# Primal-Dual Algorithm for Distributed Constrained Optimization 

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#### Abstract

The paper studies a distributed constrained optimization problem, where multiple agents connected in a network collectively minimize the sum of individual objective functions subject to a global constraint being an intersection of the local constraint sets assigned to the agents. Based on the augmented Lagrange method, a distributed primal-dual algorithm with a projection operation included is proposed to solve the problem. It is shown that with appropriately chosen constant step size, the local estimates derived at all agents asymptotically reach a consensus at an optimal solution. In addition, the value of the cost function at the time-averaged estimate converges with rate $O\left(\frac{1}{k}\right)$ to the optimal value for the unconstrained problem. By these properties the proposed primal-dual algorithm is distinguished from the existing algorithms for distributed constrained optimization. The theoretical analysis is justified by numerical simulations.


Keywords: Distributed constrained optimization, primal-dual algorithm, augmented Lagrange method, multi-agent network.

## 1. Introduction

Distributed computation and estimation recently have received much research attention, e.g., consensus problems [1, 2], distributed estimation [3], sensor localization [4], and distributed control [5, 6]. In particular, distributed optimization problems have been extensively investigated in [7]-[19], among which the distributed subgradient or gradient algorithms [7][11] belong to the primal domain methods while [12]-[19] belong to the primal-dual domain methods.

The paper considers a distributed constrained optimization problem, where $n$ agents connected in a network collectively minimize the sum of local objective functions $f(x)=\sum_{i=1}^{n} f_{i}(x)$ subject to a global constraint $\Omega_{o}=\bigcap_{i=1}^{n} \Omega_{i}$, where $\Omega_{i}$ is a convex set and $f_{i}(x)$ is a convex function in $\Omega_{i}$. Besides, $f_{i}(x)$ and $\Omega_{i}$ are the local data known to agent $i$ and cannot be shared with other agents. This problem is equivalent to a convex optimization problem with single linear coupling constraint and a convex set constraint.

The main contribution of the paper is to propose a distributed primal-dual algorithm with constant step size to solve the constrained optimization problem over the multi-agent network. The algorithm is derived on the basis of the gradient algorithm for finding saddle points of an augmented Lagrange function [21]. In an iteration each agent updates its estimate only using the local data and the information derived from the neighboring

[^0]agents. With appropriately chosen constant step size, the estimates derived at all agents are shown to reach a consensus at an optimal solution. Besides, it is found that the value of the cost function at the time-averaged estimate converges with rate $O\left(\frac{1}{k}\right)$ to the optimal value for the unconstrained problem.

A general constrained convex optimization problem is studied in [12], where the constraint sets are assumed to be compact. The problem in the random case is investigated by [10] for nonsmooth objective functions, meanwhile, the convex sets are assumed to be compact and the global constraint set is required to have a nonempty interior. Here, we study the problem in the deterministic case for smooth objective functions, while imposing weaker assumptions on the convex sets.

When there are no constraints, the problem of the paper becomes the one discussed in [7, 11, 15, 16, 17]. The estimates produced by the distributed gradient descent (DGD) algorithm with constant step size [7] converge to a neighborhood of the optimal solution. In contrast to this, our algorithm gives the accurate estimate. To solve the distributed optimization problems, some continuous-time distributed algorithms are proposed in [16, 17], while here the discrete-time distributed algorithm is investigated. The estimates generated by the fast distributed gradient algorithms [11] and by EXTRA [15] converge to an optimal solution, but in [11] each cost function is assumed to be convex with gradients being bounded and Lipschitz continuous, while EXTRA [15] only deals with unconstrained problems. Though it is shown by [20] that EXTRA [15] is also a saddle point method, the augmented Lagrange function used in [15] is different from ours. Besides, the convergence rate $O\left(\frac{1}{k}\right)$ derived here for the unconstrained case is a new result. The primal-dual algorithm proposed in the paper can be seen as an extension of EXTRA [15] to constrained problems.

The rest of the paper is organized as follows: In Section 2,
some preliminary information about graph theory and convex analysis is provided and the problem is formulated. In Section 3, a distributed primal-dual algorithm is proposed for solving the problem, while its convergence is proved in Section 4. Two numerical examples are demonstrated in Section 5, and some concluding remarks are given in Section 6.

## 2. Preliminaries and Problem Statement

We first provide some information about graph theory, convex functions, and convex sets. Then we formulate the distributed constrained optimization problem to be investigated.

### 2.1. Graph Theory

Consider a network of $n$ agents. The communication relationship among the $n$ agents is described by an undirected graph $\mathcal{G}=\left\{\mathcal{V}, \mathcal{E}_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}\right\}$, where $\mathcal{V}=\{1, \cdots, n\}$ is the node set with node $i$ representing agent $i ; \mathcal{E}_{\mathcal{G}} \subset \mathcal{V} \times \mathcal{V}$ is the undirected edge set, and the unordered pair of nodes $(i, j) \in \mathcal{E}_{\mathcal{G}}$ if and only if agent $i$ and agent $j$ can exchange information with each other; $\mathcal{A}_{\mathcal{G}}=\left[a_{i j}\right] \in \mathbb{R}^{n \times n}$ is the adjacency matrix of $\mathcal{G}$, where $a_{i j}=a_{j i}>0$ if $(i, j) \in \mathcal{E}_{\mathcal{G}}$, and $a_{i j}=0$, otherwise. Denote by $\mathcal{N}_{i}=\left\{j \in \mathcal{V}:(i, j) \in \mathcal{E}_{\mathcal{G}}\right\}$ the neighboring agents of agent $i$. The Laplacian matrix of graph $\mathcal{G}$ is defined as $\mathcal{L}_{\mathcal{G}}=\mathcal{D}_{\mathcal{G}}-\mathcal{A}_{\mathcal{G}}$, where $\mathcal{D}_{\mathcal{G}}=\operatorname{diag}\left\{\sum_{j=1}^{n} a_{1 j}, \cdots, \sum_{j=1}^{n} a_{n j}\right\}$. For a given pair $i, j \in \mathcal{V}$, if there exists a sequence of distinct nodes $i_{1}, \cdots, i_{p}$ such that $\left(i, i_{1}\right) \in \mathcal{E}_{\mathcal{G}},\left(i_{1}, i_{2}\right) \in \mathcal{E}_{\mathcal{G}}, \cdots,\left(i_{p}, j\right) \in \mathcal{E}_{\mathcal{G}}$, then $\left(i, i_{1}, \cdots, i_{p}, j\right)$ is called the undirected path between $i$ and $j$. We say that $\mathcal{G}$ is connected if there exists an undirected path between any $i, j \in \mathcal{V}$.

The following lemma presents some properties of the Laplacian matrix $\mathcal{L}$ corresponding to an undirected graph $\mathcal{G}$.

Lemma 2.1. [24] The Laplacian matrix $\mathcal{L}$ of an undirected graph $\mathcal{G}$ has the following properties:
i) $\mathcal{L}$ is symmetric and positive semi-definite;
ii) $\mathcal{L}$ has a simple zero eigenvalue with corresponding eigenvector equal to 1, and all other eigenvalues are positive if and only if the graph $\mathcal{G}$ is connected, where $\mathbf{1}$ denotes the vector of compatible dimension with all entries equal to 1 .

### 2.2. Gradient, Projection Operator and Normal Cone

For a given function $f: \mathbb{R}^{m} \rightarrow[-\infty, \infty]$, denote its domain as $\operatorname{dom}(f) \triangleq\left\{x \in \mathbb{R}^{m}: f(x)<\infty\right\}$. Let $f(\cdot)$ be a convex function, and let $x \in \operatorname{dom}(f)$. For a smooth (differentiable) function $f(\cdot)$, denote by $\nabla f(x)$ the gradient of the function $f(\cdot)$ at point $x$. Then

$$
\begin{equation*}
f(y) \geq f(x)+\langle\nabla f(x), y-x\rangle \forall y \in \operatorname{dom}(f) \tag{1}
\end{equation*}
$$

where $\langle x, y\rangle$ denotes the inner product of vectors $x$ and $y$.
For a nonempty convex set $\Omega \subset \mathbb{R}^{m}$ and a point $x \in \mathbb{R}^{m}$, we call the point in $\Omega$ that is closest to $x$ the projection of $x$ on $\Omega$ and denote it by $P_{\Omega}\{x\}$. If $\Omega \subset \mathbb{R}^{m}$ is closed, then $P_{\Omega}\{x\}$ contains only one element for any $x \in \mathbb{R}^{m}$.

Consider a convex closed set $\Omega \subset \mathbb{R}^{m}$ and a point $x \in \Omega$. Define the normal cone to $\Omega$ at $x$ as $N_{\Omega}\{x\} \triangleq\left\{v \in \mathbb{R}^{m}:\langle v, y-\right.$
$x\rangle \leq 0 \forall y \in \Omega\}$. It is shown in [22, Lemma 2.38] that the following equation holds for any $x \in \Omega$ :

$$
\begin{equation*}
N_{\Omega}\{x\}=\left\{v \in \mathbb{R}^{m}: P_{\Omega}\{x+v\}=x\right\} . \tag{2}
\end{equation*}
$$

A set $C$ is affine if it contains the lines that pass through any pairs of points $x, y \in C$ with $x \neq y$. Let $\Omega \subset \mathbb{R}^{m}$ be a nonempty convex set. We say that $x \in \mathbb{R}^{m}$ is a relative interior point of $\Omega$ if $x \in \Omega$ and there exists an open sphere $S$ centered at $x$ such that $S \cap \operatorname{aff}(\Omega) \subset \Omega$, where $\operatorname{aff}(\Omega)$ is the intersection of all affine sets containing $\Omega$. A point $x \in \mathbb{R}^{m}$ is called the interior point of $\Omega$ if $x \in \Omega$ and there exists an open sphere $S$ centered at $x$ such that $S \subset \Omega$. A pair of vectors $x^{*} \in \Omega$ and $z^{*} \in \Psi$ is called a saddle point of the function $\phi(x, z)$ in $\Omega \times \Psi$ if

$$
\phi\left(x^{*}, z\right) \leq \phi\left(x^{*}, z^{*}\right) \leq \phi\left(x, z^{*}\right) \quad \forall x \in \Omega, \quad \forall z \in \Psi
$$

These definitions can be found in [21].

### 2.3. Problem Formulation

Consider a network of $n$ agents that collectively solve the following constrained optimization problem

$$
\begin{align*}
& \operatorname{minimize} f(x)=\sum_{i=1}^{n} f_{i}(x) \\
& \text { subject to } x \in \Omega_{o}=\bigcap_{i=1}^{n} \Omega_{i}, \tag{3}
\end{align*}
$$

where $\Omega_{i} \subset \mathbb{R}^{m}$ is a closed convex set, representing the local constraint set of agent $i$, and $f_{i}(x): \mathbb{R}^{m} \rightarrow \mathbb{R}$ is a smooth convex function in $\Omega_{i}$, representing the local objective function of agent $i$. Assume that $f_{i}$ and $\Omega_{i}$ are privately known to agent $i$. We assume that there exists at least one finite solution $x^{*}$ to the problem (3). For the problem (3), denote by $f^{*}=\min _{x \in \Omega_{o}} f(x)$ the optimal value, and by $\Omega_{o}^{*}=\left\{x \in \Omega_{o}: f(x)=f^{*}\right\}$ the optimal solution set.

We use an undirected graph $\mathcal{G}=\left\{\mathcal{V}, \mathcal{E}_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}\right\}$ to describe the communication among agents. Let $\mathcal{L}$ denote the Laplacian matrix of the undirected graph $\mathcal{G}$.

Let us introduce the following conditions for the problem.
A1 $\Omega_{o}$ has at least one relative interior point.
A2 The undirected graph $\mathcal{G}$ is connected.
A3 For any $i \in \mathcal{V}, \nabla f_{i}(x)$ is locally Lipschitz continuous on $\Omega_{i}$.

## 3. Algorithm Design

We first give an equivalent form of the problem (3). Then define a distributed primal-dual algorithm with constant step size to solve the formulated problem.

### 3.1. An Equivalent Problem

Lemma 3.1. [14 Lemma 3] If A2 holds, then the problem (3) is equivalent to the following optimization problem

$$
\begin{equation*}
\text { minimize } \widetilde{f}(X)=\sum_{i=1}^{n} f_{i}\left(x_{i}\right) \tag{4}
\end{equation*}
$$

subject to $\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X=\mathbf{0}, \quad X \in \Omega$,
where $X=\operatorname{col}\left\{x_{1}, \cdots, x_{n}\right\} \triangleq\left(x_{1}^{T}, \cdots, x_{n}^{T}\right)^{T}, \Omega=\prod_{i=1}^{n} \Omega_{i} d e$ notes the Cartesian product, $\otimes$ denotes the Kronecker product, $\mathbf{I}_{m}$ denotes the identity matrix of size $m$, and $\mathbf{0}$ denotes the vector of compatible dimension with all entries equal to 0 .

Remark 3.2. Lemma 3.1 implies that solving the problem (3) is equivalent to solving the problem (4) when the underlying graph is undirected and connected. If $X^{*}=\operatorname{col}\left(x_{1}^{*}, \cdots, x_{n}^{*}\right)$ is a solution to the problem (4), i.e., $f^{*}=\widetilde{f}\left(X^{*}\right)$, then $x_{i}^{*}=x_{j}^{*}=$ $x^{*} \forall i, j \in \mathcal{V}$ for some $x^{*} \in \Omega_{o}$ by $\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X^{*}=\mathbf{0}$ and $X^{*} \in \Omega$. Thus, $\widetilde{f}\left(X^{*}\right)=\sum_{i=1}^{n} f_{i}\left(x^{*}\right)=f^{*}$, and hence $x^{*}$ is an optimal solution to the problem (3).

Define the Lagrange function $\phi(X, \Lambda)=\widetilde{f}(X)+\langle\Lambda,(\mathcal{L} \otimes$ $\left.\left.\mathbf{I}_{m}\right) X\right\rangle$, where $\Lambda \in \mathbb{R}^{m n}$ is the Lagrange multiplier vector. Then the original problem (4) can be rewritten as $\inf _{X \in \Omega} \sup _{\Lambda \in \mathbb{R}^{m n}} \phi(X, \Lambda)$, while the dual problem is defined as follows

$$
\begin{equation*}
\sup _{\Lambda \in \mathbb{R}^{m n}} \inf _{X \in \Omega} \phi(X, \Lambda) . \tag{5}
\end{equation*}
$$

Lemma 3.3. Assume A1 and A2 hold. Then $\phi(X, \Lambda)$ has at least one saddle point in $\Omega \times \mathbb{R}^{m n}$. A pair $\left(X^{*}, \Lambda^{*}\right) \in \Omega \times \mathbb{R}^{m n}$ is the primal-dual solution to the problems (4) and (5) if and only if $\left(X^{*}, \Lambda^{*}\right)$ is a saddle point of $\phi(X, \Lambda)$ in $\Omega \times \mathbb{R}^{m n}$.

Proof: Since $f_{i}(\cdot) \forall i \in \mathcal{V}$ are continuous and the problem (3) has at least one finite solution, $f^{*}$ is finite. Moreover, A1 implies that there exists a relative interior $\bar{X}$ of set $\Omega$ such that $\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \bar{X}=0$. Then by [21] Proposition 5.3.3] we know that the primal and dual optimal values are equal, i.e.,

$$
\begin{equation*}
\inf _{X \in \Omega} \sup _{\Lambda \in \mathbb{R}^{n n}} \phi(X, \Lambda)=\sup _{\Lambda \in \mathbb{R}^{n n}} \inf _{X \in \Omega} \phi(X, \Lambda), \tag{6}
\end{equation*}
$$

and there exists at least one dual optimal solution. So, by (6) we conclude that $\phi(X, \Lambda)$ has at least one saddle point in $\Omega \times \mathbb{R}^{m n}$.

Since the minimax equality (6) holds, by [21, Proposition 3.4.1] we know that $X^{*}$ is the primal optimal solution and $\Lambda^{*}$ is the dual optimal solution if and only if $\left(X^{*}, \Lambda^{*}\right)$ is a saddle point of $\phi(X, \Lambda)$ on $\Omega \times \mathbb{R}^{m n}$. This completes the proof.

### 3.2. Distributed Primal-Dual Algorithm

Denote by $x_{i, k} \in \mathbb{R}^{m}$ the estimate for the optimal solution to the problem (3) given by agent $i$ at time $k$, and by $\lambda_{i, k} \in \mathbb{R}^{m}$ the corresponding Lagrange multiplier. They are updated as follows:

$$
\begin{aligned}
x_{i, k+1}= & P_{\Omega_{i}}\left\{x_{i, k}-\alpha \nabla f_{i}\left(x_{i, k}\right)-\alpha \sum_{j=1}^{n} a_{i j}\left(\lambda_{i, k}-\lambda_{j, k}\right)\right. \\
& \left.-\alpha \sum_{j=1}^{n} a_{i j}\left(x_{i, k}-x_{j, k}\right)\right\}, \\
\lambda_{i, k+1}= & \lambda_{i, k}+\alpha \sum_{j=1}^{n} a_{i j}\left(x_{i, k}-x_{j, k}\right) .
\end{aligned}
$$

Set $X_{k}=\operatorname{col}\left\{x_{1, k}, \cdots, x_{n, k}\right\}, \Lambda_{k}=\operatorname{col}\left\{\lambda_{1, k} \cdots, \lambda_{n, k}\right\}$, and $\nabla \widetilde{f}\left(X_{k}\right)=\operatorname{col}\left\{\nabla f_{1}\left(x_{1, k}\right), \cdots, \nabla f_{n}\left(x_{n, k}\right)\right\}$. Then (7) can be written in the compact form as follows:

$$
\begin{align*}
& X_{k+1}=P_{\Omega}\left\{X_{k}-\alpha \nabla \widetilde{f}\left(X_{k}\right)-\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k}+X_{k}\right)\right\}  \tag{8}\\
& \Lambda_{k+1}=\Lambda_{k}+\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X_{k} \tag{9}
\end{align*}
$$

Note that the algorithm (8) (9) actually is the gradient algorithm for finding saddle points of the augmented Lagrange function $\widetilde{\phi}(X, \Lambda)=\phi(X, \Lambda)+\frac{1}{2}\left\langle X,\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X\right\rangle$ in $\Omega \times \mathbb{R}^{m n}$. By Lemma 3.3 we see that if the algorithm (8) (9) converges to a saddle point of the augmented Lagrange function, then it solves the original problem (4). Convergence properties of the primal-dual method have been studied extensively, see, for example, [26, 27]. In general, only a subsequence of the sequence ( $X_{k}, \Lambda_{k}$ ) converges to a saddle point of the augmented Lagrange function. To obtain the convergence of the whole sequence ( $X_{k}, \Lambda_{k}$ ), it is often to assume that the augmented Lagrangian function is strictly convex-concave. However, for the problem studied in the paper, the augmented Lagrange function is neither strongly convex in $X \in \Omega$ nor strongly concave in $\Lambda \in \mathbb{R}^{m n}$. Thus, the standard analysis of gradient methods for finding saddle points is not applicable here. Instead, we apply the Lyapunov function method to analyze convergence.

## 4. Convergence Analysis

Convergence results for the proposed primal-dual algorithm are presented in Section 4.1 with the proof given in Sections 4.2 and 4.3.

### 4.1. Main Results

By A2 from Lemma 2.1 we know that all eigenvalues of $\mathcal{L}$ are nonnegative real numbers, and zero is a simple eigenvalue. Let us write the eigenvalues of $\mathcal{L}$ in the non-decreasing order as $0=\kappa_{1}<\kappa_{2} \leq \cdots \leq \kappa_{n}$.

Set

$$
\begin{equation*}
\mathcal{W}=\left(\mathbf{I}_{n}-\alpha \mathcal{L}+\alpha^{2} \mathcal{L}^{2}\right) \otimes \mathbf{I}_{m} . \tag{10}
\end{equation*}
$$

Let $\left(X^{*}, \Lambda^{*}\right)$ be a saddle point of $\phi(X, \Lambda)$. Define

$$
V_{1}(X)=\left\langle X-X^{*}, \mathcal{W}\left(X-X^{*}\right)\right\rangle, \quad V_{2}(\Lambda)=\left\|\Lambda-\Lambda^{*}\right\|^{2} .
$$

Construct a candidate Lyapunov function as follows

$$
V(X, \Lambda)=V_{1}(X)+V_{2}(\Lambda)
$$

The following theorem shows that the local estimates derived at all agents asymptotically reach a consensus at an optimal solution to the problem (3).

Theorem 4.1. Assume A1-A3 hold. Let $\left\{x_{i, k}\right\}$ and $\left\{\lambda_{i, k}\right\}$ be produced by (7) with initial values $x_{i, 0}, \lambda_{i, 0}$. Let $\left(X^{*}, \Lambda^{*}\right)$ be a saddle point of $\phi(X, \Lambda)$ in $\Omega \times \mathbb{R}^{m n}$. Assume, in addition, that the constant step size $\alpha$ satisfies $0<\alpha \leq \frac{1}{2 \kappa_{n}}$ and $\alpha<\frac{3}{2 l_{r}}$, where $l_{r}$ is the local Lipschitz constant of $\nabla \widetilde{f}(X)$ in the compact set $\left\{X \in \Omega:\left\|X-X^{*}\right\| \leq r\right\}$ with $r$ defined by

$$
\begin{equation*}
r=\sqrt{V\left(X_{0}, \Lambda_{1}\right) / \lambda_{\min }(\mathcal{M})}, \tag{11}
\end{equation*}
$$

where $\lambda_{\text {min }}(\cdot)$ denotes the smallest eigenvalue of a symmetric matrix, and $\mathcal{M}=\operatorname{diag}\left\{I_{m n}, \mathcal{W}\right\}$. Then
(i) $V\left(X_{k}, \Lambda_{k+1}\right)$ monotonously decreases and converges,
(ii) $d_{k}=\sqrt{\left\|X_{k}-X^{*}\right\|^{2}+\left\|\Lambda_{k+1}-\Lambda^{*}\right\|^{2}} \leq r \quad \forall k \geq 0$,
(iii) $\lim _{k \rightarrow \infty} x_{i, k}=\lim _{k \rightarrow \infty} x_{j, k}=x^{*} \forall i, j \in \mathcal{V}$ for some $x^{*} \in \Omega_{o}^{*}$.

Remark 4.2. The problem considered in [14] is in the same form as the problem (3), but the local constraint is a hyperbox or hyper-sphere, which is a special case of A1. Unlike the discrete-time algorithm (7), the continuous-time distributed algorithm is proposed in [14]. Though the estimates given by all agents converge to the same optimal solution, some intermediate sequence might be unbounded, which makes the algorithm difficult to be implemented.

Denote by $\bar{X}_{k}=\frac{1}{k+1} \sum_{p=0}^{k} X_{p}$ the time-averaged estimate. In what follows, the convergence rate of the algorithm (7) for the case where $\Omega_{i}=\mathbb{R}^{m} \forall i \in \mathcal{V}$ is shown.

Theorem 4.3. Assume $\Omega_{i}=\mathbb{R}^{m} \forall i \in \mathcal{V}, A 2$, and A3 hold. Let $\left\{x_{i, k}\right\}$ and $\left\{\lambda_{i, k}\right\}$ be produced by the algorithm (7) with initial values $x_{i, 0}, \lambda_{i, 0}$. Let $\left(X^{*}, \Lambda^{*}\right)$ be a saddle point of $\phi(X, \Lambda)$ in $\Omega \times$ $\mathbb{R}^{m n}$. If $0<\alpha \leq \frac{1}{2 \kappa_{n}}$ and $\alpha<\frac{3}{2 l_{r}}$, where $l_{r}$ is the local Lipschitz constant of $\nabla \widetilde{f}(X)$ in the compact set $\left\{X \in \Omega:\left\|X-X^{*}\right\| \leq r\right\}$ with $r$ defined by (11), then
(i) $\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \bar{X}_{k}=\frac{\Lambda_{k+1}-\Lambda_{0}}{(k+1) \alpha}$,
(ii) $\widetilde{f}\left(\bar{X}_{k}\right) \leq f^{*}+\frac{1}{2 \alpha(k+1)}\left(\left\|X_{0}-X^{*}\right\|^{2}+\left\|\Lambda_{0}\right\|^{2}-\left\|X_{k+1}-X^{*}\right\|^{2}\right.$
$\left.-\left\|\Lambda_{k+1}\right\|^{2}\right)+\frac{c_{r}}{2 \alpha(k+1)} \times\left(V\left(X_{0}, \Lambda_{1}\right)-V\left(X_{k+1}, \Lambda_{k+2}\right)\right)$,
(iii) $\widetilde{f}\left(\bar{X}_{k}\right) \geq f^{*}-\frac{\left\langle\Lambda_{k+1}-\Lambda_{0}, \Lambda^{*}\right\rangle}{(k+1) \alpha}$,
where $c_{r}=1 / \lambda_{\text {min }}\left(\mathcal{W}-\frac{\alpha l_{r}}{2} I_{m n}\right)+1$.
Remark 4.4. Since $\left\|\Lambda_{k}-\Lambda^{*}\right\| \leq r \forall k \geq 0$ by Theorem 4.1 ii), $\Lambda_{k}$ is uniformly bounded in $k$ by a constant. Then by (12) we see that $\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \bar{X}_{k}$ converges to $\mathbf{0}$ with rate $O\left(\frac{1}{k}\right)$. Since Theorem 4.1 i) implies that $V\left(X_{0}, \Lambda_{1}\right)-V\left(X_{k+1}, \Lambda_{k+2}\right) \leq 0 \quad \forall k \geq 0$, by (13) (14) the value of the cost function $\widetilde{f}(\cdot)$ at $\bar{X}_{k}$ converges to the optimal value with rate $O\left(\frac{1}{k}\right)$.

Remark 4.5. Note that $\Lambda_{1}=\Lambda_{0}+\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X_{0}$. Then from Theorems 4.1 and 4.3 we see that for small enough $\alpha>0$ depending on the distance between the initial value and the optimal solution, and on the structure of the cost functions in the neighborhood of the optimal solution, the estimates given by all agents finally reach a consensus at an optimal solution. If $\nabla f_{i}(x)$ is globally Lipschitz continuous in set $\Omega_{i}$ with constant $l_{c}$ for any $i \in \mathcal{V}$, then the results given in Theorems 4.1 and 4.3 hold as well for any $\alpha$ satisfying $0<\alpha \leq \frac{1}{2 \kappa_{n}}$ and $\alpha<\frac{3}{2 l_{c}}$ but independent of the initial values.

### 4.2. Proof of Theorem 4.1

Prior to proving Theorem 4.1, we give a lemma that will be used in the proof.

Lemma 4.6. [23] Theorem 2.1.5] If $f: \mathbb{R}^{m} \rightarrow \mathbb{R}$ is a convex function whose gradient is globally Lipschitz continuous with constant $l_{c}$, then

$$
\langle x-y, \nabla f(x)-\nabla f(y)\rangle \geq \frac{1}{l_{c}}\|\nabla f(x)-\nabla f(y)\|^{2} \quad \forall x, y \in \mathbb{R}^{m}
$$

Proof of Theorem 4.1; Note that

$$
\begin{align*}
V & \left(X_{k+1}, \Lambda_{k+2}\right)-V\left(X_{k}, \Lambda_{k+1}\right) \\
= & V_{1}\left(X_{k+1}\right)-V_{1}\left(X_{k}\right)+V_{2}\left(\Lambda_{k+2}\right)-V_{2}\left(\Lambda_{k+1}\right) \\
= & \left\langle\mathcal{W}\left(X_{k+1}-X_{k}\right), X_{k+1}+X_{k}-2 X^{*}\right\rangle \\
& +\left\langle\Lambda_{k+2}-\Lambda_{k+1}, \Lambda_{k+2}+\Lambda_{k+1}-2 \Lambda^{*}\right\rangle  \tag{15}\\
= & -\left\|\Lambda_{k+2}-\Lambda_{k+1}\right\|^{2}-\left\langle X_{k+1}-X_{k}, \mathcal{W}\left(X_{k+1}-X_{k}\right)\right\rangle \\
& +2\left\langle\Lambda_{k+2}-\Lambda_{k+1}, \Lambda_{k+2}-\Lambda^{*}\right\rangle \\
& +2\left\langle X_{k+1}-X^{*}, \mathcal{W}\left(X_{k+1}-X_{k}\right)\right\rangle .
\end{align*}
$$

We now estimate the last two terms on the right hand side of (15).

Since $\left(X^{*}, \Lambda^{*}\right)$ is a saddle point of $\phi(X, \Lambda)$, by Lemma 3.3 we see that $X^{*}$ is an optimal solution to the problem (4), and hence

$$
\begin{equation*}
\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X^{*}=\mathbf{0} . \tag{16}
\end{equation*}
$$

Since $\mathcal{L}$ is symmetric, by (16) we derive

$$
\begin{aligned}
& \left\langle\Lambda_{k+2}-\Lambda_{k+1}, \Lambda_{k+2}-\Lambda^{*}\right\rangle \\
& =\left\langle\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(X_{k+1}-X^{*}\right), \Lambda_{k+2}-\Lambda^{*}\right\rangle \\
& =\left\langle\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k+2}-\Lambda^{*}\right), X_{k+1}-X^{*}\right\rangle .
\end{aligned}
$$

Thus,

$$
\begin{align*}
& \left\langle\Lambda_{k+2}-\Lambda_{k+1}, \Lambda_{k+2}-\Lambda^{*}\right\rangle+\left\langle X_{k+1}-X^{*}, \mathcal{W}\left(X_{k+1}-X_{k}\right)\right\rangle \\
& =\left\langle X_{k+1}-X^{*}, \alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k+2}-\Lambda^{*}\right)+\mathcal{W}\left(X_{k+1}-X_{k}\right)\right\rangle \tag{17}
\end{align*}
$$

From (9) we derive $\Lambda_{k+2}=\Lambda_{k}+\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X_{k}+\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X_{k+1}$, and hence by 16
$\Lambda_{k}-\Lambda^{*}=\Lambda_{k+2}-\Lambda^{*}-\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(X_{k}-X^{*}\right)-\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(X_{k+1}-X^{*}\right)$.
Then by multiplying both sides of (18) with $\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)$, from the rule of the Kronecker product $(A \otimes B)(C \otimes D)=(A C) \otimes(B D)$ we obtain

$$
\begin{align*}
& \left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k}-\Lambda^{*}\right)=\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k+2}-\Lambda^{*}\right) \\
& -\alpha\left(\mathcal{L}^{2} \otimes \mathbf{I}_{m}\right)\left(X_{k}-X^{*}\right)-\alpha\left(\mathcal{L}^{2} \otimes \mathbf{I}_{m}\right)\left(X_{k+1}-X^{*}\right) \tag{19}
\end{align*}
$$

Set

$$
\begin{equation*}
Z_{k+1}=X_{k}-\alpha \nabla \widetilde{f}\left(X_{k}\right)-\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k}+X_{k}\right)-X_{k+1} \tag{20}
\end{equation*}
$$

Then by (16) (20) we derive

$$
\begin{align*}
& X_{k+1}-X^{*}=X_{k}-X^{*}-\alpha \nabla \widetilde{f}\left(X_{k}\right)-\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(X_{k}-X^{*}\right) \\
& -\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k}-\Lambda^{*}\right)-\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}-Z_{k+1} \\
& =X_{k}-X^{*}-\alpha\left(\nabla \widetilde{f}\left(X_{k}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right)-\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(X_{k}-X^{*}\right) \\
& -\alpha\left(\nabla \widetilde{f}\left(X^{*}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}\right)-Z_{k+1} \\
& -\alpha\left(\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k+2}-\Lambda^{*}\right)-\alpha\left(\mathcal{L}^{2} \otimes \mathbf{I}_{m}\right)\left(X_{k}-X^{*}\right)\right. \\
& \left.-\alpha\left(\mathcal{L}^{2} \otimes \mathbf{I}_{m}\right)\left(X_{k+1}-X^{*}\right)\right) \\
& =\left(\left(\mathbf{I}_{n}-\alpha \mathcal{L}+\alpha^{2} \mathcal{L}^{2}\right) \otimes \mathbf{I}_{m}\right)\left(X_{k}-X^{*}\right)-Z_{k+1} \\
& -\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k+2}-\Lambda^{*}\right)-\alpha\left(\nabla \widetilde{f}\left(X_{k}\right)-\nabla \vec{f}\left(X^{*}\right)\right) \\
& -\alpha\left(\nabla \widetilde{f}\left(X^{*}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}\right)+\alpha^{2}\left(\mathcal{L}^{2} \otimes \mathbf{I}_{m}\right)\left(X_{k+1}-X^{*}\right) \tag{21}
\end{align*}
$$

Moving the last term at the right-hand side of 21 to the left and subtracting $\mathcal{W}\left(X_{k+1}-X^{*}\right)$ from both sides of (21) we derive the following recursion

$$
\begin{aligned}
& \left(\alpha \mathcal{L} \otimes \mathbf{I}_{m}-2 \alpha^{2} \mathcal{L}^{2} \otimes \mathbf{I}_{m}\right)\left(X_{k+1}-X^{*}\right) \\
& =\mathcal{W}\left(X_{k}-X_{k+1}\right)-\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k+2}-\Lambda^{*}\right)-Z_{k+1} \\
& -\alpha\left(\nabla \widetilde{f}\left(X_{k}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right)-\alpha\left(\nabla \widetilde{f}\left(X^{*}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}\right),
\end{aligned}
$$

or in the alternative form

$$
\begin{aligned}
& \mathcal{W}\left(X_{k+1}-X_{k}\right)+\alpha\left(\mathcal{L} \otimes I_{m}\right)\left(\Lambda_{k+2}-\Lambda^{*}\right) \\
& =-\left(\alpha \mathcal{L} \otimes I_{m}-2 \alpha^{2} \mathcal{L}^{2} \otimes I_{m}\right)\left(X_{k+1}-X^{*}\right)-Z_{k+1} \\
& \quad-\alpha\left(\nabla \vec{f}\left(X_{k}\right)-\nabla \vec{f}\left(X^{*}\right)\right)-\alpha\left(\nabla \vec{f}\left(X^{*}\right)+\left(\mathcal{L} \otimes I_{m}\right) \Lambda^{*}\right) .
\end{aligned}
$$

Then by 17) we derive

$$
\begin{align*}
&\left\langle\Lambda_{k+2}-\Lambda_{k+1}, \Lambda_{k+2}-\Lambda^{*}\right\rangle+\left\langle X_{k+1}-X^{*}, \mathcal{W}\left(X_{k+1}-X_{k}\right)\right\rangle \\
&=-\left\langle X_{k+1}-X^{*},\left(\left(\alpha \mathcal{L}-2 \alpha^{2} \mathcal{L}^{2}\right) \otimes \mathbf{I}_{m}\right)\left(X_{k+1}-X^{*}\right)\right\rangle \\
&-\alpha\left\langle X_{k+1}-X^{*}, \nabla \widetilde{f}\left(X_{k}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\rangle \\
&-\left\langle X_{k+1}-X^{*}, Z_{k+1}+\alpha\left(\nabla \widetilde{f}\left(X^{*}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}\right)\right\rangle . \tag{22}
\end{align*}
$$

By the definition of the saddle point we have

$$
\begin{equation*}
\phi\left(X^{*}, \Lambda\right) \leq \phi\left(X^{*}, \Lambda^{*}\right) \leq \phi\left(X, \Lambda^{*}\right) \forall X \in \Omega, \Lambda \in \mathbb{R}^{m n} . \tag{23}
\end{equation*}
$$

Therefore, $X^{*}$ minimizes $\widetilde{f}(X)+\left\langle X,\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}\right\rangle$ over $\Omega$. Since $\phi(X, \Lambda)$ is convex in $X \in \Omega$ for each $\Lambda$, by noticing $X_{k} \in \Omega \forall k \geq$ 0 from the optimal condition [21, Proposition 1.1.8] we derive

$$
\begin{equation*}
\left\langle\nabla \widetilde{f}\left(X^{*}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}, X_{k+1}-X^{*}\right\rangle \geq 0 \quad \forall k \geq 0 \tag{24}
\end{equation*}
$$

From (8) (20) it follows that $P_{\Omega}\left\{X_{k+1}+Z_{k+1}\right\}=X_{k+1}$, and hence $Z_{k+1} \in N_{\Omega}\left\{X_{k+1}\right\}$ by (2). Then by the definition of normal cone we obtain

$$
\begin{equation*}
\left\langle X_{k+1}-X^{*}, Z_{k+1}\right\rangle \geq 0 . \tag{25}
\end{equation*}
$$

Then by combining (22) (24) (25) we derive

$$
\begin{aligned}
&\left\langle\Lambda_{k+2}-\Lambda_{k+1}, \Lambda_{k+2}-\Lambda^{*}\right\rangle+\left\langle X_{k+1}-X_{k}, \mathcal{W}\left(X_{k+1}-X^{*}\right)\right\rangle \\
& \leq-\left\langle X_{k+1}-X^{*},\left(\left(\alpha \mathcal{L}-2 \alpha^{2} \mathcal{L}^{2}\right) \otimes \mathbf{I}_{m}\right)\left(X_{k+1}-X^{*}\right)\right\rangle \\
&-\alpha\left\langle X_{k+1}-X^{*}, \nabla \widetilde{f}\left(X_{k}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\rangle .
\end{aligned}
$$

This incorporating with (15) yields

$$
\begin{aligned}
& V\left(X_{k+1}, \Lambda_{k+2}\right)-V\left(X_{k}, \Lambda_{k+1}\right) \\
& \leq-\left\|\Lambda_{k+2}-\Lambda_{k+1}\right\|^{2}-\left\langle X_{k+1}-X_{k}, \mathcal{W}\left(X_{k+1}-X_{k}\right)\right\rangle \\
& \quad-2 \alpha\left\langle X_{k+1}-X^{*}, \nabla \widetilde{f}\left(X_{k}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\rangle \\
& \quad-2\left\langle X_{k+1}-X^{*},\left(\left(\alpha \mathcal{L}-2 \alpha^{2} \mathcal{L}^{2}\right) \otimes \mathbf{I}_{m}\right)\left(X_{k+1}-X^{*}\right)\right\rangle .
\end{aligned}
$$

Since $\mathcal{L}$ is symmetric, there exists an orthogonal matrix $\mathcal{U}$ such that $\mathcal{U}^{T} \mathcal{L} \mathcal{U}=\operatorname{diag}\left\{0, \kappa_{2}, \cdots, \kappa_{n}\right\}$, and hence $\mathcal{U}^{T} \mathcal{L}^{2} \mathcal{U}=$ $\operatorname{diag}\left\{0, \kappa_{2}^{2}, \cdots, \kappa_{n}^{2}\right\}$. Then by (10) we know that all possible distinct eigenvalues of $\alpha \mathcal{L}-2 \alpha^{2} \mathcal{L}^{2}$ are 0 , and $\alpha \kappa_{i}-2 \alpha^{2} \kappa_{i}^{2}, i=$ $2, \cdots, n$. If $0<\alpha \leq \frac{1}{2 \kappa_{n}}$, then $2 \alpha \kappa_{i} \leq 1 \quad \forall i=1, \cdots, n$, and hence $\alpha \kappa_{i}-2 \alpha^{2} \kappa_{i}^{2}=\alpha \kappa_{i}\left(1-2 \alpha \kappa_{i}\right) \geq 0 \quad \forall i=1, \cdots, n$. Therefore, for any $\alpha$ with $0<\alpha \leq \frac{1}{2 \kappa_{n}}$ the matrix $\alpha \mathcal{L}-2 \alpha^{2} \mathcal{L}^{2}$ is positive semi-definite, and hence by (26) we derive

$$
\begin{align*}
& V\left(X_{k+1}, \Lambda_{k+2}\right)-V\left(X_{k}, \Lambda_{k+1}\right) \\
& \leq-\left\|\Lambda_{k+2}-\Lambda_{k+1}\right\|^{2}-\left\langle X_{k+1}-X_{k}, \mathcal{W}\left(X_{k+1}-X_{k}\right)\right\rangle  \tag{27}\\
& \quad-2 \alpha\left\langle X_{k+1}-X^{*}, \nabla \bar{f}\left(X_{k}\right)-\nabla \tilde{f}\left(X^{*}\right)\right\rangle .
\end{align*}
$$

Let the constant $\alpha$ be such that $0<\alpha \leq \frac{1}{2 \kappa_{n}}$ and $\alpha<\frac{3}{2 l_{r}}$. In what follows, we show that $V\left(X_{k}, \Lambda_{k+1}\right)$ monotonously decreases and $d_{k} \leq r \quad \forall k \geq 0$ by induction.

We first show that $d_{1} \leq r$ and $V\left(X_{1}, \Lambda_{2}\right) \leq V\left(X_{0}, \Lambda_{1}\right)$. By the definition of the local Lipschitz constant $l_{r}$, we know that $\nabla \widetilde{f}(X)$ is Lipschitz continuous on the compact set $\{X \in \Omega$ : $\left.\left\|X-X^{*}\right\| \leq r\right\}$ with Lipschitz constant $l_{r}$. Since $\left\|X_{0}-X^{*}\right\| \leq r$ by the definition of $r$, from Lemma 4.6 we see

$$
\left\langle X_{0}-X^{*}, \nabla \widetilde{f}\left(X_{0}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\rangle \geq \frac{1}{l_{r}}\left\|\nabla \widetilde{f}\left(X_{0}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\|^{2}
$$

This incorporating with $x y \leq \frac{x^{2}}{4}+y^{2}$ leads to

$$
\begin{align*}
- & \left\langle X_{1}-X^{*}, \nabla \widetilde{f}\left(X_{0}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\rangle \\
= & -\left\langle X_{0}-X^{*}, \nabla \widetilde{f}\left(X_{0}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\rangle \\
& +\left\langle-X_{1}+X_{0}, \nabla \widetilde{f}\left(X_{0}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\rangle  \tag{28}\\
\leq & -\frac{1}{l_{r}}\left\|\nabla \widetilde{f}\left(X_{0}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\|^{2}+\frac{l_{4}}{4}\left\|X_{0}-X_{1}\right\|^{2} \\
& +\frac{1}{l_{r}}\left\|\nabla \widetilde{f}\left(X_{0}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\|^{2} \leq \frac{l_{r}}{4}\left\|X_{0}-X_{1}\right\|^{2} .
\end{align*}
$$

Then from here by (27) we have

$$
\begin{aligned}
& V\left(X_{1}, \Lambda_{2}\right)-V\left(X_{0}, \Lambda_{1}\right) \\
& \leq-\left\|\Lambda_{2}-\Lambda_{1}\right\|^{2}-\left\langle X_{1}-X_{0},\left(\mathcal{W}-\frac{\alpha l_{l}}{2} I_{m n}\right)\left(X_{1}-X_{0}\right)\right\rangle
\end{aligned}
$$

By (10) we know that all possible distinct eigenvalues of $\mathcal{W}-$ $\frac{\alpha l_{r}}{2} I_{m n}$ are $1-\alpha \kappa_{i}+\alpha^{2} \kappa_{i}^{2}-\frac{\alpha l_{r}}{2}, i=1, \cdots, n$. Since $\alpha<\frac{3}{2 l_{r}}$, we have $1-\alpha \kappa_{i}+\alpha^{2} \kappa_{i}^{2}-\frac{\alpha l_{r}}{2}=\left(\frac{1}{2}-\alpha \kappa_{i}\right)^{2}+\frac{3}{4}-\frac{\alpha l_{r}}{2}>0$. Thus, $\mathcal{W}-\frac{\alpha l_{r}}{2} I_{m n}$ is positive definite. As a result, $V\left(X_{1}, \Lambda_{2}\right) \leq V\left(X_{0}, \Lambda_{1}\right)$. Then $V\left(X_{1}, \Lambda_{2}\right) \leq r^{2} \lambda_{\text {min }}(\mathcal{M})$, and hence $d_{1} \leq r$.

Assume that $d_{p} \leq r$ and $V\left(X_{p}, \Lambda_{p+1}\right) \leq V\left(X_{p-1}, \Lambda_{p}\right)$ for $p=$ $1, \cdots, k$. Since 27) holds and $\left\|X_{k}-X^{*}\right\| \leq r$, similar to the case $k=0$, we can show that $V\left(X_{k+1}, \Lambda_{k+2}\right) \leq V\left(X_{k}, \Lambda_{k+1}\right)$ and $d_{k+1} \leq r$.

In summary, by the mathematical induction we conclude that $d_{k} \leq r \forall k \geq 0$, and $V\left(X_{k}, \Lambda_{k+1}\right)$ monotonously decreases.

Since $\left\|X_{k}-X^{*}\right\| \leq r \quad \forall k \geq 0$, by the same procedure for deriving (28) we obtain

$$
-\left\langle X_{k+1}-X^{*}, \nabla \widetilde{f}\left(X_{k}\right)-\nabla \widetilde{f}\left(X^{*}\right)\right\rangle \leq \frac{l_{r}}{4}\left\|X_{k}-X_{k+1}\right\|^{2} \quad \forall k \geq 0
$$

Then from here by (27) we derive

$$
\begin{align*}
& V\left(X_{k+1}, \Lambda_{k+2}\right)-V\left(X_{k}, \Lambda_{k+1}\right) \leq-\left\|\Lambda_{k+2}-\Lambda_{k+1}\right\|^{2} \\
- & \left\langle X_{k+1}-X_{k},\left(\mathcal{W}-\frac{\alpha l_{r}}{2} I_{m n}\right)\left(X_{k+1}-X_{k}\right)\right\rangle \leq 0 \tag{29}
\end{align*}
$$

Thus, we conclude that $V\left(X_{k}, \Lambda_{k+1}\right)$ converges since it is nonnegative. Summing up both sides of 29 from 0 to $p$ we derive

$$
\begin{align*}
& V\left(X_{p+1}, \Lambda_{p+2}\right)-V\left(X_{0}, \Lambda_{1}\right) \leq-\sum_{k=0}^{p}\left\|\Lambda_{k+2}-\Lambda_{k+1}\right\|^{2} \\
& \quad-\sum_{k=0}^{p}\left\langle X_{k+1}-X_{k},\left(\mathcal{W}-\frac{\alpha l_{r}}{2} I_{m n}\right)\left(X_{k+1}-X_{k}\right)\right\rangle . \tag{30}
\end{align*}
$$

Then by letting $p \rightarrow \infty$ we have

$$
\begin{align*}
& \sum_{k=0}^{\infty}\left\langle X_{k+1}-X_{k},\left(\mathcal{W}-\frac{\alpha l_{r}}{2} I_{m n}\right)\left(X_{k+1}-X_{k}\right)\right\rangle<\infty,  \tag{31}\\
& \text { and } \sum_{k=1}^{\infty}\left\|\Lambda_{k+1}-\Lambda_{k}\right\|^{2}<\infty . \tag{32}
\end{align*}
$$

Consequently, we derive $\lim _{k \rightarrow \infty}\left(X_{k+1}-X_{k}\right)=0$ by 31) since $\mathscr{W}-\frac{\alpha l_{r}}{2} I_{m n}$ is positive definite, and $\lim _{k \rightarrow \infty}\left(\Lambda_{k+1}-\Lambda_{k}\right)=0$ by (32). By convergence of $V\left(X_{k}, \Lambda_{k+1}\right)$ we conclude that $X_{k}$ and $\Lambda_{k}$ contain convergent subsequences $\left\{X_{n_{k}}\right\}$ and $\left\{\Lambda_{n_{k}}\right\}$ to some limits $X^{0}$ and $\Lambda^{0}$, respectively. Since $\lim _{k \rightarrow \infty}\left(X_{n_{k}+1}-X_{n_{k}}\right)=0$ and $\lim _{k \rightarrow \infty}\left(\Lambda_{n_{k}+1}-\Lambda_{n_{k}}\right)=0$, by noticing that $P_{\Omega}\{x\}$ is continuous in $x$ and $\lim _{k \rightarrow \infty} X_{n_{k}}=X^{0}, \lim _{k \rightarrow \infty} \Lambda_{n_{k}}=\Lambda^{0}$, from (8) (9) we derive

$$
\begin{align*}
& X^{0}=P_{\Omega}\left\{X^{0}-\alpha \nabla \widetilde{f}\left(X^{0}\right)-\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda^{0}+X^{0}\right)\right\},  \tag{33}\\
& \left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X^{0}=0 . \tag{34}
\end{align*}
$$

Then from (33) (34) by (2) we see $\alpha\left(\nabla \widetilde{f}\left(X^{0}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{0}\right)=$ $\alpha\left(\nabla \widetilde{f}\left(X^{0}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda^{0}+X^{0}\right)\right) \in-N_{\Omega}\left(X^{0}\right)$, and hence by the definition of normal cone we conclude

$$
\left\langle\nabla \widetilde{f}\left(X^{0}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{0}, X-X^{0}\right\rangle \geq 0 \quad \forall X \in \Omega
$$

Since $\phi(X, \Lambda)=\widetilde{f}(X)+\left\langle\Lambda,\left(\overline{\mathcal{L}} \otimes \mathbf{I}_{m}\right) X\right\rangle$ is convex in $X \in \Omega$ for each $\Lambda \in \mathbb{R}^{m n}$, by (1) we have

$$
\begin{aligned}
\phi\left(X, \Lambda^{0}\right) & \geq \phi\left(X^{0}, \Lambda^{0}\right)+\left\langle\nabla \widetilde{f}\left(X^{0}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{0}, X-X^{0}\right\rangle \\
& \geq \phi\left(X^{0}, \Lambda^{0}\right) \forall X \in \Omega .
\end{aligned}
$$

From (34) we see $\phi\left(X^{0}, \Lambda^{0}\right)=\phi\left(X^{0}, \Lambda\right)=\widetilde{f}\left(X^{0}\right) \forall \Lambda \in \mathbb{R}^{m n}$, and hence by definition we know ( $X^{0}, \Lambda^{0}$ ) is a saddle point of $\phi(X, \Lambda)$ in $\Omega \times \mathbb{R}^{m n}$. Thus, by Lemma 3.3 we see that $X^{0}$ is an optimal solution to the problem (4).

Since $\left\|\Lambda_{k+1}-\Lambda^{*}\right\|^{2}+\left(X_{k}-X^{*}\right)^{T} \mathcal{W}\left(X_{k}-X^{*}\right)$ converges, by setting $\left(X^{*}, \Lambda^{*}\right)=\left(X^{0}, \Lambda^{0}\right)$ from $\lim _{k \rightarrow \infty} X_{n_{k}}=X^{0}$ and $\lim _{k \rightarrow \infty} \Lambda_{n_{k}}=$ $\Lambda^{0}$ we conclude that $\left\|\Lambda_{k+1}-\Lambda^{0}\right\|^{2}+\left(X_{k}-X^{0}\right)^{T} \mathcal{W}\left(X_{k}-X^{0}\right)$ converges to zero. Therefore,

$$
\lim _{k \rightarrow \infty} X_{k}=X^{0}, \quad \lim _{k \rightarrow \infty} \Lambda_{k}=\Lambda^{0}
$$

Thus, by Remark 3.2 we conclude that $\lim _{k \rightarrow \infty} x_{i, k}=\lim _{k \rightarrow \infty} x_{j, k}=$ $x^{*} \forall i, j \in \mathcal{V}$ for some $x^{*} \in \Omega_{o}^{*}$.

### 4.3. Proof of Theorem 4.3

Proof: (i) Summing up both sides of (9) from 0 to $p$ leads to

$$
\Lambda_{p+1}-\Lambda_{0}=\sum_{k=0}^{p} \alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X_{k}=(p+1) \alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \bar{X}_{p}
$$

and hence (12) holds.
(ii) When $\Omega_{i}=\mathbb{R}^{m} \forall i \in \mathcal{V}$, the equation (8) turns to

$$
\begin{equation*}
X_{k+1}=X_{k}-\alpha \nabla \widetilde{f}\left(X_{k}\right)-\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k}+X_{k}\right) \tag{35}
\end{equation*}
$$

By (30) we see

$$
\begin{aligned}
& \sum_{k=0}^{p}\left(\left\|\Lambda_{k+2}-\Lambda_{k+1}\right\|^{2}+\left\langle X_{k+1}-X_{k},\left(\mathcal{W}-\frac{\alpha l_{r}}{2} I_{m n}\right)\left(X_{k+1}-X_{k}\right)\right\rangle\right. \\
& \leq-V\left(X_{p+1}, \Lambda_{p+2}\right)+V\left(X_{0}, \Lambda_{1}\right) .
\end{aligned}
$$

Then by noticing that $\mathcal{W}-\frac{\alpha l_{r}}{2} I_{m n}$ is positive definite we derive

$$
\begin{align*}
& \sum_{k=0}^{p}\left\|\Lambda_{k+2}-\Lambda_{k+1}\right\|^{2} \leq-V\left(X_{p+1}, \Lambda_{p+2}\right)+V\left(X_{0}, \Lambda_{1}\right),  \tag{36}\\
& \text { and } \sum_{k=0}^{p}\left\|X_{k+1}-X_{k}\right\|^{2} \leq \frac{-V\left(X_{p+1}, \Lambda_{p+2}\right)+V\left(X_{0}, \Lambda_{1}\right)}{\lambda_{\min }\left(\mathcal{W}-\frac{\alpha l_{r}}{2} I_{m n}\right)} . \tag{37}
\end{align*}
$$

By (35) we have

$$
\begin{align*}
& \left\|X_{k+1}-X^{*}\right\|^{2}=\left\|X_{k}-X^{*}\right\|^{2}+\left\|X_{k+1}-X_{k}\right\|^{2} \\
& +2\left(X_{k}-X^{*}\right)^{T}\left(X_{k+1}-X_{k}\right)=\left\|X_{k}-X^{*}\right\|^{2} \\
& +2 \alpha\left\langle X^{*}-X_{k}, \nabla \tilde{f}\left(X_{k}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right)\left(\Lambda_{k}+X_{k}\right)\right\rangle  \tag{38}\\
& +\left\|X_{k+1}-X_{k}\right\|^{2} .
\end{align*}
$$

Noticing $\phi(X, \Lambda)$ is convex in $X \in \Omega$ for any $\Lambda$, by (1) we have

$$
\begin{equation*}
\left\langle X^{*}-X_{k}, \nabla \widetilde{f}\left(X_{k}\right)+\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda_{k}\right\rangle \leq \phi\left(X^{*}, \Lambda_{k}\right)-\phi\left(X_{k}, \Lambda_{k}\right) . \tag{39}
\end{equation*}
$$

Since $\Lambda_{k}$ is bounded and $X^{*}$ is an optimal solution to the problem (4), by (16) we see $\phi\left(X^{*}, \Lambda_{k}\right)=\widetilde{f}\left(X^{*}\right)=f^{*}$. Since $\mathcal{L}$ is positive semi-definite by Lemma 2.1, from (16) it follows that

$$
\left\langle X^{*}-X_{k},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X_{k}\right\rangle=-\left\langle X_{k},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X_{k}\right\rangle \leq 0 .
$$

Thus, from here by (38) (39) we derive
$\left\|X_{k+1}-X^{*}\right\|^{2} \leq\left\|X_{k}-X^{*}\right\|^{2}+2 \alpha\left(f^{*}-\phi\left(X_{k}, \Lambda_{k}\right)\right)+\left\|X_{k+1}-X_{k}\right\|^{2}$, and hence
$\phi\left(X_{k}, \Lambda_{k}\right)-f^{*} \leq \frac{1}{2 \alpha}\left(\left\|X_{k}-X^{*}\right\|^{2}-\left\|X_{k+1}-X^{*}\right\|^{2}+\left\|X_{k+1}-X_{k}\right\|^{2}\right)$.
Summing up both sides of this inequality from 0 to $p$ for $p \geq 1$ we obtain

$$
\begin{aligned}
& \sum_{k=0}^{p}\left(\phi\left(X_{k}, \Lambda_{k}\right)-f^{*}\right) \\
& \leq \frac{1}{2 \alpha}\left(\left\|X_{0}-X^{*}\right\|^{2}-\left\|X_{p+1}-X^{*}\right\|^{2}+\sum_{k=0}^{p}\left\|X_{k+1}-X_{k}\right\|^{2}\right) .
\end{aligned}
$$

From here by the convexity of $\widetilde{f}(X)$ we derive

$$
\begin{align*}
\widetilde{f}\left(\bar{X}_{p}\right) & \leq \frac{1}{p+1} \sum_{k=0}^{p} \widetilde{f}\left(X_{k}\right) \leq \frac{1}{p+1} \sum_{k=0}^{p} \phi\left(X_{k}, \Lambda_{k}\right) \\
& -\frac{1}{p+1} \sum_{k=0}^{p} X_{k}^{T}\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda_{k} \\
& \leq f^{*}+\frac{1}{2 \alpha(p+1)}\left(\left\|X_{0}-X^{*}\right\|^{2}-\left\|X_{p+1}-X^{*}\right\|^{2}\right.  \tag{40}\\
& \left.+\sum_{k=0}^{p}\left\|X_{k+1}-X_{k}\right\|^{2}\right) \\
& -\frac{1}{p+1} \sum_{k=0}^{p}\left\langle X_{k},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda_{k}\right\rangle .
\end{align*}
$$

We now give an upper bound for $-\frac{1}{p+1} \sum_{k=0}^{p}\left\langle X_{k},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda_{k}\right\rangle$. By (9) we have

$$
\left\|\Lambda_{k+1}\right\|^{2}=\left\|\Lambda_{k}\right\|^{2}+2 \alpha\left\langle\Lambda_{k},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X_{k}\right\rangle+\left\|\alpha\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X_{k}\right\|^{2} .
$$



Figure 1: The estimates and the residual

Thus, $-\left\langle X_{k},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda_{k}\right\rangle=\frac{1}{2 \alpha}\left(\left\|\Lambda_{k}\right\|^{2}-\left\|\Lambda_{k+1}\right\|^{2}+\left\|\Lambda_{k+1}-\Lambda_{k}\right\|^{2}\right)$, and hence

$$
\begin{align*}
& -\frac{1}{p+1} \sum_{k=0}^{p}\left\langle X_{k},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda_{k}\right\rangle \\
& =\frac{1}{2 \alpha(p+1)}\left(\left\|\Lambda_{0}\right\|^{2}-\left\|\Lambda_{p+1}\right\|^{2}+\sum_{k=0}^{p}\left\|\Lambda_{k+1}-\Lambda_{k}\right\|^{2}\right) . \tag{41}
\end{align*}
$$

By substituting (41) into (40) we derive

$$
\begin{aligned}
& \widetilde{f}\left(\bar{X}_{p}\right) \leq f^{*} \\
& +\frac{1}{2 \alpha(p+1)}\left(\left\|X_{0}-X^{*}\right\|^{2}-\left\|X_{p+1}-X^{*}\right\|^{2}+\left\|\Lambda_{0}\right\|^{2}-\left\|\Lambda_{p+1}\right\|^{2}\right) \\
& +\frac{1}{2 \alpha(p+1)}\left(\sum_{k=0}^{p}\left\|X_{k+1}-X_{k}\right\|^{2}+\sum_{k=0}^{p}\left\|\Lambda_{k+1}-\Lambda_{k}\right\|^{2}\right) .
\end{aligned}
$$

Then from here by (36) (37) we obtain (13).
(iii) By (16) (23) we derive $\phi\left(\bar{X}_{p}, \Lambda^{*}\right) \geq \phi\left(X^{*}, \Lambda^{*}\right)=\widetilde{f}\left(X^{*}\right)=$ $f^{*}$, and hence for any dual solution $\Lambda^{*}$

$$
\begin{aligned}
& \widetilde{f}\left(\bar{X}_{p}\right)=\widetilde{f}\left(\bar{X}_{p}\right)+\left\langle\bar{X}_{p},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}\right\rangle-\left\langle\bar{X}_{p},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}\right\rangle \\
& =\phi\left(\bar{X}_{p}, \Lambda^{*}\right)-\left\langle\bar{X}_{p},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}\right\rangle \geq f^{*}-\left\langle\bar{X}_{p},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) \Lambda^{*}\right\rangle .
\end{aligned}
$$

Then by (12) we derive (14).

## 5. Numerical Simulations

In this section, we give two numerical examples to demonstrate the obtained theoretic results.

Example 5.1. This example shows that the primal-dual algorithm with constant step size can produce the accurate estimates for the constrained optimization problem, where gradients of the cost functions are only locally Lipschitz continuous, and the agents are equipped with different constraint sets. Besides, some of the constraint sets are not compact.


Figure 2: The communication topology

Consider an undirected network of three agents with edge set $\mathcal{E}_{\mathcal{G}}=\{(1,3),(2,3),(1,1),(2,2),(3,3)\}$. Objective functions for the agents are as follows:

$$
\begin{align*}
& f_{1}\left(x_{1}, x_{2}\right)=\frac{x_{1}^{2}}{2}+3 x_{1}+x_{2}^{2}+2 x_{2}+x_{1} x_{2}+0.5 e^{x_{1}+x_{2}}, \\
& f_{2}\left(x_{1}, x_{2}\right)=x_{1}^{2}+2 x_{1}+2 x_{2}^{2}+2 x_{2}+x_{1} x_{2}+e^{x_{2}},  \tag{42}\\
& f_{3}\left(x_{1}, x_{2}\right)=2 x_{1}^{2}+4 x_{1}+x_{2}^{2}+2 x_{2}+e^{x_{1}},
\end{align*}
$$

while the constraint sets for agents are $\Omega_{1}=\left\{\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}\right.$ : $\left.x_{1}^{2}+x_{2}^{2} \leq 2\right\}, \Omega_{2}=\left\{\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}: x_{1} \geq-1\right\}$, and $\Omega_{3}=$ $\left\{\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}: x_{2} \leq-0.5\right\}$. Denote the optimal solution by $\left(x_{1}^{*}, x_{2}^{*}\right)$, which is at the boundary of the global constraint set.

Let $\left\{x_{i, k}\right\}$ and $\left\{\lambda_{i, k}\right\}$ be produced by the algorithm (7) with initial values $x_{i, 0}=\mathbf{0}, \lambda_{i, 0}=\mathbf{0}, i=1,2,3$, and $\alpha=0.4$. Denote by $x_{i, k}^{1}$ and $x_{i, k}^{2}$ the estimates for $x_{1}^{*}$ and $x_{2}^{*}$ by agent $i$ at time $k$, respectively. Note that the primal-dual solution pair ( $X^{*}, \Lambda^{*}$ ) satisfies (33) and (34). Define the residual of the optimal condition as $r_{k}=\operatorname{col}\left\{X_{k+1}-X_{k},\left(\mathcal{L} \otimes \mathbf{I}_{m}\right) X_{k}\right\}$. The local estimates of all agents and 2-norm of the residual $r_{k}$ are shown in Figure 1. From the figure it is seen that the estimates for all agents converge to the same optimal solution.

Example 5.2. Consider a randomly generated undirected network of $n=10$ agents, where each agent has an average degree 4 . Each agent $i \in \mathcal{V}$ is assigned with a huber loss function $f_{i}: \mathbb{R} \rightarrow \mathbb{R}$ with

$$
f_{i}(x)=\left\{\begin{array}{l}
\frac{1}{2}\left(x-a_{i}\right)^{2}, \text { if }\left|x-a_{i}\right| \leq 1, \\
\left|x-a_{i}\right|-\frac{1}{2}, \text { otherwise }
\end{array}\right.
$$

For any $i \in \mathcal{V}, a_{i}$ is generated according to the uniform distribution over the interval [1.5,2.5]. The optimal solution of $f(\cdot)=\sum_{i=1}^{n} f_{i}(\cdot)$ is denoted by $x^{*}$. We compare the primal-dual algorithm (7) with the existing ones by this example.

Set $\alpha=0.8$. Denote by $x_{i, k}$ the estimate for $x^{*}$ given by agent $i$ at time $k$ with the initial values $x_{i, 0}=\mathbf{0} \forall i \in \mathcal{V}$. The simulation is for the case where the communication topology is shown in Figure 2, and the entries of the adjacency matrix $\mathcal{A}_{G}$ are Metropolis wights [25]. With this $\mathcal{A}_{\mathcal{G}}$ we carry out the simulations for the primal-dual algorithm (7) with $\lambda_{i, 0}=\mathbf{0} \forall i \in \mathcal{V}$,


Figure 3: The normalized relative error
for the DGD algorithm [7], for EXTRA [15] with constant step size $\alpha$, and for the distributed Nesterov gradient (D-NG) algorithm in [11]. The DGD algorithm runs separately for three cases: constant step size $\alpha$, diminishing step sizes $\alpha_{k}=\alpha / k^{0.75}$, and $\alpha_{k}=\alpha / k^{0.4}$. The D-NG algorithm, i.e., equations (2)-(4) in [11], is run with $c=\alpha$ and $y_{i, 0}=\mathbf{0} \forall i \in \mathcal{V}$.

Denote by $e_{k}=\frac{\left\|X_{k}-1 \otimes x^{*}\right\|}{\left\|X_{0}-1 \otimes x^{*}\right\|}$ the normalized relative error, where $X_{k}=\operatorname{col}\left\{x_{1, k}, \cdots, x_{n, k}\right\}$. The numerical results are shown in Figure 3, where the horizontal axis denotes the number of iterations $k$ and the vertical axis denotes $\log _{10}\left(e_{k}\right)$. From the figure it is seen that the DGD algorithms with decreasing step sizes converge to the optimal solution but the rate of convergence are the slowest in comparisons with other methods. It is also seen that DGD with constant step size quickly approaches to the neighborhood of the optimal solution. The estimates generated by D-NG [11], by the algorithm (7), and by EXTRA [15] all converge to the optimal solution. Besides, the algorithm (7) brings a satisfactory convergence rate for the unconstrained problem as well.

## 6. Conclusion

In the paper, a distributed primal-dual algorithm is proposed for multiple agents in a network to minimize the sum of individual cost functions subject to a global constraint, which is the intersection of the local constraints. The proposed algorithm with constant step size makes the estimates of all agents converge to the same optimal solution and achieve the convergence rate $O\left(\frac{1}{k}\right)$ when there is no constraint. The effectiveness and the priority of the proposed algorithm have been demonstrated by two numerical examples.

For further research, it is of interest to consider the primaldual algorithm for stochastic optimization, and to see if some desired properties taking place for the deterministic still remain true.

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