Distributed predictive control of continuous-time systems

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1. Introduction

Recently, intensive research efforts have focused on the development of distributed Model Predictive Control (MPC) techniques for interacting systems, see e.g. the review paper [1], the book [2], and the contributions reported in [3–9]. This interest is motivated by the increasing complexity of industrial systems and infrastructures, such as power and transport networks or hydro power plants, which require to distribute the computational effort as well as to resort to a distributed control structure to enhance the safety and reliability of the system in front of communication losses or delays.

With some notable exceptions, see e.g. [3,5,9,10], the majority of the distributed control algorithms proposed so far have been developed for discrete-time systems, possibly obtained from an underlying continuous-time model. The discrete-time framework is particularly suitable for the design of distributed MPC, since it also allows to easily develop methods based on distributed optimization approaches, see e.g. [11–16], or on agent negotiation, see e.g. [17,18]. However, some reasons motivate the effort of developing new and effective distributed MPC methods for continuous-time systems. First, control in the discrete-time framework does not allow to consider the process inter-sampling behavior in the optimization problem underlying any MPC algorithm. Also, a significant problem may concern the relationship between the discrete-time model used for distributed control and the original continuous-time one. Indeed, under exact discretization, the continuous-time system sparsity (i.e., the coupling structure between the subsystems composing the large-scale model) may be destroyed, which would significantly impact on the underlying communication graph needed for control implementation. Alternative discretization methods for overcoming this problem are nowadays available (see [19] for a discussion), at the price of introducing some approximation.

The aim of this paper is to formulate in a continuous-time framework the Distributed Predictive Control (DPC) algorithm recently developed for discrete-time plant models in [20] and extended to the tracking and output feedback cases in [21,22]. DPC is based on a non-iterative scheme where the future state and control reference trajectories are transmitted among neighboring systems, i.e. systems with direct couplings through their state or control variables, and the differences between these trajectories and the true ones are interpreted as disturbances to be rejected by a proper robust control method. Therefore in DPC it is not necessary for each subsystem to know the dynamical models governing the trajectories of the other subsystems and the transmission of information is limited. It is also worth noting that the rationale of DPC is very similar to the MPC algorithms presently employed in industry, where reference trajectories tailored on the dynamics of the system under control are used.

The paper is organized as follows. In Section 2 the problem is formulated and the main assumptions on the process model are introduced. In Section 3 we present the continuous-time DPC algorithm and its properties, while some implementation issues are tackled in Section 4. A simulation example is discussed in

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Section 5 and in Section 6 we draw some conclusions, while in Appendix A we present the proof of the main result.

Notation. A matrix is Hurwitz if all its eigenvalues have negative real part. The short-hand $\mathbf{v} = (v_1, \ldots, v_s)$ denotes a column vector with s (not necessarily scalar) components $v_1, \ldots, v_s$. The symbol $\ominus$ denotes the Pontryagin difference, while $\oplus$ denotes the Minkowski sum and $\bigoplus_{i=1}^{M} A_i = A_1 \oplus \cdots \oplus A_M$. For a continuous-time variable $s(t)$ and a given interval $\mathscr{I} \subseteq \mathbb{R}_+$ ($\mathscr{I}$ can be open or closed), the trajectory $s(t)$ with $t \in \mathscr{I}$ is denoted with $s(\mathscr{I})$. Finally, $\mathscr{B}^{(dim)}_{\bar{\rho}_E}(0)$ is a ball of radius $\bar{\rho}_E > 0$ centered at the origin in the $\mathbb{R}^{dim}$ space.

2. Partitioned continuous-time systems

Consider a process made by M interacting systems described by the continuous time linear models

$$\dot{x}^{[i]}(t) = A_{ii}\, x^{[i]}(t) + B_{ii} u^{[i]}(t) + \sum_{j \neq i} \{A_{ij} x^{[j]}(t) + B_{ij} u^{[j]}(t)\} \tag{1}$$

where $x^{[i]}(t) \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ and $u^{[i]}(t) \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ are the state and input vectors, respectively, of the ith system ($i = 1, \ldots, M$), while $\mathbb{X}_i$ and $\mathbb{U}_i$ are convex neighborhoods of the origin. Letting $\mathbf{x}(t) = (x^{[1]}(t), \ldots, x^{[M]}(t))$, $\mathbf{u}(t) = (u^{[1]}(t), \ldots, u^{[M]}(t))$, the ensemble of systems (1) can be written in the collective form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{2}$$

where the block matrices $\mathbf{A}$ and $\mathbf{B}$ are composed by the systems' matrices A_{ij}, B_{ij}, $i, j = 1, \ldots, M$. Analogously, it is possible to define $\mathbb{X} = \prod_{i=1}^{M} \mathbb{X}_i \subseteq \mathbb{R}^n$ and $\mathbb{U} = \prod_{i=1}^{M} \mathbb{U}_i \subseteq \mathbb{R}^m$, where $n = \sum_{i=1}^{M} n_i$ and $m = \sum_{i=1}^{M} m_i$, which are convex by convexity of $\mathbb{X}_i$ and $\mathbb{U}_i$. In the following, subsystem j will be defined as a *neighbor* of subsystem i if and only if the state and/or the input of j affects the dynamics of subsystem i, i.e. iff $A_{ij} \neq 0$ and/or $B_{ij} \neq 0$, and $\mathscr{N}_i$ will denote the set of neighbors of subsystem i (which excludes i).

Concerning systems (1), the following stabilizability assumption is introduced.

Assumption 1. There exist matrices $\bar{K}_i \in R^{m_i \times n_i}$, $i = 1, \ldots, M$, such that $\bar{F}_{ii} = (A_{ii} + B_{ii}\bar{K}_i)$ are Hurwitz.

Define $\bar{\mathbf{K}} = \mathrm{diag}(\bar{K}_1, \ldots, \bar{K}_M)$.

As for the collective system (2), the following assumption on decentralized stabilizability is made:

Assumption 2. There exists a block-diagonal matrix $\mathbf{K}^c = \mathrm{diag}(K_1^c, \ldots, K_M^c)$, with $K_i^c \in R^{m_i \times n_i}$, $i = 1, \ldots, M$ such that: (i) $\mathbf{A} + \mathbf{B}\mathbf{K}^c$ is Hurwitz, (ii) $F_{ii} = (A_{ii} + B_{ii}K_i^c)$ is Hurwitz, $i = 1, \ldots, M$.

Note that Assumption 2 implies Assumption 1 which can be trivially satisfied by setting $\bar{K}_i = K_i^c$. However, since $\bar{K}_i$ and K_i^c play different roles in the design algorithm to be presented, it can be useful to allow them to be different for performance enhancement.

3. DPC for continuous-time systems

The continuous-time distributed control law described in the following is composed by two terms, the first one is a standard state feedback, while the second one is computed by an MPC-based distributed control algorithm running with sampling period T and at sampling times $t_k = kT$, $k \in \mathbb{N}$. For simplicity of notation, given the sampling instant t_k, the sampling time instant $t_k + hT$ will be denoted by t_{k+h}.

3.1. Models: perturbed, nominal and auxiliary

Similarly to the discrete-time DPC algorithm presented in [20], we assume that at any time instant t_k each subsystem i transmits to its neighbors its future state and input reference trajectories, to be later specified, $\tilde{x}^{[i]}(t)$ and $\tilde{u}^{[i]}(t)$, $t \in [t_k, t_k + (N-1)T)$. Moreover, by adding suitable constraints in the MPC formulation, each subsystem will be able to guarantee that its state and control trajectories lie in specified time-invariant neighborhoods of the reference trajectories, i.e, for all $t \in [t_k, t_{k+N-1})$, $x^{[i]}(t) \in \tilde{x}^{[i]}(t) \oplus \mathscr{E}_i$ and $u^{[i]}(t) \in \tilde{u}^{[i]}(t) \oplus \mathscr{U}_i$, where $0 \in \mathscr{E}_i$ and $0 \in \mathscr{U}_i$. It is possible to rewrite (1) as the perturbed model

$$\dot{x}^{[i]}(t) = A_{ii}\, x^{[i]}(t) + B_{ii} u^{[i]}(t) + \sum_{j \in \mathscr{N}_i} (A_{ij} \tilde{x}^{[j]}(t) + B_{ij} \tilde{u}^{[j]}(t)) + w^{[i]}(t) \tag{3}$$

where the term $\sum_{j \in \mathscr{N}_i} (A_{ij}\tilde{x}^{[j]}(t) + B_{ij}\tilde{u}^{[j]}(t))$ can be interpreted as a disturbance, known in advance over the future prediction horizon of length $(N-1)T$ (i.e., for all $t \in [t_k, t_k + (N-1)T)$), to be suitably compensated. On the other hand, $w^{[i]}(t) = \sum_{j \in \mathscr{N}_i} (A_{ij} (x^{[j]}(t) - \tilde{x}^{[j]}(t)) + B_{ij}(u^{[j]}(t) - \tilde{u}^{[j]}(t))) \in \mathbb{W}_i$ is a bounded unknown disturbance (i.e., $\mathbb{W}_i = \bigoplus_{j \in \mathscr{N}_i} \{A_{ij}\mathscr{E}_i \oplus B_{ij}\mathscr{U}_i\}$) to be rejected.

For the statement of the individual MPC sub-problems (i.e., denoted i-DPC problems), we rely on a continuous-time (see, e.g., [23]) version of the robust MPC algorithm presented in [24] for constrained discrete-time linear systems with bounded disturbances. As a preliminary step, define the ith subsystem *nominal model* obtained from equation (3) by neglecting the disturbance $w^{[i]}(t)$:

$$\dot{\hat{x}}^{[i]}(t) = A_{ii}\, \hat{x}^{[i]}(t) + B_{ii} \hat{u}^{[i]}(t) + \sum_{j \in \mathscr{N}_i} (A_{ij} \tilde{x}^{[j]}(t) + B_{ij} \tilde{u}^{[j]}(t)). \tag{4}$$

The control law for the ith perturbed subsystem (3) is given by

$$u^{[i]}(t) = \hat{u}^{[i]}(t) + K_i^c (x^{[i]}(t) - \hat{x}^{[i]}(t)) \tag{5}$$

where K_i^c is the feedback gain satisfying Assumption 2. Letting $z^{[i]}(t) = x^{[i]}(t) - \hat{x}^{[i]}(t)$, from (3) and (5), one obtains

$$\dot{z}^{[i]}(t) = F_{ii} z^{[i]}(t) + w^{[i]}(t) \tag{6}$$

where $w^{[i]}(t) \in \mathbb{W}_i$. Since $\mathbb{W}_i$ is bounded and F_{ii} is Hurwitz, it is possible to define the robust positively invariant (RPI) set Z_i for (6) (see, for example, [25] and [26]) such that, for all $z^{[i]}(t_k) \in Z_i$, then $z^{[i]}(t) \in Z_i$ for all $t \geq t_k$. Here, we assume that the sets Z_i are such that there exist non-empty sets

$$\hat{\mathbb{X}}_i \subseteq \mathbb{X}_i \ominus Z_i, \ \hat{\mathbb{U}}_i \subseteq \mathbb{U}_i \ominus K_i^c Z_i. \tag{7}$$

Given Z_i, define the neighborhoods of the origin E_i and U_i, $i = 1, \ldots, M$ such that

$$E_i \oplus Z_i \subseteq \mathscr{E}_i, \ U_i \oplus K_i^c Z_i \subseteq \mathscr{U}_i. \tag{8}$$

Recall now that $\tilde{x}^{[j]}(t)$ and $\tilde{u}^{[j]}(t)$, $i \in \mathscr{N}_i$ are available to subsystem i for $t \in [t_k, t_k + (N-1)T)$. In view of this, differently from the discrete-time DPC algorithm discussed in [20], it is not possible to compute the final segment of the nominal trajectory for $t \in [t_k + (N-1)T, t_k + NT)$ using (4). Then, with reference to the time interval $t \in [t_{k+N-1}, t_{k+N}]$, we define an *auxiliary "decentralized" model*, obtained from equation (4) by neglecting the known disturbance term, i.e.,

$$\dot{\bar{x}}^{[i]}(t) = A_{ii}\, \bar{x}^{[i]}(t) + B_{ii} \bar{u}^{[i]}(t). \tag{9}$$

Similarly to (5), the term $\hat{u}^{[i]}(t)$ is set as follows

$$\hat{u}^{[i]}(t) = \bar{u}^{[i]}(t) + \bar{K}_i (\hat{x}^{[i]}(t) - \bar{x}^{[i]}(t)) \tag{10}$$

where $\bar{K}_i$ is the feedback gain satisfying Assumption 1. Letting $s^{[i]}(t) = \hat{x}^{[i]}(t) - \bar{x}^{[i]}(t)$, from (4), (9) and (10) one has

$$\dot{s}^{[i]}(t) = \bar{F}_{ii}s^{[i]}(t) + \bar{w}^{[i]}(t) \tag{11}$$

where $\bar{w}^{[i]}(t) = \sum_{j\in\mathcal{N}_i}\{A_{ij}\tilde{x}^{[j]}(t) + B_{ij}\tilde{u}^{[j]}(t)\}$. Assuming that it is possible to guarantee (by properly-defined constraints) that, for suitably defined sets $\bar{\mathcal{E}}_i$ and $\bar{\mathcal{U}}_i$ and for $t \in [t_{k+N-1}, t_{k+N})$, $\tilde{x}^{[i]}(t) \in \bar{\mathcal{E}}_i$ and $\tilde{u}^{[i]}(t) \in \bar{\mathcal{U}}_i$, then $\bar{w}^{[i]}(t) \in \bar{\mathbb{W}}_i = \bigoplus_{j\in\mathcal{N}_i}\{A_{ij}\bar{\mathcal{E}}_j \oplus B_{ij}\bar{\mathcal{U}}_j\}$. Since $\bar{\mathbb{W}}_i$ is bounded and $\bar{F}_{ii}$ is Hurwitz, it is possible to define a further robust positively invariant (RPI) set S_i for (11). The sets $\bar{\mathcal{E}}_i$ and $\bar{\mathcal{U}}_i$ must satisfy $\bar{\mathcal{E}}_i \oplus S_i \subseteq \hat{\mathbb{X}}_i$, $\bar{\mathcal{U}}_i \oplus \bar{K}_iS_i \subseteq \hat{\mathbb{U}}_i$, $S_i \subseteq E_i$ and $\bar{K}_iS_i \subseteq U_i$. Although not strictly necessary for the derivation of the properties of the proposed DPC method, for simplicity in the following it will be assumed that the feed-forward term $\bar{u}^{[i]}(t)$ is a piecewise constant signal, i.e., $\bar{u}^{[i]}(t) = \bar{u}^{[i]}(t_{k+N-1})$ for all $t \in [t_{k+N-1}, t_{k+N})$.

3.2. Statement of the *i*-DPC problems

At any time instant t_k, given the future reference trajectories $\tilde{x}^{[j]}(t), \tilde{u}^{[j]}(t)$, $t \in [t_k, t_{k+N-1})$, $j \in \mathcal{N}_i \cup \{i\}$, for system $i = 1, \dots M$ we define the following i-**DPC problem**

$$\min_{\hat{x}^{[i]}(t_k),\hat{u}^{[i]}([t_k,t_{k+N-1})),\bar{x}^{[i]}(t_{k+N-1}),\bar{u}^{[i]}(t_{k+N-1})} V_i^N \tag{12}$$

subject to (4), (9), to

$$x^{[i]}(t_k) - \hat{x}^{[i]}(t_k) \in Z_i \tag{13}$$

$$\hat{x}^{[i]}(t) - \tilde{x}^{[i]}(t) \in E_i \tag{14}$$

$$\hat{u}^{[i]}(t) - \tilde{u}^{[i]}(t) \in U_i \tag{15}$$

$$\hat{x}^{[i]}(t) \in \hat{\mathbb{X}}_i \tag{16}$$

$$\hat{u}^{[i]}(t) \in \hat{\mathbb{U}}_i \tag{17}$$

for all $t \in [t_k, t_{k+N-1})$, to

$$\hat{x}^{[i]}(t_{k+N-1}) - \bar{x}^{[i]}(t_{k+N-1}) \in S_i \tag{18}$$

$$\bar{x}^{[i]}(t) \in \bar{\mathcal{E}}_i \tag{19}$$

$$\bar{u}^{[i]}(t) \in \bar{\mathcal{U}}_i \tag{20}$$

for all $t \in [t_{k+N-1}, t_{k+N})$, and to the terminal constraint

$$\bar{x}^{[i]}(t_{k+N}) \in \bar{\mathbb{X}}_i^F \tag{21}$$

where $\bar{\mathbb{X}}_i^F$ is a terminal set related to the ith nominal subsystem (9), specified in the following section. The cost function V_i^N is defined as follows.

$$\begin{aligned}V_i^N &= \frac{1}{2}\int_{t_k}^{t_{k+N-1}} (\|\hat{x}^{[i]}(t)\|^2_{\hat{Q}_i} + \|\hat{u}^{[i]}(t)\|^2_{\hat{R}_i})dt \\ &+ \frac{\lambda}{2}\int_{t_{k+N-1}}^{t_{k+N}} (\|\bar{x}^{[i]}(t)\|^2_{\bar{Q}_i} + \|\bar{u}^{[i]}(t)\|^2_{\bar{R}_i})dt \\ &+ \frac{1}{2}\|\hat{x}^{[i]}(t_{k+N-1})\|^2_{\hat{P}_i} + \frac{\lambda}{2}\|\bar{x}^{[i]}(t_{k+N})\|^2_{\bar{P}_i}\end{aligned} \tag{22}$$

where λ is a positive constant and the symmetric, positive definite matrices $\hat{Q}_i$, $\bar{Q}_i$, $\hat{R}_i$, $\bar{R}_i$, $\hat{P}_i$, and $\bar{P}_i$ are design parameters to be chosen as specified later. It is worth remarking that, differently from the discrete-time case, where the terminal cost on the final state is easily included, here it is necessary to weight the continuous state and control evolution for $t \in [t_{k+N-1}, t_{k+N}]$. This motivates the presence of the terms weighted by λ in (22), and the definition of the system (9) in addition to the nominal model (4). Denoting by $X_i(t_k) = (\hat{x}^{[i]}(t_k), \hat{u}^{[i]}([t_k, t_{k+N-1})), \bar{x}^{[i]}(t_{k+N-1}), \bar{u}^{[i]}(t_{k+N-1}))$ the arguments of the cost function V_i^N, the optimal solution to the i-DPC problem at time t_k is the 4-uple $X_i(t_k|t_k) = (\hat{x}^{[i]}(t_k|t_k), \hat{u}^{[i]}([t_k, t_{k+N-1})|t_k), \bar{x}^{[i]}(t_{k+N-1}|t_k), \bar{u}^{[i]}(t_{k+N-1}|t_k))$. The signal $\hat{x}^{[i]}(t|t_k)$, $t \in [t_k, t_{k+N-1}]$ (respectively $\bar{x}^{[i]}(t|t_k)$, $t \in [t_{k+N-1}, t_{k+N}]$) is the solution to (4) (respectively to (9)) obtained with $\hat{x}^{[i]}(t_k|t_k)$ as initial condition and $\hat{u}([t_k, t_{k+N-1})|t_k)$ as input sequence (respectively with $\bar{x}^{[i]}(t_{k+N-1}|t_k)$ as initial condition and $\bar{u}(t_{k+N-1}|t_k)$ as constant input). According to (5), the control law for the system (1), for $t \in [t_k, t_{k+1})$, is given by

$$u^{[i]}(t) = \hat{u}^{[i]}(t|t_k) + K_i^c(x^{[i]}(t) - \hat{x}^{[i]}(t|t_k)). \tag{23}$$

Finally, the reference trajectories $\tilde{x}^{[i]}(t)$ and $\tilde{u}^{[i]}(t)$ are incrementally defined as follows. For $t \in [t_{k+N-1}, t_{k+N})$, we set

$$\tilde{x}^{[i]}(t) = \bar{x}^{[i]}(t|t_k) \tag{24a}$$

$$\tilde{u}^{[i]}(t) = \bar{u}^{[i]}(t|t_k). \tag{24b}$$

Then, these pieces of trajectories are transmitted to the subsystems j such that $i \in \mathcal{N}_j$, i.e., which need their knowledge to compute the future predictions $\hat{x}^{[j]}(t)$.

Some remarks are due.

(I) In the optimization problem (12) it has been assumed that $\hat{u}^{[i]}([t_k, t_{k+N-1}))$ is a generic function of time. However, for computational reasons, it is usually more convenient to resort to parameterized functions and to optimize with respect to the corresponding parameters.

(II) As discussed, the pieces of continuous-time trajectories $\tilde{x}^{[i]}(t)$ and $\tilde{u}^{[i]}(t)$, $t \in [t_{k+N-1}, t_{k+N})$, must be communicated in a neighbor-to-neighbor fashion. To do so, two possible strategies can be adopted: (i) if we rely on suitable approximations, e.g., by means of orthonormal basis functions, we can transmit a continuous-time piece of trajectory by transmitting a suitable parameter vector; (ii) if we allow subsystem j to know also the dynamical model governing the subsystem i, with $i \in \mathcal{N}_j$, by solely transmitting $\bar{x}^{[i]}(t_{k+N-1}|t_k)$ and $\bar{u}^{[i]}(t_{k+N-1}|t_k)$, the whole $\bar{x}^{[i]}(t|t_k)$, $t \in [t_{k+N-1}, t_{k+N})$, can be exactly reconstructed by subsystem j itself.

(III) Note that the dynamics and the settling time of the real state trajectories are strongly dependent on how the reference trajectories are initialized and on the length N of the prediction horizon, in view of the fact that the reference trajectories are updated as in (24) and cannot be redefined at each iteration step. This makes the initialization phase crucial for enhanced performance. Future work will be especially devoted to this, in line with the results presented in [27] for the discrete-time case.

3.3. Properties of DPC

Before we establish the main stability and convergence properties of the proposed distributed control scheme, we formally state the main requirements of the sets introduced in the previous sections.

Assumption 3. Given the sets $\mathcal{E}_i$, $\mathcal{U}_i$, and the RPI sets Z_i for equation (6), there exists a real positive constant $\bar{\rho}_E > 0$ such that $Z_i \oplus \mathcal{B}^{(n_i)}_{\bar{\rho}_E}(0) \subseteq \mathcal{E}_i$ and $K_i^c Z_i \oplus \mathcal{B}^{(m_i)}_{\bar{\rho}_E}(0) \subseteq \mathcal{U}_i$ for all $i = 1, \dots, M$.

Assumption 4. For each $i = 1, \dots, M$

(i) the set Z_i satisfies $Z_i \subset \mathbb{X}_i$ and $K_i^c Z_i \subset \mathbb{U}_i$;
(ii) the sets S_i, $\bar{\mathcal{E}}_i$, and $\bar{\mathcal{U}}_i$ satisfy the following inclusions: $\bar{\mathcal{E}}_i \oplus S_i \subseteq \hat{\mathbb{X}}_i$, $\bar{\mathcal{U}}_i \oplus \bar{K}_iS_i \subseteq \hat{\mathbb{U}}_i$, $S_i \subseteq E_i$ and $\bar{K}_iS_i \subseteq U_i$.

A comment on Assumptions 3–4 is due. Concerning the sets Z_i, $\mathscr{E}_i$, and $\mathscr{U}_i$, $i = 1, \ldots, M$, Assumption 3, i.e., $Z_i \subset \mathscr{E}_i$ and $K_i^c Z_i \subset \mathscr{U}_i$ is required for suitably defining E_i and U_i, see (8); on the other hand, Assumption 4(i), i.e., $Z_i \subset \mathbb{X}_i$ and $K_i^c Z_i \subset \mathbb{U}_i$, is required for properly introducing $\hat{\mathbb{X}}_i$ and $\hat{\mathbb{U}}_i$, see (7). Finally, Assumption 4(ii), concerning sets S_i, $\bar{\mathscr{E}}_i$, and $\bar{\mathscr{U}}_i$, i.e., that $\bar{\mathscr{E}}_i \oplus S_i \subseteq \hat{\mathbb{X}}_i$, $\bar{\mathscr{U}}_i \oplus \bar{K}_i S_i \subseteq \hat{\mathbb{U}}_i$, $S_i \subseteq E_i$ and $\bar{K}_i S_i \subseteq U_i$, is required for properly constraining the final segment of the nominal and reference trajectories.

Note that, concerning Assumption 3, it is not guaranteed that a suitable choice of $\mathscr{E}_i$ and $\mathscr{U}_i$ exists. On the other hand, Assumption 4 is constructive: indeed, provided that sets $\mathscr{E}_i$, $\mathscr{U}_i$, $\bar{\mathscr{E}}_i$, and $\bar{\mathscr{U}}_i$ are scaled by a suitable factor, Assumption 4 can always be verified.

Now we need to define the set of admissible initial conditions $\mathbf{x}(t_0) = (x^{[1]}(t_0), \ldots, x^{[M]}(t_0))$ and initial reference trajectories $\tilde{x}^{[j]}(t)$, $\tilde{u}^{[j]}(t)$, for all $j = 1 \ldots, M$ and $t \in [t_0, t_{N-1})$.

Definition 1. Letting $\mathbf{x} = (x^{[1]}, \ldots, x^{[M]})$, denote by

$$\begin{aligned}\mathbb{X}^N := \big\{\mathbf{x} :\ & \text{if } x^{[i]}(t_0) = x^{[i]} \text{ for all } i = 1, \ldots, M \\ & \text{then } \exists(\tilde{x}^{[1]}(t), \ldots, \tilde{x}^{[M]}(t)), (\tilde{u}^{[1]}(t), \ldots, \tilde{u}^{[M]}(t)) \\ & \text{for all } t \in [t_0, t_{N-1}), X_i(t_0|t_0) \text{ such that (4), (9),} \\ & \text{and (13)–(21) are satisfied for all } i = 1, \ldots, M\big\}\end{aligned}$$

the feasibility region for all the i-DPC problems. Moreover, for each $\mathbf{x} \in \mathbb{X}^N$, let

$$\begin{aligned}\tilde{\mathbb{X}}_{\mathbf{x}} := \big\{& (\tilde{x}^{[1]}(t), \ldots, \tilde{x}^{[M]}(t)), (\tilde{u}^{[1]}(t), \ldots, \tilde{u}^{[M]}(t)) \text{ for} \\ & \text{all } t \in [t_0, t_{N-1}) \text{ if } x^{[i]}(t_0) = x^{[i]} \text{ for all } i = 1, \ldots, M \\ & \text{then } \exists\, X_i(t_0|t_0) \text{ such that (4), (9),} \\ & \text{and (13)–(21) are satisfied for all } i = 1, \ldots, M\big\}\end{aligned}$$

be the region of feasible initial reference trajectories.

We are now in the position to state the following result

Theorem 1. *Let Assumptions 1–4 be satisfied; then, there exist computable design parameters λ, $\hat{Q}_i$, $\bar{Q}_i$, $\hat{R}_i$, $\bar{R}_i$, $\hat{P}_i$, $\bar{P}_i$ such that, for any initial reference trajectories in $\tilde{\mathbb{X}}_{\mathbf{x}(t_0)}$, the trajectory $\mathbf{x}(t)$, starting from any initial condition $\mathbf{x}(t_0) \in \mathbb{X}^N$, asymptotically converges to the origin.*

A detailed discussion on how to select the design parameters and the sets of interest is reported in the following section.

4. Tuning of the design parameters

In this section we show how to compute design parameters which guarantee that Theorem 1 holds.

4.1. Choice of the control gains K_i^c, $\overline{K}_i$

The control laws (5), (10) require the knowledge of the gains K_i^c and $\overline{K}_i$ satisfying Assumptions 1 and 2. While the terms $\overline{K}_i$ can be computed with any standard synthesis method provided that the pair (A_{ii}, B_i) is stabilizable, the computation of $\mathbf{K}^c = \operatorname{diag}(K_1^c, \ldots, K_M^c)$ is more difficult, since both a collective and a number of local stability conditions must be fulfilled. For instance, this problem can be easily tackled, similarly to [28,29], in a centralized fashion by defining two block diagonal matrices $\mathbf{P} = (P_1, \ldots, P_M)$, $P_i \in \mathbb{R}^{n_i \times n_i}$, and $\mathbf{Y} = \operatorname{diag}(Y_1, \ldots, Y_M)$, $Y_i \in \mathbb{R}^{m_i \times n_i}$, and by solving the following set of LMI's

$$\begin{cases}\mathbf{P} \succ 0 \\ P_i \succ 0, \quad i = 1, \ldots, M \\ \mathbf{P}\mathbf{A}^T + \mathbf{A}\mathbf{P} + \mathbf{Y}^T\mathbf{B}^T + \mathbf{B}\mathbf{Y} \prec 0 \\ P_i A_{ii}^T + A_{ii} P_i + Y_i^T B_{ii}^T + B_{ii} Y_i \prec 0, \quad i = 1, \ldots, M\end{cases} \tag{25}$$

Then, $\mathbf{K}^c = \mathbf{Y}\mathbf{P}^{-1}$ is the required stabilizing block diagonal matrix.

This approach is not scalable and could become critical in case of very large-scale systems. An alternative and possibly completely distributed strategy can be adopted, similar to the one discussed e.g., in [30]: (i) stabilize each subsystem separately (as if it were decoupled from the rest), possibly using robust design methods to minimize the mutual interactions between subsystems; (ii) check decentralized stability using a small-gain like condition, e.g., based on vector Lyapunov functions [30], that can be applied in a distributed fashion. As remarked in [30], this method proved to be effective in a number of applications, especially those involving *weakly-interacting* subsystems; moreover, similarly to [31], a similar approach can lead to *plug-and-play* control system design and implementation.

It is important to mention that also the fulfillment of Assumption 3 directly depends on how K_i^c, $i = 1, \ldots, M$, are selected. In our framework, Assumption 3 should be verified after the definition of the local gains K_i^c, discussed above. Similarly to the discrete-time case (see [31]), however, it could be possible to devise a unique (possibly distributed) design procedure for addressing Assumptions 1–3 at the same time. Future work will be devoted to this issue.

4.2. Choice of $\bar{Q}_i$, $\bar{R}_i$, $\bar{P}_i$, $\bar{\mathbb{X}}_i^F$

In order to define matrices $\bar{Q}_i$, $\bar{R}_i$, $\bar{P}_i$, and the invariant set $\bar{\mathbb{X}}_i^F$, we must preliminarily define the auxiliary control law for the system (9), which must be consistent with the simplifying assumption that $\bar{u}^{[i]}(t)$ is piecewise constant. Assuming that the terminal constraint $\bar{x}^{[i]}(t_{k+N}) \in \bar{\mathbb{X}}_i^F$ is verified, we define the auxiliary control law, to be applied to system (9) for all $t \in [t_{k+N}, t_{k+N+1})$, as

$$\bar{u}^{[i]}(t) = \bar{u}^{[i]}(t_{k+N}) = K_i^d \bar{x}^{[i]}(t_{t+N}) \tag{26}$$

where the gain K_i^d must stabilize the continuous-time system (9). Denoting, for all $\eta \in [0, T]$, $A_{ii}^{zoh}(\eta) = e^{A_{ii}\eta}$ and $B_{ii}^{zoh}(\eta) = \int_0^\eta e^{A_{ii}(\eta-\nu)} B_{ii} d\nu$ and given $\bar{x}^{[i]}(t_{k+N})$, for all $t \in [t_{k+N}, t_{k+N+1}]$ one has

$$\begin{aligned}\bar{x}^{[i]}(t) &= F_{ii}^{zoh}(t - t_{k+N})\bar{x}^{[i]}(t_{k+N}) \\ \bar{x}^{[i]}(t_{k+N+1}) &= F_{ii}^d \bar{x}^{[i]}(t_{k+N})\end{aligned}$$

where $F_{ii}^{zoh}(\eta) = A_{ii}^{zoh}(\eta) + B_{ii}^{zoh}(\eta) K_i^d$ and $F_{ii}^d = F_{ii}^{zoh}(T)$. Therefore, the gains K_i^d can be computed with any standard stabilization method to guarantee that F_{ii}^d is Schur. This procedure allows also one to resort to the results reported in [32], Lemma 1. Specifically, given the symmetric weighting matrices $\bar{Q}_i > 0$ and $\bar{R}_i > 0$ appearing in (22) and which can be chosen as free design parameters, define two constants $\gamma_{i1} > 0$, $\gamma_{i2} > 0$ in such a way that

$$\gamma_{i1} > \lambda_M(\bar{Q}_i) \tag{27a}$$

$$\gamma_{i2} > T\|K_i^d\|^2 \lambda_M(\bar{R}_i). \tag{27b}$$

Furthermore, define a matrix Q_i^* in such a way that $\lambda_m(Q_i^*) > \gamma_{i1}$. Let the symmetric matrix $\bar{P}_i$ be the unique positive definite solution of the following Lyapunov equation

$$(F_{ii}^d)^T \bar{P}_i F_{ii}^d - \bar{P}_i + \tilde{Q}_i = 0 \tag{28}$$

where $\tilde{Q}_i = \int_0^T (F_{ii}^{zoh}(\eta))^T Q^* F_{ii}^{zoh}(\eta) d\eta + \gamma_{i2} I$. Then, for each pair of sets $\bar{\mathscr{E}}_i$, $\bar{\mathscr{U}}_i$, it is proven in [32] that there exist a sampling period $T \in [0, +\infty)$ and a constant $c_i > 0$ such that the set

$$\bar{\mathbb{X}}_i^F(K_i^d, T) = \{\bar{x}^{[i]} \mid \|\bar{x}^{[i]}\|_{\bar{P}_i}^2 \le c_i\} \tag{29}$$

satisfies, for all $\bar{x}^{[i]}(t_{k+N}) \in \bar{\mathbb{X}}_i^F$ and for all $t \in [t_{k+N}, t_{k+N+1})$, the conditions

$$\bar{x}^{[i]}(t) \in \bar{\mathscr{E}}_i, \ \ K_d\bar{x}^{[i]}(t_{k+N}) \in \bar{\mathscr{U}}_i \tag{30a}$$

$$\|\bar{x}^{[i]}(t_{k+N+1})\|^2_{\hat{P}_i} - \|\bar{x}^{[i]}(t_{k+N})\|^2_{\hat{P}_i} \le -\gamma_{i1}\int_{t_{k+N}}^{t_{k+N+1}} \|\bar{x}^{[i]}(\eta)\|^2 d\eta - \gamma_{i2}\|\bar{x}^{[i]}(t_{k+N})\|^2. \tag{30b}$$

Letting

$$\bar{l}_i(\bar{x}^{[i]}(t), \bar{u}^{[i]}(t)) = \frac{\lambda}{2}\int_t^{t+T} (\|\bar{x}^{[i]}(\eta)\|^2_{\bar{Q}_i} + \|\bar{u}^{[i]}(\eta)\|^2_{\bar{R}_i})d\eta \tag{31a}$$

$$\bar{V}_i^F(\bar{x}^{[i]}(t)) = \frac{\lambda}{2}\|\bar{x}^{[i]}(t)\|^2_{\bar{P}_i} \tag{31b}$$

from the definition of $\gamma_{i1} > 0$, $\gamma_{i2} > 0$ and $\bar{V}_i^F$, and recalling (26), (31b) implies that $\bar{x}^{[i]}(t_{k+N+1}) \in \bar{\mathbb{X}}_i^F$ and

$$\begin{aligned}&\bar{V}_i^F(\bar{x}^{[i]}(t_{k+N+1})) - \bar{V}_i^F(\bar{x}^{[i]}(t_{k+N}))\\ &\quad\le -\frac{\lambda}{2}\int_{t_{k+N}}^{t_{k+N+1}} (\|\bar{x}^{[i]}(\eta)\|^2_{\bar{Q}_i} + \|\bar{u}^{[i]}(\eta)\|^2_{\bar{R}_i})d\eta\\ &\quad\le -\bar{l}_i(\bar{x}^{[i]}(t_{k+N}), \bar{u}^{[i]}(t_{k+N})).\end{aligned} \tag{32}$$

Therefore, since properties (30a)–(32) are required to establish the main properties of the method (see the proof of Theorem 1), for any pair $\bar{Q}_i, \bar{R}_i$ it is required to choose the weights $\bar{P}_i$ in (22) according to (28) and the terminal set $\bar{\mathbb{X}}_i^F$ in (21) according to (29).

4.3. Choice of $\hat{Q}_i$, $\hat{R}_i$, $\hat{P}_i$, λ

The symmetric, positive definite matrices $\hat{Q}_i$, $\hat{R}_i$ can be freely chosen according to specific design criteria, while, in order to guarantee the stability properties of Theorem 1, given an arbitrary constant $\alpha > 1$, the matrix $\hat{P}_i$ must be computed to satisfy the following Lyapunov equation:

$$\bar{\Phi}_x^{[i]}(T)^T\hat{P}_i\bar{\Phi}_x^{[i]}(T) - \hat{P}_i + \mathscr{Q}_x^{[i]} + \alpha I_n = 0 \tag{33}$$

where

$$\mathscr{Q}_x^{[i]} = \int_0^T \bar{\Phi}_x^{[i]}(\eta)^T\hat{Q}_i\bar{\Phi}_x^{[i]}(\eta) + \bar{\Phi}_u^{[i]}(\eta)^T\hat{R}_i\bar{\Phi}_u^{[i]}(\eta)d\eta$$

and, for all $\eta = [0, T]$, $\bar{\Phi}_x^{[i]}(\eta) = e^{\bar{F}_{ii}\eta}$ and $\bar{\Phi}_u^{[i]}(\eta) = \bar{K}_i\bar{\Phi}_x^{[i]}(\eta)$.

For the tuning of scalar λ, there exists a positive number $\bar{\lambda} > 0$ such that, if $\lambda \ge \bar{\lambda}$, then the convergence of the scheme is guaranteed. For a numerical assessment of $\bar{\lambda}$, see the discussion in Appendix A.3.

5. Simulation example

Assume we have to regulate the levels y_i, $i = 1, \ldots, 5$ of the five flotation tanks system proposed in [33], where a flow of pulp q enters into the first one. The tanks are connected in cascade with control valves between subsequent reservoirs (Fig. 1), and the manipulated inputs are the signals to the valves v_i, $i = 1, \ldots, 5$.

The dynamic model of the levels inside the five tanks is [33]:

$$\begin{aligned}\pi r^2\frac{dy_1}{dt} &= q - k_1v_1\sqrt{y_1 - y_2 + h_1}\\ \pi r^2\frac{dy_2}{dt} &= k_1v_1\sqrt{y_1 - y_2 + h_1} - k_2v_2\sqrt{y_2 - y_3 + h_2}\\ \pi r^2\frac{dy_3}{dt} &= k_2v_2\sqrt{y_2 - y_3 + h_2} - k_3v_3\sqrt{y_3 - y_4 + h_3}\\ \pi r^2\frac{dy_4}{dt} &= k_3v_3\sqrt{y_3 - y_4 + h_3} - k_4v_4\sqrt{y_4 - y_5 + h_4}\\ \pi r^2\frac{dy_5}{dt} &= k_4v_4\sqrt{y_4 - y_5 + h_4} - k_5v_5\sqrt{y_5 + h_5}\end{aligned} \tag{34}$$

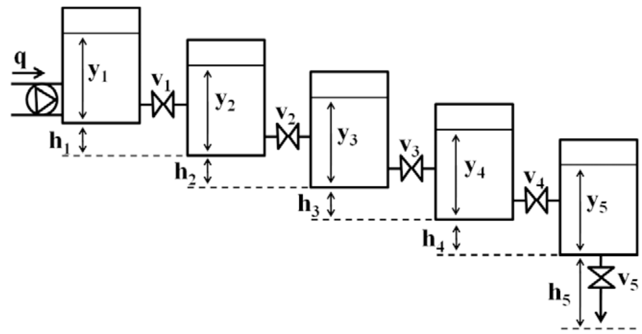

Fig. 1. Schematic representation of the flotation tanks.

where r is radius of the tanks, k_i, $i = 1, \ldots, 5$ are the valves coefficients and h_i, $i = 1, \ldots, 5$ are the physical height differences between subsequent tanks. We set $r = 1$ m, $k_i = 0.1$ m$^{2.5}$/Vs, $i = 1, \ldots, 5$ and $h_i = 0.5$ m, $i = 1, \ldots, 5$. The value for the inlet flow is $q = 0.1$ m^3/s. To linearize system (34), we considered the equilibrium point where $\bar{y}_i = 2$ m, $i = 1, \ldots, 5$, and, correspondingly, $\bar{v}_i = 1.4142$ V, $i = 1, \ldots, 4$ and $\bar{v}_5 = 0.6325$ V. Let $\delta y_i = y_i - \bar{y}_i$, $i = 1, \ldots, 5$, $\delta v_i = v_i - \bar{v}_i$, $i = 1, \ldots, 5$, $\mathbf{x} = (\delta y_1, \delta y_2, \delta y_3, \delta y_4, \delta y_5)$ and $\mathbf{u} = (\delta v_1, \delta v_2, \delta v_3, \delta v_4, \delta v_5)$.

The partition of inputs and states, for $i = 1, \ldots, 5$ is:

$$x^{[i]} = \delta y_i, \qquad u^{[1]} = \delta v_1.$$

The constraints on the inputs and the states of the linearized system, for $i = 1, \ldots, 5$, have been set as:

$$x^{[i]}_{min} = -1, \qquad x^{[i]}_{max} = 1, \qquad u^{[i]}_{min} = -\bar{v}_i, \qquad u^{[i]}_{max} = 3 - \bar{v}_i.$$

The inputs have been parameterized as piece-wise constant signals, where the adopted sampling period is 5 s. The weighting matrices, for $i = 1, \ldots, 5$, are $\bar{Q}_i = \bar{R}_i = \hat{Q}_i = \hat{R}_i = 1$. The initial state and control reference trajectories have been computed using an iterative method similar to the one described in [20].

In Fig. 2 the state trajectories are depicted, computed with the continuous-time nonlinear model in simulation. Fig. 3 shows the applied inputs. In order to evaluate the results obtained with the distributed continuous-time method here proposed, Figs. 2–3 also show the transients of the state and control variables obtained with (i) the most popular industrial implementation of MPC, i.e, centralized MPC – denoted cMPC – based on a linearized discrete-time model of the plant and (ii) the DPC scheme proposed in [20] for discrete-time systems, where the model used in the control design is discretized using the approximate method discussed in [19]. From these plots it is apparent that the performance deterioration with respect to the centralized approach mainly consists of a slightly larger settling time and an overshoot of some state variables. On the other hand, a slight degradation of the results is witnessed in case of discrete-time DPC, which would significantly aggravate as the sampling time increases.

6. Conclusions

In this paper we have presented a novel non-cooperative distributed predictive control algorithm for continuous-time systems based on robust MPC, whose convergence properties have been proved. The continuous-time approach proposed in this paper represents a significant improvement of the method described in [20] also in view of an easier tuning of the main design parameters (e.g., the weights in the cost function) which can guarantee stability and convergence properties. A realistic case study is used for testing the performance of the algorithm. Future work will focus on the initialization phase, where computationally scalable automatic tools are needed for the definition of auxiliary control gains, sets and reference trajectories: this could eventually pave the way for plug-and-play implementations.

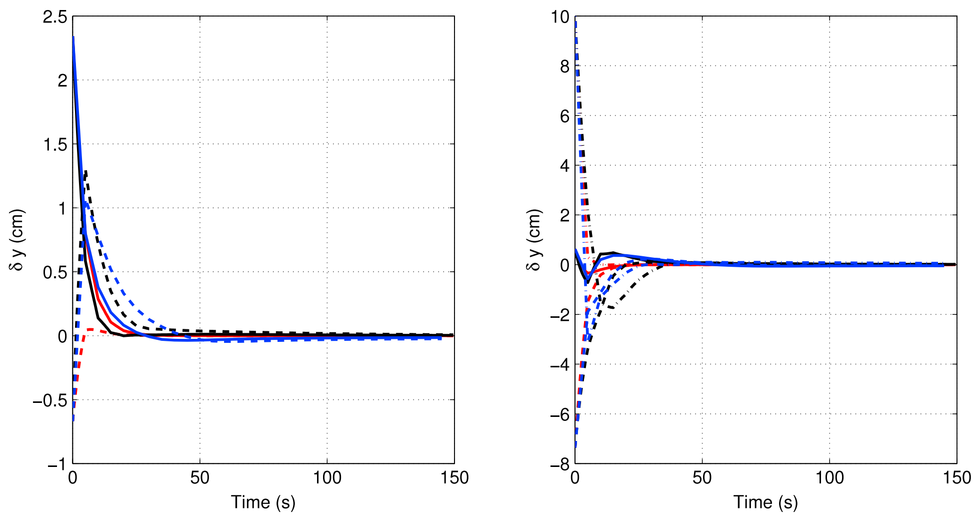

Fig. 2. Trajectories of the states $x^{[1]}$ (solid lines), $x^{[2]}$ (dashed lines), on the left, and $x^{[3]}$ (solid lines), $x^{[4]}$ (dashed lines), $x^{[5]}$ (dash–dot lines), on the right, obtained with continuous-time DPC (black lines), with cMPC (red lines), and with discrete-time DPC (blue lines) for the control of the floating tanks. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

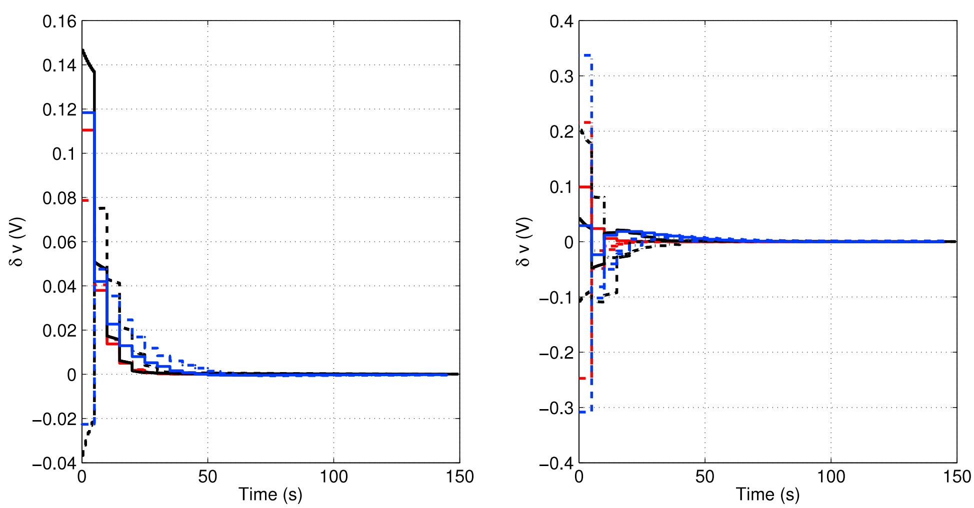

Fig. 3. Inputs $u^{[1]}$ (solid lines), $u^{[2]}$ (dashed lines), on the left, and $u^{[3]}$ (solid lines), $u^{[4]}$ (dashed lines), $u^{[5]}$ (dash–dot lines), on the right, obtained with continuous-time DPC (black lines), with cMPC (red lines), and with discrete-time DPC (blue lines) for the control of the floating tanks. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Appendix. Proof of Theorem 1

A.1. Proof of recursive feasibility

First we prove that, given for all $i = 1, \dots, M$, an optimal feasible solution $X_i(t_k|t_k)$ to (12) at time t_k, the 4-uple

$$
\begin{aligned}
X_i(t_{k+1}|t_k) = {} & (\hat{x}^{[i]}(t_{k+1}|t_k), (\hat{u}^{[i]}([t_{k+1}, t_{k+N-1})|t_k), \bar{u}^{[i]}(t|t_k) \\
& + \bar{K}_i(\hat{x}^{[i]}(t|t_k) - \bar{x}^{[i]}(t|t_k)), t \in [t_{k+N-1}, t_{k+N})), \bar{x}^{[i]}(t_{k+N}|t_k), \\
& K_i^d \bar{x}^{[i]}(t_{k+N}|t_k))
\end{aligned} \tag{A.1}
$$

is a feasible solution to (12) at time t_{k+1}. Recall that, according to (24), after the solution to (12) is computed at time t_k, each subsystem j transmits $\tilde{x}^{[j]}([t_{k+N-1}, t_{k+N})) = \bar{x}^{[j]}([t_{k+N-1}, t_{k+N})|t_k)$ and $\tilde{u}^{[j]}([t_{k+N-1}, t_{k+N})) = \bar{u}^{[i]}([t_{k+N-1}, t_{k+N})|t_k)$ to all the subsystems i satisfying $j \in \mathcal{N}_i$.

Importantly, in (A.1), for $t \in [t_{k+N-1}, t_{k+N})$, the trajectory $\hat{x}^{[i]}(t|t_k)$ is computed, by subsystem i, according to system (4), with $\hat{u}^{[i]}(t) = \bar{u}^{[i]}(t|t_k) + \bar{K}_i(\hat{x}^{[i]}(t|t_k) - \bar{x}^{[i]}(t|t_k))$. Therefore it results that, for all $t \in [t_{k+N-1}, t_{k+N})$

$$
\begin{aligned}
\dot{\hat{x}}^{[i]}(t|t_k) &= A_{ii}\hat{x}^{[i]}(t|t_k) + B_{ii}(\bar{u}^{[i]}(t|t_k) + \bar{K}_i(\hat{x}^{[i]}(t|t_k) - \bar{x}^{[i]}(t|t_k))) \\
&\quad + \sum_{j \in \mathcal{N}_i} (A_{ij}\tilde{x}^{[j]}(t|t_k) + B_{ij}\tilde{u}^{[j]}(t|t_k)) \\
&= (A_{ii} + B_{ii}\bar{K}_i)\hat{x}^{[i]}(t|t_k) - B_{ii}\bar{K}_i\bar{x}^{[i]}(t|t_k) \\
&\quad + \sum_{j \in \mathcal{N}_i} A_{ij}\bar{x}^{[j]}(t|t_k) + \sum_{j=1}^{M} B_{ij}\bar{u}^{[j]}(t|t_k).
\end{aligned} \tag{A.2}
$$

On the other hand, the trajectory $\bar{x}^{[i]}(t|t_k)$, for all $t \in [t_{k+N}, t_{k+N+1}]$, is computed according to (9) with $\bar{u}^{[i]}(t|t_k) = K_i^d\bar{x}^{[i]}(t_{k+N}|t_k)$, and therefore $\bar{x}^{[i]}(t|t_k) = F_i^{zoh}(t - t_{k+N})\bar{x}^{[i]}(t_{k+N}|t_k)$.

From (13) $x^{[i]}(t_k) - \hat{x}^{[i]}(t_k) \in Z_i$ and, from (14)–(15), for $t \in [t_k, t_{k+1})$, it is guaranteed that $\hat{x}^{[j]}(t) - \tilde{x}^{[j]}(t) \in E_j$, $\hat{u}^{[j]}(t) - \tilde{u}^{[j]}(t) \in U_j$ for all $j \in \mathcal{N}_i$ and $w^{[i]}(t) \in \mathbb{W}_i$. Therefore, in view of the invariance of Z_i with respect to (6), it holds that $x^{[i]}(t_{k+1}) - \hat{x}^{[i]}(t_{k+1}|t_k) \in Z_i$.

For $t \in [t_{k+1}, t_{k+N-1})$, constraints (14)–(17) are verified in view of the feasibility of (12) at time t_k.

For $t \in [t_{k+N-1}, t_{k+N})$, recalling (A.2) we have that

$$\dot{\hat{x}}^{[i]}(t|t_k) - \dot{\bar{x}}^{[i]}(t|t_k) = (A_{ii} + B_{ii}\bar{K}_i)(\hat{x}^{[i]}(t|t_k) - \bar{x}^{[i]}(t|t_k)) + \sum_{j\in\mathcal{N}_i} \left(A_{ij}\bar{x}^{[j]}(t|t_k) + B_{ij}\bar{u}^{[j]}(t|t_k)\right) \tag{A.3}$$

and recall also that $\tilde{x}^{[i]}(t) = \bar{x}^{[i]}(t|t_k)$ and $\tilde{u}^{[i]}(t) = \bar{u}^{[i]}(t|t_k)$ for all $i = 1, \ldots, M$. In view of (18) $\hat{x}^{[i]}(t_{k+N-1}|t_k) - \bar{x}^{[i]}(t_{k+N-1}|t_k) \in S_i$ and from (19)–(20), it is guaranteed that $\sum_{j\in\mathcal{N}_i}(A_{ij}\bar{x}^{[j]}(t|t_k) + B_{ij}\bar{u}^{[j]}(t|t_k)) \in \bar{\mathbb{W}}_i$ for all $j \in \mathcal{N}_i$. In view of the invariance of S_i with respect to (11), it holds that $\hat{x}^{[i]}(t|t_k) - \bar{x}^{[i]}(t|t_k) = \hat{x}^{[i]}(t|t_k) - \tilde{x}^{[i]}(t) \in S_i$. Furthermore, since $\hat{u}^{[i]}(t|t_k) - \bar{u}^{[i]}(t|t_k) = \hat{u}^{[i]}(t|t_k) - \tilde{u}^{[i]}(t) \in \bar{K}_i S_i$ and, in view of Assumption 4, $S_i \subseteq E_i$ and $\bar{K}_i S_i \subseteq U_i$, then (14) and (15) are also verified for $t \in [t_{k+N-1}, t_{k+N})$. This also proves that $\hat{x}^{[i]}(t_{k+N}|t_k) - \bar{x}^{[i]}(t_{k+N}|t_k) \in S_i$ and that (18) is satisfied. Moreover, being $\bar{\mathscr{E}}_i \oplus S_i \subseteq \hat{\mathbb{X}}_i$ and $\bar{\mathscr{U}}_i \oplus \bar{K}_i S_i \subseteq \hat{\mathbb{U}}_i$ from Assumption 4, constraints (16) and (17) are verified for $t \in [t_{k+N-1}, t_{k+N})$.

Finally, note that $\bar{x}^{[i]}(t_{k+N}) \in \bar{\mathbb{X}}_i^F$ in view of (21) and of the definition (29) of $\bar{\mathbb{X}}_i^F$ and (30b), the constraints (19), (20), (21) are also verified at time t_{k+1}.

In view of this, the 4-uple $X_i(t_{k+1}|t_k)$ is a feasible solution to (12) at time t_{k+1}.

This implies that, given the optimal solution $X_i^*(t_{k+1})$ to the problem (12) at time t_{k+1} (which is proved to exist, provided that (12) is feasible at time t_k), for all $i = 1, \ldots, M$ it holds that, by optimality

$$V_i^{N*}(x(t_{k+1})) = V_i^N(X_i^*(t_{k+1})) \le V_i^N(X_i(t_{k+1}|t_k)). \tag{A.4}$$

A.2. The collective problem

To prove the convergence to zero of the solution, we now define the collective problem, equivalent to the one considered in the previous sections. Define the vectors $\hat{\mathbf{x}}(t) = (\hat{x}^{[1]}(t), \ldots, \hat{x}^{[M]}(t))$, $\bar{\mathbf{x}}(t) = (\bar{x}^{[1]}(t), \ldots, \bar{x}^{[M]}(t))$, $\tilde{\mathbf{x}}(t) = (\tilde{x}^{[1]}(t), \ldots, \tilde{x}^{[M]}(t))$, $\hat{\mathbf{u}}(t) = (\hat{u}^{[1]}(t), \ldots, \hat{u}^{[M]}(t))$, $\bar{\mathbf{u}}(t) = (\bar{u}^{[1]}(t), \ldots, \bar{u}^{[M]}(t))$, $\tilde{\mathbf{u}}(t) = (\tilde{u}^{[1]}(t), \ldots, \tilde{u}^{[M]}(t))$, $\mathbf{w}(t) = (w^{[1]}(t), \ldots, w^{[M]}(t))$, $\bar{\mathbf{w}}(t) = (\bar{w}^{[1]}(t), \ldots, \bar{w}^{[M]}(t))$, $\mathbf{z}(t) = (z^{[1]}(t), \ldots, z^{[M]}(t))$, $\mathbf{s}(t) = (s^{[1]}(t), \ldots, s^{[M]}(t))$. and the matrices $\mathbf{A}^* = \mathrm{diag}(A_{11}, \ldots, A_{MM})$, $\mathbf{B}^* = \mathrm{diag}(B_{11}, \ldots, B_{MM})$, $\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{A}^*$, $\tilde{\mathbf{B}} = \mathbf{B} - \mathbf{B}^*$. Collectively, we write equations (3), (4) and (9) as

$$\dot{\mathbf{x}}(t) = \mathbf{A}^*\mathbf{x}(t) + \mathbf{B}^*\mathbf{u}(t) + \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\tilde{\mathbf{u}}(t) + \mathbf{w}(t) \tag{A.5}$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}^*\hat{\mathbf{x}}(t) + \mathbf{B}^*\hat{\mathbf{u}}(t) + \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\tilde{\mathbf{u}}(t) \tag{A.6}$$

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{A}^*\bar{\mathbf{x}}(t) + \mathbf{B}^*\bar{\mathbf{u}}(t). \tag{A.7}$$

In view of (5) and (10), $\mathbf{u}(t) = \hat{\mathbf{u}}(t) + \mathbf{K}^c(\mathbf{x}(t) - \hat{\mathbf{x}}(t))$ and $\hat{\mathbf{u}}(t) = \bar{\mathbf{u}}(t) + \bar{\mathbf{K}}(\hat{\mathbf{x}}(t) - \bar{\mathbf{x}}(t))$. From this, and in view of (6) and (11),

$$\dot{\mathbf{z}}(t) = (\mathbf{A}^* + \mathbf{B}^*\mathbf{K}^c)\mathbf{z}(t) + \mathbf{w}(t) \tag{A.8}$$

$$\dot{\mathbf{s}}(t) = (\mathbf{A}^* + \mathbf{B}^*\bar{\mathbf{K}})\mathbf{s}(t) + \bar{\mathbf{w}}(t). \tag{A.9}$$

Minimizing (12) at time t_k for all $i = 1, \ldots, M$ is equivalent to solve the following collective minimization problem

$$\mathbf{V}^{N*}(\mathbf{x}(t_k)) = \min_{\mathbf{X}(t_k)} \mathbf{V}^N(\mathbf{X}(t_k)) \tag{A.10}$$

where $\mathbf{X}(t_k) = (X_1(t_k), \ldots, X_M(t_k))$, subject to the dynamic constraints (A.6), (A.7) and

$$\mathbf{x}(t_k) - \hat{\mathbf{x}}(t_k) \in \mathbb{Z} = \prod_{i=1}^M Z_i \tag{A.11a}$$

$$\hat{\mathbf{x}}(t) - \tilde{\mathbf{x}}(t) \in \mathbb{E} = \prod_{i=1}^M E_i \tag{A.11b}$$

$$\hat{\mathbf{u}}(t) - \tilde{\mathbf{u}}(t) \in \tilde{\mathbb{U}} = \prod_{i=1}^M U_i \tag{A.11c}$$

$$\hat{\mathbf{x}}(t) \in \hat{\mathbb{X}} = \prod_{i=1}^M \hat{\mathbb{X}}_i \tag{A.11d}$$

$$\hat{\mathbf{u}}(t) \in \hat{\mathbb{U}} = \prod_{i=1}^M \hat{\mathbb{U}}_i \tag{A.11e}$$

for all $t \in [t_k, t_{k+N-1})$, to

$$\hat{\mathbf{x}}(t_{k+N-1}) - \bar{\mathbf{x}}(t_{k+N-1}) \in \mathbb{S} = \prod_{i=1}^M S_i \tag{A.12}$$

$$\bar{\mathbf{x}}(t) \in \bar{\mathscr{E}} = \prod_{i=1}^M \bar{\mathscr{E}}_i \tag{A.13}$$

$$\bar{\mathbf{x}}(t) \in \bar{\mathscr{U}} = \prod_{i=1}^M \bar{\mathscr{U}}_i \tag{A.14}$$

and the terminal constraint

$$\bar{\mathbf{x}}(t_{k+N}) \in \bar{\mathbb{X}}^F = \prod_{i=1}^M \bar{\mathscr{X}}_i^F. \tag{A.15}$$

The collective cost function $\mathbf{V}^N$ is

$$\mathbf{V}^N = \sum_{h=0}^{N-2} \hat{\mathbf{l}}(\hat{\mathbf{x}}(t_{k+h}), \hat{\mathbf{u}}(t_{k+h})) + \hat{\mathbf{V}}^F\left(\hat{\mathbf{x}}(t_{k+N-1})\right) + \bar{\mathbf{l}}(\bar{\mathbf{x}}(t_{k+N-1}), \bar{\mathbf{u}}(t_{k+N-1})) + \bar{\mathbf{V}}^F\left(\bar{\mathbf{x}}(t_{k+N})\right)$$

where, from (31):

$$\hat{\mathbf{l}}(\hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t)) = \frac{1}{2}\int_t^{t+T} (\|\hat{\mathbf{x}}(\eta)\|^2_{\hat{\mathbf{Q}}} + \|\hat{\mathbf{u}}(\eta)\|^2_{\hat{\mathbf{R}}})d\eta \tag{A.16a}$$

$$\bar{\mathbf{l}}(\bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t)) = \frac{\lambda}{2}\int_t^{t+T} (\|\bar{\mathbf{x}}(\eta)\|^2_{\bar{\mathbf{Q}}} + \|\bar{\mathbf{u}}(\eta)\|^2_{\bar{\mathbf{R}}})d\eta \tag{A.16b}$$

$$\hat{\mathbf{V}}^F(\hat{\mathbf{x}}(t)) = \frac{1}{2}\|\hat{\mathbf{x}}(t)\|^2_{\hat{\mathbf{P}}} \tag{A.16c}$$

$$\bar{\mathbf{V}}^F(\bar{\mathbf{x}}(t)) = \frac{\lambda}{2}\|\bar{\mathbf{x}}(t)\|^2_{\bar{\mathbf{P}}} \tag{A.16d}$$

and $\hat{\mathbf{Q}} = \mathrm{diag}(\hat{Q}_1, \ldots, \hat{Q}_M)$, $\hat{\mathbf{R}} = \mathrm{diag}(\hat{R}_1, \ldots, \hat{R}_M)$, $\hat{\mathbf{P}} = \mathrm{diag}(\hat{P}_1, \ldots, \hat{P}_M)$, $\bar{\mathbf{Q}} = \mathrm{diag}(\bar{Q}_1, \ldots, \bar{Q}_M)$, $\bar{\mathbf{R}} = \mathrm{diag}(\bar{R}_1, \ldots, \bar{R}_M)$, and $\bar{\mathbf{P}} = \mathrm{diag}(\bar{P}_1, \ldots, \bar{P}_M)$.

A.3. Proof of convergence

Denote with $\mathbf{X}(t_k|t_k) = (X_1(t_k|t_k), \ldots, X_M(t_k|t_k))$ the optimal solution to (A.10) at time t_k, and with $\mathbf{X}(t_{k+1}|t_k) = (X_1(t_{k+1}|t_k), \ldots, X_M(t_{k+1}|t_k))$ the feasible (non-optimal) solution to (A.10) at time t_{k+1}, where $X_i(t_{k+1}|t_k)$ is defined in (A.1), for all $i = 1, \ldots, M$. From (A.4) we have that

$$\mathbf{V}^{N*}(\mathbf{x}(t_{k+1})) - \mathbf{V}^{N*}(\mathbf{x}(t_k)) \le \mathbf{V}^N(\mathbf{X}(t_{k+1}|t_k)) - \mathbf{V}^N(\mathbf{X}(t_k|t_k)) \le -\hat{\mathbf{l}}(\hat{\mathbf{x}}(t_k|t_k), \hat{\mathbf{u}}(t_k|t_k)) + (a) + (b) \tag{A.17}$$

where

$$(a) = \hat{\mathbf{l}}(\hat{\mathbf{x}}(t_{k+N-1}|t_k), \hat{\mathbf{u}}(t_{k+N-1}|t_k)) + \hat{\mathbf{V}}^F(\hat{\mathbf{x}}(t_{k+N}|t_k)) - \bar{\mathbf{l}}(\bar{\mathbf{x}}(t_{k+N-1}|t_k), \bar{\mathbf{u}}(t_{k+N-1}|t_k)) - \hat{\mathbf{V}}^F(\hat{\mathbf{x}}(t_{k+N-1}|t_k))$$

$$(b) = \bar{\mathbf{l}}(\bar{\mathbf{x}}(t_{k+N}|t_k), \bar{\mathbf{u}}(t_{k+N}|t_k)) + \bar{\mathbf{V}}^F(\bar{\mathbf{x}}(t_{k+N+1}|t_k)) - \bar{\mathbf{V}}^F(\bar{\mathbf{x}}(t_{k+N}|t_k)).$$

Consider first term (b). If matrices $\bar{P}_i$, $i = 1, \ldots, M$, are chosen as the solutions to the Lyapunov equations (28) then, from (32), for all $i = 1, \ldots, M$, $\bar{V}_i^F(\bar{x}_i(t_{k+N+1}|t_k)) - \bar{V}_i^F(\bar{x}_i(t_{k+N}|t_k)) \le -\bar{l}_i(\bar{x}_i(t_{k+N}|t_k), \bar{u}_i(t_{k+N}|t_k))$ which, collectively, implies that $(b) \le 0$.

Considering now term (a), define the following collective quantities: $\mathbf{F}^* = \operatorname{diag}(\bar{F}_{11}, \ldots, \bar{F}_{MM})$, $\mathbf{A}^{zoh}(\eta) = \operatorname{diag}(A_{11}^{zoh}(\eta), \ldots, A_{MM}^{zoh}(\eta))$, $\mathbf{B}^{zoh}(\eta) = \operatorname{diag}(B_{11}^{zoh}(\eta), \ldots, B_{MM}^{zoh}(\eta))$.

Since $\bar{\mathbf{u}}(t|t_k)$ is constant for all $t \in [t_{k+N-1}, t_{k+N})$, and recalling (A.2), for $t \in [t_{k+N-1}, t_{k+N}]$ it results that

$$\begin{aligned}\bar{\mathbf{x}}(t|t_k) &= \mathbf{A}^{zoh}(t - t_{k+N-1})\bar{\mathbf{x}}(t_{k+N-1}|t_k)\\ &\quad + \mathbf{B}^{zoh}(t - t_{k+N-1})\bar{\mathbf{u}}(t_{k+N-1}|t_k)\end{aligned} \tag{A.18a}$$

and, from (A.2)

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t|t_k) &= \bar{\mathbf{F}}^*\hat{\mathbf{x}}(t|t_k) + (\tilde{\mathbf{A}} - \mathbf{B}^*\bar{\mathbf{K}})\bar{\mathbf{x}}(t|t_k)\mathbf{B}\bar{\mathbf{u}}(t|t_k)\\ &= \bar{\mathbf{F}}^*\hat{\mathbf{x}}(t|t_k) + (\tilde{\mathbf{A}} - \mathbf{B}^*\bar{\mathbf{K}})\mathbf{A}^{zoh}(t - t_{k+N-1})\bar{\mathbf{x}}(t_{k+N-1}|t_k)\\ &\quad + \left((\tilde{\mathbf{A}} - \mathbf{B}^*\bar{\mathbf{K}})\mathbf{B}^{zoh}(t - t_{k+N-1}) + \mathbf{B}\right)\bar{\mathbf{u}}(t|t_k).\end{aligned} \tag{A.18b}$$

Therefore, solving (A.18b), we obtain

$$\begin{aligned}\hat{\mathbf{x}}(t|t_k) &= \bar{\boldsymbol{\Phi}}_x(t - t_{k+N-1})\hat{\mathbf{x}}(t_{k+N-1}|t_k)\\ &\quad + \bar{\boldsymbol{\Gamma}}_{1x}(t - t_{k+N-1})\bar{\mathbf{x}}(t_{k+N-1}|t_k)\\ &\quad + \bar{\boldsymbol{\Gamma}}_{2x}(t - t_{k+N-1})\bar{\mathbf{u}}(t_{k+N-1}|t_k)\end{aligned} \tag{A.19}$$

$$\begin{aligned}\hat{\mathbf{u}}(t|t_k) &= \bar{\boldsymbol{\Phi}}_u(t - t_{k+N-1})\hat{\mathbf{x}}(t_{k+N-1}|t_k)\\ &\quad + \bar{\boldsymbol{\Gamma}}_{1u}(t - t_{k+N-1})\bar{\mathbf{x}}(t_{k+N-1}|t_k)\\ &\quad + \bar{\boldsymbol{\Gamma}}_{2u}(t - t_{k+N-1})\bar{\mathbf{u}}(t_{k+N-1}|t_k)\end{aligned} \tag{A.20}$$

where $\bar{\boldsymbol{\Phi}}_x(\eta) = e^{\mathbf{F}^*\eta}$, $\bar{\boldsymbol{\Gamma}}_{1x}(\eta) = \int_0^\eta e^{\mathbf{F}^*(\eta-\nu)}(\tilde{\mathbf{A}} - \mathbf{B}^*\bar{\mathbf{K}})\mathbf{A}^{zoh}(\nu)d\nu$, $\bar{\boldsymbol{\Gamma}}_{2x}(\eta) = \int_0^\eta e^{\mathbf{F}^*(\eta-\nu)}((\tilde{\mathbf{A}} - \mathbf{B}^*\bar{\mathbf{K}})\mathbf{B}^{zoh}(\nu) + \mathbf{B})d\nu$, $\bar{\boldsymbol{\Phi}}_u(\eta) = \bar{\mathbf{K}}\bar{\boldsymbol{\Phi}}_x(\eta)$, $\bar{\boldsymbol{\Gamma}}_{1u}(\eta) = \bar{\mathbf{K}}(\bar{\boldsymbol{\Gamma}}_{1x}(\eta) - \mathbf{A}^{zoh}(\eta))$, $\bar{\boldsymbol{\Gamma}}_{2u}(\eta) = \mathbf{I} + \bar{\mathbf{K}}(\bar{\boldsymbol{\Gamma}}_{2x}(\eta) - \mathbf{B}^{zoh}(\eta))$. Denote, for brevity, $\hat{\mathbf{x}}_{k+N-1} = \hat{\mathbf{x}}(t_{k+N-1}|t_k)$, $\mathbf{v}_{k+N-1} = (\bar{\mathbf{x}}(t_{k+N-1}|t_k), \bar{\mathbf{u}}(t_{k+N-1}|t_k))$, and

$$\begin{aligned}\bar{\boldsymbol{\Gamma}}_x(\eta) &= \left[\bar{\boldsymbol{\Gamma}}_{1x}(\eta)\ \bar{\boldsymbol{\Gamma}}_{2x}(\eta)\right]\\ \bar{\boldsymbol{\Gamma}}_u(\eta) &= \left[\bar{\boldsymbol{\Gamma}}_{1u}(\eta)\ \bar{\boldsymbol{\Gamma}}_{2u}(\eta)\right]\\ \bar{\mathbf{A}}_x(\eta) &= \left[\mathbf{A}^{zoh}(\eta)\ \mathbf{B}^{zoh}(\eta)\right]\\ \bar{\mathbf{A}}_u(\eta) &= \left[\mathbf{0}\ \mathbf{I}\right].\end{aligned}$$

Then, in view of (A.18a), (A.19), and (A.20) we compute the elements of term (a) as

$$\begin{aligned}&\hat{\mathbf{l}}(\hat{\mathbf{x}}(t_{k+N-1}|t_k), \hat{\mathbf{u}}(t_{k+N-1}|t_k))\\ &= \frac{1}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2_{\int_0^T \bar{\boldsymbol{\Phi}}_x(\eta)^T\hat{\mathbf{Q}}\bar{\boldsymbol{\Phi}}_x(\eta) + \bar{\boldsymbol{\Phi}}_u(\eta)^T\hat{\mathbf{R}}\bar{\boldsymbol{\Phi}}_u(\eta)d\eta}\\ &\quad + \frac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\int_0^T \bar{\boldsymbol{\Gamma}}_x(\eta)^T\hat{\mathbf{Q}}\bar{\boldsymbol{\Gamma}}_x(\eta) + \bar{\boldsymbol{\Gamma}}_u(\eta)^T\hat{\mathbf{R}}\bar{\boldsymbol{\Gamma}}_u(\eta)d\eta}\\ &\quad + \hat{\mathbf{x}}_{k+N-1}^T\left(\int_0^T \bar{\boldsymbol{\Phi}}_x(\eta)^T\hat{\mathbf{Q}}\bar{\boldsymbol{\Gamma}}_x(\eta)\right.\\ &\quad \left. + \ \bar{\boldsymbol{\Phi}}_u(\eta)^T\hat{\mathbf{R}}\bar{\boldsymbol{\Gamma}}_u(\eta)d\eta\right)\mathbf{v}_{k+N-1}\end{aligned} \tag{A.21a}$$

$$\begin{aligned}\hat{\mathbf{V}}^F(\hat{\mathbf{x}}(t_{k+N}|t_k)) &= \frac{1}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2_{\bar{\boldsymbol{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\boldsymbol{\Phi}}_x(T)}\\ &\quad + \frac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\bar{\boldsymbol{\Gamma}}_x(T)^T\hat{\mathbf{P}}\bar{\boldsymbol{\Gamma}}_x(T)}\\ &\quad + \hat{\mathbf{x}}_{k+N-1}^T\bar{\boldsymbol{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\boldsymbol{\Gamma}}_x(T)\mathbf{v}_{k+N-1}\end{aligned} \tag{A.21b}$$

$$\begin{aligned}&\bar{\mathbf{l}}(\bar{\mathbf{x}}(t_{k+N-1}|t_k), \bar{\mathbf{u}}(t_{k+N-1}|t_k))\\ &= \frac{\lambda}{2}\|\mathbf{v}_{k+N-1}\|^2_{\int_0^T \bar{\mathbf{A}}_x(\eta)^T\bar{\mathbf{Q}}\bar{\mathbf{A}}_x(\eta) + \bar{\mathbf{A}}_u(\eta)^T\bar{\mathbf{R}}\bar{\mathbf{A}}_u(\eta)d\eta}.\end{aligned} \tag{A.21c}$$

Define, for simplicity:

$$\mathscr{Q}_v = \int_0^T \left(\bar{\boldsymbol{\Gamma}}_x(\eta)^T\hat{\mathbf{Q}}\bar{\boldsymbol{\Gamma}}_x(\eta) + \bar{\boldsymbol{\Gamma}}_u(\eta)^T\hat{\mathbf{R}}\bar{\boldsymbol{\Gamma}}_u(\eta)\right)d\eta \tag{A.22a}$$

$$\mathscr{S}_{xv} = \int_0^T \left(\bar{\boldsymbol{\Phi}}_x(\eta)^T\hat{\mathbf{Q}}\bar{\boldsymbol{\Gamma}}_x(\eta) + \bar{\boldsymbol{\Phi}}_u(\eta)^T\hat{\mathbf{R}}\bar{\boldsymbol{\Gamma}}_u(\eta)\right)d\eta \tag{A.22b}$$

$$\mathscr{R}_v = \int_0^T \left(\bar{\mathbf{A}}_x(\eta)^T\bar{\mathbf{Q}}\bar{\mathbf{A}}_x(\eta) + \bar{\mathbf{A}}_u(\eta)^T\bar{\mathbf{R}}\bar{\mathbf{A}}_u(\eta)\right)d\eta. \tag{A.22c}$$

Therefore

$$\begin{aligned}(a) \quad = \quad &\frac{1}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2_{\bar{\boldsymbol{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\boldsymbol{\Phi}}_x(T) - \hat{\mathbf{P}} + \mathscr{Q}_x}\\ &+ \frac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\bar{\boldsymbol{\Gamma}}_x(T)^T\hat{\mathbf{P}}\bar{\boldsymbol{\Gamma}}_x(T) + \mathscr{Q}_v - \lambda\mathscr{R}_v}\\ &+ \hat{\mathbf{x}}_{k+N-1}^T(\bar{\boldsymbol{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\boldsymbol{\Gamma}}_x(T) + \mathscr{S}_{xv})\mathbf{v}_{k+N-1}.\end{aligned} \tag{A.23}$$

Recall that $\hat{\mathbf{P}}$ is the block-diagonal matrix whose blocks $\hat{P}_i$ satisfy (33) for all $i = 1, \ldots, M$, i.e., such that $\hat{\mathbf{P}}$ satisfies

$$\bar{\boldsymbol{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\boldsymbol{\Phi}}_x(T) - \hat{\mathbf{P}} + \mathscr{Q}_x + \alpha\mathbf{I} = 0 \tag{A.24}$$

where $\alpha > 1$ is an arbitrary scalar parameter. The following procedure is proposed for defining a suitable scalar λ.

(I) Define $\mathscr{S}_{Pxv} = \bar{\boldsymbol{\Phi}}_x(T)^T\hat{\mathbf{P}}\bar{\boldsymbol{\Gamma}}_x(T) + \mathscr{S}_{xv}$ and an arbitrary scalar $\beta > 0$ such that

$$\beta\mathbf{I} \ge \mathscr{S}_{Pxv}^T\mathscr{S}_{Pxv} \tag{A.25}$$

or equivalently

$$\beta \ge \|\mathscr{S}_{Pxv}\|_2^2. \tag{A.26}$$

(II) Define $\bar{\lambda}$ as the smallest value of $\lambda > 0$ satisfying

$$\lambda\mathscr{R}_v - \bar{\boldsymbol{\Gamma}}_x(T)^T\hat{\mathbf{P}}\bar{\boldsymbol{\Gamma}}_x(T) - \mathscr{Q}_v \ge \beta\mathbf{I}. \tag{A.27}$$

Note that, given $\hat{\mathbf{P}}$ and $\beta > 0$, since $\mathscr{R}_v > 0$, it is always possible to define $\bar{\lambda} > 0$. Finally, set $\lambda > \bar{\lambda}$.

According to the sketched procedure and in view of (A.24) and (A.27), from (A.23) we can write

$$\begin{aligned}(a) \le &-\frac{\alpha}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2 - \frac{\beta}{2}\|\mathbf{v}_{k+N-1}\|^2\\ &+ \hat{\mathbf{x}}_{k+N-1}^T\mathscr{S}_{Pxv}\mathbf{v}_{k+N-1}.\end{aligned} \tag{A.28}$$

Since

$$\begin{aligned}0 \le \frac{1}{2}\|\hat{\mathbf{x}}_{k+N-1} - \mathscr{S}_{Pxv}\mathbf{v}_{k+N-1}\|^2 &= \frac{1}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2\\ + \frac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\mathscr{S}_{Pxv}^T\mathscr{S}_{Pxv}} &- \hat{\mathbf{x}}_{k+N-1}^T\mathscr{S}_{Pxv}\mathbf{v}_{k+N-1}\end{aligned}$$

it follows that

$$\hat{\mathbf{x}}_{k+N-1}^T\mathscr{S}_{Pxv}\mathbf{v}_{k+N-1} \le \frac{1}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2 + \frac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\mathscr{S}_{Pxv}^T\mathscr{S}_{Pxv}}$$

and, from (A.28)

$$(a) \le \frac{1-\alpha}{2}\|\hat{\mathbf{x}}_{k+N-1}\|^2 + \frac{1}{2}\|\mathbf{v}_{k+N-1}\|^2_{\mathscr{S}_{Pxv}^T\mathscr{S}_{Pxv} - \beta\mathbf{I}}. \tag{A.29}$$

Therefore, since β satisfies (A.25) and $\alpha > 1$, then $(a) \le 0$. From (A.17), and having proved that both $(a) \le 0$ and $(b) \le 0$, we obtain that

$$\mathbf{V}^{N*}(\mathbf{x}(t_{k+1})) - \mathbf{V}^{N*}(\mathbf{x}(t_k)) \le -\hat{\mathbf{l}}(\hat{\mathbf{x}}(t_k|t_k), \hat{\mathbf{u}}(t_k|t_k)). \tag{A.30}$$

Therefore $\hat{\mathbf{l}}(\hat{\mathbf{x}}(t_k|t_k), \hat{\mathbf{u}}(t_k|t_k)) \to 0$ as $k \to \infty$. Under suitable smoothness assumptions on $\hat{\mathbf{u}}(t|t_k)$ and $\hat{\mathbf{x}}(t|t_k)$ and since $\hat{\mathbf{Q}} > 0$ and $\hat{\mathbf{R}} > 0$, it follows that $\hat{\mathbf{x}}([t_k, t_{k+1})|t_k) \to 0$ and $\hat{\mathbf{u}}([t_k, t_{k+1})|t_k) \to 0$ as $k \to \infty$.

Recalling now system (2) where, for all $k \in \mathbb{N}$, $t \in [t_k, t_{k+1})$, $\mathbf{u}(t) = \hat{\mathbf{u}}(t|t_k) + \mathbf{K}^c(\mathbf{x}(t|t_k) - \hat{\mathbf{x}}(t))$. We can write

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{B}\mathbf{K}^c)\mathbf{x}(t) + \mathbf{B}(\hat{\mathbf{u}}(t|t_k) - \mathbf{K}^c\mathbf{x}(t|t_k)).$$

Since $\mathbf{B}(\hat{\mathbf{u}}(t|t_k) - \mathbf{K}^c\mathbf{x}(t|t_k))$ is an asymptotically vanishing term and since $\mathbf{A} + \mathbf{B}\mathbf{K}^c$ is Hurwitz in view of Assumption 2, we obtain that $\mathbf{x}(t) \to 0$ as $t \to \infty$.

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