Primal-Dual Algorithm for Distributed Constrained Optimization

Jinlong Lei, Han-Fu Chen, and Hai-Tao Fang

The Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences

Abstract

The paper studies a distributed constrained optimization problem, where multiple agents connected in a network collectively minimize the sum of individual objective functions subject to a global constraint being an intersection of the local constraint step size, the local estimates derived at all agents asymptotically preach a consensus at an optimal solution. In addition, the value of the cost function at the time-averaged agorithm with reproposed primal-dual algorithm with appropriately chosen constant step size, the local estimates derived at all agents asymptotically reach a consensus at an optimal solution. In addition, the value of the cost function at the time-averaged agorithm is distinguished from the existing algorithms for distributed constrained optimization. The theoretical analysis is justified by numerical simulations. *Keywords:* Distributed constrained optimization primal-dual algorithm, augmented Lagrange method, multi-agent network. **Collection Distributed computation and estimation recently have represented in a network collectively minimize the sum of local distributed optimization [7]. To paper considers a distributed optimization problem (12). The paper considers a distributed optimization problem, where** *n* **agents connected in a network collectively minimize the sum of local distributed constraint of f_{11} (12). The paper considers a distributed constraint of f_{11} (2). The paper considers a distributed constraint do primization problem, where** *n* **agents connected in a network collectively minimize the sum of local discributed from f_{12} (2) for non-minimize the sum of local discriber functions f_{12} = f_{11}^{-1} f_{12}. Since the constraint g_{12} = f_{12}^{-1} f_{12} subject to a global constraint g_{12} = f_{12}^{-1} f_{12} since the constraint g_{12} = f_{12}^{-1} f_{12}. Since the main contribution of the paper is to propose a distributed agorithm with constant step size to solve the constraint step size (7] to converge to aneighborhood of the optimal-dual algo** The paper studies a distributed constrained optimization problem, where multiple agents connected in a network collectively minimize the sum of individual objective functions subject to a global constraint being an intersection of the local constraint sets

strained optimization problem over the multi-agent network. The algorithm is derived on the basis of the gradient algorithm for finding saddle points of an augmented Lagrange function [21]. In an iteration each agent updates its estimate only using the local data and the information derived from the neighboring

investigated. The estimates generated by the fast distributed gradient algorithms [11] and by EXTRA [15] converge to an optimal solution, but in [11] each cost function is assumed to be convex with gradients being bounded and Lipschitz continuous, while EXTRA [15] only deals with unconstrained problems. Though it is shown by [20] that EXTRA [15] is also a saddle point method, the augmented Lagrange function used in [15] is different from ours. Besides, the convergence rate $O(\frac{1}{L})$ derived here for the unconstrained case is a new result. The primal-dual algorithm proposed in the paper can be seen as an extension of EXTRA [15] to constrained problems.

The rest of the paper is organized as follows: In Section 2,

This work was supported by the NSFC under Grants 61273193, 61120106011, 61134013, 61174143, the 973 program of China under grant No.2014CB845301, and the National Center for Mathematics and Interdisciplinary Sciences, Chinese Academy of Sciences.

Email address: leijinlong11@mails.ucas.ac.cn,

hfchen@iss.ac.cn, htfang@iss.ac.cn. (Jinlong Lei, Han-Fu Chen, and Hai-Tao Fang)

some preliminary information about graph theory and convex analysis is provided and the problem is formulated. In Section 3, a distributed primal-dual algorithm is proposed for solving the problem, while its convergence is proved in Section 4. Two numerical examples are demonstrated in Section 5, and some concluding remarks are given in Section 6.

2. Preliminaries and Problem Statement

We first provide some information about graph theory, convex functions, and convex sets. Then we formulate the distributed constrained optimization problem to be investigated.

2.1. Graph Theory

Consider a network of *n* agents. The communication relationship among the n agents is described by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}\}$, where $\mathcal{V} = \{1, \dots, n\}$ is the node set with node *i* representing agent *i*; $\mathcal{E}_{\mathcal{G}} \subset \mathcal{V} \times \mathcal{V}$ is the undirected edge set, and the unordered pair of nodes $(i, j) \in \mathcal{E}_{G}$ if and only if agent i and agent j can exchange information with each other; $\mathcal{A}_{\mathcal{G}} = [a_{ij}] \in \mathbb{R}^{n \times n}$ is the adjacency matrix of \mathcal{G} , where $a_{ij} = a_{ji} > 0$ if $(i, j) \in \mathcal{E}_{\mathcal{G}}$, and $a_{ij} = 0$, otherwise. Denote by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}_G\}$ the neighboring agents of agent i. The Laplacian matrix of graph G is defined as $\mathcal{L}_{\mathcal{G}} = \mathcal{D}_{\mathcal{G}} - \mathcal{A}_{\mathcal{G}}$, where $\mathcal{D}_{\mathcal{G}} = diag\{\sum_{j=1}^{n} a_{1j}, \cdots, \sum_{j=1}^{n} a_{nj}\}$. For a given pair $i, j \in \mathcal{V}$, if there exists a sequence of distinct nodes i_1, \cdots, i_p such that $(i, i_1) \in \mathcal{E}_{\mathcal{G}}, (i_1, i_2) \in \mathcal{E}_{\mathcal{G}}, \cdots, (i_p, j) \in \mathcal{E}_{\mathcal{G}},$ then (i, i_1, \dots, i_p, j) is called the undirected path between *i* and j. We say that G is connected if there exists an undirected path between any $i, j \in \mathcal{V}$.

The following lemma presents some properties of the Laplacian matrix \mathcal{L} corresponding to an undirected graph \mathcal{G} .

Lemma 2.1. [24] The Laplacian matrix \mathcal{L} of an undirected graph \mathcal{G} has the following properties:

i) L is symmetric and positive semi-definite;

ii) \mathcal{L} has a simple zero eigenvalue with corresponding eigenvector equal to 1, and all other eigenvalues are positive if and only if the graph \mathcal{G} is connected, where 1 denotes the vector of compatible dimension with all entries equal to 1.

2.2. Gradient, Projection Operator and Normal Cone

For a given function $f : \mathbb{R}^m \to [-\infty, \infty]$, denote its domain as dom $(f) \triangleq \{x \in \mathbb{R}^m : f(x) < \infty\}$. Let $f(\cdot)$ be a convex function, and let $x \in \text{dom}(f)$. For a smooth (differentiable) function $f(\cdot)$, denote by $\nabla f(x)$ the gradient of the function $f(\cdot)$ at point x. Then

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle \ \forall y \in \text{dom}(f), \tag{1}$$

where $\langle x, y \rangle$ denotes the inner product of vectors *x* and *y*.

For a nonempty convex set $\Omega \subset \mathbb{R}^m$ and a point $x \in \mathbb{R}^m$, we call the point in Ω that is closest to x the projection of x on Ω and denote it by $P_{\Omega}\{x\}$. If $\Omega \subset \mathbb{R}^m$ is closed, then $P_{\Omega}\{x\}$ contains only one element for any $x \in \mathbb{R}^m$.

Consider a convex closed set $\Omega \subset \mathbb{R}^m$ and a point $x \in \Omega$. Define the normal cone to Ω at x as $N_{\Omega}\{x\} \triangleq \{v \in \mathbb{R}^m : \langle v, y - v \rangle \}$ $x \ge 0 \quad \forall y \in \Omega$. It is shown in [22, Lemma 2.38] that the following equation holds for any $x \in \Omega$:

$$N_{\Omega}\{x\} = \{v \in \mathbb{R}^m : P_{\Omega}\{x+v\} = x\}.$$
 (2)

A set *C* is affine if it contains the lines that pass through any pairs of points $x, y \in C$ with $x \neq y$. Let $\Omega \subset \mathbb{R}^m$ be a nonempty convex set. We say that $x \in \mathbb{R}^m$ is a relative interior point of Ω if $x \in \Omega$ and there exists an open sphere *S* centered at *x* such that $S \cap \operatorname{aff}(\Omega) \subset \Omega$, where $\operatorname{aff}(\Omega)$ is the intersection of all affine sets containing Ω . A point $x \in \mathbb{R}^m$ is called the interior point of Ω if $x \in \Omega$ and there exists an open sphere *S* centered at *x* such that $S \subset \Omega$. A pair of vectors $x^* \in \Omega$ and $z^* \in \Psi$ is called a saddle point of the function $\phi(x, z)$ in $\Omega \times \Psi$ if

 $\phi(x^*, z) \le \phi(x^*, z^*) \le \phi(x, z^*) \quad \forall x \in \Omega, \ \forall z \in \Psi.$

These definitions can be found in [21].

2.3. Problem Formulation

Consider a network of n agents that collectively solve the following constrained optimization problem

minimize
$$f(x) = \sum_{i=1}^{n} f_i(x)$$

subject to $x \in \Omega_o = \bigcap_{i=1}^{n} \Omega_i$, (3)

where $\Omega_i \subset \mathbb{R}^m$ is a closed convex set, representing the local constraint set of agent *i*, and $f_i(x) : \mathbb{R}^m \to \mathbb{R}$ is a smooth convex function in Ω_i , representing the local objective function of agent *i*. Assume that f_i and Ω_i are privately known to agent *i*. We assume that there exists at least one finite solution x^* to the problem (3). For the problem (3), denote by $f^* = \min_{x \in \Omega_o} f(x)$ the optimal value, and by $\Omega_o^* = \{x \in \Omega_o : f(x) = f^*\}$ the optimal solution set.

We use an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}\}$ to describe the communication among agents. Let \mathcal{L} denote the Laplacian matrix of the undirected graph \mathcal{G} .

Let us introduce the following conditions for the problem.

- A1 Ω_o has at least one relative interior point.
- A2 The undirected graph G is connected.
- A3 For any $i \in \mathcal{V}$, $\nabla f_i(x)$ is locally Lipschitz continuous on Ω_i .

3. Algorithm Design

We first give an equivalent form of the problem (3). Then define a distributed primal-dual algorithm with constant step size to solve the formulated problem.

3.1. An Equivalent Problem

Lemma 3.1. [14, Lemma 3] If A2 holds, then the problem (3) is equivalent to the following optimization problem

minimize
$$\widetilde{f}(X) = \sum_{i=1}^{n} f_i(x_i)$$

subject to $(\mathcal{L} \otimes \mathbf{I}_m)X = \mathbf{0}, \ X \in \Omega,$ (4)

where $X = col\{x_1, \dots, x_n\} \triangleq (x_1^T, \dots, x_n^T)^T$, $\Omega = \prod_{i=1}^n \Omega_i$ denotes the Cartesian product, \otimes denotes the Kronecker product, \mathbf{I}_m denotes the identity matrix of size m, and $\mathbf{0}$ denotes the vector of compatible dimension with all entries equal to 0.

Remark 3.2. Lemma 3.1 implies that solving the problem (3) is equivalent to solving the problem (4) when the underlying graph is undirected and connected. If $X^* = col\{x_1^*, \dots, x_n^*\}$ is a solution to the problem (4), i.e., $f^* = \tilde{f}(X^*)$, then $x_i^* = x_j^* = x^* \forall i, j \in \mathcal{V}$ for some $x^* \in \Omega_o$ by $(\mathcal{L} \otimes \mathbf{I}_m)X^* = \mathbf{0}$ and $X^* \in \Omega$. Thus, $\tilde{f}(X^*) = \sum_{i=1}^n f_i(x^*) = f^*$, and hence x^* is an optimal solution to the problem (3).

Define the Lagrange function $\phi(X, \Lambda) = \overline{f}(X) + \langle \Lambda, (\mathcal{L} \otimes \mathbf{I}_m)X \rangle$, where $\Lambda \in \mathbb{R}^{mn}$ is the Lagrange multiplier vector. Then the original problem (4) can be rewritten as $\inf_{X \in \Omega} \sup_{\Lambda \in \mathbb{R}^{mn}} \phi(X, \Lambda)$, while the dual problem is defined as follows

while the dual problem is defined as follows

$$\sup_{\Lambda \in \mathbb{R}^{mn}} \inf_{X \in \Omega} \phi(X, \Lambda).$$
 (5)

Lemma 3.3. Assume A1 and A2 hold. Then $\phi(X, \Lambda)$ has at least one saddle point in $\Omega \times \mathbb{R}^{mn}$. A pair $(X^*, \Lambda^*) \in \Omega \times \mathbb{R}^{mn}$ is the primal-dual solution to the problems (4) and (5) if and only if (X^*, Λ^*) is a saddle point of $\phi(X, \Lambda)$ in $\Omega \times \mathbb{R}^{mn}$.

Proof: Since $f_i(\cdot) \forall i \in \mathcal{V}$ are continuous and the problem (3) has at least one finite solution, f^* is finite. Moreover, A1 implies that there exists a relative interior \bar{X} of set Ω such that $(\mathcal{L} \otimes \mathbf{I}_m)\bar{X} = 0$. Then by [21, Proposition 5.3.3] we know that the primal and dual optimal values are equal, i.e.,

$$\inf_{X \in \Omega} \sup_{\Lambda \in \mathbb{R}^{mn}} \phi(X, \Lambda) = \sup_{\Lambda \in \mathbb{R}^{mn}} \inf_{X \in \Omega} \phi(X, \Lambda), \tag{6}$$

and there exists at least one dual optimal solution. So, by (6) we conclude that $\phi(X, \Lambda)$ has at least one saddle point in $\Omega \times \mathbb{R}^{mn}$.

Since the minimax equality (6) holds, by [21, Proposition 3.4.1] we know that X^* is the primal optimal solution and Λ^* is the dual optimal solution if and only if (X^*, Λ^*) is a saddle point of $\phi(X, \Lambda)$ on $\Omega \times \mathbb{R}^{mn}$. This completes the proof.

3.2. Distributed Primal-Dual Algorithm

Denote by $x_{i,k} \in \mathbb{R}^m$ the estimate for the optimal solution to the problem (3) given by agent *i* at time *k*, and by $\lambda_{i,k} \in \mathbb{R}^m$ the corresponding Lagrange multiplier. They are updated as follows:

$$\begin{aligned} x_{i,k+1} &= P_{\Omega_i} \{ x_{i,k} - \alpha \nabla f_i(x_{i,k}) - \alpha \sum_{j=1}^n a_{ij} (\lambda_{i,k} - \lambda_{j,k}) \\ &- \alpha \sum_{j=1}^n a_{ij} (x_{i,k} - x_{j,k}) \}, \quad (7) \\ \lambda_{i,k+1} &= \lambda_{i,k} + \alpha \sum_{j=1}^n a_{ij} (x_{i,k} - x_{j,k}). \end{aligned}$$

Set $X_k = col\{x_{1,k}, \dots, x_{n,k}\}$, $\Lambda_k = col\{\lambda_{1,k}, \dots, \lambda_{n,k}\}$, and $\nabla \widetilde{f}(X_k) = col\{\nabla f_1(x_{1,k}), \dots, \nabla f_n(x_{n,k})\}$. Then (7) can be written in the compact form as follows:

$$X_{k+1} = P_{\Omega}\{X_k - \alpha \nabla \widetilde{f}(X_k) - \alpha(\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_k + X_k)\}, \quad (8)$$

$$\Lambda_{k+1} = \Lambda_k + \alpha(\mathcal{L} \otimes \mathbf{I}_m) X_k.$$
(9)

Note that the algorithm (8) (9) actually is the gradient algorithm for finding saddle points of the augmented Lagrange function $\widetilde{\phi}(X,\Lambda) = \phi(X,\Lambda) + \frac{1}{2} \langle X, (\mathcal{L} \otimes \mathbf{I}_m) X \rangle$ in $\Omega \times \mathbb{R}^{mn}$. By Lemma 3.3 we see that if the algorithm (8) (9) converges to a saddle point of the augmented Lagrange function, then it solves the original problem (4). Convergence properties of the primal-dual method have been studied extensively, see, for example, [26, 27]. In general, only a subsequence of the sequence (X_k, Λ_k) converges to a saddle point of the augmented Lagrange function. To obtain the convergence of the whole sequence (X_k, Λ_k) , it is often to assume that the augmented Lagrangian function is strictly convex-concave. However, for the problem studied in the paper, the augmented Lagrange function is neither strongly convex in $X \in \Omega$ nor strongly concave in $\Lambda \in \mathbb{R}^{mn}$. Thus, the standard analysis of gradient methods for finding saddle points is not applicable here. Instead, we apply the Lyapunov function method to analyze convergence.

4. Convergence Analysis

Convergence results for the proposed primal-dual algorithm are presented in Section 4.1 with the proof given in Sections 4.2 and 4.3.

4.1. Main Results

By A2 from Lemma 2.1 we know that all eigenvalues of \mathcal{L} are nonnegative real numbers, and zero is a simple eigenvalue. Let us write the eigenvalues of \mathcal{L} in the non-decreasing order as $0 = \kappa_1 < \kappa_2 \leq \cdots \leq \kappa_n$. Set

$$\mathcal{W} = (\mathbf{I}_n - \alpha \mathcal{L} + \alpha^2 \mathcal{L}^2) \otimes \mathbf{I}_m.$$
(10)

Let (X^*, Λ^*) be a saddle point of $\phi(X, \Lambda)$. Define

 $V_1(X) = \langle X - X^*, \mathcal{W}(X - X^*) \rangle, \quad V_2(\Lambda) = \|\Lambda - \Lambda^*\|^2.$

Construct a candidate Lyapunov function as follows

$$V(X, \Lambda) = V_1(X) + V_2(\Lambda).$$

The following theorem shows that the local estimates derived at all agents asymptotically reach a consensus at an optimal solution to the problem (3).

Theorem 4.1. Assume A1-A3 hold. Let $\{x_{i,k}\}$ and $\{\lambda_{i,k}\}$ be produced by (7) with initial values $x_{i,0}$, $\lambda_{i,0}$. Let (X^*, Λ^*) be a saddle point of $\phi(X, \Lambda)$ in $\Omega \times \mathbb{R}^{mn}$. Assume, in addition, that the constant step size α satisfies $0 < \alpha \leq \frac{1}{2\kappa_n}$ and $\alpha < \frac{3}{2l_r}$, where l_r is the local Lipschitz constant of $\nabla f(X)$ in the compact set $\{X \in \Omega : ||X - X^*|| \leq r\}$ with r defined by

$$r = \sqrt{V(X_0, \Lambda_1)/\lambda_{min}(\mathcal{M})},$$
(11)

where $\lambda_{min}(\cdot)$ denotes the smallest eigenvalue of a symmetric matrix, and $\mathcal{M} = diag\{I_{mn}, \mathcal{W}\}$. Then

(i)
$$V(X_k, \Lambda_{k+1})$$
 monotonously decreases and converges,
(ii) $d_k = \sqrt{||X_k - X^*||^2 + ||\Lambda_{k+1} - \Lambda^*||^2} \le r \quad \forall k \ge 0,$
(iii) $\lim_{k \to \infty} x_{i,k} = \lim_{k \to \infty} x_{j,k} = x^* \quad \forall i, j \in \mathcal{V} \text{ for some } x^* \in \Omega_o^*.$

Remark 4.2. The problem considered in [14] is in the same form as the problem (3), but the local constraint is a hyperbox or hyper-sphere, which is a special case of A1. Unlike the discrete-time algorithm (7), the continuous-time distributed algorithm is proposed in [14]. Though the estimates given by all agents converge to the same optimal solution, some intermediate sequence might be unbounded, which makes the algorithm difficult to be implemented.

Denote by $\bar{X}_k = \frac{1}{k+1} \sum_{p=0}^k X_p$ the time-averaged estimate. In what follows, the convergence rate of the algorithm (7) for the case where $\Omega_i = \mathbb{R}^m \quad \forall i \in \mathcal{V}$ is shown.

Theorem 4.3. Assume $\Omega_i = \mathbb{R}^m \ \forall i \in \mathcal{V}$, A2, and A3 hold. Let $\{x_{i,k}\}$ and $\{\lambda_{i,k}\}$ be produced by the algorithm (7) with initial values $x_{i,0}$, $\lambda_{i,0}$. Let (X^*, Λ^*) be a saddle point of $\phi(X, \Lambda)$ in $\Omega \times \mathbb{R}^{mn}$. If $0 < \alpha \leq \frac{1}{2\kappa_n}$ and $\alpha < \frac{3}{2l_r}$, where l_r is the local Lipschitz constant of $\nabla \widetilde{f}(X)$ in the compact set $\{X \in \Omega : ||X - X^*|| \leq r\}$ with r defined by (11), then

(i)
$$(\mathcal{L} \otimes \mathbf{I}_m) \bar{X}_k = \frac{\Lambda_{k+1} - \Lambda_0}{(k+1)\alpha},$$
 (12)

(*ii*)
$$\tilde{f}(\bar{X}_k) \le f^* + \frac{1}{2\alpha(k+1)} (||X_0 - X^*||^2 + ||\Lambda_0||^2 - ||X_{k+1} - X^*||^2)$$

$$-\|\Lambda_{k+1}\|^{2} + \frac{c_{r}}{2\alpha(k+1)} \times \left(V(X_{0},\Lambda_{1}) - V(X_{k+1},\Lambda_{k+2})\right), \quad (13)$$

$$(iii) \ \widetilde{f}(\bar{X}_k) \ge f^* - \frac{\langle \Lambda_{k+1} - \Lambda_0, \Lambda^* \rangle}{(k+1)\alpha}, \tag{14}$$

where $c_r = 1/\lambda_{min}(\mathcal{W} - \frac{\alpha l_r}{2}I_{mn}) + 1$.

Remark 4.4. Since $||\Lambda_k - \Lambda^*|| \le r \ \forall k \ge 0$ by Theorem 4.1(ii), Λ_k is uniformly bounded in k by a constant. Then by (12) we see that $(\mathcal{L} \otimes \mathbf{I}_m) \overline{X}_k$ converges to **0** with rate $O(\frac{1}{k})$. Since Theorem 4.1(i) implies that $V(X_0, \Lambda_1) - V(X_{k+1}, \Lambda_{k+2}) \le 0 \ \forall k \ge 0$, by (13)(14) the value of the cost function $\widetilde{f}(\cdot)$ at \overline{X}_k converges to the optimal value with rate $O(\frac{1}{k})$.

Remark 4.5. Note that $\Lambda_1 = \Lambda_0 + \alpha(\mathcal{L} \otimes \mathbf{I}_m)X_0$. Then from Theorems 4.1 and 4.3 we see that for small enough $\alpha > 0$ depending on the distance between the initial value and the optimal solution, and on the structure of the cost functions in the neighborhood of the optimal solution, the estimates given by all agents finally reach a consensus at an optimal solution. If $\nabla f_i(x)$ is globally Lipschitz continuous in set Ω_i with constant l_c for any $i \in \mathcal{V}$, then the results given in Theorems 4.1 and 4.3 hold as well for any α satisfying $0 < \alpha \leq \frac{1}{2\kappa_n}$ and $\alpha < \frac{3}{2l_c}$ but independent of the initial values.

4.2. Proof of Theorem 4.1

Prior to proving Theorem 4.1, we give a lemma that will be used in the proof.

Lemma 4.6. [23, Theorem 2.1.5] If $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function whose gradient is globally Lipschitz continuous with constant l_c , then

$$\langle x - y, \nabla f(x) - \nabla f(y) \rangle \ge \frac{1}{l_c} \|\nabla f(x) - \nabla f(y)\|^2 \quad \forall x, y \in \mathbb{R}^m.$$

Proof of Theorem 4.1: Note that

$$V(X_{k+1}, \Lambda_{k+2}) - V(X_k, \Lambda_{k+1}) = V_1(X_{k+1}) - V_1(X_k) + V_2(\Lambda_{k+2}) - V_2(\Lambda_{k+1}) = \langle \mathcal{W}(X_{k+1} - X_k), X_{k+1} + X_k - 2X^* \rangle + \langle \Lambda_{k+2} - \Lambda_{k+1}, \Lambda_{k+2} + \Lambda_{k+1} - 2\Lambda^* \rangle$$
(15)
$$= -\|\Lambda_{k+2} - \Lambda_{k+1}\|^2 - \langle X_{k+1} - X_k, \mathcal{W}(X_{k+1} - X_k) \rangle + 2\langle \Lambda_{k+2} - \Lambda_{k+1}, \Lambda_{k+2} - \Lambda^* \rangle + 2\langle X_{k+1} - X^*, \mathcal{W}(X_{k+1} - X_k) \rangle.$$

We now estimate the last two terms on the right hand side of (15).

Since (X^*, Λ^*) is a saddle point of $\phi(X, \Lambda)$, by Lemma 3.3 we see that X^* is an optimal solution to the problem (4), and hence

$$(\mathcal{L} \otimes \mathbf{I}_m) X^* = \mathbf{0}. \tag{16}$$

Since \mathcal{L} is symmetric, by (9) (16) we derive

$$\begin{aligned} \langle \Lambda_{k+2} - \Lambda_{k+1}, \Lambda_{k+2} - \Lambda^* \rangle \\ &= \langle \alpha(\mathcal{L} \otimes \mathbf{I}_m)(X_{k+1} - X^*), \Lambda_{k+2} - \Lambda^* \rangle \\ &= \langle \alpha(\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_{k+2} - \Lambda^*), X_{k+1} - X^* \rangle. \end{aligned}$$

Thus,

$$\langle \Lambda_{k+2} - \Lambda_{k+1}, \Lambda_{k+2} - \Lambda^* \rangle + \langle X_{k+1} - X^*, \mathcal{W}(X_{k+1} - X_k) \rangle = \langle X_{k+1} - X^*, \alpha(\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_{k+2} - \Lambda^*) + \mathcal{W}(X_{k+1} - X_k) \rangle.$$
(17)

From (9) we derive $\Lambda_{k+2} = \Lambda_k + \alpha(\mathcal{L} \otimes \mathbf{I}_m) X_k + \alpha(\mathcal{L} \otimes \mathbf{I}_m) X_{k+1}$, and hence by (16)

$$\Lambda_k - \Lambda^* = \Lambda_{k+2} - \Lambda^* - \alpha(\mathcal{L} \otimes \mathbf{I}_m)(X_k - X^*) - \alpha(\mathcal{L} \otimes \mathbf{I}_m)(X_{k+1} - X^*).$$
(18)

Then by multiplying both sides of (18) with $(\mathcal{L} \otimes \mathbf{I}_m)$, from the rule of the Kronecker product $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ we obtain

$$(\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_k - \Lambda^*) = (\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_{k+2} - \Lambda^*) -\alpha(\mathcal{L}^2 \otimes \mathbf{I}_m)(X_k - X^*) - \alpha(\mathcal{L}^2 \otimes \mathbf{I}_m)(X_{k+1} - X^*).$$
(19)

Set

$$Z_{k+1} = X_k - \alpha \nabla \widetilde{f}(X_k) - \alpha(\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_k + X_k) - X_{k+1}.$$
 (20)

Then by (16) (19) (20) we derive

$$\begin{aligned} X_{k+1} - X^* &= X_k - X^* - \alpha \nabla f(X_k) - \alpha(\mathcal{L} \otimes \mathbf{I}_m)(X_k - X^*) \\ -\alpha(\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_k - \Lambda^*) - \alpha(\mathcal{L} \otimes \mathbf{I}_m)\Lambda^* - Z_{k+1} \\ &= X_k - X^* - \alpha(\nabla \widetilde{f}(X_k) - \nabla \widetilde{f}(X^*)) - \alpha(\mathcal{L} \otimes \mathbf{I}_m)(X_k - X^*) \\ -\alpha(\nabla \widetilde{f}(X^*) + (\mathcal{L} \otimes \mathbf{I}_m)\Lambda^*) - Z_{k+1} \\ -\alpha\Big((\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_{k+2} - \Lambda^*) - \alpha(\mathcal{L}^2 \otimes \mathbf{I}_m)(X_k - X^*) \\ -\alpha(\mathcal{L}^2 \otimes \mathbf{I}_m)(X_{k+1} - X^*)\Big) \\ &= ((\mathbf{I}_n - \alpha \mathcal{L} + \alpha^2 \mathcal{L}^2) \otimes \mathbf{I}_m)(X_k - X^*) - Z_{k+1} \\ -\alpha(\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_{k+2} - \Lambda^*) - \alpha(\nabla \widetilde{f}(X_k) - \nabla \widetilde{f}(X^*)) \\ -\alpha(\nabla \widetilde{f}(X^*) + (\mathcal{L} \otimes \mathbf{I}_m)\Lambda^*) + \alpha^2(\mathcal{L}^2 \otimes \mathbf{I}_m)(X_{k+1} - X^*). \end{aligned}$$

Moving the last term at the right-hand side of (21) to the left and subtracting $\mathcal{W}(X_{k+1} - X^*)$ from both sides of (21) we derive the following recursion

$$\begin{aligned} &(\alpha \mathcal{L} \otimes \mathbf{I}_m - 2\alpha^2 \mathcal{L}^2 \otimes \mathbf{I}_m)(X_{k+1} - X^*) \\ &= \mathcal{W}(X_k - X_{k+1}) - \alpha(\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_{k+2} - \Lambda^*) - Z_{k+1} \\ &- \alpha(\nabla \widetilde{f}(X_k) - \nabla \widetilde{f}(X^*)) - \alpha(\nabla \widetilde{f}(X^*) + (\mathcal{L} \otimes \mathbf{I}_m)\Lambda^*), \end{aligned}$$

or in the alternative form

$$\begin{aligned} \mathcal{W}(X_{k+1} - X_k) &+ \alpha(\mathcal{L} \otimes I_m)(\Lambda_{k+2} - \Lambda^*) \\ &= -(\alpha \mathcal{L} \otimes I_m - 2\alpha^2 \mathcal{L}^2 \otimes I_m)(X_{k+1} - X^*) - Z_{k+1} \\ &- \alpha(\nabla \widetilde{f}(X_k) - \nabla \widetilde{f}(X^*)) - \alpha(\nabla \widetilde{f}(X^*) + (\mathcal{L} \otimes I_m)\Lambda^*). \end{aligned}$$

Then by (17) we derive

$$\langle \Lambda_{k+2} - \Lambda_{k+1}, \Lambda_{k+2} - \Lambda^* \rangle + \langle X_{k+1} - X^*, \mathcal{W}(X_{k+1} - X_k) \rangle = -\langle X_{k+1} - X^*, ((\alpha \mathcal{L} - 2\alpha^2 \mathcal{L}^2) \otimes \mathbf{I}_m)(X_{k+1} - X^*) \rangle - \alpha \langle X_{k+1} - X^*, \nabla \widetilde{f}(X_k) - \nabla \widetilde{f}(X^*) \rangle - \langle X_{k+1} - X^*, Z_{k+1} + \alpha (\nabla \widetilde{f}(X^*) + (\mathcal{L} \otimes \mathbf{I}_m)\Lambda^*) \rangle.$$

$$(22)$$

By the definition of the saddle point we have

$$\phi(X^*, \Lambda) \le \phi(X^*, \Lambda^*) \le \phi(X, \Lambda^*) \quad \forall X \in \Omega, \Lambda \in \mathbb{R}^{mn}.$$
(23)

Therefore, X^* minimizes $f(X) + \langle X, (\mathcal{L} \otimes \mathbf{I}_m) \Lambda^* \rangle$ over Ω . Since $\phi(X, \Lambda)$ is convex in $X \in \Omega$ for each Λ , by noticing $X_k \in \Omega \ \forall k \ge 0$ from the optimal condition [21, Proposition 1.1.8] we derive

$$\langle \nabla \widetilde{f}(X^*) + (\mathcal{L} \otimes \mathbf{I}_m) \Lambda^*, X_{k+1} - X^* \rangle \ge 0 \quad \forall k \ge 0.$$
 (24)

From (8) (20) it follows that $P_{\Omega}\{X_{k+1} + Z_{k+1}\} = X_{k+1}$, and hence $Z_{k+1} \in N_{\Omega}\{X_{k+1}\}$ by (2). Then by the definition of normal cone we obtain

$$\langle X_{k+1} - X^*, Z_{k+1} \rangle \ge 0.$$
 (25)

Then by combining (22) (24) (25) we derive

$$\begin{aligned} &\langle \Lambda_{k+2} - \Lambda_{k+1}, \Lambda_{k+2} - \Lambda^* \rangle + \langle X_{k+1} - X_k, \mathcal{W}(X_{k+1} - X^*) \rangle \\ &\leq -\langle X_{k+1} - X^*, ((\alpha \mathcal{L} - 2\alpha^2 \mathcal{L}^2) \otimes \mathbf{I}_m)(X_{k+1} - X^*) \rangle \\ &- \alpha \langle X_{k+1} - X^*, \nabla \widetilde{f}(X_k) - \nabla \widetilde{f}(X^*) \rangle. \end{aligned}$$

This incorporating with (15) yields

$$V(X_{k+1}, \Lambda_{k+2}) - V(X_k, \Lambda_{k+1}) \\ \leq - \|\Lambda_{k+2} - \Lambda_{k+1}\|^2 - \langle X_{k+1} - X_k, \mathcal{W}(X_{k+1} - X_k) \rangle \\ - 2\alpha \langle X_{k+1} - X^*, \nabla \widetilde{f}(X_k) - \nabla \widetilde{f}(X^*) \rangle \\ - 2 \langle X_{k+1} - X^*, ((\alpha \mathcal{L} - 2\alpha^2 \mathcal{L}^2) \otimes \mathbf{I}_m)(X_{k+1} - X^*) \rangle.$$
(26)

Since \mathcal{L} is symmetric, there exists an orthogonal matrix \mathcal{U} such that $\mathcal{U}^T \mathcal{L} \mathcal{U} = diag\{0, \kappa_2, \cdots, \kappa_n\}$, and hence $\mathcal{U}^T \mathcal{L}^2 \mathcal{U} = diag\{0, \kappa_2^2, \cdots, \kappa_n^2\}$. Then by (10) we know that all possible distinct eigenvalues of $\alpha \mathcal{L} - 2\alpha^2 \mathcal{L}^2$ are 0, and $\alpha \kappa_i - 2\alpha^2 \kappa_i^2$, $i = 2, \cdots, n$. If $0 < \alpha \leq \frac{1}{2\kappa_n}$, then $2\alpha\kappa_i \leq 1 \quad \forall i = 1, \cdots, n$, and hence $\alpha\kappa_i - 2\alpha^2\kappa_i^2 = \alpha\kappa_i(1 - 2\alpha\kappa_i) \geq 0 \quad \forall i = 1, \cdots, n$. Therefore, for any α with $0 < \alpha \leq \frac{1}{2\kappa_n}$ the matrix $\alpha \mathcal{L} - 2\alpha^2 \mathcal{L}^2$ is positive semi-definite, and hence by (26) we derive

$$V(X_{k+1}, \Lambda_{k+2}) - V(X_k, \Lambda_{k+1}) \\ \leq -\|\Lambda_{k+2} - \Lambda_{k+1}\|^2 - \langle X_{k+1} - X_k, \mathcal{W}(X_{k+1} - X_k) \rangle$$
(27)
$$- 2\alpha \langle X_{k+1} - X^*, \nabla \widetilde{f}(X_k) - \nabla \widetilde{f}(X^*) \rangle.$$

Let the constant α be such that $0 < \alpha \leq \frac{1}{2\kappa_n}$ and $\alpha < \frac{3}{2l_r}$. In what follows, we show that $V(X_k, \Lambda_{k+1})$ monotonously decreases and $d_k \leq r \quad \forall k \geq 0$ by induction.

We first show that $d_1 \leq r$ and $V(X_1, \Lambda_2) \leq V(X_0, \Lambda_1)$. By the definition of the local Lipschitz constant l_r , we know that $\nabla \widetilde{f}(X)$ is Lipschitz continuous on the compact set $\{X \in \Omega :$ $\|X - X^*\| \leq r\}$ with Lipschitz constant l_r . Since $\|X_0 - X^*\| \leq r$ by the definition of *r*, from Lemma 4.6 we see

$$\langle X_0 - X^*, \nabla \widetilde{f}(X_0) - \nabla \widetilde{f}(X^*) \rangle \ge \frac{1}{l_r} ||\nabla \widetilde{f}(X_0) - \nabla \widetilde{f}(X^*)||^2.$$

This incorporating with $xy \le \frac{x^2}{4} + y^2$ leads to

$$\begin{aligned} -\langle X_1 - X^*, \nabla \widetilde{f}(X_0) - \nabla \widetilde{f}(X^*) \rangle \\ &= -\langle X_0 - X^*, \nabla \widetilde{f}(X_0) - \nabla \widetilde{f}(X^*) \rangle \\ &+ \langle -X_1 + X_0, \nabla \widetilde{f}(X_0) - \nabla \widetilde{f}(X^*) \rangle \\ &\leq -\frac{1}{l_r} \| \nabla \widetilde{f}(X_0) - \nabla \widetilde{f}(X^*) \|^2 + \frac{l_r}{4} \| X_0 - X_1 \|^2 \\ &+ \frac{1}{l_r} \| \nabla \widetilde{f}(X_0) - \nabla \widetilde{f}(X^*) \|^2 \leq \frac{l_r}{4} \| X_0 - X_1 \|^2. \end{aligned}$$

$$(28)$$

Then from here by (27) we have

$$V(X_1, \Lambda_2) - V(X_0, \Lambda_1) \le -\|\Lambda_2 - \Lambda_1\|^2 - \langle X_1 - X_0, (\mathcal{W} - \frac{\alpha l_r}{2} I_{mn})(X_1 - X_0) \rangle.$$

By (10) we know that all possible distinct eigenvalues of $\mathcal{W} - \frac{\alpha l_r}{2}I_{mn}$ are $1-\alpha\kappa_i+\alpha^2\kappa_i^2-\frac{\alpha l_r}{2}, i=1,\cdots,n$. Since $\alpha < \frac{3}{2l_r}$, we have $1-\alpha\kappa_i+\alpha^2\kappa_i^2-\frac{\alpha l_r}{2}=(\frac{1}{2}-\alpha\kappa_i)^2+\frac{3}{4}-\frac{\alpha l_r}{2}>0$. Thus, $\mathcal{W}-\frac{\alpha l_r}{2}I_{mn}$ is positive definite. As a result, $V(X_1,\Lambda_2) \leq V(X_0,\Lambda_1)$. Then $V(X_1,\Lambda_2) \leq r^2\lambda_{min}(\mathcal{M})$, and hence $d_1 \leq r$.

Assume that $d_p \leq r$ and $V(X_p, \Lambda_{p+1}) \leq V(X_{p-1}, \Lambda_p)$ for $p = 1, \dots, k$. Since (27) holds and $||X_k - X^*|| \leq r$, similar to the case k = 0, we can show that $V(X_{k+1}, \Lambda_{k+2}) \leq V(X_k, \Lambda_{k+1})$ and $d_{k+1} \leq r$.

In summary, by the mathematical induction we conclude that $d_k \le r \quad \forall k \ge 0$, and $V(X_k, \Lambda_{k+1})$ monotonously decreases.

Since $||X_k - X^*|| \le r \quad \forall k \ge 0$, by the same procedure for deriving (28) we obtain

$$-\langle X_{k+1} - X^*, \nabla \widetilde{f}(X_k) - \nabla \widetilde{f}(X^*) \rangle \le \frac{l_r}{4} ||X_k - X_{k+1}||^2 \quad \forall k \ge 0.$$

Then from here by (27) we derive

$$V(X_{k+1}, \Lambda_{k+2}) - V(X_k, \Lambda_{k+1}) \le -\|\Lambda_{k+2} - \Lambda_{k+1}\|^2 - \langle X_{k+1} - X_k, (\mathcal{W} - \frac{\alpha l_r}{2} I_{mn})(X_{k+1} - X_k) \rangle \le 0.$$
(29)

Thus, we conclude that $V(X_k, \Lambda_{k+1})$ converges since it is nonnegative. Summing up both sides of (29) from 0 to *p* we derive

$$V(X_{p+1}, \Lambda_{p+2}) - V(X_0, \Lambda_1) \le -\sum_{k=0}^p \|\Lambda_{k+2} - \Lambda_{k+1}\|^2 -\sum_{k=0}^p \langle X_{k+1} - X_k, (\mathcal{W} - \frac{\alpha l_r}{2} I_{mn})(X_{k+1} - X_k) \rangle.$$
(30)

Then by letting $p \to \infty$ we have

$$\sum_{k=0}^{\infty} \langle X_{k+1} - X_k, (\mathcal{W} - \frac{\alpha l_r}{2} I_{mn})(X_{k+1} - X_k) \rangle < \infty, \qquad (31)$$

and
$$\sum_{k=1}^{\infty} \|\Lambda_{k+1} - \Lambda_k\|^2 < \infty.$$
(32)

Consequently, we derive $\lim_{k\to\infty} (X_{k+1} - X_k) = 0$ by (31) since $\mathcal{W} - \frac{\alpha l_r}{2} I_{mn}$ is positive definite, and $\lim_{k\to\infty} (\Lambda_{k+1} - \Lambda_k) = 0$ by (32). By convergence of $V(X_k, \Lambda_{k+1})$ we conclude that X_k and Λ_k contain convergent subsequences $\{X_{n_k}\}$ and $\{\Lambda_{n_k}\}$ to some limits X^0 and Λ^0 , respectively. Since $\lim_{k\to\infty} (X_{n_k+1} - X_{n_k}) = 0$ and $\lim_{k\to\infty} (\Lambda_{n_k+1} - \Lambda_{n_k}) = 0$, by noticing that $P_{\Omega}\{x\}$ is continuous in x and $\lim_{k\to\infty} X_{n_k} = X^0$, $\lim_{k\to\infty} \Lambda_{n_k} = \Lambda^0$, from (8) (9) we derive

$$X^{0} = P_{\Omega} \{ X^{0} - \alpha \nabla \widetilde{f}(X^{0}) - \alpha(\mathcal{L} \otimes \mathbf{I}_{m}) (\Lambda^{0} + X^{0}) \},$$
(33)

$$(\mathcal{L} \otimes \mathbf{I}_m) X^0 = 0. \tag{34}$$

Then from (33) (34) by (2) we see $\alpha(\nabla \widetilde{f}(X^0) + (\mathcal{L} \otimes \mathbf{I}_m)\Lambda^0) = \alpha(\nabla \widetilde{f}(X^0) + (\mathcal{L} \otimes \mathbf{I}_m)(\Lambda^0 + X^0)) \in -N_{\Omega}(X^0)$, and hence by the definition of normal cone we conclude

$$\langle \nabla \widetilde{f}(X^0) + (\mathcal{L} \otimes \mathbf{I}_m) \Lambda^0, X - X^0 \rangle \ge 0 \quad \forall X \in \Omega.$$

Since $\phi(X, \Lambda) = \tilde{f}(X) + \langle \Lambda, (\bar{\mathcal{L}} \otimes \mathbf{I}_m) X \rangle$ is convex in $X \in \Omega$ for each $\Lambda \in \mathbb{R}^{mn}$, by (1) we have

$$\phi(X, \Lambda^0) \geq \phi(X^0, \Lambda^0) + \langle \nabla f(X^0) + (\mathcal{L} \otimes \mathbf{I}_m) \Lambda^0, X - X^0 \rangle \geq \phi(X^0, \Lambda^0) \quad \forall X \in \Omega.$$

From (34) we see $\phi(X^0, \Lambda^0) = \phi(X^0, \Lambda) = \tilde{f}(X^0) \quad \forall \Lambda \in \mathbb{R}^{mn}$, and hence by definition we know (X^0, Λ^0) is a saddle point of $\phi(X, \Lambda)$ in $\Omega \times \mathbb{R}^{mn}$. Thus, by Lemma 3.3 we see that X^0 is an optimal solution to the problem (4).

Since $\|\Lambda_{k+1} - \Lambda^*\|^2 + (X_k - X^*)^T \mathcal{W}(X_k - X^*)$ converges, by setting $(X^*, \Lambda^*) = (X^0, \Lambda^0)$ from $\lim_{k \to \infty} X_{n_k} = X^0$ and $\lim_{k \to \infty} \Lambda_{n_k} = \Lambda^0$ we conclude that $\|\Lambda_{k+1} - \Lambda^0\|^2 + (X_k - X^0)^T \mathcal{W}(X_k - X^0)$ converges to zero. Therefore,

$$\lim_{k\to\infty} X_k = X^0, \quad \lim_{k\to\infty} \Lambda_k = \Lambda^0.$$

Thus, by Remark 3.2 we conclude that $\lim_{k \to \infty} x_{i,k} = \lim_{k \to \infty} x_{j,k} = x^* \quad \forall i, j \in \mathcal{V} \text{ for some } x^* \in \Omega_o^*.$

4.3. Proof of Theorem 4.3

Proof: (i) Summing up both sides of (9) from 0 to p leads to

$$\Lambda_{p+1} - \Lambda_0 = \sum_{k=0}^p \alpha(\mathcal{L} \otimes \mathbf{I}_m) X_k = (p+1) \alpha(\mathcal{L} \otimes \mathbf{I}_m) \bar{X}_p,$$

and hence (12) holds.

(ii) When $\Omega_i = \mathbb{R}^m \ \forall i \in \mathcal{V}$, the equation (8) turns to

$$X_{k+1} = X_k - \alpha \nabla f(X_k) - \alpha (\mathcal{L} \otimes \mathbf{I}_m) (\Lambda_k + X_k).$$
(35)

By (30) we see

$$\sum_{k=0}^{p} \left(\|\Lambda_{k+2} - \Lambda_{k+1}\|^2 + \langle X_{k+1} - X_k, (\mathcal{W} - \frac{\alpha l_r}{2} I_{mn})(X_{k+1} - X_k) \right)$$

$$\leq -V(X_{p+1}, \Lambda_{p+2}) + V(X_0, \Lambda_1).$$

Then by noticing that $\mathcal{W} - \frac{\alpha l_r}{2} I_{mn}$ is positive definite we derive

$$\sum_{k=0}^{p} \|\Lambda_{k+2} - \Lambda_{k+1}\|^2 \le -V(X_{p+1}, \Lambda_{p+2}) + V(X_0, \Lambda_1), \quad (36)$$

and
$$\sum_{k=0}^{p} ||X_{k+1} - X_k||^2 \le \frac{-V(X_{p+1}, \Lambda_{p+2}) + V(X_0, \Lambda_1)}{\lambda_{min}(\mathcal{W} - \frac{\alpha l_r}{2}I_{mn})}.$$
 (37)

By (35) we have

$$\begin{aligned} \|X_{k+1} - X^*\|^2 &= \|X_k - X^*\|^2 + \|X_{k+1} - X_k\|^2 \\ &+ 2(X_k - X^*)^T (X_{k+1} - X_k) = \|X_k - X^*\|^2 \\ &+ 2\alpha \langle X^* - X_k, \nabla \widetilde{f}(X_k) + (\mathcal{L} \otimes \mathbf{I}_m)(\Lambda_k + X_k) \rangle \\ &+ \|X_{k+1} - X_k\|^2. \end{aligned}$$
(38)

Noticing $\phi(X, \Lambda)$ is convex in $X \in \Omega$ for any Λ , by (1) we have

$$\langle X^* - X_k, \nabla \tilde{f}(X_k) + (\mathcal{L} \otimes \mathbf{I}_m) \Lambda_k \rangle \le \phi(X^*, \Lambda_k) - \phi(X_k, \Lambda_k).$$
(39)

Since Λ_k is bounded and X^* is an optimal solution to the problem (4), by (16) we see $\phi(X^*, \Lambda_k) = \tilde{f}(X^*) = f^*$. Since \mathcal{L} is positive semi-definite by Lemma 2.1, from (16) it follows that

$$\langle X^* - X_k, (\mathcal{L} \otimes \mathbf{I}_m) X_k \rangle = -\langle X_k, (\mathcal{L} \otimes \mathbf{I}_m) X_k \rangle \leq 0.$$

Thus, from here by (38) (39) we derive

$$||X_{k+1} - X^*||^2 \le ||X_k - X^*||^2 + 2\alpha(f^* - \phi(X_k, \Lambda_k)) + ||X_{k+1} - X_k||^2,$$

and hence

$$\phi(X_k, \Lambda_k) - f^* \leq \frac{1}{2\alpha} (||X_k - X^*||^2 - ||X_{k+1} - X^*||^2 + ||X_{k+1} - X_k||^2).$$

Summing up both sides of this inequality from 0 to p for $p \ge 1$ we obtain

$$\begin{split} &\sum_{k=0}^{p} (\phi(X_k, \Lambda_k) - f^*) \\ &\leq \frac{1}{2\alpha} (\|X_0 - X^*\|^2 - \|X_{p+1} - X^*\|^2 + \sum_{k=0}^{p} \|X_{k+1} - X_k\|^2). \end{split}$$

From here by the convexity of $\tilde{f}(X)$ we derive

$$\widetilde{f}(\overline{X}_{p}) \leq \frac{1}{p+1} \sum_{k=0}^{p} \widetilde{f}(X_{k}) \leq \frac{1}{p+1} \sum_{k=0}^{p} \phi(X_{k}, \Lambda_{k})
- \frac{1}{p+1} \sum_{k=0}^{p} X_{k}^{T}(\mathcal{L} \otimes \mathbf{I}_{m}) \Lambda_{k}
\leq f^{*} + \frac{1}{2\alpha(p+1)} (||X_{0} - X^{*}||^{2} - ||X_{p+1} - X^{*}||^{2}
+ \sum_{k=0}^{p} ||X_{k+1} - X_{k}||^{2})
- \frac{1}{p+1} \sum_{k=0}^{p} \langle X_{k}, (\mathcal{L} \otimes \mathbf{I}_{m}) \Lambda_{k} \rangle.$$
(40)

We now give an upper bound for $-\frac{1}{p+1}\sum_{k=0}^{p}\langle X_k, (\mathcal{L}\otimes \mathbf{I}_m)\Lambda_k \rangle$. By (9) we have

$$\|\Lambda_{k+1}\|^2 = \|\Lambda_k\|^2 + 2\alpha \langle \Lambda_k, (\mathcal{L} \otimes \mathbf{I}_m) X_k \rangle + \|\alpha(\mathcal{L} \otimes \mathbf{I}_m) X_k\|^2.$$



Figure 1: The estimates and the residual

Thus, $-\langle X_k, (\mathcal{L} \otimes \mathbf{I}_m) \Lambda_k \rangle = \frac{1}{2\alpha} (\|\Lambda_k\|^2 - \|\Lambda_{k+1}\|^2 + \|\Lambda_{k+1} - \Lambda_k\|^2)$, and hence

$$-\frac{1}{p+1}\sum_{k=0}^{p} \langle X_{k}, (\mathcal{L} \otimes \mathbf{I}_{m})\Lambda_{k} \rangle$$

$$=\frac{1}{2\alpha(p+1)} (\|\Lambda_{0}\|^{2} - \|\Lambda_{p+1}\|^{2} + \sum_{k=0}^{p} \|\Lambda_{k+1} - \Lambda_{k}\|^{2}).$$
(41)

By substituting (41) into (40) we derive

$$\begin{split} f(X_p) &\leq f^* \\ &+ \frac{1}{2\alpha(p+1)} \Big(\|X_0 - X^*\|^2 - \|X_{p+1} - X^*\|^2 + \|\Lambda_0\|^2 - \|\Lambda_{p+1}\|^2 \Big) \\ &+ \frac{1}{2\alpha(p+1)} \Big(\sum_{k=0}^p \|X_{k+1} - X_k\|^2 + \sum_{k=0}^p \|\Lambda_{k+1} - \Lambda_k\|^2 \Big). \end{split}$$

Then from here by (36)(37) we obtain (13).

(iii) By (16) (23) we derive $\phi(\bar{X}_p, \Lambda^*) \ge \phi(X^*, \Lambda^*) = \tilde{f}(X^*) = f^*$, and hence for any dual solution Λ^*

$$\begin{split} \widetilde{f}(\bar{X}_p) &= \widetilde{f}(\bar{X}_p) + \langle \bar{X}_p, (\mathcal{L} \otimes \mathbf{I}_m) \Lambda^* \rangle - \langle \bar{X}_p, (\mathcal{L} \otimes \mathbf{I}_m) \Lambda^* \rangle \\ &= \phi(\bar{X}_p, \Lambda^*) - \langle \bar{X}_p, (\mathcal{L} \otimes \mathbf{I}_m) \Lambda^* \rangle \geq f^* - \langle \bar{X}_p, (\mathcal{L} \otimes \mathbf{I}_m) \Lambda^* \rangle. \end{split}$$

Then by (12) we derive (14).

5. Numerical Simulations

In this section, we give two numerical examples to demonstrate the obtained theoretic results.

Example 5.1. This example shows that the primal-dual algorithm with constant step size can produce the accurate estimates for the constrained optimization problem, where gradients of the cost functions are only locally Lipschitz continuous, and the agents are equipped with different constraint sets. Besides, some of the constraint sets are not compact.



Figure 2: The communication topology

Consider an undirected network of three agents with edge set $\mathcal{E}_{\mathcal{G}} = \{(1, 3), (2, 3), (1, 1), (2, 2), (3, 3)\}$. Objective functions for the agents are as follows:

$$f_1(x_1, x_2) = \frac{x_1^2}{2} + 3x_1 + x_2^2 + 2x_2 + x_1x_2 + 0.5e^{x_1 + x_2},$$

$$f_2(x_1, x_2) = x_1^2 + 2x_1 + 2x_2^2 + 2x_2 + x_1x_2 + e^{x_2},$$

$$f_3(x_1, x_2) = 2x_1^2 + 4x_1 + x_2^2 + 2x_2 + e^{x_1},$$
(42)

while the constraint sets for agents are $\Omega_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 2\}$, $\Omega_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \ge -1\}$, and $\Omega_3 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \le -0.5\}$. Denote the optimal solution by (x_1^*, x_2^*) , which is at the boundary of the global constraint set.

Let $\{x_{i,k}\}$ and $\{\lambda_{i,k}\}$ be produced by the algorithm (7) with initial values $x_{i,0} = \mathbf{0}$, $\lambda_{i,0} = \mathbf{0}$, i = 1, 2, 3, and $\alpha = 0.4$. Denote by $x_{i,k}^1$ and $x_{i,k}^2$ the estimates for x_1^* and x_2^* by agent *i* at time *k*, respectively. Note that the primal-dual solution pair (X^*, Λ^*) satisfies (33) and (34). Define the residual of the optimal condition as $r_k = col\{X_{k+1} - X_k, (\mathcal{L} \otimes \mathbf{I}_m)X_k\}$. The local estimates of all agents and 2-norm of the residual r_k are shown in Figure 1. From the figure it is seen that the estimates for all agents converge to the same optimal solution.

Example 5.2. Consider a randomly generated undirected network of n = 10 agents, where each agent has an average degree 4. Each agent $i \in \mathcal{V}$ is assigned with a huber loss function $f_i : \mathbb{R} \to \mathbb{R}$ with

$$f_i(x) = \begin{cases} \frac{1}{2}(x-a_i)^2, \text{ if } |x-a_i| \le 1, \\ |x-a_i| - \frac{1}{2}, \text{ otherwise.} \end{cases}$$

For any $i \in \mathcal{V}$, a_i is generated according to the uniform distribution over the interval [1.5, 2.5]. The optimal solution of $f(\cdot) = \sum_{i=1}^{n} f_i(\cdot)$ is denoted by x^* . We compare the primal-dual algorithm (7) with the existing ones by this example.

Set $\alpha = 0.8$. Denote by $x_{i,k}$ the estimate for x^* given by agent i at time k with the initial values $x_{i,0} = \mathbf{0} \quad \forall i \in \mathcal{V}$. The simulation is for the case where the communication topology is shown in Figure 2, and the entries of the adjacency matrix $\mathcal{R}_{\mathcal{G}}$ are Metropolis wights [25]. With this $\mathcal{R}_{\mathcal{G}}$ we carry out the simulations for the primal-dual algorithm (7) with $\lambda_{i,0} = \mathbf{0} \quad \forall i \in \mathcal{V}$,



Figure 3: The normalized relative error

for the DGD algorithm [7], for EXTRA [15] with constant step size α , and for the distributed Nesterov gradient (D-NG) algorithm in [11]. The DGD algorithm runs separately for three cases: constant step size α , diminishing step sizes $\alpha_k = \alpha/k^{0.75}$, and $\alpha_k = \alpha/k^{0.4}$. The D-NG algorithm, i.e., equations (2)-(4) in [11], is run with $c = \alpha$ and $y_{i,0} = \mathbf{0} \quad \forall i \in \mathcal{V}$.

[11], is run with $c = \alpha$ and $y_{i,0} = \mathbf{0} \forall i \in \mathcal{V}$. Denote by $e_k = \frac{\|X_k - 1 \otimes x^*\|}{\|X_0 - 1 \otimes x^*\|}$ the normalized relative error, where $X_k = col\{x_{1,k}, \dots, x_{n,k}\}$. The numerical results are shown in Figure 3, where the horizontal axis denotes the number of iterations *k* and the vertical axis denotes $\log_{10}(e_k)$. From the figure it is seen that the DGD algorithms with decreasing step sizes converge to the optimal solution but the rate of convergence are the slowest in comparisons with other methods. It is also seen that DGD with constant step size quickly approaches to the neighborhood of the optimal solution. The estimates generated by D-NG [11], by the algorithm (7), and by EXTRA [15] all converge to the optimal solution. Besides, the algorithm (7) brings a satisfactory convergence rate for the unconstrained problem as well.

6. Conclusion

In the paper, a distributed primal-dual algorithm is proposed for multiple agents in a network to minimize the sum of individual cost functions subject to a global constraint, which is the intersection of the local constraints. The proposed algorithm with constant step size makes the estimates of all agents converge to the same optimal solution and achieve the convergence rate $O(\frac{1}{k})$ when there is no constraint. The effectiveness and the priority of the proposed algorithm have been demonstrated by two numerical examples.

For further research, it is of interest to consider the primaldual algorithm for stochastic optimization, and to see if some desired properties taking place for the deterministic still remain true.

References

 R. Olfati-Saber, and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

- [2] W. Cao, J. Zhang, and W. Ren, "Leader-follower consensus of linear multiagent systems with unknown external disturbances," *Systems & Control Letters*, vol. 82, pp. 64-70, 2015.
- [3] S. Kar, J. M. F. Moura, and K. Ramanan, "Distributed parameter estimation in sensor networks: nonlinear observation models and imperfect communication," *IEEE Trans. Inf. Theory*, vol. 58, no. 6, pp. 3575–3605, 2012.
- [4] U. A. Khan, S. Kar, and J. M. F. Moura, "Distributed sensor localization in random environments using minimal number of anchor nodes," *IEEE Trans. Singnal Processing*, vol. 57, no. 5, pp. 2000–2016, 2009.
- [5] K. You, Z. Li, and L. Xie, "Consensus condition for linear multi-agent systems over randomly switching topologies," *Automatica*, vol. 49, no. 10, pp. 3125–3132, 2013.
- [6] B. Johansson, A. Speranzon, M. Johansson, and K. H. Johansson, "On decentralized negotiation of optimal consensus," *Automatic*, vol. 44, no. 4, pp. 1175-1179, 2008.
- [7] A. Nedić, and A. Ozdaglar, "Distributed subgradient methods for multiagent optimization," *IEEE Trans. Autom. Control*, vol. 54, no. 1, pp.48-61, 2009.
- [8] I. Lobel, and A. Ozdaglar, "Distributed subgradient methods for convex optimization over random networks," *IEEE Trans. Autom. Control*, vol. 56, no. 6, pp. 1291-1306, 2011.
- [9] S. S. Ram, A. Nedić, and V. V. Veeravalli, "Distributed stochastic subgradient projection algorithms for convex optimization," *J Optim Theorey Appl*, vol. 147, pp. 516-545, 2010.
- [10] K. Srivastava, and A. Nedić, "Distributed asynchronous constrained stochastic optimization," *IEEE Journal of Selected Topics in Signal Processing*, vol. 5, no. 4, pp. 772-790, 2011.
- [11] D. Jakovetic, J. Xavier, and J. M. F. Moura, "Fast distributed gradient methods," *IEEE Trans. Autom. Control*, vol. 59, no. 5, pp. 1131-1146, 2014.
- [12] T. H. Chang, A. Nedić, and A. Scaglione, "Distributed constrained optimization by consensus-based primal-dual perturbation method," *IEEE Trans. Autom. Control*, vol. 59, no. 6, pp. 1524-1538,2014.
- [13] M. Zhu, and S. Martínez, "On distributed convex optimization under inequality and equality constraints" *IEEE Trans. Autom. Control*, vol. 57, no. 1, pp. 151-164, 2012.
- [14] Q. Liu, and J. Wang, "A second-order multi-agent network for bounded constrained distributed optimization," *IEEE Trans. Autom. Control*, vol. 60, no. 12, pp. 3310–3315, 2015.
- [15] W. Shi, Q. Ling, G. Wu, and W. Yin, "EXTRA: An exact first-order algorithm for decentralized consensus optimization," *SIAM Journal on Optimization*, vol. 25, no. 2, pp. 944-966, 2015
- [16] J. Wang, and N. Elia, "Control approach to distributed optimization," *Allerton Conference*, pp. 557–561, 2010.
- [17] J. Wang, and N. Elia, "A control perspective for centralized and distributed convex optimization," CDC-ECC, pp. 3800–3805, 2011.
- [18] X. Zeng, P. Yi, and Y. Hong, "Distributed continuous-time clgorithm for constrained convex optimizations via nonsmooth analysis approach", arXiv:1510.07386.
- [19] P. Yi, Y. Hong, and F. Liu, "Distributed gradient algorithm for constrained optimization with application to load sharing in power systems," *Systems & Control Letters*, vol. 83, pp. 45–52, 2015.
- [20] A. Mokhtari, and A. Ribeiro, "DSA: decentralized double stochastic averaging gradient algorithm", arXiv:1506.04216v1, 2015.
- [21] D. P. Bertsekas, Convex Optimization Theory, Athena Scientific and Tsinghua University Press, 2010.
- [22] A. Ruszczynski, Nonlinear Optimization, Princeton University Press, New Jersey, 2006.
- [23] Y. Nesterov, Introductory Lectures on Convex Programming Volume I: Basic Course, 1998.
- [24] C. D. Godsil and G. Royle, Algebraic Graph Theory. New York: Springer-Verlag, 2001.
- [25] L. Xiao, S. Boyd, and S. Lall, "Distributed average consensus with timevarying Metropolis weights," *Preprint submitted to Automatic*, June, 2006.
- [26] H. Uzawa, "Iterative methods in concave programming," in Studies in Linear and Nonlinear Programming, K. Arrow, L. Hurwicz, and H. Uzawa, Eds. Stanford, CA: Stanford Univ. Press, 1958, pp. 154–165.
- [27] A. Nedić and A. Ozdaglar, "Subgradeint methods for saddle-point problems," J. Optim. Theory Appl., vol. 142, pp. 205–228, 2009.