Stochastic minimum-energy control

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Abstract

We give the solution to the minimum-energy control problem for linear stochastic systems. The problem is as follows: given an exactly controllable system, find the control process with the minimum expected energy that transfers the system from a given initial state to a desired final state. The solution is found in terms of a certain forward-backward stochastic differential equation of Hamiltonian type.

Keywords: Exact controllability, Minimum-energy control, Hamiltonian system.

1. Introduction

The notion of *controllability* was introduced by Kalman [6] and it characterizes the ability of controls to transfer a system from a given initial state to a desired final state. When the system is completely controllable, there are many controls that can achieve such a transfer of the system state. This naturally leads to the problem of choosing the "best" control that performs this task. Another contribution of Kalman [6] was the solution of this problem for linear deterministic systems and using the quadratic cost as an optimality criterion. These kinds of problems are know as the *minimum-energy control* problems (see also [7], [8], [3], [10]).

In this paper we formulate and solve the minimum-energy control problem for linear *stochastic* systems driven by a Brownian motion. The notion of controllability that we adapt is that of *exact controllability*, as introduced by Peng [13] (see also [11], [12], [5], [14]). This notion of controllability is a

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faithful extension of Kalman's notion of complete controllability to stochastic systems. The difference between these two definitions is that in the case of exact controllability the terminal state can be a random variable rather than a fixed number. This makes the stochastic minimum-energy control problem considerably harder than in deterministic setting.

The precise formulation of the stochastic minimum-energy control problem is given in the next section. This is followed by the proof of solvability for a Hamiltonian system and its relation with exact controllability. Section 4 contains the solution to the stochastic minimum energy control problem. As an extension of this result, we give the solution to the stochastic linear-quadratic (LQ) regulator problem with a fixed final state in the final section.

2. Problem formulation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ be a given complete filtered probability space on which the scalar standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathcal{F}_t$ is the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. If $\xi : \Omega \to \mathbb{R}^n$ is an $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}[|\xi|^2] < \infty$, we write $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$. If $f : [0, T] \times \Omega \to \mathbb{R}^n$ is an $\{\mathcal{F}_t\}_{t\geq 0}$ adapted process and if $\mathbb{E} \int_0^T |f(t)|^2 dt < \infty$, we write $f(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$; if $f(\cdot)$ has a.s. continuous sample paths and $\mathbb{E} \sup_{t\in[0,T]} |f(t)|^2 < \infty$, we write $f(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))$; if $f(\cdot)$ is uniformly bounded (i.e. $\mathrm{esssup}_{t\in[0,T]} |f(t)| < \infty$), we write $f(\cdot) \in L^\infty(0, T; \mathbb{R}^n)$.

Consider the linear stochastic control system:

$$
\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]dt + [C(t)x(t) + D(t)u(t)]dW(t) \\ \\ x(0) = x_0 \in \mathbb{R}^n, \quad \text{is given.} \end{cases} \tag{2.1}
$$

We assume that $A(\cdot), C(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times n})$, and $B(\cdot), D(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times m})$. If the control process $u(\cdot)$ belongs to $L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$, then (2.1) has a unique strong solution $x(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))$ (see, e.g. Theorem 1.6.14 of [19]).

For a given $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$, we are interested in the following subset of control processes:

$$
\mathcal{U}_\xi \equiv \left\{ u(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) : x(T) = \xi \quad a.s. \right\}.
$$

Minimum-energy control problem. *Let* $R(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{m\times m})$ *be a given symmetric matrix such that* $R(t) > 0$, *a.e.* $t \in [0,T]$. *For any given* $x_0 \in \mathbb{R}^n$ *and* $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ *find the control process* $u(\cdot) \in \mathcal{U}_\xi$ *that minimizes the cost functional*

$$J(u(\cdot)) = \mathbb{E} \int_0^T u'(t)R(t)u(t)dt. \tag{2.2}$$

This is clearly the stochastic version of the Kalman's minimum energy control problem. A related problem was considered by Klamka [9]. However, in [9] only the linear stochastic systems with *additive* noise are considered, whereas (2.1) has a *multiplicative* noise. Our approach to solving the stochastic minimum-energy control problem is different from that of [9], where an operator-theoretic method was used, whereas here we base our approach on a forward-backward stochastic differential equation of a Hamiltonian type.

In order to ensure that the set $\mathcal{U}_\xi$ is not empty, we make some assumptions on the controllability of (2.1). Out of the many possible notions of controllability for stochastic systems, we employ the notion of *exact controllability* as introduced by Peng [13].

Definition 1. *System (2.1) is called exactly controllable at time* $T > 0$ *if for any* $x_0 \in \mathbb{R}^n$ *and* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$, *there exists at least one control* $u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, *such that the corresponding trajectory* $x(\cdot)$ *satisfies the initial condition* $x(0) = x_0$ *and the terminal condition* $x(T) = \xi$, *a.s..*

We solve the minimum-energy control problem under the following two assumptions.

(A1) The system (2.1) is exactly controllable at time $T > 0$.

(A2) There exists an *invertible* matrix $M(\cdot) \in L^{\infty}(0,T;\mathbb{R}^{m\times m})$ such that $D(t)M(t) = [I, 0]$.

Assumption A1 ensures that the set $\mathcal{U}_\xi$ is not empty. Assumption A2 implies that $m \geq n$, i. e. the number of control inputs to the system is at least as large as the number of the states of the system. This may appear as a strong assumption when compared with the minimum-energy control problem of deterministic systems. However, at least when the matrix $D(\cdot)$ has continuous coefficients, this assumption is implied by assumption A1. Indeed, by Proposition 2.1. of [13], a *necessary* condition for exact controllability at time T of the system (2.1) is that $rank\ D(t) = n$, $\forall t \in [0,T]$. Then from the Doležal's theorem [4], it follows that there exists the matrix $M(\cdot)$ in assumption A2.

We now reformulate the minimum-energy control problem in a more convenient form. Let the processes $z(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ and $v(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m-n})$ be such that

$$u(t) = M(t) \begin{bmatrix} z(t) \\ \\ v(t) \end{bmatrix}. \tag{2.3}$$

Let the matrices $G(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $F(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times(m-n)})$, $H_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $H_2(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times(m-n)})$, $H_3(\cdot) \in L^\infty(0,T;\mathbb{R}^{(m-n)^2})$, be such that

$$B(t)M(t) = \begin{bmatrix} G(t) & F(t) \end{bmatrix}, \quad M'(t)R(t)M(t) = \begin{bmatrix} H_1(t) & H_2(t) \\ \\ H_2'(t) & H_3(t) \end{bmatrix}. \tag{2.4}$$

Due to the symmetric nature of the matrix $R(\cdot)$, the matrices $H_1(\cdot)$ and $H_3(\cdot)$ are also symmetric. Moreover, due to the positive definiteness of $R(\cdot)$ and the Schur's lemma, it holds that

$$H_3(t) > 0, \quad a.e. \quad t \in [0,T],$$

$$H_1(t) - H_2'(t)H_3^{-1}(t)H_2(t) > 0, \quad a.e. \quad t \in [0,T].$$

Equation (2.1) and the cost functional (2.2) can now be written as

$$\begin{cases} dx(t) = [A(t)x(t) + F(t)v(t) + G(t)z(t)]dt + [C(t)x(t) + z(t)]dW(t), \\ \\ x(0) = x_0 \in \mathbb{R}^n, \quad \text{is given,} \end{cases} \tag{2.5}$$

$$J(v(\cdot),z(\cdot)) = \mathbb{E}\int_0^T [z'(t)H_1(t)z(t) + 2v'(t)H_2'(t)z(t) + v'(t)H_3(t)v(t)]dt. \tag{2.6}$$

To each element of the set $\mathcal{U}_\xi$ it corresponds a pair of processes $(v(\cdot),z(\cdot))$ from the set

$$\mathcal{A}_\xi \equiv \left\{ v(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m-n}), z(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) : x(T) = \xi \quad a.s. \right\}.$$

In this reformulation, the minimum-energy control problem is:

$$\begin{cases} \min_{(v(\cdot),z(\cdot))\in\mathcal{A}_\xi} J(v(\cdot),z(\cdot)), \\ \\ s.t. \quad (2.5). \end{cases} \tag{2.7}$$

Before we proceed to its solution, let us state a useful necessary and sufficient condition for the exact controllability of (2.5). It is a slight modification of the result in [11], and we thus omit the proof.

Proposition 1. *Let $E(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times m})$ be any symmetric matrix such that $E(t) > 0$, a.e. $t \in [0,T]$. Also let $\Phi(\cdot)$ be the unique solution to the equation*

$$\begin{cases} d\Phi(t) = -\Phi(t)[A(t) - G(t)C(t)]dt - \Phi(t)G(t)dW(t), \\ \\ \Phi(0) = I. \end{cases}$$

The system (2.5) is exactly controllable at time T if and only if

$$rank \quad \left[\mathbb{E}\int_0^T \Phi(t)F(t)E(t)F'(t)\Phi'(t)dt\right] = n. \tag{2.8}$$

3. Hamiltonian system of equations

The following forward-backward stochastic differential equation of Hamiltonian type appears naturally in the next section:

$$\begin{cases} dX(t) = \{A(t)X(t) + F(t)H_3^{-1}(t)F'(t)Y(t) + [G(t) - F(t)H_3^{-1}(t)H_2'(t)]Z(t)\}dt \\ \\ \qquad +[C(t)X(t) + Z(t)]dW(t), \\ \\ dY(t) = -[A'(t) - C'(t)G'(t) + C'(t)H_2(t)H_3^{-1}(t)F'(t)]Y(t)dt \\ \\ \qquad -C'(t)[H_1(t) - H_2(t)H_3^{-1}(t)H_2'(t)]Z(t)dt \\ \\ \qquad +\{[-G'(t) + H_2(t)H_3^{-1}(t)F'(t)]Y(t) + [H_1(t) - H_2(t)H_3^{-1}(t)H_2'(t)]Z(t)\}dW(t), \\ \\ X(0) = x_0, \quad X(T) = \xi, \quad Y(0) = K, \end{cases} \tag{3.1}$$

where we can choose the vector $K \in \mathbb{R}^n$. To simplify the notation, we introduce the matrices:

$$\begin{aligned}
\bar{A}(t) &\equiv A(t) - G(t)C(t) + F(t)H_3^{-1}(t)H_2'(t)C(t), \\
\bar{B}(t) &\equiv G(t) - F(t)H_3^{-1}(t)H_2'(t), \\
\bar{H}(t) &\equiv H_1(t) - H_2(t)H_3^{-1}(t)H_2'(t).
\end{aligned}$$

By defining $\bar{X}(t) \equiv -X(t)$ and $\bar{Z}(t) \equiv -[C(t)X(t) + Z(t)]$, we can rewrite (3.1) as

$$\begin{cases}
d\bar{X}(t) = [\bar{A}(t)\bar{X}(t) - F(t)H_3^{-1}(t)F'(t)Y(t) + \bar{B}(t)\bar{Z}(t)]dt + \bar{Z}(t)dW(t), \\
dY(t) = [-\bar{A}'(t)Y(t) - C'(t)\bar{H}(t)C(t)\bar{X}(t) + C'(t)\bar{H}(t)\bar{Z}(t)]dt \\
\qquad\qquad +[-\bar{B}'(t)Y(t) + \bar{H}(t)C(t)\bar{X}(t) - \bar{H}(t)\bar{Z}(t)]dW(t), \\
\bar{X}(0) = -x_0, \quad \bar{X}(T) = -\xi, \quad Y(0) = K.
\end{cases} \tag{3.2}$$

This forward-backward stochastic differential equation is similar to the Hamiltonian system of stochastic LQ control problem [19]. Two main differences are that here the initial value of of $\bar{X}(\cdot)$ is fixed and the vector K can be chosen. We thus seek the solution *quadruple* $(\bar{X}(t), Y(t), \bar{Z}(t), K)$, rather than the solution triple $(\bar{X}(t), Y(t), \bar{Z}(t))$, as is usually the case with Hamiltonian systems.

Theorem 1. *There exists a unique solution quadruple* $(\bar{X}(\cdot), Y(\cdot), \bar{Z}(\cdot), K) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times \mathbb{R}^n$ *to* (3.2) *for any* $x_0 \in \mathbb{R}^n$ *and any* $\xi \in L^2_{\mathcal{F}}(\Omega, \mathbb{P}, \mathcal{F}; \mathbb{R}^n)$*, if and only if the system* (2.5) *is exactly controllable at time* T*. In this case,* $X(\cdot)$ *and* $Z(\cdot)$ *in terms of* $Y(\cdot)$*, and the explicit formula for* K*, are given by*

$$\bar{X}(t) = \bar{P}(t)Y(t) + p(t), \tag{3.3}$$

$$\begin{aligned}
\bar{Z}(t) &= [I + \bar{P}(t)\bar{H}(t)]^{-1}[\bar{P}(t)\bar{H}(t)C(t)\bar{P}(t) - \bar{P}(t)\bar{B}'(t)]Y(t) \\
&+ [I + \bar{P}(t)\bar{H}(t)]^{-1}[\bar{P}(t)\bar{H}(t)C(t)p(t) + q(t)],
\end{aligned} \tag{3.4}$$

$$K = \bar{P}^{-1}(0)\left\{x_0 - \mathbb{E}[\mathcal{P}(T)\xi]\right\}. \tag{3.5}$$

Here $\bar{P}(\cdot)$ is the unique solution of the Riccati equation

$$\begin{cases} \dot{\bar{P}}(t) - A(t)\bar{P}(t) - \bar{P}(t)A'(t) + F(t)H_3^{-1}(t)F'(t) + \bar{B}(t)\bar{H}^{-1}(t)\bar{B}'(t) \\ -[\bar{P}(t)C'(t) - \bar{B}\bar{H}^{-1}][\bar{H}^{-1}(t) + \bar{P}(t)]^{-1}[\bar{P}(t)C'(t) - \bar{B}\bar{H}^{-1}]' = 0, \\ \bar{P}(T) = 0, \end{cases} \tag{3.6}$$

whereas $(p(\cdot), q(\cdot))$ are the unique solution pair of the following linear backward stochastic differential equation

$$\begin{cases} dp(t) = [\mathcal{B}_1(t)p(t) + \mathcal{B}_2(t)q(t)]dt + q(t)dW(t), \\ p(T) = \xi, \\ \mathcal{B}_1(t) \equiv \bar{A}(t) + \bar{P}(t)C'(t)\bar{H}(t)C(t) \\ \qquad\qquad +[\bar{B}(t) - \bar{P}(t)C'(t)\bar{H}(t)][I + \bar{P}(t)\bar{H}(t)]^{-1}\bar{P}(t)\bar{H}(t)C(t), \\ \mathcal{B}_2(t) \equiv [\bar{B}(t) - \bar{P}(t)C'(t)\bar{H}(t)][I + \bar{P}(t)\bar{H}(t)]^{-1}. \end{cases} \tag{3.7}$$

Finally, the process $\mathcal{P}(\cdot)$ is the unique solution to the stochastic differential equation

$$\begin{cases} d\mathcal{P}(t) = -\mathcal{P}(t)\mathcal{B}_1(t)dt - \mathcal{P}(t)\mathcal{B}_2(t)dW(t), \\ \mathcal{P}(0) = I. \end{cases} \tag{3.8}$$

Proof. (*Positivity of $\bar{P}(0)$*) The Riccati differential equation (3.6) has a unique solution $\bar{P}(t) \geq 0$, $\forall t \in [0, T]$ (see, e.g., Theorem 3.1 of [1]). We show that $\bar{P}(0) > 0$ if and only if (2.5) is exactly controllable at time T. Thus consider the stochastic control system

$$\begin{cases} d\widetilde{x}(t) = [-A'(t)\widetilde{x}(t) + C'(t)\widetilde{u}(t)]dt - \widetilde{u}(t)dW(t), \\ \widetilde{x}(0) = \widetilde{x}_0 \neq 0, \end{cases} \tag{3.9}$$

and the associated cost functional

$$\begin{aligned} \widetilde{J}(\widetilde{u}(\cdot)) &= \mathbb{E}\int_0^T \widetilde{x}'(t)[F(t)H_3^{-1}(t)F'(t) + \bar{B}(t)\bar{H}^{-1}(t)\bar{B}'(t)]\widetilde{x}(t)dt \\ &+ \mathbb{E}\int_0^T [-2\widetilde{x}'(t)\bar{B}(t)\bar{H}^{-1}(t)\widetilde{u}(t) + \widetilde{u}'(t)\bar{H}^{-1}(t)\widetilde{u}(t)]dt. \end{aligned} \tag{3.10}$$

From Theorem 2.2 of [1], it follows that $\widetilde{x}_0'\bar{P}(0)\widetilde{x}_0 = \min_{\tilde{u}(\cdot)} \widetilde{J}(\widetilde{u}(\cdot))$. Introducing the new control $\tilde{v}(t) \equiv \tilde{u}(t) - \bar{B}'\tilde{x}(t)$ transforms (3.9) and (3.10) into

$$\begin{cases} d\widetilde{x}(t) = [-A'(t) + C'(t)\bar{B}'(t)]\widetilde{x}(t)dt + C'(t)\widetilde{v}(t)dt - [\bar{B}'\tilde{x}(t) + \widetilde{v}(t)]dW(t), \\ \\ \widetilde{x}(0) = \widetilde{x}_0 \neq 0, \end{cases} \tag{3.11}$$

$$\widetilde{J}(\widetilde{v}(\cdot)) = \mathbb{E}\int_0^T [\widetilde{x}'(t)F(t)H_3^{-1}(t)F'(t)\widetilde{x}(t) + \widetilde{v}'(t)\bar{H}^{-1}(t)v(t)]dt. \tag{3.12}$$

To prove the *sufficiency*, let us assume the opposite, i.e. that $\widetilde{x}_0'\bar{P}(0)\widetilde{x}_0 = 0$ and the system (2.5) is exactly controllable at time T. From (3.12), and the fact that $\bar{H}^{-1}(t) > 0$, *a.e.* $t \in [0,T]$, we conclude that in order to minimize (3.12) it is necessary to have $\widetilde{v}(t) = 0$, *a.e.* $t \in [0,T]$ *a.s.*. For such a $\widetilde{v}(\cdot)$, the solution to (3.11) becomes $\widetilde{x}(t) = \widetilde{\Phi}(t)x_0$, with $\widetilde{\Phi}(t)$ being the solution to the stochastic differential equation

$$\begin{cases} d\widetilde{\Phi}(t) = [-A'(t) + C'(t)\bar{B}'(t)]\widetilde{\Phi}(t)dt - \bar{B}'\widetilde{\Phi}(t)dW(t), \\ \\ \widetilde{\Phi}(0) = I, \end{cases} \tag{3.13}$$

The cost (3.12) now becomes

$$0 = \widetilde{x}_0'\left[\mathbb{E}\int_0^T \widetilde{\Phi}'(t)F(t)H_3^{-1}(t)F'(t)\widetilde{\Phi}(t)dt\right]\widetilde{x}_0, \tag{3.14}$$

which means condition (2.8) does not hold. Since (2.8) is necessary for the exact controllability of (2.5), we have a contradiction.

To prove the *necessity*, let us assume that $\bar{P}(0) > 0$ and the system (2.5) is not exactly controllable at time T. Taking $\widetilde{v}(t) = 0$ *a.e.* $t \in [0,T]$ *a.s.*, the cost (3.12) becomes the right-hand side of (3.14), and thus there exists $\widetilde{x}_0 \neq 0$ such that equation (3.14) holds. This means that $\min_{\tilde{v}(\cdot)} \widetilde{J}(\widetilde{v}(\cdot)) = 0$, which contradicts the fact that is should be $\widetilde{x}_0'\bar{P}(0)\widetilde{x}_0 > 0$.

(*Solvability of (3.2)*) Here we follow Yong [17], [18], in seeking the relation

$$\bar{X}(t) \;\; = \;\; \bar{P}(t)Y(t) + p(t), \tag{3.15}$$

which implies that the differential of $\bar{X}$ should be

$$d\bar{X}(t) = \dot{\bar{P}}(t)Y(t) + \bar{P}(t)dY(t) + dp(t)$$

$$= \dot{\bar{P}}(t)Y(t)dt + \bar{P}(t)[-\bar{A}'(t)Y(t) - C'(t)\bar{H}(t)C(t)\bar{X}(t) + C'(t)\bar{H}(t)\bar{Z}(t)]dt$$

$$+[\mathcal{B}_1(t)p(t) + \mathcal{B}_2(t)q(t)]dt$$

$$+\bar{P}(t)[-\bar{B}'(t)Y(t) + \bar{H}(t)C(t)\bar{X}(t) - \bar{H}(t)\bar{Z}(t)]dW(t) + q(t)dW(t)$$

By comparing this differential with $d\bar{X}(t)$ in (3.2), we conclude that for *a.e.* $t \in [0,T]$ *a.s.*, we must have

$$\begin{cases} \bar{A}(t)\bar{X}(t) - F(t)H_3^{-1}(t)F'(t)Y(t) + \bar{B}(t)\bar{Z}(t) = \mathcal{B}_1(t)p(t) + \mathcal{B}_2(t)q(t) \\ +\dot{\bar{P}}(t)Y(t) + \bar{P}(t)[-\bar{A}'(t)Y(t) - C'(t)\bar{H}(t)C(t)\bar{X}(t) + C'(t)\bar{H}(t)\bar{Z}(t)], \end{cases} \tag{3.16}$$

$$\bar{Z}(t) = \bar{P}(t)[-\bar{B}'(t)Y(t) + \bar{H}(t)C(t)\bar{X}(t) - \bar{H}(t)\bar{Z}(t)] + q(t). \tag{3.17}$$

Since the matrix $[I + \bar{P}(t)\bar{H}(t)] = [\bar{H}^{-1}(t) + \bar{P}(t)]\bar{H}(t)$ is invertible, from (3.17) we obtain (3.4). Substituting such a $\bar{Z}(t)$ into (3.16) gives

$$\begin{cases} \bar{A}(t)[\bar{P}(t)Y(t) + p(t)] - F(t)H_3^{-1}(t)F'(t)Y(t) \\ +\bar{B}(t)[I + P(t)\bar{H}(t)]^{-1}[\bar{P}(t)\bar{H}(t)C(t)\bar{P}(t) - \bar{P}(t)\bar{B}'(t)]Y(t) \\ +\bar{B}(t)[I + P(t)\bar{H}(t)]^{-1}[\bar{P}(t)\bar{H}(t)C(t)p(t) + q(t)] \\ = \mathcal{B}_1(t)p(t) + \mathcal{B}_2(t)q(t) \\ +\dot{\bar{P}}(t)Y(t) + \bar{P}(t)[-\bar{A}'(t)Y(t) - C'(t)\bar{H}(t)C(t)(\bar{P}(t)Y(t) + p(t))] \\ +\bar{P}(t)C'(t)\bar{H}(t)[I + P(t)\bar{H}(t)]^{-1}[\bar{P}(t)\bar{H}(t)C(t)\bar{P}(t) - \bar{P}(t)\bar{B}'(t)]Y(t) \\ +\bar{P}(t)C'(t)\bar{H}(t)[I + P(t)\bar{H}(t)]^{-1}[\bar{P}(t)\bar{H}(t)C(t)p(t) + q(t)], \end{cases} \tag{3.18}$$

which holds due to our assumptions on $\bar{P}(t)$ and $p(t)$. Substituting (3.15) and (3.4) into the equation for $Y(t)$ in (3.2), shows that it is a linear stochastic differential equation with a unique solution for any $K \in \mathbb{R}^n$. This proves the existence of a unique solution triple $(\bar{X}(\cdot), Y(\cdot), \bar{Z}(\cdot))$ of (3.2) for any ξ. In order to ensure that $\bar{X}(0) = x_0$ for any $x_0 \in \mathbb{R}^n$, it is necessary and sufficient to have $x_0 = \bar{P}(0)K + p(0)$. This equation has a unique solution for any ξ and x_0 if and only if $\bar{P}(0)$ is invertible, which we proved is equivalent with the exact controllability of (2.5). □

4. Minimum-energy control

In this section we give the solution to the minimum-energy control problem (2.7). Let us first prove two useful lemmas.

Lemma 1. *Let $(v_1(\cdot), z_1(\cdot)) \in \mathcal{A}_\xi$ and $(v_2(\cdot), z_2(\cdot)) \in \mathcal{A}_\xi$ be any two pairs of admissible controls. Then*

$$\mathbb{E}\int_0^T \Phi(t)F(t)[v_1(t) - v_2(t)]dt = 0. \tag{4.1}$$

Proof. By Itô's product rule, for any admissible pair $(v(\cdot), z(\cdot)) \in \mathcal{A}_\xi$, we obtain

$$\begin{aligned} d[\Phi(t)x(t)] &= [d\Phi(t)]x(t) + \Phi(t)dx(t) - \Phi(t)G(t)[C(t)x(t) + z(t)]dt \\ &= -\Phi(t)[A(t) - G(t)C(t)]x(t)dt - \Phi(t)G(t)x(t)dW(t) \\ &+ \Phi(t)[A(t)x(t) + F(t)v(t) + G(t)z(t)]dt \\ &+ \Phi(t)[C(t)x(t) + z(t)]dW(t) - \Phi(t)G(t)[C(t)x(t) + z(t)]dt \\ &= \Phi(t)F(t)v(t)dt + \Phi(t)\{z(t) + [C(t) - G(t)]x(t)\}dW(t). \end{aligned} \tag{4.2}$$

Denoting by $x^{(1)}(\cdot)$ and $x^{(2)}(\cdot)$ the solutions to (2.5) corresponding to $(v_1(\cdot), z_1(\cdot))$ and $(v_2(\cdot), z_2(\cdot))$, respectively, we obtain

$$\Phi(T)\xi - x_0 = \int_0^T \Phi(t)F(t)v_1(t)dt + \int_0^T \Phi(t)\{z_1(t) + [C(t) - G(t)]x^{(1)}(t)\}dW(t),$$

$$\Phi(T)\xi - x_0 = \int_0^T \Phi(t)F(t)v_2(t)dt + \int_0^T \Phi(t)\{z_2(t) + [C(t) - G(t)]x^{(2)}(t)\}dW(t).$$

The difference of the above two equations is

$$\begin{aligned}
0 &= \int_0^T \Phi(t)F(t)[v_1(t)-v_2(t)]dt \\
&+ \int_0^T \Phi(t)\{z_1(t)-z_2(t)+[C(t)-G(t)][x^{(1)}(t)-x^{(2)}(t)]\}dW(t).
\end{aligned}$$

The conclusion follows by taking the expectation of both sides. □

Consider an $\mathbb{R}^n$-valued stochastic process $\Gamma(\cdot)$ defined as the solution to the stochastic differential equation

$$\begin{cases} d\Gamma(t) = \Gamma_1(t)dt + \Gamma_2(t)dW(t) \\ \Gamma(0) = 0, \end{cases} \tag{4.3}$$

where $\Gamma_2(\cdot)$ is any process in $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$, and

$$\Gamma_1(t) \equiv -[\Phi^{-1}(t)]'[C(t)-G(t)]'\Phi'(t)\Gamma_2(t).$$

Lemma 2. *Let $(v_1(\cdot),z_1(\cdot)) \in \mathcal{A}_\xi$ and $(v_2(\cdot),z_2(\cdot)) \in \mathcal{A}_\xi$ be any two pairs of admissible controls. Then*

$$\mathbb{E}\int_0^T \Gamma_2'(t)\Phi(t)[z_2(t)-z_1(t)]dt = -\mathbb{E}\int_0^T \Gamma'(t)\Phi(t)F(t)[v_2(t)-v_1(t)]dt. \tag{4.4}$$

Proof. For any pair of admissible controls $(v(\cdot),z(\cdot)) \in \mathcal{A}_\xi$, by Itô's product rule and (4.2), we obtain

$$\begin{aligned}
d[\Gamma'(t)\Phi(t)x(t)] &= [d\Gamma'(t)]\Phi(t)x(t) + \Gamma'(t)d[\Phi(t)x(t)] + \Gamma_2'(t)\Phi(t)\{z(t)+[C(t)-G(t)]x(t)\}dt \\
&= \Gamma_1'(t)\Phi(t)x(t)dt + \Gamma_2'(t)\Phi(t)x(t)dW(t) \\
&+ \Gamma'(t)\Phi(t)F(t)v(t)dt + \Gamma'(t)\Phi(t)\{z(t)+[C(t)-G(t)]x(t)\}dW(t) \\
&+ \Gamma_2'(t)\Phi(t)\{z(t)+[C(t)-G(t)]x(t)\}dt \\
&= [\Gamma'(t)\Phi(t)F(t)v(t) + \Gamma_2'(t)\Phi(t)z(t)]dt \\
&+ \Gamma_2'(t)\Phi(t)x(t)dW(t) + \Gamma'(t)\Phi(t)\{z(t)+[C(t)-G(t)]x(t)\}dW(t).
\end{aligned}$$

The rest of the proof proceeds as in the previous lemma. $\square$

Theorem 2. *(Minimum-energy control) There exists a unique solution to the problem (2.7) given by*

$$v^*(t) = H_3^{-1}(t)[F'(t)Y(t) - H_2'Z(t)], \tag{4.5}$$

$$z^*(t) = Z(t). \tag{4.6}$$

Proof. We first show that $(v^*(\cdot), z^*(\cdot)) \in \mathcal{A}_\xi$. By choosing the process $\Gamma_2(\cdot)$ in (4.3) as

$$\Gamma_2(t) = [\Phi'(t)]^{-1}[H_1(t)z^*(t) + H_2(t)v^*(t)],$$

the process $\Gamma_1(\cdot)$ and the equation for $\Gamma(\cdot)$ become

$$\Gamma_1(t) = -[\Phi'(t)]^{-1}[C(t) - G(t)]'[H_1(t)z^*(t) + H_2(t)v^*(t)],$$

$$\begin{cases} d\Gamma(t) = -[\Phi'(t)]^{-1}[C(t) - G(t)]'[H_1(t)z^*(t) + H_2(t)v^*(t)]dt \\ \qquad\qquad +[\Phi'(t)]^{-1}[H_1(t)z^*(t) + H_2(t)v^*(t)]dW(t) \\ \Gamma(0) = 0, \end{cases} \tag{4.7}$$

Let $\bar{Y}(\cdot) \equiv \Phi'(\cdot)[\Gamma(\cdot) + K]$. We show that $\bar{Y}(\cdot)$ is in fact the process $Y(\cdot)$ of the previous section. Note that $\bar{Y}(0) = \Phi'(0)[\Gamma(0) + K] = K = Y(0)$. The differential of $\bar{Y}(\cdot)$ is:

$$\begin{aligned} d\bar{Y}(t) &= [d\Phi'(t)][\Gamma(t) + K] + \Phi'(t)d\Gamma(t) - G'[H_1z^*(t) + H_2(t)v^*(t)]dt \\ &= -[A(t) - G(t)C(t)]'\Phi'(t)[\Gamma(t) + K]dt - G'(t)\Phi'(t)[\Gamma(t) + K]dW(t) \\ &- [C'(t) - G'(t)][H_1(t)z^*(t) + H_2(t)v^*(t)]dt + [H_1(t)z^*(t) + H_2(t)v^*(t)]dW(t) \\ &- G'(t)[H_1(t)z^*(t) + H_2(t)v^*(t)]dt. \end{aligned}$$

After substituting the expressions for $v^*(\cdot)$ and $z^*(\cdot)$, this equation becomes

$$\begin{aligned} d\bar{Y}(t) &= \{-[A'(t) - C'(t)G'(t)]\bar{Y}(t) - C'(t)H_2(t)H_3^{-1}(t)F'(t)Y(t)\}dt \\ &- C'(t)[H_1(t) - H_2(t)H_3^{-1}H_2'(t)]Z(t)dt \\ &+ \{-G'(t)\bar{Y}(t) + H_2(t)H_3^{-1}(t)F'(t)Y(t) + [H_1(t) - H_2(t)H_3^{-1}(t)H_2'(t)]Z(t)\}dW(t), \end{aligned}$$

which is satisfied by the process $Y(\cdot)$. Therefore, $\bar{Y}(t) = Y(t)$. Substituting (4.5) and (4.6) in (2.5) makes it clear that the equations for $x(t)$ and $X(t)$ are the same, and in particular $x(T) = \xi$ a.s.. Hence, $(v^*(\cdot), z^*(\cdot)) \in \mathcal{A}_\xi$.

We now focus in proving that $(v^*(\cdot), z^*(\cdot))$ are the unique optimal controls. By using equation (4.4), for any control pair $(v(\cdot), z(\cdot)) \in \mathcal{A}_\xi$, we obtain:

$$\begin{aligned} &\mathbb{E}\int_0^T \{[H_1(t)z^*(t) + H_2(t)v^*(t)]'[z(t) - z^*(t)] + [H_2'(t)z^*(t) + H_3(t)v^*(t)]'[v(t) - v^*(t)]\}dt \\ &= \mathbb{E}\int_0^T \{-\Gamma'(t)\Phi(t)F(t)[v(t) - v^*(t)] + [H_2'(t)z^*(t) + H_3(t)v^*(t)]'[v(t) - v^*(t)]\}dt \\ &= \mathbb{E}\int_0^T [H_2'(t)z^*(t) + H_3(t)v^*(t) - F'(t)\Phi'(t)\Gamma(t)]'[v(t) - v^*(t)]dt \\ &= \mathbb{E}\int_0^T K'\Phi(t)F(t)[v(t) - v^*(t)]dt = 0, \end{aligned} \tag{4.8}$$

where the last equality is due to (4.1). For any control pair $(v(\cdot), z(\cdot)) \in \mathcal{A}_\xi$ we have

$$\begin{aligned} J(v(\cdot), z(\cdot)) &= \mathbb{E}\int_0^T [z'(t)H_1(t)z(t) + 2v'(t)H_2'(t)z(t) + v'(t)H_3(t)v(t)]dt \\ &= \mathbb{E}\int_0^T [z(t) - z^*(t) + z^*(t)]'H_1(t)[z(t) - z^*(t) + z^*(t)]dt \\ &+ \mathbb{E}\int_0^T [v(t) - v^*(t) + v^*(t)]'H_2'(t)[z(t) - z^*(t) + z^*(t)]dt \\ &+ \mathbb{E}\int_0^T [v(t) - v^*(t) + v^*(t)]'H_3(t)[v(t) - v^*(t) + v^*(t)]dt \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\int_0^T \{[z(t)-z^*(t)]'H_1(t)[z(t)-z^*(t)] + 2[z(t)-z^*(t)]'H_1(t)z^*(t) + (z^*(t))'H_1(t)z^*(t)\}dt \\
&+ 2\mathbb{E}\int_0^T \{[v(t)-v^*(t)]'H_2'(t)[z(t)-z^*(t)] + [v(t)-v^*(t)]'H_2'(t)z^*(t)\}dt \\
&+ 2\mathbb{E}\int_0^T \{(v^*(t))'H_2'(t)[z(t)-z^*(t)] + (v^*(t))'H_2'(t)z^*(t)\}dt \\
&+ \mathbb{E}\int_0^T \{[v(t)-v^*(t)]'H_3(t)[v(t)-v^*(t)] + 2(v^*(t))'H_3(t)[v(t)-v^*(t)] + (v^*(t))'H_3(t)v^*(t)\}dt \\
&= J(v^*(\cdot),z^*(\cdot)) + \mathbb{E}\int_0^T \begin{bmatrix} z(t)-z^*(t) \\ v(t)-v^*(t) \end{bmatrix}' \begin{bmatrix} H_1(t) & H_2(t) \\ H_2'(t) & H_3(t) \end{bmatrix} \begin{bmatrix} z(t)-z^*(t) \\ v(t)-v^*(t) \end{bmatrix} dt \\
&+ 2\mathbb{E}\int_0^T \{[z(t)-z^*(t)]'[H_1(t)z^*(t)+H_2(t)v^*(t)] + [v(t)-v^*(t)]'[H_2'(t)z^*(t)+H_3(t)v^*(t)]\}dt,
\end{aligned}
$$

Due to (4.8), for any $(v(\cdot),z(\cdot)) \in \mathcal{A}_\xi$ we have

$$
\begin{aligned}
J(v(\cdot),z(\cdot)) &= J(v^*(\cdot),z^*(\cdot)) + \mathbb{E}\int_0^T \begin{bmatrix} z(t)-z^*(t) \\ v(t)-v^*(t) \end{bmatrix}' \begin{bmatrix} H_1(t) & H_2(t) \\ H_2'(t) & H_3(t) \end{bmatrix} \begin{bmatrix} z(t)-z^*(t) \\ v(t)-v^*(t) \end{bmatrix} dt \\
&\geq J(v^*(\cdot),z^*(\cdot)),
\end{aligned}
$$

with equality if and only if $v(t) = v^*(t)$, *a.e.* $t \in [0,T]$ *a.s.*, and $z(t) = z^*(t)$, *a.e.* $t \in [0,T]$ *a.s.*. $\square$

5. Stochastic LQ regulator with a fixed final state

Ever since its introduction by Kalman [6], the LQ regulator has been studied extensively in both deterministic [2], [3], and stochastic [15], [16], [19], settings. The version of this regulator with a *fixed final state* [10] is the minimum-energy control problem with the cost functional that has a penalty on the state as well as on the control. As an extension of our results on minimum-energy control, we give the solution to the following *stochastic* LQ regulator problem with a fixed final state.

LQ regulator with a fixed final state. *Let $Q(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$ be a given symmetric matrix such that $Q(t) \geq 0$, a.e. $t \in [0,T]$. For any given*

$x_0 \in \mathbb{R}^n$ and $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$ find the control process $u(\cdot) \in \mathcal{U}_\xi$ that minimizes the cost functional

$$\widehat{J}(u(\cdot)) = \mathbb{E}\int_0^T [x'(t)Q(t)x(t) + u'(t)R(t)u(t)]dt. \tag{5.1}$$

We solve this problem by transforming it into an equivalent minimum-energy control problem of the previous sections. Consider the Riccati differential equation

$$\begin{cases} \dot{P}(t) + P(t)A(t) + A'(t)P(t) + C'(t)P(t)C(t) + Q(t) \\ -[P(t)B(t) + C'(t)P(t)D(t)][D'(t)P(t)D(t) + R(t)]^{-1}[B'(t)P(t) + D'(t)P(t)C(t)] = 0, \\ P(T) = 0, \\ D'(t)P(t)D(t) + R(t) > 0, \end{cases}$$

which has a unique solution (see, e.g., [15], [16], [19], [1]). We introduce the following matrices for notational convenience:

$$\widehat{A}(t) \equiv A(t) - B(t)[D'(t)P(t)D(t) + R(t)]^{-1}[B'(t)P(t) + D'(t)P(t)C(t)],$$

$$\widehat{B}(t) \equiv B(t),$$

$$\widehat{C}(t) \equiv C(t) - D(t)[D'(t)P(t)D(t) + R(t)]^{-1}[B'(t)P(t) + D'(t)P(t)C(t)],$$

$$\widehat{D}(t) \equiv D(t),$$

$$\widehat{R}(t) \equiv D'(t)P(t)D(t) + R(t).$$

By introducing a new control $\widehat{u}(t) \equiv u(t)+[D'(t)P(t)D(t)+R(t)]^{-1}[B'(t)P(t)+D'(t)P(t)C(t)]x(t)$, we can rewrite the state equation (2.1) as

$$\begin{cases} dx(t) = [\widehat{A}(t)x(t) + \widehat{B}(t)\widehat{u}(t)]dt + [\widehat{C}(t)x(t) + \widehat{D}(t)\widehat{u}(t)]dW(t) \\ x(0) = x_0 \in \mathbb{R}^n, \quad \text{is given.} \end{cases} \tag{5.2}$$

Using the completion of squares method of stochastic LQ control [19], we can rewrite the cost functional (5.1) as

$$\widehat{J}(\widehat{u}(\cdot)) = x_0' P(0) x_0 + \mathbb{E} \int_0^T \widehat{u}'(t) \widehat{R}(t) \widehat{u}(t) dt. \tag{5.3}$$

Apart from the obvious change of notation, the problem of minimizing (5.3) subject to (5.2) and $\widehat{u}(\cdot) \in \mathcal{A}_\xi$, is the stochastic minimum-energy control problem. Hence, its solution can be obtained by applying the results of the previous sections.

6. Conclusions

A stochastic version of the classical minimum-energy control problem is formulated. The system is driven by a Brownian motion and all coefficients can be time-varying. By assuming the exact controllability of the system, complete solution is given. The minimum-energy control problem is crucial is solving several optimal control problems involving terminal state constraints and quadratic criteria. One such problem is the stochastic LQ regulator with a fixed final state, and it has been solved as an application of our results on minimum-energy control. We expect that the method proposed in this paper will prove useful in tackling more general minimum-energy control problems.

References

[1] Ait Rami, M., Moore, J. B., and Zhou, X. Y., Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls, *IEEE Transactions on Automatic Control*, 40 (2001), 428-440.

[2] Anderson, B. D. O., and Moore, J., *Optimal control: linear quadratic methods*, Prentice Hall, 1990.

[3] Athans, M., and Falb, P. L., *Optimal control: an introduction to the theory and its applications*, McGraw-Hill, 1966.

[4] Doležal, V., The existence of a continuous basis of a certain linear susbspace of E_r, which depends on a parameter, *Časopis pro pęstováni matematiky*, 89, pp. 466-469, 1964.

[5] Gashi, B., and A. A. Pantelous, Linear backward stochastic differential equations of descriptor type: regular systems, *Journal of Stochastic Analysis and Applications*, 31 (2013), 142-166.

[6] Kalman, R. E., Contributions to the theory of optimal control, *Boletin-Sociedad Matematica Mexicana*, pp. 102-119, 1960.

[7] Kalman, R. E., Ho, Y. C., and Narendra, K. S., Controllability of linear dynamical systems, *Contributions to differential equations*, Vol. 1, No. 2, pp. 189-213, 1962.

[8] Kalman, R. E., The theory of optimal control and the calculus of variations, *Mathematical Optimization Techniques*, R. Bellman, ed., Univ. of California Press, pp. 309-331, 1963.

[9] Klamka, J., Stochastic controllability and minimum energy control of systems with multiple delays in control, *Applied Mathematics and Computation*, 206, pp. 704-715, 2008.

[10] Lewis, F., and Syrmos, V. L., *Optimal control*, Second edition, John Wiley & Sons, 1995.

[11] Liu, F., and Peng, S., On controllability for stochastic control systems when the coefficient is time-variant, *J Syst. Sci. Complex*, 23 (2010), pp. 270-278.

[12] Liu, Y., and Peng, S., Infinite horizon backward stochastic differential equation and exponential convergence index assignment of stochastic control systems, *Automatica*, 38, pp. 1417-1423, 2002.

[13] Peng, S., Backward stochastic differential equations and exact controllability of stochastic control systems, *Progress in natural science*, Vol. 4, No. 3, pp. 274-284, Jun. 1994.

[14] Wang, Y., BSDEs with general filtration driven by Lvy processes, and an application in stochastic controllability, *Systems & Control Letters*, 62, pp. 242-247, 2013.

[15] Wonham, W. M., On a matrix Riccati equation of stochastic control, *SIAM J. Control*, 6, pp. 681-697, 1968.

[16] Wonham, W. M., Random differential equations in control theory, *Probabilistic Methods in Applied Mathematics*, Vol.2, Academic Press, 1970.

[17] Yong, J., Linear forward-backward stochastic differential equations, *Applied Mathematics & Optimization*, 39, pp. 93-119, 1999.

[18] Yong, J., Linear forward-backward stochastic differential equations with random coefficients, *Probab. Theorey Relat. Fields*, 135, pp. 53-83, 2006.

[19] Yong, J., and Zhou, X., Y., *Stochastic controls: Hamiltonian systems and HJB equations*, Springer,1999.