https://doi.org/10.1016/j.sysconle.2017.09.008

Characterization of Closed-loop Equilibrium Solutions for Dynamic Mean-variance Optimization Problems

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Abstract

Herein, we study dynamic mean-variance portfolio optimization problems, and provide an intrinsic characterization of the introduced closed-loop equilibrium solutions for the first time. Comparing to those related papers, our new viewpoint sheds new light on this topic. More precisely, the approach proposed here not only differs from existing literature, but also avoids conventional complex convergence arguments. By using the obtained equivalent conditions, we prove that this optimization problem indeed admits a unique pair of equilibrium solution.

Keywords: Stochastic linear quadratic problems, time-inconsistency, dynamic mean-variance portfolio optimization, closed-loop equilibrium solutions.

1. Introduction

The pioneering Markowitz's Nobel-prize-winning work on mean-variance portfolio selection, proposed in [11] for single-period setting, has laid down the foundation of modern investment portfolio theory. Later, there emerges quite a lot of papers extending it into multi-period setting. Nevertheless, the multi-period case may demonstrate the so-called time-inconsistency property, i.e. the "optimal" strategy based on today may not keep optimality tomorrow. To overcome this difficulty, one feasible approach is to discuss the *pre-committed* strategy for which the solutions are actually verified to be optimal only at the initial time (e.g. [19] and the references therein).

In this paper, we would like to discuss this problem from another viewpoint. Since it is not feasible to give the exact definition of "optimality", one may investigate the time inconsistency within a game-theoretic framework and analyze the time-consistent equilibrium solution. This basic idea firstly appeared in [12] and later developed in [1] under the mean-variance framework. For more general case, the authors in [2] and [3] examined this problem from the view of equilibrium value functions. They formally derived an extended HJB equations, and then proved the verification theorem (e.g. Theorem 3.1 of [3] or Theorem 2.3 of [4]) in a rigorous manner. When returning back to mean-variance problem (see [4]), they proposed some state-dependent risk aversion parameters, which enhances well the solutions' economic meaning. We emphasize that in the above mentioned papers the continuous time formation is justified as some limit of discrete time case. This routine also appears in [5], where mean-variance portfolio problem for semi-martingale model was studied. At the same time, another multi-person differential games approach was firstly shown in [15], [17], where a new class of equilibrium HJB equations/sytems of Riccati equations are introduced. Unlike [2], [4], [3], [5], they directly chose to investigate in the continuous time setting, make partition on the time intervals and use tricks of forward-backward stochastic differential equations (FBSDEs, in short). Eventually, we would like to mention some other papers on time inconsistent problems, like the same optimization problem with open-loop equilibrium solutions in [8], some insurance problems in e.g. [10], [14], [18], and the consumption-investment problems in [6], [7].

For above-mentioned papers, we emphasize several points. In the first place, the authors in [2], [4], [15], [3] and the following-up references all set up general time inconsistent frameworks, and then applied the obtained results into particular cases. For our mean-variance problems, we observe that this indirect manner may not fully make use of linear quadratic structure. In the second place, the procedures of deriving HJB equations or Riccati equations are quite complicated, see [2], [4], [15], [3], [17]. Actually, after examining firstly the discrete case, they had to rely on some other delicate convergence arguments to obtain the desired conclusions under the continuous time setting. In the third place, it seems that the existing papers did not provide enough details on the uniqueness issue of closed-loop equilibrium solutions for mean-variance selection problems, even though the equilibrium solutions indeed exist with the help of verification theorem (e.g. Theorem 2.3 in [4],

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or Theorem 3.1 in [3]). At this point, it is worthy pointing out the work of [9] which studied the uniqueness issue of open-loop equilibrium solutions.

Motivated by these facts, in this paper we make an attempt to develop dynamic mean-variance optimization problems from different perspectives. More precisely, by introducing two proper mean-field BSDEs with conditional expectation and some subtle decoupling tricks, we give a new characterization of closed-loop equilibrium solutions. Here we emphasize that the obtained necessary conditions help us investigate the uniqueness of closed-loop equilibrium solutions. Unlike the existing literature, here we choose to face linear quadratic structure directly. One advantage lies in the fact that we can adapt the decoupling tricks (e.g. [16]) into our setting. Moreover, through out this paper neither discrete time setting nor time duration partition tricks are needed, not to mention complex convergence arguments. Eventually, we unify the two different frameworks in both [1] and [4] and improve well the corresponding results.

At this moment, we provide some comparisons with papers related to ours. Firstly, the decoupling ideas also appeared in [8] and [9], where open-loop equilibrium solutions were introduced and discussed. However, our procedures here essentially differ from theirs due to the introducing of Lemma 3.1. Moreover, it seems that the techniques here are applicable in the open-loop equilibrium solution case as well. Secondly, the author in [5] also obtained some equivalent conclusions (see Theorem 4.6, Theorem 4.7 there). Nevertheless, their results are a little implicit and the introduced equilibrium solutions do not have uniqueness, see their Example 4.14.

The reminder of this paper is structured as follows. Section 2 formulates our model and presents the preliminary notations. Section 3 is devoted to the study of necessary and sufficient conditions for the existence of equilibrium solutions, as well as two more special cases. Section 4 concludes the paper.

2. Preliminary notations and model formulation

For $H := \mathbb{R}^n, \mathbb{R}^{n\times m}$, etc., $0 \le s < t \le T$, define

$$L^2_{\mathcal{F}_t}(\Omega; H) := \Big\{ X : \Omega \to H \;\Big|\; X \text{ is } \mathcal{F}_t \text{ measurable}, \mathbb{E}|X|^2 < \infty \Big\},$$

$$L^2_{\mathbb{F}}(s,t;H) := \Big\{ X : [s,t]\times\Omega \to H \;\Big|\; X(\cdot) \text{ is measurable and } \mathbb{F}\text{-adapted}, \;\; \mathbb{E}\Big[\int_s^t |X(r)|^2 dr\Big] < \infty \Big\},$$

$$L^2_{\mathbb{F}}\Big(\Omega; C([s,t];H)\Big) := \Big\{ X : [s,t]\times\Omega \to H \;\Big|\; X(\cdot) \text{ is measurable, } \mathbb{F}\text{-adapted, has continuous paths, } \mathbb{E}\Big(\sup_{r\in[s,t]} |X(r)|^2\Big) < \infty \Big\}.$$

Consider a financial market for which $m+1$ assets are traded continuously on $[0,T]$. One asset is the risky-free bond whose price evolves as

$$\begin{cases} dS_0(s) &= r(s)S_0(s)ds, \quad s \in [0,T], \\ S_0(0) &= s_0 > 0, \end{cases}$$

with $r(\cdot) > 0$ being the risk-free return rate. Denote by $\{W(t), t \ge 0\}$ a standard $\mathbb{F}$-adapted m-dimensional Brownian motion on a filtered complete probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\ge0})$ with $\mathbb{F} = \{\mathcal{F}_t\}_{t\ge0}$. We model the m risky assets by

$$\begin{cases} dS_i(s) = S_i(s)\Big\{ b_i(s)ds + \sum_{j=1}^m \sigma_{ij}(s)dW^j(s) \Big\}, \quad s \in [0,T], \\ S_i(0) = s_i > 0, \quad i = 1,2,\cdots,m, \end{cases}$$

where $b_i(\cdot)$ is the expected return rate of risky asset i, $\sigma_{ij}(\cdot)$ is the corresponding volatility rate. In the following, we assume that $r(\cdot)$, $b(\cdot) := (b_1(\cdot),\cdots,b_m(\cdot))$ and $\sigma(\cdot) := (\sigma_{ij}(\cdot))_{1\le i,j\le m}$ are deterministic and continuous on $[0,T]$ such that

$$b_i(\cdot) > r(\cdot), \quad i = 1,2,\cdots,m, \quad \sigma(\cdot)\sigma^T(\cdot) \ge \delta I, \quad \delta > 0. \tag{1}$$

Suppose the investor has an initial capital $x_0 > 0$ to invest and his/her wealth $x(s)$ should satisfy

$$\begin{cases} dX(s) &= r(s)X(s)ds + \sum_{i=1}^m (b_i(s) - r(s))u_i(s)ds \\ &\quad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(s)u_i(s)dW^j(s), \quad 0 \le s \le T, \\ X(0) &= x_0, \end{cases} \tag{2}$$

where $u_i(\cdot)$, $i = 1,2\cdots,m$ is the capital invested in the i-th risky asset. Here, we call $u(\cdot) := (u_1(\cdot),\cdots,u_m(\cdot))^\top \in L^2_{\mathbb{F}}(0,T,\mathbb{R}^m)$ the allocation strategy. Defining $\beta(\cdot) := (b_1(\cdot) - r(\cdot),\cdots,b_m(\cdot) - r(\cdot))$ and the risk premium by $\theta \equiv (\theta_1(\cdot),\cdots,\theta_m(\cdot)) := \beta(\cdot)(\sigma(\cdot)^\top)^{-1}$, equation (2) becomes

$$\begin{cases} dX(s) &= r(s)X(s)ds + \beta(s)u(s)ds + u(s)^\top\sigma(s)dW(s), \\ X(0) &= x_0. \end{cases} \tag{3}$$

Varying the initial time, we are interested in the following wealth dynamics starting from t with capital x,

$$\begin{cases} dX(s) &= r(s)X(s)ds + \beta(s)u(s)ds + u(s)^\top\sigma(s)dW(s), \\ X(t) &= x. \end{cases} \tag{4}$$

At anytime t, the objective of a mean-variance portfolio choice model is to choose an allocation strategy to minimize

$$J(t,x,u) = \mathrm{Var}_{t,x}\Big[X(T)\Big] - [\gamma_1 x + \gamma_2]\mathbb{E}_{t,x}\Big[X(T)\Big], \tag{5}$$

where $\mathbb{E}_{t,x}[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$, γ_i denotes the weight between conditional variance and conditional expectation. As to the dependence of x, we refer the reader to [4] and [8] for more details and related explanations. Given $(\Theta^*(\cdot), \varphi^*(\cdot)) \in C([0,T];\mathbb{R}^m) \times C([0,T];\mathbb{R}^m)$, $(t,x) \in [0,T]\times\mathbb{R}$, $\varepsilon > 0$, we define

$$\begin{aligned} &u^{v,\varepsilon}(s) := \Theta^*(s)X^{v,\varepsilon}(s) + \varphi^*(s) + vI_{[t,t+\varepsilon]}(s), \quad s \in [t,T], \\ &u^*(s) := \Theta^*(s)X^*(s) + \varphi^*(s), \quad s \in [0,T], \end{aligned} \tag{6}$$

where $v \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^m)$, $X^{v,\varepsilon}(\cdot)$, $X^*(\cdot)$ is associated with $u^{v,\varepsilon}(\cdot)$, $u^*(\cdot)$. It is easy to see that

$$\begin{aligned} &X^{v,\varepsilon}(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R})), \quad u^{v,\varepsilon}(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m), \\ &X^*(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R})), \quad u^*(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m). \end{aligned} \tag{7}$$

Definition 2.1. *Given (6), we call* $(\Theta^*(\cdot), \varphi^*(\cdot))$ *a closed-loop equilibrium solution if*

$$\liminf_{\epsilon \downarrow 0} \frac{J\Big(t, X^*(t); u^{v,\varepsilon}(\cdot)\Big) - J\Big(t, X^*(t); u^*(\cdot)\Big)}{\epsilon} \geq 0.$$

In this case, $u^*(\cdot)$ *is called closed-loop equilibrium control.*

The closed-loop equilibrium solution adopted here is similar in spirit to those in [4], [7] and [17], but essentially distinct from the open-loop strategy proposed in [8], [9].

3. A characterization of closed-loop equilibrium strategy

3.1. A preliminary lemma

Here we prove one result which plays crucial role in the following discussions. To begin with, for any $r,\ t \in [0, T]$, we consider the following mean-field FBSDE,

$$\begin{cases} X(r) = \xi + \displaystyle\int_t^r \Big[A(s)X(s) + B(s)u(s)\Big]ds \\ \qquad\qquad + \displaystyle\int_t^r \Big[C(s)X(s) + u(s)^\top D(s)\Big]dW(s), \\ Y(r,t) = GX(T) + \bar{G}\mathbb{E}_t X(T) + \displaystyle\int_r^T \Big[A(s)Y(s,t) \\ \qquad\qquad + C(s)Z(s,t)\Big]ds - \displaystyle\int_r^T Z(s,t)^\top dW(s). \end{cases} \tag{8}$$

(H1) Suppose $u(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$, $\xi \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R})$ with $t \in [0, T]$, G, $\bar{G}$ are constants, $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are respectively $\mathbb{R}$, $\mathbb{R}^{1\times m}$, $\mathbb{R}^{1\times m}$, $\mathbb{R}^{m\times m}$-valued deterministic and continuous functions on $[0, T]$.

For any $t \in [0, T]$, under (H1), (8) admits a unique triple of $(X(\cdot), Y(\cdot, t), Z(\cdot, t)) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R})) \times L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R})) \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$. Inspired by the decoupling tricks in e.g. [16], we are about to give explicit expressions of Y, Z in terms of X. Before the rigorous arguments, we would like to give some formal and intuitive deduction. For $T \geq s \geq t \geq 0$, suppose

$$Y(s,t) = P_1(s)X(s) + P_2(s)\mathbb{E}_t X(s) + P_3(s,t), \tag{9}$$

where P_1, P_2 are deterministic, $P_3(\cdot, t)$ is $\mathbb{F}$-adapted process such that $P_1(T) = G$, $P_2(T) = \bar{G}$, $P_3(T, t) = 0$,

$$\begin{aligned} &dP_i(s) = \Pi_i(s)ds, \quad i = 1, 2, \\ &dP_3(s,t) = \Pi_3(s,t)ds + \Lambda_3(s,t)^\top dW(s), \quad s \geq t. \end{aligned}$$

Here $\Pi_i(\cdot)$ are undetermined coefficients. Using Itô formula to $P_1(\cdot)X(\cdot)$ and $P_2(\cdot)\mathbb{E}_t X(\cdot)$, one has

$$\begin{aligned} d\Big[P_1 X\Big] &= \Big[[AP_1 + \Pi_1]X + P_1 Bu\Big]ds + P_1(CX + u^\top D)dW(s), \\ d\Big[P_2\mathbb{E}_t X\Big] &= \Big[[AP_2 + \Pi_2]\mathbb{E}_t X + P_2 B\mathbb{E}_t u\Big]ds, \quad s \in [t, T], \end{aligned}$$

which then implies that

$$\begin{aligned} dY = \Big[&[AP_1 + \Pi_1]X + P_1 Bu + [AP_2 + \Pi_2]\mathbb{E}_t X \\ &+ P_2 B\mathbb{E}_t u + \Pi_3\Big]ds + \Big[P_1(CX + u^\top D) + \Lambda_3^\top\Big]dW(s). \end{aligned} \tag{10}$$

Comparing the diffusion terms in backward equations of (8) and (10), to guarantee (9) we should require

$$Z(s,t)^\top = \Big[P_1(s)(C(s)X(s) + u^\top(s)D(s)) + \Lambda_3(s,t)^\top\Big]. \tag{11}$$

Hence

$$AY + CZ = A[P_1 X + P_2\mathbb{E}_t X + P_3] + C\Big[P_1(C^\top X + D^\top u) + \Lambda_3\Big].$$

For the drift terms between (8) and (10), to guarantee (9) we also have

$$\begin{aligned} &-A\Big[P_1 X + P_2\mathbb{E}_t X + P_3\Big] - C\Big[P_1(C^\top X + D^\top u) + \Lambda_3\Big] \\ &= \Big[\Big[AP_1 + \Pi_1\Big]X + P_1 Bu + \Big[AP_2 + \Pi_2\Big]\mathbb{E}_t X + P_2 B\mathbb{E}_t u + \Pi_3\Big], \end{aligned}$$

which can be rewritten as,

$$\begin{aligned} \Big[2AP_2 + \Pi_2\Big]\mathbb{E}_t X + \Big[CC^\top P_1 + 2AP_1 + \Pi_1\Big]X + \Big[P_1 Bu \\ + P_2 B\mathbb{E}_t u + \Pi_3 + P_3 A + CP_1 D^\top u + C\Lambda_3\Big] = 0. \end{aligned}$$

In this case, by choosing $\Pi_1 = -[CC^\top + 2A]P_1$, $\Pi_2 = -2AP_2$, and

$$\Pi_3 = -\Big[P_1 Bu + P_2 B\mathbb{E}_t u + P_3 A + CP_1 D^\top u + C\Lambda_3\Big],$$

we then end up with three equations of

$$\begin{aligned} &dP_1 = -[CC^\top + 2A]P_1 ds, \quad P_1(T) = G, \\ &dP_2 = -2AP_2 ds, \quad P_2(T) = \bar{G}, \\ &dP_3 = \Big[-AP_3 - C\Lambda_3 - \Big[P_1(B + CD^\top)u + P_2 B\mathbb{E}_t u\Big]\Big]ds \\ &\qquad + \Lambda_3 dW(s), \quad P_3(T,t) = 0. \end{aligned} \tag{12}$$

To sum up, we give a rigorous statement next.

Lemma 3.1. Given (8), suppose (H1) is true. Then there are $P_1(\cdot)$, $P_2(\cdot)$, $P_3(\cdot)$ satisfying (12) such that $(Y(\cdot, t), Z(\cdot, t))$ defined in (9) and (11) solves (8).

Proof. Notice that (H1) implies that there exist unique processes $P_1(\cdot)$, $P_2(\cdot) \in C([0, T]; \mathbb{R})$ satisfying equations in (12). Consequently, for $u(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$, the following is true,

$$\Big[P_1(B + CD^\top)u + P_2 B\mathbb{E}_t u\Big] \in L^2_{\mathbb{F}}(t, T; \mathbb{R}).$$

The standard BSDE theory indicates the existence and uniqueness of $(P_3(\cdot, t), \Lambda_3(\cdot, t)) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R})) \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$ with any $t \in [0, T]$. As a result, we define two processes (Y, Z) in (9), (11). By repeating above arguments from (9) to (12), one can see that

$$(Y(\cdot, t), Z(\cdot, t)) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R})) \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$$

with $t \in [0, T]$ satisfies the backward equation in (8). $\square$

3.2. The main results

Now we state the first main result of this paper.

Theorem 3.1. Given wealth equation (3) and cost functional (5), suppose all the coefficients are continuous and deterministic such that (1) is true. Then the time inconsistent control problem admits a closed-loop equilibrium solution $(\Theta^*, \varphi^*) \in C([0,T];\mathbb{R}^m) \times C([0,T];\mathbb{R}^m)$ if and only if $\Theta^*(\cdot)$ solves the following equation

$$\Theta^*(t) = -\Big[\sigma(t)\sigma(t)^\top\Big]^{-1}\beta(t)^\top\Big\{1 - \exp\Big[-\int_t^T [\Theta^*]^\top \sigma\sigma^\top \Theta^* ds\Big] - \frac{\gamma_1}{2}\exp\Big[\int_t^T [-r - \beta\Theta^* - [\Theta^*]^\top\sigma\sigma^\top\Theta^*(s)]ds\Big]\Big\}, \tag{13}$$

and for $P_1(\cdot),\ P_2(\cdot)$ satisfying (22), $\varphi^*(\cdot)$ can be shown as,

$$\varphi^*(t) = \frac{\gamma_2\big[\sigma(t)\sigma(t)^\top\big]^{-1}\beta(t)^\top}{2P_1(t)}\exp\Big[\int_t^T \Big[r - [1 + \frac{P_2}{P_1}]\beta(\sigma\sigma^\top)^{-1}\beta^\top\Big]ds\Big],\quad t \in [0,T]. \tag{14}$$

In order to prove this result, we need some preparations. Given $\Theta^*(\cdot) \in C([0,T];\mathbb{R}^m)$, $X^*(\cdot)$, $X^{v,\varepsilon}(\cdot)$ in (7), in what follows, $\bar{B}(\cdot)$ and $\bar{D}(\cdot)$ stand for

$$\bar{B}(\cdot) := r(\cdot) + \beta(\cdot)\Theta^*(\cdot) \in \mathbb{R},\quad \bar{D}(\cdot) := \Theta^*(\cdot)^\top\sigma(\cdot) \in \mathbb{R}^{1\times m}, \tag{15}$$

and $X_1^{v,\varepsilon}(\cdot) := X^{v,\varepsilon}(\cdot) - X^*(\cdot)$, i.e. for $s \in [t,T]$,

$$\begin{cases} dX_1^{v,\varepsilon}(s) = \Big[\bar{B}(s)X_1^{v,\varepsilon}(s) + \beta(s)vI_{[t,t+\varepsilon]}(s)\Big]ds \\ \qquad\qquad + \Big[\bar{D}(s)X_1^{v,\varepsilon}(s) + v^\top\sigma(s)I_{[t,t+\varepsilon]}(s)\Big]dW(s), \\ X_1^{v,\varepsilon}(t) = 0. \end{cases}$$

We introduce BSDEs with parameter t, i.e. for $s \in [t,T]$,

$$\begin{cases} Y^*(s,t) = X^*(T) - \mathbb{E}_tX^*(T) + \displaystyle\int_s^T \Big[\bar{B}(r)Y^*(r,t) \\ \qquad\qquad + \bar{D}(r)Z^*(r,t)\Big]dr - \displaystyle\int_s^T Z^*(r,t)^\top dW(r), \\ Y_1^{v,\varepsilon}(s,t) = X_1^{v,\varepsilon}(T) - \mathbb{E}_tX_1^{v,\varepsilon}(T) + \displaystyle\int_s^T \Big[\bar{B}(r)Y_1^{v,\varepsilon}(r,t) \\ \qquad\qquad + \bar{D}(r)Z_1^{v,\varepsilon}(r,t)\Big]dr - \displaystyle\int_s^T Z_1^{v,\varepsilon}(r,t)^\top dW(r), \end{cases} \tag{16}$$

where $\mathbb{E}_tX^*(T) := \mathbb{E}^{\mathcal{F}_t}X^*(T)$. If there exists equilibrium solution (Θ^*, v^*), then (16) admit unique pairs of solutions

$$(Y^*, Z^*),\ (Y^{v,\varepsilon}, Z^{v,\varepsilon}) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R})) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^m). \tag{17}$$

Lemma 3.2. Given (Θ^*, φ^*) and (6), for $t \in [0,T]$ and any solutions (Y^*, Z^*), $(Y_1^{v,\varepsilon}, Z_1^{v,\varepsilon})$ in the sense of (17), one has

$$\begin{aligned} J\big(t, X^*(t); u^{v,\varepsilon}(\cdot)\big) - J\big(t, X^*(t); u^*(\cdot)\big) &= -\Big[\gamma_1 x + \gamma_2\Big]\mathbb{E}_tX_1^{v,\varepsilon}(T) \\ &\quad + \mathbb{E}_t\int_t^{t+\varepsilon}\Big\{\beta(s)[Y_1^{v,\varepsilon}(s,t) + 2Y^*(s,t)] \\ &\quad + [Z_1^{v,\varepsilon}(s,t)^\top + 2Z^*(s,t)^\top]\sigma(s)^\top\Big\}vds. \end{aligned}$$

Proof. By the definition of $X^*(\cdot),\ X^{v,\varepsilon}(\cdot),\ X_1^{v,\varepsilon}(\cdot)$, we know that

$$\begin{aligned} &J(t, X^*(t); u^{v,\varepsilon}(\cdot)) - J\big(t, X^*(t); u^*(\cdot)\big) \\ &= \mathbb{E}_t\Big[\Big[X_1^{v,\varepsilon}(T) - \mathbb{E}_tX_1^{v,\varepsilon}(T)\Big]\cdot X_1^{v,\varepsilon}(T)\Big] \\ &\quad + \mathbb{E}_t\Big[\Big(2\Big[X^*(T) - \mathbb{E}_tX^*(T)\Big] - \gamma_1 x - \gamma_2\Big)\cdot X_1^{v,\varepsilon}(T)\Big]. \end{aligned} \tag{18}$$

We use Itô's formula to $Y^*(\cdot,t)X_1^{v,\varepsilon}(\cdot)$ on $[t+\varepsilon, T]$ and $[t, t+\varepsilon]$, and thus deduce that (with s being omitted for simplicity),

$$\begin{aligned} d\Big[Y^*(t)X_1^{v,\varepsilon}\Big] &= \Big[Z^*(t)^\top X_1^{v,\varepsilon} + Y^*(t)\bar{D}X_1^{v,\varepsilon}\Big]dW(s),\quad s \in [t+\varepsilon, T], \\ d\Big[Y^*(t)X_1^{v,\varepsilon}\Big] &= \Big[Z^*(t)^\top X_1^{v,\varepsilon} + Y^*(t)[\bar{D}(s)X_1^{v,\varepsilon} + v^\top\sigma]\Big]dW(s) \\ &\quad + \Big[Y^*(t)\beta + Z^*(t)^\top\sigma^\top\Big]v,\quad s \in [t, t+\varepsilon], \end{aligned}$$

where $\bar{D}(\cdot)$ is defined in (15). Therefore, by the integrability of $Z^*(\cdot,t),\ Y^*(\cdot,t),\ X^{v,\varepsilon}(\cdot)$,

$$\begin{aligned} &\mathbb{E}_t\Big[Y^*(T,t)X_1^{v,\varepsilon}(T)\Big] \\ &= \mathbb{E}_t\int_t^{t+\varepsilon}\Big[\beta(s)Y^*(s,t) + Z^*(s,t)^\top\sigma(s)^\top\Big]vds. \end{aligned} \tag{19}$$

Similarly we can also obtain that,

$$\begin{aligned} &\mathbb{E}_t\Big[Y_1^{v,\varepsilon}(T,t)X_1^{v,\varepsilon}(T)\Big] \\ &= \mathbb{E}_t\int_t^{t+\varepsilon}\Big[\beta(s)Y_1^{v,\varepsilon}(s,t) + Z_1^{v,\varepsilon}(s,t)^\top\sigma(s)^\top\Big]vds. \end{aligned} \tag{20}$$

Therefore, the desired conclusion is easy to see. ☐

In order to deal with $(Y^*(\cdot,t), Z^*(\cdot,t))$ in Lemma 3.2, we would like to make use of previous Lemma 3.1.

Lemma 3.3. Given $(\Theta^*(\cdot), \varphi^*(\cdot))$, and $(Y^*(\cdot,t), Z^*(\cdot,t))$ in (16), $\bar{B},\ \bar{D}$ in (15), we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\Big[\frac{1}{\varepsilon}\mathbb{E}_t\int_t^{t+\varepsilon} 2[\beta(s)Y^*(s,t) + Z^*(s,t)^\top\sigma(s)^\top]vds\Big] \\ &= 2\Big[\Big\{[P_1(t) + P_2(t)]\beta(t) + P_1(t)\bar{D}(t)\sigma(t)^\top\Big\}x \\ &\quad + \beta(t)\widetilde{P}_3(t) + P_1(t)\varphi^*(t)^\top\sigma(t)\sigma(t)^\top\Big]v \end{aligned} \tag{21}$$

where $t \in [0,T]$, and for $P_1(T) = 1,\ P_2(T) = -1,\ \widetilde{P}_3(T) = 0$, $P_1(\cdot),\ P_2(\cdot),\ P_3(\cdot)$ satisfy

$$\begin{aligned} &dP_1(s) = -\Big[\bar{D}\bar{D}^\top + 2\bar{B}\Big]P_1 ds,\quad dP_2(s) = -2\bar{B}P_2 ds, \\ &d\widetilde{P}_3 = \Big[-\bar{B}\widetilde{P}_3 - \Big[P_1(\beta + \bar{D}\sigma^\top) + P_2\beta\Big]\varphi^*\Big]ds,\quad s \in [0,T]. \end{aligned} \tag{22}$$

Proof. Recall that $X^*(t) = x$, $\varphi^* \in C([0,T];\mathbb{R}^m)$ and

$$\begin{aligned} dX^*(s) &= [\bar{B}(s)X^*(s) + \beta(s)\varphi^*(s)]ds \\ &\quad + [\bar{D}(s)X^*(s) + \varphi^*(s)^\top\sigma(s)]dW(s),\quad s \in [0,T], \end{aligned}$$

it follows from Lemma 3.1 that the following (Y^*, Z^*) satisfy the first equation in (16),

$$\begin{aligned} Y^*(s,t) &= P_1(s)X^*(s) + P_2(s)\mathbb{E}_tX^*(s) + \widetilde{P}_3(s,t), \\ Z^*(s,t)^\top &= P_1(s)[\bar{D}(s)X^*(s) + \varphi^*(s)^\top\sigma(s)] + \widetilde{\Lambda}_3(s,t)^\top, \end{aligned}$$

where $P_1(\cdot)$, $P_2(\cdot)$ and $(\widetilde{P}_3(\cdot,t), \widetilde{\Lambda}_3(\cdot,t))$ respectively solves the equations in (22) and for $\widetilde{P}_3(T,t)=0$,

$$d\widetilde{P}_3(s,t) = \Big[-\bar{B}(s)\widetilde{P}_3(s,t) - \bar{D}(s)\widetilde{\Lambda}_3(s,t) - \Big[P_1(s)[\beta(s) + \bar{D}(s)\sigma(s)^\top]\varphi^*(s) + P_2(s)\beta(s)\mathbb{E}_t\varphi^*(s)\Big]\Big]ds + \widetilde{\Lambda}_3(s,t)dW(s).$$

Notice that $\varphi^*(\cdot)$ is deterministic, hence $\widetilde{\Lambda}_3(\cdot,t)=0$ and $\widetilde{P}_3(\cdot,t)$ is independent of t, and we thus obtain the third one in (22). Therefore,

$$\begin{aligned}&\beta(t)Y^*(t,t) + Z^*(t,t)^\top\sigma(t)^\top\\ &= \beta(t)[P_1(t)+P_2(t)]x + P_1(t)\Theta^*(t)^\top\sigma(t)\sigma(t)^\top x\\ &\quad+\beta(t)\widetilde{P}_3(t) + P_1(t)\varphi^*(t)^\top\sigma(t)\sigma(t)^\top,\quad t\in[0,T].\end{aligned}\tag{23}$$

As to $P_1(\cdot)$, $P_2(\cdot)$, they are bounded such that for any $t\in[0,T]$,

$$\begin{aligned}&|P_1(t)|\leq \exp\Big[\int_0^T\Big[2|r(s)+\beta(s)\Theta^*(s)| + |\Theta^*(s)^\top\sigma(s)|^2\Big]ds\Big],\\ &|P_2(t)|\leq \exp\Big[\int_0^T\Big[2|r(s)+\beta(s)\Theta^*(s)|\Big]ds\Big].\end{aligned}\tag{24}$$

Let us use the temporary notation $\mathcal{P}$ for $[P_1+P_2]\beta + P_1\bar{D}\sigma^\top$. We know that

$$\begin{aligned}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathcal{P}(s)v\mathbb{E}_tX^*(s)ds &= \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathcal{P}(s)v[\mathbb{E}_tX^*(s)-X^*(t)]ds\\ &\quad+\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathcal{P}(s)vds\cdot x\end{aligned}$$

By the continuity of $\mathcal{P}(\cdot)$, for any $t\in[0,T]$, one has

$$\begin{aligned}&\Big|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathcal{P}(s)v[\mathbb{E}_tX^*(s)-X^*(t)]ds\Big|\\ &\leq\Big|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}|\mathcal{P}(s)v|^2ds\Big|^{\frac12}\Big|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}[\mathbb{E}_tX^*(s)-X^*(t)]^2ds\Big|^{\frac12}\\ &\leq|\mathcal{P}(t)v|\cdot\mathbb{E}_t\sup_{s\in[t,t+\varepsilon]}|X^*(s)-X^*(t)|^2\to0,\quad\varepsilon\to0.\end{aligned}$$

As to the second term, it is easy to see that

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathcal{P}(s)vds\cdot x\to\mathcal{P}(t)v\cdot x,\quad\varepsilon\to0,\quad\forall t\in[0,T].$$

As a result,

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathcal{P}(s)v\mathbb{E}_tX^*(s)ds\to\mathcal{P}(t)v\cdot x,\quad\varepsilon\to0,\quad\forall t\in[0,T].$$

Recall that $\varphi^*(\cdot)\in C([0,T];\mathbb{R}^m)$, using Lebesgue differentiation theorem, we then obtain the desired conclusion with any $t\in[0,T]$. ◻

The following result is concerned with $(Y_1^{v,\varepsilon}(\cdot,\cdot), Z_1^{v,\varepsilon}(\cdot,\cdot))$.

Lemma 3.4. Given $(\Theta^*(\cdot),\varphi^*(\cdot))$ and $(Y^{v,\varepsilon}(\cdot,t), Z^{v,\varepsilon}(\cdot,t))$ in (16), $P_1(\cdot)$ in (22), we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}_t\int_t^{t+\varepsilon}[\beta(s)Y_1^{v,\varepsilon}(s,t) + Z_1^{v,\varepsilon}(s,t)^\top\sigma(s)^\top]vds\\ &=P_1(t)v^\top\sigma(t)\sigma(t)^\top v.\ \text{a.s.}\end{aligned}\tag{25}$$

Proof. We separate the discussions on $[t,T]$ into two parts. Firstly let us look at the case of $[t+\varepsilon,T]$. By Lemma 3.1, for $P_1(\cdot)$, $P_2(\cdot)$ in (22) and $s\in[t+\varepsilon,T]$, the following processes satisfy the second equation of (16) on $[t+\varepsilon,T]$,

$$\begin{aligned}&Y_1^{v,\varepsilon}(s,t) := P_1(s)X_1^{v,\varepsilon}(s) + P_2(s)\mathbb{E}_t[X_1^{v,\varepsilon}(s)],\\ &Z_1^{v,\varepsilon}(s,t)^\top := P_1(s)\bar{D}(s)X_1^{v,\varepsilon}(s).\end{aligned}\tag{26}$$

For the case on $[t,t+\varepsilon]$, let us firstly look at

$$d\widetilde{P}_1^{v,\varepsilon} = \Big[-\bar{B}\widetilde{P}_1^{v,\varepsilon} - \Big[P_1[\beta+\bar{D}\sigma^\top]+P_2\beta\Big]\Big]ds,\tag{27}$$

with $s\in[t,t+\varepsilon]$, $\widetilde{P}^{v,\varepsilon}(t+\varepsilon)=0$. Obviously, there exists a unique $\widetilde{P}_1^{v,\varepsilon}(\cdot)\in C([t,t+\varepsilon];\mathbb{R}^{1\times m})$. By Lemma 3.1, the following pair of processes satisfy the second equation of (16) on $[t,t+\varepsilon]$,

$$\begin{aligned}&Y_1^{v,\varepsilon}(s,t) := P_1(s)X_1^{v,\varepsilon}(s) + P_2(s)\mathbb{E}_t[X_1^{v,\varepsilon}(s)] + \widetilde{P}_1^{v,\varepsilon}(s)v,\\ &Z_1^{v,\varepsilon}(s,t)^\top := P_1(s)\Big[v^\top\sigma(s)+\bar{D}(s)X_1^{v,\varepsilon}(s)\Big],\quad s\in[t,t+\varepsilon].\end{aligned}\tag{28}$$

As a result,

$$\begin{aligned}&\beta(s)Y_1^{v,\varepsilon}(s,t) + Z_1^{v,\varepsilon}(s,t)^\top\sigma(s)^\top\\ &=[\beta(s)+\bar{D}(s)\sigma(s)^\top]P_1(s)X_1^{v,\varepsilon}(s) + \widetilde{P}_1^{v,\varepsilon}(s)v\beta(s)\\ &\quad+\beta(s)P_2(s)\mathbb{E}_t[X_1^{v,\varepsilon}(s)] + P_1(s)v^\top\sigma(s)\sigma(s)^\top.\end{aligned}\tag{29}$$

By the definition of $\widetilde{P}_1^{v,\varepsilon}(\cdot)$, it is easy to see that

$$\frac{1}{\varepsilon}\mathbb{E}_t\int_t^{t+\varepsilon}\widetilde{P}_1^{v,\varepsilon}(s)v\beta(s)vds\to0,\quad\varepsilon\to0,\quad\forall t\in[0,T].\tag{30}$$

On the other hand, as to $X_1^{v,\varepsilon}(\cdot)$, a standard estimate is

$$\mathbb{E}_t\Big[\sup_{t\in[t,t+\varepsilon]}|X_1^{v,\varepsilon}(s)|^2\Big]\leq C\mathbb{E}_t\Big[\int_t^{t+\varepsilon}|\beta(r)v|dr\Big]^2 + C\mathbb{E}_t\int_t^{t+\varepsilon}|v^\top\sigma(r)|^2dr.$$

Therefore, $\mathbb{E}_t\Big[\sup_{t\in[t,t+\varepsilon)}|X_1^{v,\varepsilon}(s)|^2\Big]=O(\varepsilon)$, from which one can see that, for any $t\in[0,T)$, as $\varepsilon\to0$, (with s be omitted)

$$\frac{1}{\varepsilon}\mathbb{E}_t\int_t^{t+\varepsilon}\Big[\Big[\beta+\bar{D}\sigma^\top\Big]P_1+\beta P_2\Big]X_1^{v,\varepsilon}vds\to0.\ \text{a.s.}\tag{31}$$

Our conclusion can be derived by (29), (30), (31). ◻

Proof of Theorem 3.1. [1] Firstly we discuss the necessity part. To this end, let us look at the term $[\gamma_1x+\gamma_2]\mathbb{E}_t[X_1^{v,\varepsilon}(T)]$. Recall $\bar{B}(\cdot)$ in (15), after some calculations, for any $s\geq t$ we have

$$\begin{aligned}&[\gamma_1x+\gamma_2]\mathbb{E}_t\Big\{X_1^{v,\varepsilon}(T)\Big\}\\ &=[\gamma_1x+\gamma_2]\int_t^{t+\varepsilon}\exp\Big[\int_r^T\bar{B}(s)ds\Big]\beta(r)vdr.\end{aligned}\tag{32}$$

Using Lemma 3.2, 3.3 and 3.4, as well as Lebesgue differentiation theorem, we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\Big[J(t,X^*(t);u^{v,\varepsilon}(\cdot)) - J\Big(t,X^*(t);u^*(\cdot)\Big)\Big]\\ &=P_1(t)v^\top\sigma(t)\sigma(t)^\top v + F_1(t)v\cdot x + F_2(t)v,\end{aligned}\tag{33}$$

where $F_1(\cdot)$, $F_2(\cdot)$ are defined as

$$F_1(t):=-\gamma_1\exp\Big[\int_t^T\bar{B}ds\Big]\beta+2[P_1+P_2]\beta+2P_1\bar{D}\sigma^\top,\ t\in[0,T]$$
$$F_2(t):=2\beta\widetilde{P}_3+2P_1[\varphi^*]^\top\sigma\sigma^\top-\gamma_2\exp\Big[\int_t^T\bar{B}ds\Big]\beta,\ t\in[0,T].$$

Notice that by the existence of $P_1(\cdot)$ and condition (1), one has $P_1(t)\sigma(t)\sigma(t)^\top\geq\delta_1>0$ with $t\in[0,T]$. Moreover, the existence of equilibrium strategy in the spirit of Definition 2.1 implies that the right hand of (33) is nonnegative for any $\mathcal{F}_t$-measurable v. Hence $F_1(t)\cdot x+F_2(t)=0$. Using the arbitrariness of $x\in\mathbb{R}$, for any $t\in[0,T]$, one then ends up with,

$$\begin{aligned}\Theta^*(t)&=-\Big[\sigma(t)\sigma(t)^\top P_1(t)\Big]^{-1}\beta(t)^\top\Big\{P_2(t)+P_1(t)\\&\quad-\frac{\gamma_1}{2}\exp\Big[\int_t^T[r(s)+\beta(s)\Theta^*(s)]ds\Big]\Big\},\\\varphi^*(t)&=-\Big[\sigma(t)\sigma(t)^\top P_1(t)\Big]^{-1}\beta(t)^\top\Big\{\widetilde{P}_3(t)\\&\quad-\frac{\gamma_2}{2}\exp\Big[\int_t^T[r(s)+\beta(s)\Theta^*(s)]ds\Big]\Big\}.\end{aligned}\tag{34}$$

Notice that we can obtain equation (13) by substituting the expressions of $P_1(\cdot)$, $P_2(\cdot)$ into (34). As to $\varphi^*(\cdot)$, we put $\varphi^*(\cdot)$ back into the third equation of (22). Recall that $\mathcal{P}:=\Big\{[P_1+P_2]\beta+P_1\bar{D}\sigma^\top\Big\}$ in the proof of Lemma 3.3, we then have

$$\begin{aligned}d\widetilde{P}_3=\Big\{&-\Big[\bar{B}-\mathcal{P}\big(\sigma\sigma^\top P_1\big)^{-1}\beta^\top\Big]\widetilde{P}_3-\frac{\gamma_2}{2}\mathcal{P}\big(\sigma\sigma^\top P_1\big)^{-1}\\&\cdot\beta^\top\exp\Big[\int_s^T\bar{B}(r)dr\Big]\Big\}ds,\quad s\in[t,T].\end{aligned}$$

Since $\widetilde{P}_3(T)=0$, one can obtain

$$\begin{aligned}\widetilde{P}_3(t)=-\Big\{&\exp\Big[\int_t^T[-\mathcal{P}(r)\big(\sigma(r)\sigma(r)^\top P_1(r)\big)^{-1}\beta(r)^\top]dr\Big]-1\Big\}\\&\cdot\frac{\gamma_2}{2}\exp\Big[\int_t^T\bar{B}(r)dr\Big].\end{aligned}$$

Plugging it back into $\varphi^*(\cdot)$ of (34), we then finish the necessity results.

[2] We turn to look at the sufficiency issue. Suppose there exist $(\Theta^*(\cdot),\varphi^*(\cdot))\in C([0,T];\mathbb{R}^m)\times C([0,T];\mathbb{R}^m)$ satisfy (13), (14). Hence there exist bounded and continuous functions $P_1(\cdot)$, $P_2(\cdot)$ satisfying (22). In this case one can rewrite (13) as the first expression in (34). Moreover, for some $\delta_1>0$, one has $P_1(t)\sigma(t)\sigma(t)^\top\geq\delta_1$. Hence repeating the related arguments in the previous necessity proof, one can derive that

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\Big[J(t,X^*(t);u^{v,\varepsilon}(\cdot))-J\big(t,X^*(t);u^*(\cdot)\big)\Big]\geq0.\tag{35}$$

This implies that $(\Theta^*(\cdot),\varphi^*(\cdot))$ is an equilibrium strategy in the sense of Definition 2.1. □

Given $\Theta^*(\cdot)$, if we denote by (with s being omitted)

$$\begin{aligned}K(t):=-1+\exp\Big[-\int_t^T[\Theta^*]^\top\sigma\sigma^\top\Theta^*ds\Big]\\+\frac{\gamma_1}{2}\exp\Big[\int_t^T[-r-\beta\Theta^*-[\Theta^*]^\top\sigma\sigma^\top\Theta^*]ds\Big],\end{aligned}\tag{36}$$

with $t\in[0,T]$, then $\Theta^*(\cdot)=\Big[\sigma(\cdot)\sigma(\cdot)^\top\Big]^{-1}\beta(\cdot)^\top K(\cdot)$. Putting it back into (36), we can obtain

$$\begin{aligned}K(t)=-1+\exp\Big[-\int_t^TK^\top\Phi Kds\Big]\\+\frac{\gamma_1}{2}\exp\Big[\int_t^T[-r-\Phi K-K^\top\Phi K]ds\Big],\end{aligned}\tag{37}$$

where $\Phi(\cdot):=\beta(\cdot)[\sigma(\cdot)\sigma(\cdot)^\top]^{-1}\beta(\cdot)^\top$. Notice that $K(\cdot)$ is one-dimensional. Mimicking Theorem 4.7 in [4], one has

Proposition 3.1. If $r(\cdot)$, $\beta(\cdot)$, $\sigma(\cdot)$ are continuous deterministic functions such that (1) is true. Then equation (37) admits a unique solution $K(\cdot)\in C([0,T];\mathbb{R})$.

Considering the equivalent relation between (13) and (37), the previous result implies that (13) indeed admits a solution $\Theta^*(\cdot)\in C([0,T];\mathbb{R}^m)$. By means of $\Theta^*(\cdot)$ we can thus represent $\varphi^*(\cdot)$ via (14), and therefore obtain the existence of closed-loop equilibrium solution. However, so far we have not touched the uniqueness issue. As a result, next we would like to give some related details.

Theorem 3.2. If the time inconsistent mean-variance optimization problem admits a closed-loop equilibrium strategy $(\Theta^*(\cdot),\varphi^*(\cdot))$, then it is unique.

Proof. Suppose there are two equilibrium strategy (Θ_i^*,φ_i^*), $i=1,2$, in the sense of Definition 2.1. From the necessary conditions in Theorem 3.1, both of $\Theta_i^*(\cdot)$ satisfy

$$\begin{aligned}\Theta(t)=-\Big[\sigma(t)\sigma(t)^\top\Big]^{-1}\beta(t)^\top\Big\{1-\exp\Big[-\int_t^T\Theta^\top\sigma\sigma^\top\Theta ds\Big]\\-\frac{\gamma_1}{2}\exp\Big[\int_t^T[-r-\beta\Theta-\Theta^\top\sigma\sigma^\top\Theta]ds\Big]\Big\}.\end{aligned}\tag{38}$$

Let $\Theta^*=\Big[\sigma\sigma^\top\Big]^{-1}\beta^\top K$, one then obtain the same equation as (37). Using the uniqueness in Proposition 3.1, we can see that $\Theta_1^*(t)=\Theta_2^*(t)$ with any $t\in[0,T]$. The conclusion of $\varphi_1^*(\cdot)=\varphi_2^*(\cdot)$ follows naturally from (14) and $\Theta_1^*(\cdot)=\Theta_2^*(\cdot)$. □

3.3. Two special cases

In this part, let us point out two special cases.

Case I: For (5), suppose $\gamma_2=0$. Given $\Theta^*(\cdot)\in C([0,T];\mathbb{R}^m)$, it is easy to see that

$$\alpha(\cdot):=\exp\Big[\int_\cdot^T[r(s)+\beta(s)\Theta^*(s)]ds\Big]\in C([0,T];\mathbb{R})$$

is the unique solution of

$$d(\alpha(s)) = -\alpha(s)[r(s) + \beta(s)\Theta^*(s)]ds, \quad s \in [0, T], \quad \alpha(T) = 1.$$

Moreover, some simple calculations show that $-\alpha^2(\cdot) = P_2(\cdot)$. As a result, we can rewrite the first equation in (34), or (13) as

$$\Theta^*(t) = \Big[\sigma(t)\sigma(t)^\top P_1(t)\Big]^{-1}\beta(t)^\top\Big\{\alpha^2(t) - P_1(t) + \frac{\gamma_1}{2}\alpha(t)\Big\}.$$

Notice that the expression of $\Theta^*(\cdot)$ degenerates into the one in Proposition 4.5 of [4] if $r(\cdot)$, $\beta(\cdot)$, $\sigma(\cdot)$ are time-independent and $m = 1$. As to φ^*, it equals to zero due to $\gamma_2 = 0$ and (14). According to Proposition 3.1 and Theorem 3.2, the mean-variance optimization problem under this framework admits a unique closed-loop equilibrium solution in the sense of Definition 2.1.

Case II: For (5), suppose $\gamma_1 = 0$. From (34), or (13), we have

$$\Theta^*(t) = -\Big[\sigma(t)\sigma(t)^\top P_1(t)\Big]^{-1}\beta(t)^\top\Big\{P_2(t) + P_1(t)\Big\}. \quad (39)$$

We claim that both $\Theta^*(\cdot)$ and $P_1(\cdot) + P_2(\cdot)$ equal to zero. Actually, from (22), we know that $\widehat{P} := P_1 + P_2$ satisfies $\widehat{P}(T) = 0$,

$$d\widehat{P} + \Big[2\bar{B}\widehat{P} + [\Theta^*]^\top\sigma\sigma^\top\Theta^* P_1\Big]ds = 0, \quad s \in [0, T]. \quad (40)$$

Since $\Theta^* \in C([0, T]; \mathbb{R}^m)$, $P_1(\cdot) \in C([0, T]; \mathbb{R})$, there exists a unique solution $\widehat{P}(\cdot)$ satisfying (40). Putting $\Theta^*(\cdot)$ of (39) into (40), we then end up with

$$d\widehat{P} + \Big[2\bar{B}\widehat{P} + \beta\Big[\sigma\sigma^\top P_1\Big]^{-1}\beta^\top[\widehat{P}]^2\Big]ds = 0, \quad s \in [0, T].$$

Since $\widehat{P}(T) = 0$, hence $\widehat{P}(\cdot) = 0$ is a solution of (40). By its uniqueness, $\widehat{P}(\cdot) = \Big[P_1(\cdot) + P_2(\cdot)\Big] = 0$. Consequently, from (39), we know that $\Theta^*(\cdot) = 0$. Therefore, by (22)

$$P_1(t) = -P_2(t) = \exp\Big[\int_t^T 2r(s)ds\Big], \quad t \in [0, T].$$

Putting $P_1(\cdot)$, $P_2(\cdot)$ back into (14), we then arrive at

$$\varphi^*(t) = \frac{\gamma_2}{2}\Big[\sigma(t)\sigma(t)^\top\Big]^{-1}\beta(t)^\top \exp\Big[-\int_t^T r(s)ds\Big].$$

When $r(\cdot)$, $\beta(\cdot), \sigma(\cdot)$ are time-dependent and $m = 1$, then

$$\varphi^*(t) = \frac{\gamma_2\beta}{2\sigma^2}\exp\Big[-r[T - t]\Big].$$

This expression coincides with Proposition 1 in [1] or Proposition 3.1 in [4]. The existence and uniqueness of closed-loop equilibrium in the spirit of Definition 2.1 can be derived via Proposition 3.1 and Theorem 3.2 as well.

4. Concluding remarks

In this paper we develop the theory of dynamic mean-variance portfolio problems with new techniques and tricks. When the coefficients are deterministic, we introduce closed-loop equilibrium solutions and establish the equivalent conditions of their existence. As a result, we prove that this optimization problem indeed admits a unique pair of equilibrium solutions, which extends and improves the results in existing papers. Related study with random coefficients is under consideration. We hope to do some relevant research in future.

Acknowledgement

This research of the first author was supported by Hongkong RGC under grant PolyU 5006/13P. The research of the second author was supported by 519913, 15209614, 15224215 and 15255416. The research of the third author was supported by the NSF of China under grant 11231007, 11301298, 11401404 and 11471231, China Postdoctoral Science Foundation(2014M562321).

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