Stability analysis of linear systems subject to regenerative switchings

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Abstract

This paper investigates the stability of switched linear systems whose switching signal is modeled as a stochastic process called a regenerative process. We show that the mean stability of such a switched system is characterized by the spectral radius of a matrix. The matrix is obtained by taking the expectation of the transition matrix of the system on one cycle of the underlying regenerative process. The characterization generalizes Floquet's theorem for the stability analysis of linear time-periodic systems. We illustrate the result with the stability analysis of a linear system with a failure-prone controller under periodic maintenance.

Keywords: Switched linear system, regenerative process, mean stability, periodic maintenance

1. Introduction

The stability analysis of stochastic switched linear systems has attracted a significant amount of attention in the last two decades. In particular, a lot of effort has been put on their mean stability, which requires that the power of the norm of the state variable converges to zero in expectation. Some early results on the mean stability of switched linear systems with an independent and identically distributed (i.i.d.) switching signal can be found in [1, 2]. The stability characterizations [3, 4] of linear systems subject to switching

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by homogeneous Markov process now form the basis of the various types of optimal control of so-called Markov jump linear systems [5]. The stability characterizations of switched linear systems driven by an extension of homogeneous Markov processes called homogeneous semi-Markov processes [6] are available in [7, 8].

It is known that homogeneous Markov processes having certain irreducibility and recurrence properties and also discrete-time i.i.d. stochastic processes are special cases of a more general class of stochastic processes called *regenerative processes* [9]. Firstly introduced by Smith [10], regenerative processes have found applications especially in queuing systems [11] and network reliability analysis [12]. As we will see later in Example 2, regenerative processes are also suitable to describe a controlled system under periodic maintenance [13, 14, 15]. Despite the above facts, as far as we are aware of, no effort has been made to investigate switched linear systems with a regenerative switching signal in the literature of systems and control theory.

The aim of this paper is to give the characterization of the mean stability of a switched linear system with a regenerative switching signal, which we call a *regenerative switched linear system*. We show that, if the exponent of the mean stability is even or the system is positive [7, 16], then the mean stability of the system is characterized by the spectral radius of a matrix. The matrix is obtained as the expected value of the lift [17] of the transition matrix of the system over one cycle of the underlying regenerative process. The proof makes use of a stability-preserving discretization of the system at the embedded renewal process of the underlying regenerative process. The characterization in particular generalizes well-known Floquet's theorem for the stability analysis of linear time-periodic systems [18].

This paper is organized as follows. After preparing necessary notations and conventions, in Section 2 we recall the definition of regenerative processes and then introduce regenerative switched linear systems. Then Section 3 presents the main result of this paper, which is followed by an example. The proof of the main result is given in Section 4. Then Section 5 discusses the discrete-time case.

1.1. Mathematical preliminaries

Let (Ω, \mathcal{M}, P) be a probability space. For an integrable random variable X on Ω its expected value is denoted by E[X]. The random variables that appear in this paper will be assumed to be integrable.

When $x \in \mathbb{R}^n$ is nonnegative entrywise we write $x \ge 0$. The standard Euclidean norm on \mathbb{R}^n is denoted by $\|\cdot\|$. For $m \ge 1$, the *m*-norm on \mathbb{R}^n is defined by $\|x\|_m = (\sum_{i=1}^n |x_i|^m)^{1/m}$. The symbol 1_n denotes the column vector of length *n* whose entries are all 1. It is easy to see $\|x\|_1 = 1_n^{\top} x$ if $x \ge 0$. Let *I* and *O* denote the identity and the zero matrices, respectively. We say that $A \in \mathbb{R}^{n \times n}$ is Schur stable if its spectral radius $\rho(A)$ is less than one.

The *m*-lift of $x \in \mathbb{R}^n$, denoted by $x^{[m]}$, is defined [17] as the real vector of length $n_m = \binom{n+m-1}{m}$ with its elements being the lexicographically ordered monomials $\sqrt{\alpha!} x^{\alpha}$ indexed by all the possible exponents $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1, \ldots, m\}^n$ such that $\alpha_1 + \cdots + \alpha_n = m$, where $\alpha! := m!/(\alpha_1! \cdots \alpha_n!)$. It holds [17] that

$$\|x^{[m]}\| = \|x\|^m.$$
(1)

We then define $A^{[m]} \in \mathbb{R}^{n_m \times n_m}$ as the unique matrix [17] satisfying $(Ax)^{[m]} = A^{[m]}x^{[m]}$ for every $x \in \mathbb{R}^n$. For any matrix B it holds that

$$(AB)^{[m]} = A^{[m]}B^{[m]} (2)$$

provided the product AB is well defined. We also define $A_{[m]} \in \mathbb{R}^{n_m \times n_m}$ as the unique real matrix [19, 20] such that, for every \mathbb{R}^n -valued differentiable function x on \mathbb{R} satisfying dx/dt = Ax, it holds that $dx^{[m]}/dt = A_{[m]}x^{[m]}$. It is easy to check that

$$(e^{At})^{[m]} = e^{A_{[m]}t}$$
(3)

for every $t \ge 0$.

2. Regenerative switched linear systems

Let us first recall the definition of regenerative stochastic processes [9]. Throughout this paper we fix an underlying probability space (Ω, \mathcal{M}, P) .

Definition 1. A stochastic process $\sigma = {\sigma_t}_{t\geq 0}$ is called a *regenerative process* if there exists a random variable $R_1 > 0$, called a regeneration epoch, such that the following statements hold.

- $\{\sigma_{t+R_1}\}_{t\geq 0}$ is independent of $\{\{\sigma_t\}_{t< R_1}, R_1\};$
- $\{\sigma_{t+R_1}\}_{t\geq 0}$ is stochastically equivalent to $\{\sigma_t\}_{t\geq 0}$.

In the following we quote some consequences of the above definition from [9]. By repeatedly applying the definition, one can obtain a sequence of independent and identically distributed random variables $\{R_k\}_{k\geq 1}$ called *cy*cle lengths, which can be used to break σ into independent and identically distributed cycles $\{\sigma_t\}_{0\leq t< R_1}, \{\sigma_t\}_{R_1\leq t< R_1+R_2}, \ldots$ Then the stochastic process $\{Z_k\}_{k\geq 1}$ defined by $Z_k = R_1 + \cdots + R_k$ is called the *embedded renewal* process of σ . Throughout this paper, for the sake of convenience, we set $Z_0 = 0$ and call $\{Z_k\}_{k\geq 0}$ as the embedded renewal process of σ .

The next example presents a regenerative process that is not a homogeneous Markov process and is of a systems and control theoretical interest.

Example 2. Consider a dynamical system with a failure-prone controller [21]. Let us model the controlled system as a switched system with the two modes $\{1,2\} = \{$ Non-failure, Failure $\}$. Instead of assuming that the transition of the mode can be described by a homogeneous Markov process (see, e.g., [22, 23]), let us consider the scenario when the controlled system is under periodic maintenance, which is commonly employed in the literature from reliability theory [13, 14, 15].

Let the stochastic process $\{Z_k\}_{k\geq 0}$ represent the times at which a maintenance is performed. For simplicity we assume that $\sigma_{Z_k} = 1$ with probability one for every $k \geq 0$, i.e., that every maintenance repairs a failure with probability one within a negligible time period. We set $Z_0 = 0$. Furthermore we assume that $R_k = Z_k - Z_{k-1}$ equals $T + \Delta_k$, where T > 0 is a constant and $\{\Delta_k\}_{k=1}^{\infty}$ are independent and identically distributed random variables. T represents the designed period of the maintenance and Δ_k models its random perturbation. We assume that the length of the time for which the process σ stays at mode 1 after the reset at $t = Z_k$ follows an exponential distribution with parameter $\lambda > 0$. In other words we are assuming that, on any interval of a sufficiently small length h, the probability of the occurrence of a failure is approximately equal to λh . Then σ is clearly a regenerative process with a regeneration epoch R_1 and the embedded renewal process $\{Z_k\}_{k\geq 0}$.

Notice that σ is not a Markov process because the length of the time while mode 2 is active depends on past information, i.e., the length of the last time interval while mode 1 was active.

Then we introduce the class of switched linear systems studied in this paper. Let $\{\sigma_t\}_{t>0}$ be a regenerative process that can take values in a set S.

Let $\{A_s\}_{s\in\mathcal{S}}$ be a family of real $n \times n$ matrices indexed by \mathcal{S} . Then we call the stochastic differential equation

$$\Sigma : \frac{dx}{dt} = A_{\sigma_t} x(t)$$

as a regenerative switched linear system. We assume that $x(0) = x_0 \in \mathbb{R}^n$ is a constant vector.

The stability of Σ is defined in the following standard manner.

Definition 3. Let m be a positive integer.

- Σ is said to be *exponentially mth mean stable* if there exist $\alpha > 0$ and $\beta > 0$ such that $E[||x(t)||^m] \leq \alpha e^{-\beta t} ||x_0||^m$ for all x_0 and $t \geq 0$.
- Σ is said to be stochastically mth mean stable if $\int_0^\infty E[||x(t)||^m] dt < \infty$ for any x_0 .

We also introduce the notion of positivity for Σ following [7, 16].

Definition 4. We say that Σ is *positive* if $x_0 \ge 0$ implies $x(t) \ge 0$ with probability one for every $t \ge 0$.

For Σ to be positive it is clearly sufficient that all the matrices A_s are Metzler, i.e., the off-diagonal entries of each A_s are all nonnegative [18]. However it is not necessary, as illustrated in the following non-trivial example.

Example 5. Consider a switched linear system with $S = \{1, 2\}$ and

$$A_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since $e^{A_1t} = I + (e^t - 1)A_1$, a simple calculation shows the existence of T > 0such that if $t \ge T$ then, for every $x_0 \ge 0$, the vector $e^{A_1t}x_0$ is in the sector $S = \{x \in \mathbb{R}^2 : x \ge 0, \arg x \ge 1\}$. Then we construct a regenerative process σ as follows. Set $R_1 = T + 1$ and let h follow the uniform distribution on [T, T + 1]. Define σ on the first cycle $[0, R_1)$ by

$$\sigma_t = \begin{cases} 1, & 0 \le t \le h \\ 2, & h \le t < R_1 \end{cases}$$

and extend this definition to the whole interval $[0, \infty)$ regeneratively. We can see that the above defined σ is a regenerative process as in Example 2. Then, since the second mode decreases the argument of the state vector at most $R_1 - h < 1$, we can see that x(t) stays in the positive orthant for every $t \ge 0$ whenever $x_0 \ge 0$. Therefore Σ is positive although A_2 is not Metzler.

3. Stability characterization

This section states the characterization of the mean stability of regenerative switched linear systems and also presents an example to illustrate the result. We state the next assumption.

Assumption 6.

A1. Either m is even or Σ is positive.

A2. R_1 is essentially bounded.

A3. The set $\{A_s\}_{s \in S}$ is bounded.

A1 covers mean square stability (m = 2), which has been the central stability notion of stochastic switched linear systems in the literature [3, 8, 24]. The second condition A2 on the boundedness of cycle lengths is crucial. Similar assumptions were employed for the stability analysis of semi-Markov jump linear systems [7] and stochastic hybrid systems with renewal transitions [8]. A3 is only to ensure that the state variable does not diverge in a finite time and thus is not restrictive.

In order to state the main result we need fundamental matrices [18] of the system Σ . For all $\omega \in \Omega$, $t_0 \ge 0$, and $t \ge t_0$ let us define $\Phi(\omega; t_0, t) \in \mathbb{R}^{n \times n}$ by the differential equation

$$\frac{\partial \Phi}{\partial t} = A_{\sigma_t(\omega)} \Phi(\omega; t_0, t), \ \Phi(\omega; t_0, t_0) = I_n.$$

Then define the $\mathbb{R}^{n \times n}$ -valued random variables $\{M_k\}_{k \ge 0}$ by

$$M_k(\omega) := \Phi(\omega; Z_k(\omega), Z_{k+1}(\omega)), \tag{4}$$

which expresses the transition of x from $t = Z_k$ to $t = Z_{k+1}$.

The next theorem is the main result of this paper.

Theorem 7. The following statements are equivalent.

- 1. Σ is exponentially mth mean stable.
- 2. Σ is stochastically mth mean stable.
- 3. $E[M_0^{[m]}]$ is Schur stable.

Based on the theorem and continuing from Example 2, the next example presents the stability analysis of a linear time-invariant system with a failureprone controller under periodic maintenance.

Example 8. Consider the internally unstable linear time-invariant system dx/dt = Ax + Bu with the failure-prone controller

$$u(t) = \begin{cases} 0, & \text{a fault is occuring} \\ Kx(t), & \text{otherwise} \end{cases}$$

where

$$A = \begin{bmatrix} -0.4 & 0.2 \\ -0.1 & 0.5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, K = \begin{bmatrix} -0.1 & -1.6 \end{bmatrix}.$$

The stabilizing feedback gain K is obtained by solving a linear quadratic regulator problem. We assume that the transition between the modes $\{1, 2\} =$ $\{\text{Non-failure, Failure}\}$ follows the regenerative process σ described in Example 2. With this labeling we have $A_1 = A + BK$ and $A_2 = A$. For simplicity we set $\lambda = 1$ and also suppose that each Δ_k follows the uniform distribution on the interval [-0.1T, 0.1T] independently.

Let the random variable h denote the first time in $[0, R_1)$ when the transition to mode 2 occurs. We set $h = R_1$ when a transition does not occur on the interval. Then one can see $M_0 = e^{A_2 \max(0, R_1 - h)} e^{A_1 \min(R_1, h)}$. If we let $\bar{A}_i := (A_i)_{[m]}$ (i = 1, 2) then equations (3) and (2) show

$$M_0^{[m]} = e^{\bar{A}_2 \max(0, R_1 - h)} e^{\bar{A}_1 \min(R_1, h)}.$$

Here we recall that for square matrices F_1, F_2 with the same dimensions and $t \ge 0$ it holds that [25]

$$\exp\left(\begin{bmatrix}F_1 & I\\ O & F_2\end{bmatrix}t\right) = \begin{bmatrix}* & \int_0^t e^{(t-\tau)F_1}e^{\tau F_2} d\tau\\ 0 & *\end{bmatrix}.$$

Using this identity and the independence of h and R_1 , since h follows the



Figure 1: Spectral radius of $E[\Phi^{[2]}]$ as T varies

exponential distribution with mean 1, we can show that

$$\begin{split} E[M_0^{[m]}] &= \int_{0.9T}^{1.1T} \int_0^\infty e^{\bar{A}_2 \max(0,t-s)} e^{\bar{A}_1 \min(t,s)} e^{-s} ds \, \frac{dt}{0.2T} \\ &= \frac{5}{T} \int_{0.9T}^{1.1T} \int_0^t e^{\bar{A}_2(t-s)} e^{(\bar{A}_1-I)s} \, ds \, dt + \frac{5}{T} \int_{0.9T}^{1.1T} \int_t^\infty e^{\bar{A}_1 t} e^{-s} \, ds \, dt \\ &= \frac{5}{T} \begin{bmatrix} I & O \end{bmatrix} \int_{0.9T}^{1.1T} \exp\left(\begin{bmatrix} \bar{A}_2 & I \\ O & \bar{A}_1 - I \end{bmatrix} t\right) \, dt \begin{bmatrix} O \\ I \end{bmatrix} \\ &+ \frac{5}{T} \int_{0.9T}^{1.1T} e^{(\bar{A}_1 - I)t} \, dt. \end{split}$$

Figure 1 shows the graph of the spectral radius of $E[M_0^{[2]}]$ as T varies from 0 to 2. As is expected, instability is caused by making the period of the maintenance longer. We can see that, by Theorem 7, Σ is mean square stable if and only if T < 1.55. The computation of the matrix $E[M_0^{[2]}]$ is performed with MATLAB. Figure 2 shows 20 sample paths of $||x(t)||^2$ when T = 1.25.

Remark 9. Theorem 7 extends celebrated Floquet's theory [18] for the stability analysis of linear time-periodic systems. In fact, if the process σ is a periodic function with the period T then we have $\rho(E[M_0^{[m]}]) = \rho(M_0^{[m]}) =$ $\rho(M_0)^m$. Therefore, by Theorem 7, the linear time-periodic system Σ is stable in the standard sense if and only if $\rho(M_0) < 1$, which is the main consequence of Floquet's theory.



Figure 2: 20 sample paths of $||x(t)||^2$

4. Proof of the main result

The proof of Theorem 7 is based on the discretization of Σ at the embedded renewal process of the underlying regenerative process. In order to analyze the stability of the discretization, in the next section we first present the stability analysis of discrete-time switched linear systems with i.i.d. parameters. Then in Section 4.2 we give the proof of Theorem 7.

4.1. Stability of discrete-time linear systems with i.i.d. parameters

Let $\{F_k\}_{k\geq 0}$ be independent and identically distributed random variables following a distribution μ on $\mathbb{R}^{n\times n}$. Consider the discrete-time switched linear system

$$\Sigma_{\mu} : x_d(k+1) = F_k x_d(k), \ k \ge 0$$

where $x_d(0) = x_0 \in \mathbb{R}^n$ is a constant vector. The mean stability of Σ_{μ} is introduced as follows.

Definition 10. Let m be a positive integer.

• Σ_{μ} is said to be *exponentially mth mean stable* if there exist $\alpha > 0$ and $\beta > 0$ such that

$$E[\|x_d(k)\|^m] \le \alpha e^{-\beta k} \|x_0\|^m$$
(5)

for all x_0 and $k \ge 0$.

• Σ_{μ} is said to be stochastically mth mean stable if $\sum_{k=0}^{\infty} E[||x_d(k)||^m] < \infty$ for any x_0 .

Also we define the positivity of Σ_{μ} in the same way as for Σ .

Definition 11. We say that Σ_{μ} is positive if $x_0 \ge 0$ implies $x(k) \ge 0$ with probability one for every $k \ge 0$.

When we check the exponential mean stability of a positive Σ_{μ} , we can without loss of generality assume that its initial state is nonnegative.

Lemma 12. Assume that Σ_{μ} is positive. Then Σ_{μ} is exponentially mth mean stable if and only if there exist $\alpha > 0$ and $\beta > 0$ such that (5) holds for all $x_0 \ge 0$ and $k \ge 0$.

Proof. See the proof of [7, Lemma 3.6].

The next proposition characterizes the stability of Σ_{μ} in terms of the spectral radius of a matrix.

Proposition 13. Assume that either

- a. m is even or
- b. Σ_{μ} is positive.

Then the following conditions are equivalent.

- 1. Σ_{μ} is exponentially mth mean stable.
- 2. Σ_{μ} is stochastically mth mean stable.
- 3. $E[F_0^{[m]}]$ is Schur stable.

Proof. We shall show the cycle $[1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1]$. One can easily see $[1 \Rightarrow 2]$.

 $[2 \Rightarrow 3]$: Let $x(\cdot; x_0)$ denote the trajectory of Σ_{μ} with the initial state x_0 . Since the identity (2) shows $E[x(k+1; x_0)^{[m]}] = E[F_0^{[m]}]E[x(k; x_0)^{[m]}]$, an induction with respect to k yields

$$E[x(k;x_0)^{[m]}] = E[F_0^{[m]}]^k x_0^{[m]}.$$
(6)

Now assume that Σ_{μ} is stochastically *m*th mean stable. Let λ be an eigenvalue of $E[F_0^{[m]}]$ with a corresponding eigenvector $v \in \mathbb{C}^{n_m}$. Since the set $\{x^{[m]}: x \in \mathbb{R}^n\}$ spans \mathbb{R}^{n_m} ([7, Lemma 1.5]), there exist $y_1, \ldots, y_\ell \in \mathbb{R}^n$ and $c_1, \ldots, c_\ell \in \{1, \sqrt{-1}\}$ such that $v = \sum_{i=1}^{\ell} c_i y_i^{[m]}$. Multiplying $E[F_0^{[m]}]^k$ to this equation we obtain $\lambda^k v = \sum_{i=1}^{\ell} c_i E[x(k; y_i)^{[m]}]$ by (6). Therefore, by the

triangle inequality and (1), we can see $|\lambda|^k ||v|| \leq \sum_{i=1}^{\ell} E[||x(k; y_i)||^m]$. Since the right hand side of this inequality is summable with respect to k by the stochastic mth mean stability of Σ_{μ} and also $||v|| \neq 0$, we conclude $|\lambda| < 1$.

 $[3 \Rightarrow 1]$: Assume that $E[F_0^{[m]}]$ is Schur stable. By (6), there exist $\alpha > 0$ and $\beta > 0$ such that

$$\|E[x(k)^{[m]}]\| \le \alpha e^{-\beta k} \|x_0^{[m]}\| = \alpha e^{-\beta k} \|x_0\|^m$$
(7)

for every $k \ge 0$. We shall show that Σ_{μ} is exponentially *m*th mean stable. Let x_0 and $k \ge 0$ be arbitrary and write y = x(k). We consider the two cases a and b separately. First assume that *m* is even. Take positive constants C_1 and C_2 such that $C_1 \|\cdot\|_1 \le \|\cdot\| \le C_2 \|\cdot\|_m$. Then

$$E[\|y\|^{m}] \le C_{2}^{m} E[\|y\|_{m}^{m}] = C_{2}^{m} \sum_{i=1}^{n} |E[y_{i}^{m}]|.$$
(8)

Since the random vector $y^{[m]}$ has all the monomials y_i^m $(i = 1, \dots, m)$, we can see that $\sum_{i=1}^n |E[y_i^m]| \le ||E[y]^{[m]}||_1 \le C_1^{-1} ||E[y]^{[m]}||$. Therefore this inequality together with (8) and (7) shows that Σ_{μ} is exponentially *m*th mean stable.

Next assume that Σ_{μ} is positive. Notice that, by Lemma 12, without loss of generality we can assume $x_0 \geq 0$, which implies $y \geq 0$ and hence $y^{[m]} \geq 0$ with probability one. Let us take a positive constant C_3 such that $\|\cdot\| \leq C_3 \|\cdot\|_1$. Then we have $\|y\|^m = \|y^{[m]}\| \leq C_3 \|y^{[m]}\|_1 = C_3 \mathbf{1}_{n_m}^{\top} y^{[m]}$ with probability one. Therefore, the Schwartz inequality shows $E[\|y\|^m] \leq C_3 \mathbf{1}_{n_m}^{\top} E[y^{[m]}] \leq C_3 \|\mathbf{1}_{n_m}\| \|E[y^{[m]}]\|$. This inequality and (7) prove the exponential *m*th mean stability of Σ_{μ} .

Remark 14. Proposition 13 improves the stability condition in [26] by reducing the computational cost for checking mean stability. The size n_m of the matrix $E[F_0^{[m]}]$ is far less than the size n^m of the matrix used in [26, Theorem 5.1]. Also the proof presented above is simpler than the proof of [26, Theorem 5.1], which needs the approximation of μ by a sequence of finitely supported probability measures.

4.2. Proof of the main result

Let Σ be a regenerative switched linear system satisfying the conditions A1 to A3 and let x be the trajectory of Σ . Then the discretized process

 ${x_d(k)}_{k\geq 0}$ given by $x_d(k) = x(Z_k)$ is clearly the solution of the discrete-time system

$$\mathcal{S}\Sigma: x_d(k+1) = M_k x_d(k), \ k \ge 0$$

where M_k is defined by (4). Proposition 13 immediately gives the next corollary on the stability of $S\Sigma$.

Corollary 15. The following conditions are equivalent.

- 1. $S\Sigma$ is exponentially mth mean stable.
- 2. $S\Sigma$ is stochastically mth mean stable.
- 3. $E[M_0^{[m]}]$ is Schur stable.

Proof. The random variables $\{M_k\}_{k=0}^{\infty}$ are independent and identically distributed by the definition of regenerating processes. Also A1 automatically ensures that one of the conditions a and b in Proposition 13 is satisfied. \Box

We will also need the next lemma to prove the main result.

Lemma 16. There exists C > 1 such that

$$C^{-1} \|x(Z_k)\| \le \|x(t)\| \le C \|x(Z_k)\|$$
(9)

for all $k \geq 0$ and $t \in [Z_k, Z_{k+1}]$.

Proof. Let $t \in [Z_k, Z_{k+1}]$. By A3 there exists a constant, say, M > 0, such that $||A_s|| \leq M$ for every $s \in S$. Since $x(t) = \int_{Z_k}^t A_{\sigma_\tau} x(\tau) d\tau + x(Z_k)$ we have $||x(t)|| \leq \int_{Z_k}^t M ||x(\tau)|| d\tau + ||x(Z_k)||$. Then Gronwall's inequality and A2 shows $||x(t)|| \leq e^{M||R_1||} ||x(Z_k)||$, where $||R_1||$ denotes the essential supremum of R_1 . Similarly we can show $e^{-M||R_1||} ||x(Z_k)|| \leq ||x(t)||$. This completes the proof.

Now we prove the main result of this paper.

Proof of Theorem 7. We shall show the cycle $[1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1]$. It is obvious to prove $[1 \Rightarrow 2]$.

 $[2 \Rightarrow 3]$: By inequality (9) and the definition of regenerative processes, for every k we can show

$$E\left[\int_{Z_{k}}^{Z_{k+1}} \|x(t)\|^{m} dt\right] \geq C^{-m} E\left[\|x(Z_{k})\|^{m} \int_{Z_{k}}^{Z_{k+1}} dt\right]$$

= $C^{-m} E[R_{k+1}] E[\|x_{d}(k)\|^{m}]$
= $C^{-m} E[R_{1}] E[\|x_{d}(k)\|^{m}],$ (10)

where we used the definition of regenerative processes. Since Fubini's theorem shows $\int_0^\infty E[\|x(t)\|^m] dt = \sum_{k=0}^\infty E\left[\int_{Z_k}^{Z_{k+1}} \|x(t)\|^m dt\right]$, taking the summation about k in (10) yields

$$C^{-m}E[R_1]\sum_{k=0}^{\infty}E[\|x_d(k)\|^m] \le \int_0^{\infty}E[\|x(t)\|^m]\,dt < \infty.$$

Therefore $S\Sigma$ is stochastically *m*th mean stable because both *C* and $E[R_1]$ are positive. Hence Corollary 15 implies that $E[M_0^{[m]}]$ is Schur stable.

 $[3 \Rightarrow 1]$: Here we employ the idea used in the proof of the sufficiency part for [7, Theorem 2.5]. Assume that $E[M_0^{[m]}]$ is Schur stable. Then $S\Sigma$ is exponentially *m*th mean stable by Corollary 15. Let x_0 and $t \ge 0$ be arbitrary. Let us define $k_t = \max\{k \in \mathbb{N} : Z_k \le t\}$. Since $Z_{k_t} \le t < Z_{k_{t+1}}$, the inequality (9) gives $||x(t)|| \le C ||x(Z_{k_t})|| = C ||x_d(k_t)||$. Therefore

$$E[\|x(t)\|^{m}] \le C^{m} E[\|x_{d}(k_{t})\|^{m}].$$
(11)

On the other hand, since A2 shows $t < Z_{k_t+1} \leq ||R_1||(k_t+1)$ we have $k_t > ||R_1||^{-1}t - 1$. This implies $||x_d(k_t)||^m \leq \sum_{k > ||R_1||^{-1}t-1} ||x_d(k)||^m$. Taking the expectation in this inequality and then using the *m*th mean stability of $S\Sigma$ we obtain $E[||x_d(k_t)||^m] \leq \alpha' e^{-\beta' t} ||x_0||^m$, where $\alpha' = \alpha e^{\beta}/(1 - e^{-\beta})$ and $\beta' = \beta/||R_1||$ are positive constants. This inequality and (11) show the *m*th mean exponential stability of Σ .

5. Discrete-time case

This section briefly discusses the stability characterization of regenerative switched linear systems in discrete-time. Let $\sigma = {\{\sigma_k\}_{k=0}^{\infty}}$ be a regenerative

process taking values in a set S and defined on the set of nonnegative integers $\{0, 1, ...\}$. Let $\{A_s\}_{s \in S}$ be a family of $n \times n$ real matrices. Consider the discrete-time regenerative switched linear system

$$\Sigma_d : x(k+1) = A_{\sigma_k} x(k), \ k \ge 0.$$

The exponential and stochastic mean stability of Σ_d are defined as in Definition 10. In addition to the assumptions A1 to A3 we place the next assumption on Σ_d :

A4. A_s is invertible for each $s \in \mathcal{S}$ and the set $\{A_s^{-1}\}_{s \in \mathcal{S}}$ is bounded.

For each $k \ge 0$ we define the transition matrix $M_{d,k}$ for Σ_d representing the transition of x from $t = Z_k$ to $t = Z_{k+1}$ in the same way as we defined M_k for continuous-time regenerative switched linear systems in (4). The next theorem is a discrete-time counterpart of Theorem 7.

Theorem 17. The following statements are equivalent.

- 1. Σ_d is exponentially mth mean stable.
- 2. Σ_d is stochastically mth mean stable.
- 3. $E[M_{d,0}^{[m]}]$ is Schur stable.

Proof. Let $\{Z_k\}_{k=0}^{\infty}$ be the embedded renewal process of σ . Using A3 and A4 we can show the existence of a constant C > 1 such that, for every $k \ge 0$, if $Z_k \le \ell \le Z_{k+1}$ then $C^{-1} ||x_d(Z_k)|| \le ||x_d(\ell)|| \le C ||x_d(Z_k)||$. Then we can prove the desired equivalence in the same way as in the proof of Theorem 7. The details are omitted.

6. Conclusion

In this paper we investigated the mean stability of regenerative switched linear systems. A necessary and sufficient condition for the mth mean stability of regenerative switched linear systems was established under the assumption that either m is even or the system is positive and that the length of each cycle of the underlying regenerative process is essentially bounded. The proof used a discretization of the system at the embedded renewal process of the underlying regenerative process. A numerical example was presented to illustrate the result.

References

- [1] R. E. Kalman, Analysis and synthesis of linear systems operating on randomly sampled data, Ph.D. thesis, Columbia University (1957).
- [2] J. Bertram, P. Sarachik, Stability of circuits with randomly time-varying parameters, IRE Trans. Circuit Theory 6 (1959) 260–270.
- [3] X. Feng, K. A. Loparo, Y. Ji, H. J. Chizeck, Stochastic stability properties of jump linear systems, IEEE Trans. Automat. Control 31 (1992) 38–53.
- [4] O. Costa, M. Fragoso, Stability results for discrete-time linear systems with Markovian jumping parameters, J. Math. Anal. Appl. 179 (1993) 154–178.
- [5] O. L. Costa, M. D. Fragoso, M. G. Todorov, Continuous-time Markov Jump Linear Systems, Springer, 2013.
- [6] J. Janssen, R. Manca, Applied Semi-Markov Processes, Springer-Verlag, Berlin, 2006.
- [7] M. Ogura, C. F. Martin, Stability analysis of positive semi-Markovian jump linear systems with state resets, SIAM J. Control Optim. (accepted). arXiv:1309.2720 [math.OC]
- [8] D. Antunes, J. P. Hespanha, C. Silvestre, Stochastic hybrid systems with renewal transitions: moment analysis with application to networked control systems with delays, SIAM J. Control Optim. 51 (2013) 1481–1499.
- [9] K. Sigman, R. Wolff, A review of regenerative processes, SIAM Rev. 35 (1993) 269–288.
- [10] W. L. Smith, Regenerative stochastic processes, Proceedings of the Royal Soc. A: Math., Phys. Eng. Sci. 232 (1955) 6–31.
- [11] J. Zhang, B. Zwart, Steady state approximations of limited processor sharing queues in heavy traffic, Queueing Syst. 60 (2008) 227–246.
- [12] D. Logothetis, K. Trivedi, The effect of detection and restoration times for error recovery in communication networks, J. Netw. Syst. Manag. 5 (1997) 173–195.

- [13] R. V. Canfield, Cost optimization of periodic preventive maintenance, IEEE Trans. Reliab. 35 (1986) 78–81.
- [14] T. Nakagawa, Periodic and sequential preventive maintenance policies, J. Appl. Probab. 23 (1986) 536–542.
- [15] I. Gertsbakh, Reliability Theory: With Applications to Preventive Maintenance, Springer, 2000.
- [16] P. Bolzern, P. Colaneri, G. De Nicolao, Stochastic stability of Positive Markov Jump Linear Systems, Automatica. doi:10.1016/j. automatica.2014.02.016.
- [17] P. A. Parrilo, A. Jadbabaie, Approximation of the joint spectral radius using sum of squares, Linear Algebra Appl. 428 (2008) 2385–2402.
- [18] L. Farina, S. Rinaldi, Positive Linear Systems: Theory and Applications, Wiley-Interscience, 2000.
- [19] R. W. Brockett, Lie theory and control systems defined on spheres, SIAM J. Appl. Math. 25 (1973) 213–225.
- [20] A. Barkin, A. Zelentsovsky, Method of power transformations for analysis of stability of nonlinear control systems, Systems Control Lett. 3 (1983) 303–310.
- [21] Y. Zhang, J. Jiang, Bibliographical review on reconfigurable faulttolerant control systems, Annual Rev. Control 32 (2008) 229–252.
- [22] P. Bolzern, P. Colaneri, G. De Nicolao, Markov Jump Linear Systems with switching transition rates: Mean square stability with dwell-time, Automatica 46 (2010) 1081–1088.
- [23] M. Mariton, Detection delays, false alarm rates and the reconfiguration of control systems, Int. J. Control 49 (1989) 981–992.
- [24] Y. Fang, K. A. Loparo, Stochastic stability of jump linear systems, IEEE Trans. Automat. Control 47 (2002) 1204–1208.
- [25] T. Chen, B. A. Francis, Optimal Sampled-Data Control Systems, Springer, 1994.

[26] M. Ogura, C. F. Martin, Generalized joint spectral radius and stability of switching systems, Linear Algebra Appl. 439 (2013) 2222–2239.