(\mu)$  is defined by (5.3). In particular, the  $p$ th moment of the solution of Eq. (2.2) with initial data  $u^0$  will converge exponentially with probability one to zero if and only if

$$p\lambda_{n_0} - \sup \left\{ \sum_{i=1}^N g(i)\mu(i) - I(\mu) \right\} > 0.$$

(2) For any  $u^0$  (which is independent of  $r(\cdot)$  and  $B(\cdot)$  as assumed throughout this paper),

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}(\|u(t)\|^p)) \leq -p\lambda_1 + \sup \left\{ \sum_{i=1}^N g(i)\mu(i) - I(\mu) \right\}. \quad (5.5)$$

In particular, the solution of Eq. (2.2) is exponentially stable in the  $p$ th moment if

$$-p\lambda_1 + \sup \left\{ \sum_{i=1}^N g(i)\mu(i) - I(\mu) \right\} < 0.$$

*Proof.* Recall that the solution of Eq. (2.2) has the explicit form

$$u(t, x) = v(t, x) \exp \left\{ -\frac{1}{2} \int_0^t \beta^2(r(s)) ds + \int_0^t \beta(r(s)) dB(s) \right\}, \quad (5.6)$$

where

$$v(t, x) = \sum_{n=1}^{\infty} \exp \left\{ -\lambda_n t + \int_0^t \alpha(r(s)) ds \right\} u_n^0 e_n(x).$$

It is well known that almost every sample path of the Markov chain  $r(\cdot)$  is a right continuous step function with a finite number of sample jumps in any finite subinterval of  $\mathbb{R}_+ := [0, \infty)$ . Hence there is a sequence of finite stopping times  $0 = \tau_0 < \tau_1 < \dots < \tau_k \uparrow \infty$  such that  $r(t) = \sum_{k=0}^{\infty} r(\tau_k) I_{[\tau_k, \tau_{k+1})}(t)$ ,  $t \geq 0$ . For any integer  $k > 0$  and  $t \geq 0$ , we compute

$$\begin{aligned} \|u(t \wedge \tau_k)\|^p &= \|v(t \wedge \tau_k)\|^p \exp \left\{ -\frac{p}{2} \int_0^{t \wedge \tau_k} \beta^2(r(s)) ds + \int_0^{t \wedge \tau_k} p\beta(r(s)) dB(s) \right\} \\ &= \|v(t \wedge \tau_k)\|^p \xi(t \wedge \tau_k) \exp \left\{ -\frac{p^2}{2} \int_0^{t \wedge \tau_k} \beta^2(r(s)) ds + \int_0^{t \wedge \tau_k} p\beta(r(s)) dB(s) \right\} \\ &= \|v(t \wedge \tau_k)\|^p \xi(t \wedge \tau_k) \prod_{j=0}^{k-1} \zeta_j(t), \end{aligned}$$

where

$$\xi(t) = \exp \left\{ \frac{p(p-1)}{2} \int_0^t \beta^2(r(s)) ds \right\}$$

and

$$\zeta_j(t) = \exp \left\{ -\frac{1}{2} p^2 \beta^2(r(t \wedge \tau_j))(t \wedge \tau_{j+1} - t \wedge \tau_j) + p\beta(r(t \wedge \tau_j))[B(t \wedge \tau_{j+1}) - B(t \wedge \tau_j)] \right\}.$$

Letting  $\mathcal{G}_t = \sigma(\{r(u)\}_{u \geq 0}, \{B(s)\}_{0 \leq s \leq t})$ , by properties of conditional expectations, we have

$$\begin{aligned} \mathbb{E}(\|u(t \wedge \tau_k)\|^p) &= \mathbb{E} \left( \|v(t \wedge \tau_k)\|^p \xi(t \wedge \tau_k) \prod_{j=0}^{k-1} \zeta_j(t) \right) \\ &= \mathbb{E} \left\{ \mathbb{E} \left( \|v(t \wedge \tau_k)\|^p \xi(t \wedge \tau_k) \prod_{j=0}^{k-1} \zeta_j(t) \middle| \mathcal{G}_{t \wedge \tau_{k-1}} \right) \right\} \\ &= \mathbb{E} \left\{ \left[ \|v(t \wedge \tau_k)\|^p \xi(t \wedge \tau_k) \prod_{j=0}^{k-2} \zeta_j(t) \right] \mathbb{E} \left( \zeta_{k-1}(t) \middle| \mathcal{G}_{t \wedge \tau_{k-1}} \right) \right\}. \end{aligned}$$

Clearly,

$$\begin{aligned}\mathbb{E}(\zeta_{k-1}(t)|\mathcal{G}_{t\wedge\tau_{k-1}}) &= \mathbb{E}\left(\sum_{i\in\mathbb{S}} I_{\{r(t\wedge\tau_{k-1})=i\}}\zeta_{k-1}^i(t)\middle|\mathcal{G}_{t\wedge\tau_{k-1}}\right) \\ &= \sum_{i\in\mathbb{S}} I_{\{r(t\wedge\tau_{k-1})=i\}}\mathbb{E}(\zeta_{k-1}^i(t)|\mathcal{G}_{t\wedge\tau_{k-1}}),\end{aligned}$$

where

$$\zeta_{k-1}^i(t) = \exp\left\{-\frac{1}{2}p^2\beta_i^2(t\wedge\tau_k - t\wedge\tau_{k-1}) + p\beta_i[B(t\wedge\tau_k) - B(t\wedge\tau_{k-1})]\right\}, \quad i \in \mathbb{S}.$$

Noting that  $t\wedge\tau_k - t\wedge\tau_{k-1}$  is  $\mathcal{G}_{t\wedge\tau_{k-1}}$ -measurable and  $B(t_2) - B(t_1)$  is independent of  $\mathcal{G}_{t\wedge\tau_{k-1}}$  whenever  $t_2 \geq t_1 \geq t\wedge\tau_{k-1}$ , by [10, Lemma 3.2, p104], we can derive that

$$\begin{aligned}\mathbb{E}(\zeta_{k-1}^i(t)|\mathcal{G}_{t\wedge\tau_{k-1}}) &= \exp\left\{-\frac{1}{2}p^2\beta_i^2(t\wedge\tau_k - t\wedge\tau_{k-1})\right\}\mathbb{E}\exp(p\beta_i[B(t\wedge\tau_k) - B(t\wedge\tau_{k-1})])|\mathcal{G}_{t\wedge\tau_{k-1}} \\ &= \exp\left\{-\frac{1}{2}p^2\beta_i^2(t\wedge\tau_k - t\wedge\tau_{k-1})\right\}\mathbb{E}\exp(p\beta_i[B(t_2) - B(t_1)])|_{t_2=t\wedge\tau_k, t_1=t\wedge\tau_{k-1}} \\ &= \exp\left\{-\frac{1}{2}p^2\beta_i^2(t\wedge\tau_k - t\wedge\tau_{k-1})\right\}\exp\left\{\frac{1}{2}p^2\beta_i^2(t\wedge\tau_k - t\wedge\tau_{k-1})\right\} \\ &= 1.\end{aligned}$$

Hence,

$$\mathbb{E}(\|u(t\wedge\tau_k)\|^p) = \mathbb{E}\left(\|v(t\wedge\tau_k)\|^p\xi(t\wedge\tau_k)\prod_{j=0}^{k-2}\zeta_j(t)\right).$$

Repeating this procedure, we obtain that

$$\mathbb{E}(\|u(t\wedge\tau_k)\|^p) = \mathbb{E}(\|v(t\wedge\tau_k)\|^p\xi(t\wedge\tau_k)).$$

Letting  $k \rightarrow \infty$  gives

$$\mathbb{E}(\|u(t)\|^p) = \mathbb{E}(\|v(t)\|^p\xi(t)). \tag{5.7}$$

(1) The initial data  $u^0$  is deterministic and  $u^0 \neq 0$ . Using (3.2), we compute

$$\begin{aligned}\mathbb{E}(\|v(t)\|^p\xi(t)) &= \mathbb{E}\left(\left\|\sum_{n=1}^{\infty}\exp\left\{-\lambda_n t + \int_0^t \alpha(r(s))ds\right\}u_n^0 e_n\right\|^p\xi(t)\right) \\ &= \mathbb{E}\left(\left\|\sum_{n=n_0}^{\infty}\exp\left\{-\lambda_n t + \int_0^t \alpha(r(s))ds\right\}u_n^0 e_n\right\|^p\xi(t)\right) \\ &= \mathbb{E}\left(\left(\sum_{n=n_0}^{\infty}\left|\exp\left\{-\lambda_n t + \int_0^t \alpha(r(s))ds\right\}u_n^0\right|^2\right)^{\frac{p}{2}}\xi(t)\right) \\ &\leq \|u^0\|^p\mathbb{E}\left(\exp\left\{-p\lambda_{n_0}t + \int_0^t g(r(s))ds\right\}\right),\end{aligned} \tag{5.8}$$

and

$$\mathbb{E}\left(\|v(t)\|^p \xi(t)\right) \geq |u_n^0|^p \mathbb{E}\left(\exp\left\{-p\lambda_{n_0}t + \int_0^t g(r(s))ds\right\}\right).$$

Therefore, by (5.7)

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}(\|u(t)\|^p)) = -p\lambda_{n_0} + \lim_{t \rightarrow \infty} \frac{1}{t} \log\left(\mathbb{E}\left(\exp\left\{\int_0^t g(r(s))ds\right\}\right)\right). \quad (5.9)$$

Furthermore, note that

$$\begin{aligned} \int_0^t g(r(s))ds &= \int_0^t g(r(s))1_{\{r(s) \in \mathbb{S}\}}ds \\ &= t \sum_{i=1}^N g(i) \frac{1}{t} \int_0^t 1_{\{i\}}(r(s))ds \\ &= t \sum_{i=1}^N g(i) L_t(\omega, i). \end{aligned}$$

Then it follows from Lemma 5.1 that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log\left(\mathbb{E}\left(\exp\left\{\int_0^t g(r(s))ds\right\}\right)\right) = \sup\left\{\sum_{i=1}^N g(i)\mu(i) - I(\mu)\right\},$$

where the supremum is taken over probability measures  $\mu$  on  $\mathbb{S}$  and  $I(\mu)$  is defined by (5.1).

(2) For any  $u^0$  which is independent of  $r(\cdot)$  and  $B(\cdot)$ , we observe from (5.8) that

$$\begin{aligned} \mathbb{E}(\|v(t)\|^p \xi(t)) &\leq \mathbb{E}\left(\exp\left\{-p\lambda_1 t + \int_0^t g(r(s))ds\right\} \|u^0\|^p\right) \\ &= \mathbb{E}\left(\exp\left\{-p\lambda_1 t + \int_0^t g(r(s))ds\right\}\right) \mathbb{E}(\|u^0\|^p). \end{aligned}$$

Hence the desired assertion (5.5) follows from Lemma 5.1.  $\square$

**Remark 5.3.** Since the Markov chain  $r(t)$  with  $Q$ -matrix  $(\gamma_{ij})_{N \times N}$ ,  $i, j \in \mathbb{S}$  is assumed to be irreducible throughout the paper, there exists a unique invariant measure  $\pi = (\pi_1, \pi_2, \dots, \pi_N)$  such that

$$\sum_{j=1}^N \pi_j \gamma_{ij} = 0 \text{ and } \sum_{j=1}^N \pi_j = 1.$$

Letting  $\mu$  be the invariant measure  $\pi$  and  $u_i, i \in \mathbb{S}$  be constants, together with the nonnegative property of rate function, it follows that  $I(\pi)=0$ . Therefore,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}(\|u(t)\|^p)) \geq -p\lambda_{n_0} + \sum_{i=1}^N g(i)\pi(i)$$

whenever  $u_0$  is deterministic.

**Remark 5.4.** Let us reconsider Example 4.1, where  $r(t)$  is a right-continuous Markov chain taking values in state space  $\mathbb{S} = \{1, 2\}$  with the generator  $\Gamma = (\gamma_{ij})_{2 \times 2}$ :

$$-\gamma_{11} = \gamma_{12} = 1, \quad -\gamma_{22} = \gamma_{21} = q > 0.$$

By computation, the invariant measure of the Markov chain is  $\pi = (\pi_1, \pi_2) = (q/(q+1), 1/(q+1))$ . In what follows, we assume that, for some probability measure  $\mu_0 = (\theta, 1-\theta)$ ,  $0 < \theta < 1$ , on the state space  $\mathbb{S}$ ,

$$a\theta + b(1-\theta) - I(\mu_0) > a\pi_1 + b\pi_2,$$

where  $g(1) = a > 0$ ,  $g(2) = b > 0$  and  $g(i)$ ,  $i = 1, 2$  is defined in Theorem 5.1, that is,

$$\begin{aligned} (a-b)\theta + b &> a\pi_1 + b\pi_2 + \theta + (1-\theta)q - \inf_{u_1, u_2 > 0} \left[ \theta \frac{u_2}{u_1} + (1-\theta)q \frac{u_1}{u_2} \right] \\ &= a\pi_1 + b\pi_2 + \theta + (1-\theta)q - 2\sqrt{\theta(1-\theta)q}. \end{aligned}$$

Recalling  $\pi_1 = q/(q+1)$ ,  $\pi_2 = 1/(q+1)$  and letting  $\mu = \frac{1}{2}$ , then the previous inequality can be rewritten as

$$a + b > (q^2 - 2q^{\frac{3}{2}} + 2q - 2q^{\frac{1}{2}} + 1)/(3q + 1). \quad (5.10)$$

Since the right hand side of the inequality above tends to zero whenever  $q \rightarrow 1$ , then we can choose suitable  $q$  to satisfy (5.10) and further get

$$-p\lambda_{n_0} + \sup_{\mu} \left\{ \sum_{i=1}^2 g(i)\mu(i) - I(\mu) \right\} > -p\lambda_{n_0} + \sum_{i=1}^2 g(i)\pi_i - I(\pi).$$

Since  $I(\pi) = 0$  and  $\lambda_{n_0} = 1$ , the solution of Eq. (4.1) will explode exponentially with probability one to  $\infty$  provided that  $-p + \sum_{i=1}^2 g(i)\pi_i > 0$ .

So far we have assumed the Brownian motion is a scalar one. However, our theory can easily be generalized to cope with a hybrid stochastic heat equation driven by a multi-dimensional Brownian motion of the form as Xie [17].

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = Au(t,x) + \alpha(r(t))u(t,x) + \sum_{i=1}^m \beta_i(r(t))u(t,x)\dot{B}_i(t), & t \geq 0, x \in \mathcal{O}; \\ u(t,x) = 0, & t \geq 0, x \in \partial\mathcal{O}; \\ u(0) = u^0, \end{cases} \quad (5.11)$$

where  $(B_1(t), \dots, B_m(t))$  is an  $m$ -dimensional Brownian motion,  $\alpha, \beta_i$  are mappings from  $\mathbb{S} \rightarrow \mathbb{R}$  and we write  $\beta_{ij} = \beta_i(j)$  for any  $j \in \mathbb{S}$ . Following the similar arguments to that of Theorems 4.1 and 5.1, we can derive the following theorems.

**Theorem 5.2.** The solution  $\{u(t)\}_{t \geq 0}$  of Eq. (5.11) has the following properties:

- (1) If  $u^0$  is deterministic and  $u^0 \neq 0$ , the sample Lyapunov exponent

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\|u(t)\|) = -\left( \lambda_{n_0} - \sum_{i=1}^N \pi_i \left( \alpha_i - \frac{1}{2} \sum_{j=1}^m \beta_{ij}^2 \right) \right) \quad \text{a.s.}$$

In particular, the solution of Eq. (5.11) with initial data  $u^0$  will converge to zero exponentially with probability one if and only if

$$\lambda_{n_0} > \sum_{i=1}^N \pi_i \left( \alpha_i - \frac{1}{2} \sum_{j=1}^m \beta_{ij}^2 \right).$$

(2) For any  $u^0$ , the sample Lyapunov exponent

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\|u(t)\|) \leq - \left( \lambda_1 - \sum_{i=1}^N \pi_i \left( \alpha_i - \frac{1}{2} \sum_{j=1}^m \beta_{ij}^2 \right) \right) \quad \text{a.s.}$$

In particular, the solution of Eq.(5.11) is almost surely exponentially stable if

$$\lambda_1 > \sum_{i=1}^N \pi_i \left( \alpha_i - \frac{1}{2} \sum_{j=1}^m \beta_{ij}^2 \right).$$

**Theorem 5.3.** The solution  $\{u(t)\}_{t \geq 0}$  of Eq. (5.11) has the following properties:

(1) If  $u^0$  is deterministic and  $u^0 \neq 0$ , the  $p$ th moment Lyapunov exponent is

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}(\|u(t)\|^p)) = -p\lambda_{n_0} + \sup_{\mu} \left\{ \sum_{i=1}^N \left( p\alpha + \frac{p(p-1)}{2} \sum_{j=1}^m \beta_{ij}^2 \right) \mu(i) - I(\mu) \right\},$$

where the supremum is taken over probability measures  $\mu$  on  $\mathbb{S}$  and  $I(\mu)$  is defined by (5.1). In particular, the  $p$ th moment of the solution of Eq. (5.11) with initial data  $u^0$  will converge exponentially to zero if and only if

$$p\lambda_{n_0} - \sup_{\mu} \left\{ \sum_{i=1}^N \left( p\alpha + \frac{p(p-1)}{2} \sum_{j=1}^m \beta_{ij}^2 \right) \mu(i) - I(\mu) \right\} > 0.$$

(1) For any  $u^0$ , the  $p$ th moment Lyapunov exponent

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}(\|u(t)\|^p)) \leq -p\lambda_1 + \sup_{\mu} \left\{ \sum_{i=1}^N \left( p\alpha + \frac{p(p-1)}{2} \sum_{j=1}^m \beta_{ij}^2 \right) \mu(i) - I(\mu) \right\}.$$

In particular, the solution of Eq. (5.11) is exponentially stable in  $p$ th moment if

$$-p\lambda_1 + \sup_{\mu} \left\{ \sum_{i=1}^N \left( p\alpha + \frac{p(p-1)}{2} \sum_{j=1}^m \beta_{ij}^2 \right) \mu(i) - I(\mu) \right\} < 0.$$

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