

Distributed Resource Allocation Over Random Networks Based on Stochastic Approximation [☆]

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Abstract

In this paper, a stochastic approximation (SA) based distributed algorithm is proposed to solve the resource allocation (RA) with uncertainties. In this problem, a group of agents cooperatively optimize a separable optimization problem with a linear network resource constraint and allocation feasibility constraints, where the global objective function is the sum of agents' local objective functions. Each agent can only get noisy observations of its local function's gradient and its local resource, which cannot be shared by other agents or transmitted to a center. Moreover, there are communication uncertainties such as time-varying topologies (described by random graphs) and additive channel noises. To solve the RA, we propose an SA-based distributed algorithm, and prove that agents can collaboratively achieve the optimal allocation with probability one by virtue of ordinary differential equation (ODE) method for SA. Finally, simulations related to the demand response management in power systems verify the effectiveness of the proposed algorithm.

Keywords: Distributed optimization, Resource allocation, Stochastic approximation, Random graph, Demand response

1. Introduction

Resource allocation (RA) problem is to allocate the network resource among a group of agents while optimizing certain performance index. It has drawn much research attention in many areas, such as the media access control in communication networks [1], signal processing in [2], and load demand management in [3]. Hence, various RA models and RA algorithms have been proposed (see [1]-[6] and the references therein). However, most of existing algorithms need a center to collect the data over networks or to coordinate computation processes among all agents.

In fact, the center-free distributed optimization algorithms have attracted more and more research attention in recent years [7]-[13]. In various network optimization problems, the optimal decisions are made based on the whole network data, which, however, are collected and stored by each individual agent of the network. The distributed optimization algorithm keeps the data distributed through the network when seeking the optimal decision, and hence eliminates the "one-to-all" communication burden and protects agents' privacy. Distributed optimization also endows each individual agent with autonomy and reactivity by allowing it to formulate its local objective function and constraints with its local data. From the network viewpoint, the robustness to single point failure and the network scalability can be enhanced with distributed design. Following the

seminal work [5] of RA in large-scale networks along with the distributed optimization work in [7]-[13], various center-free distributed algorithms for RA have been proposed recently in [14]-[17].

Stochastic approximation (SA) has been adopted in distributed optimization algorithms to address various kinds of uncertainties or to improve the computation efficiency. In [8], an SA-based distributed algorithm was proposed when each agent can only get the noisy observations of its local gradient, which extended the traditional SA optimization methods (see [18]) to distributed settings. In [19], an SA algorithm was given for distributed root seeking problem under noisy observations, which was also a generalization of distributed optimization problems. In practice, noisy gradient observations also exist in the zero-order distributed optimization algorithm as in [20], and randomized data sample was considered to reduce the computational complexity in optimization with "big data", resorting to SA for theoretical analysis (see [21]). Besides, SA algorithms were also adopted for distributed optimization to handle uncertainties in communication systems in [9, 10], and [22].

Nevertheless, the existing distributed works of RA in [14]-[17] have not considered various stochastic uncertainties related to information sharing or data observations. Since the problem data is distributed throughout the network, each agent needs to share its local information with other agents through a communication network, which may involve various of uncertainties. Firstly, the communication network may switch due to packet loss, media access control, or energy constraint. To describe uncertainties of communication topologies, different from the deterministic switching graphs in [7, 12] and [13], we adopt random graph models like [9, 10, 23] and [24] here. Secondly,

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the information shared through the network may not be accurate or may be corrupted by random noises due to quantization errors or channel fading (referring to [13], [22] and [24]). On the other hand, noises can also be actively added to the shared information for privacy protection as discussed in [25]. Moreover, agents may not get the exact local gradient or resource information due to measurement or observation noises.

Main contributions of the paper are summarized as follows. (i) A novel center-free distributed algorithm is proposed to handle the RA problem, where each agent only utilizes noisy observations of its local gradient and resource information, and noisy neighboring information shared through the randomly switching networks. (ii) The estimates are shown to converge to the optimal allocation with probability one based on the ODE method for SA algorithm. (iii) The proposed model and algorithm are applied to distributed multi-periods demand response management in power systems, along with simulations to show the effectiveness.

The remainder of the paper is organized as follows. The RA problem is formulated and an SA-based distributed algorithm is proposed in Section 2. Then the convergence result for the distributed algorithm is established in Section 3, while simulation studies are shown in Section 4. Finally, the concluding remarks are given in Section 5.

2. Problem Formulation and Proposed Algorithm

Firstly, we show related notations and preliminaries about convex analysis. Denote $\mathbf{1}_m = (1, \dots, 1)^T \in \mathbf{R}^m$ and $\mathbf{0}_m = (0, \dots, 0)^T \in \mathbf{R}^m$. $\text{col}\{x_1, \dots, x_n\} = (x_1^T, \dots, x_n^T)^T$ stacks the vectors x_1, \dots, x_n . I_n denotes the identity matrix in $\mathbf{R}^{n \times n}$. For a matrix $A = [a_{ij}]$, a_{ij} or A_{ij} stands for the matrix entry in the i th row and j th column of A . \otimes denotes the Kronecker product. Denote $\ker\{A\}$ and $\text{range}\{A\}$ as the null space and range space of matrix A , respectively.

For a nonempty closed convex set $\Omega \subset \mathbf{R}^m$ and a point $x \in \mathbf{R}^m$, denote $P_\Omega(x)$ as the point in Ω that is closest to x , and call it the projection of x on Ω . $P_\Omega(x)$ contains only one element for any $x \in \mathbf{R}^m$, and satisfies

$$\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\| \quad \forall x, y \in \mathbf{R}^m. \quad (1)$$

For a convex set $\Omega \subset \mathbf{R}^m$ and a point $x \in \Omega$, define the normal cone to Ω at x as $N_\Omega(x) \triangleq \{v \in \mathbf{R}^m : \langle v, y - x \rangle \leq 0 \quad \forall y \in \Omega\}$.

In the following two subsections, we formulate the distributed RA problem with the data observation and communication network models, and propose an SA-based distributed algorithm.

2.1. Problem Formulation

Consider a group of agents $\mathcal{N} = \{1, \dots, n\}$ that cooperatively decide the optimal network resource allocation (RA), formulated as follows:

$$\begin{aligned} \min_{x_i \in \mathbf{R}^m, i \in \mathcal{N}} \quad & \sum_{i \in \mathcal{N}} f_i(x_i), \\ \text{subject to} \quad & \sum_{i \in \mathcal{N}} x_i = \sum_{i \in \mathcal{N}} d_i, \quad x_i \in \Omega_i, i \in \mathcal{N} \end{aligned} \quad (2)$$

The local allocation variable $x_i \in \mathbf{R}^m$ is decided by agent i , which is also associated with a local objective function $f_i(x_i)$. d_i is the local resource data, and can only be observed by agent i . The resource of the whole network is the sum of all local resources, i.e., $\sum_{i \in \mathcal{N}} d_i$. Ω_i is the local allocation feasibility constraint of agent i , and cannot be known by other agents. Furthermore, Ω_i is determined by p_i inequality constraints: $\Omega_i = \{x \in \mathbf{R}^m : q_{ij}(x) \leq 0, \forall j = 1, \dots, p_i\}$, where $q_{ij}(\cdot)$, $j = 1, \dots, p_i$ are continuously differentiable convex functions on \mathbf{R}^m . Therefore, RA problem (2) is to find an allocation that minimizes the sum of local objective functions while satisfying the network resource constraint and the allocation feasibility constraints. The following assumptions can also be found in [1]-[6].

Assumption 1. *Problem (2) has a finite optimal solution. For any $i \in \mathcal{N}$, $f_i(x_i)$ is differentiable strictly convex function, and moreover, its gradient is globally Lipschitz continuous, i.e., there exists a constant $l_c > 0$ such that $\|\nabla f_i(x) - \nabla f_i(y)\| \leq l_c \|x - y\|, \forall x, y \in \mathbf{R}^m$.*

The following constraint qualification assumption can be found in [27].

Assumption 2. *For any $i \in \mathcal{N}$, the set Ω_i is closed convex set and has nonempty interior points, and $\{\nabla q_{ij}(x), j \in \mathcal{I}_i(x)\}$ is linearly independent, where $\mathcal{I}_i(x) = \{j : q_{ij}(x) = 0\}$.*

The data observation model for agent i at time k is given as follows: agent i can get the noisy observation of its gradient $\nabla f_i(x_i)$ at given testing point $x_i(k)$ corrupted with noise $v_i(k)$ (that is, $\nabla f_i(x_i(k)) + v_i(k)$) and the noisy local resource information corrupted with noise $\delta_i(k)$ (that is, $d_i + \delta_i(k)$). The stochastic gradient model should be taken into consideration in the following three cases:

(i) Stochastic optimization: Agent i 's local objective function takes the expectation form as $f_i(x_i) = E_{\phi_i}[g(x_i, \phi_i)] = \int_{\Phi_i} g(x_i, \phi_i) d\mathbb{P}(\phi_i)$, where ϕ_i is a random vector supported on set $\Phi_i \in \mathbf{R}^d$ with probability distribution \mathbb{P} , and $g_i : \mathbf{R}^m \times \Phi_i \rightarrow \mathbf{R}$. It is more practical to utilize noisy gradient $\nabla g_i(x_i, \phi_i)$ given sampling ϕ_i rather than exact gradient by performing multi-value integral at each iteration. In fact, the SA algorithm in [18] and DSA algorithm in [8] considered this kind of gradient noise.

(ii) Zero-order optimization: When agent i can only get the value of $f_i(x_i)$ given the testing point $x_i(k)$, the gradient estimation methods, such as the Kiefer-Wolfowitz method in [26] and the randomized coordinate estimation in [20], can lead to noisy gradient observations.

(iii) Randomized data sample: If the local objective functions are constructed with "big data", a noisy gradient based on randomly sampled data is an alternative to the exact gradient, which may reduce the overall iteration computational complexity (see [21]).

Given the local data observations, it is important and practical to solve (2) in a distributed way, where the agents need to share the local information with neighbors through switching networks and noisy channels.

As we know, switching communication networks can be modeled by random graphs, e.g., [9], [10]. Denote a realization of the random graph at time k as $\mathcal{G}(k) = (\mathcal{N}, \mathcal{E}(k))$, where $\mathcal{E}(k) \subset \mathcal{N} \times \mathcal{N}$ is the edge set at time k . If agent i can get information from agent j at time k , then $(j, i) \in \mathcal{E}(k)$ and agent j belongs to agent i 's neighbor set $\mathcal{N}_i(k) = \{j | (j, i) \in \mathcal{E}(k)\}$ at time k . Define adjacency matrix $A(k) = [a_{ij}(k)]$ of $\mathcal{G}(k)$ with $a_{ij}(k) = 1$ if $j \in \mathcal{N}_i(k)$, and $a_{ij}(k) = 0$ otherwise. Denote by $Deg(k) = \text{diag}\{\sum_{j=1}^n a_{1j}(k), \dots, \sum_{j=1}^n a_{nj}(k)\}$ the degree matrix, and by $L(k) = Deg(k) - A(k)$ the Laplacian matrix of $\mathcal{G}(k)$.

The following assumption is given for the random graphs $\{\mathcal{G}(k)\}_{k \geq 1}$ (referring to [9]).

Assumption 3. $\{L(k)\}$ is an i.i.d. sequence with mean denoted by $\bar{L} = E[L(k)]$. Besides, \bar{L} is symmetric with $s_2(\bar{L}) > 0$, where $s_2(\bar{L})$ denotes the secondly smallest eigenvalue of \bar{L} .

Remark 2.1. Note that Assumption 3 does not require the communication graph to be connected or undirected at any time instance. Only the mean graph is required to be undirected and connected, which ensures that the local information can reach any other agents in the average sense. The gossip model in [23] and the broadcast model in [10] are also consistent with Assumption 3.

2.2. SA-based Distributed Algorithm

It is time to propose an SA-based distributed algorithm, based on assumptions on data observations and communication noises.

Denote $x_i(k)$ as agent i 's estimate for its local optimal allocation at time k , and denote $\lambda_i(k)$, $z_i(k)$ as the auxiliary variables of agent i . The agents share their auxiliary variables through the communication network at each iteration. If $(j, i) \in \mathcal{E}(k)$, then agent i can get the noisy information of $\{\lambda_j(k), z_j(k)\}$, corrupted with noise $\zeta_{ij}(k)$ and $\epsilon_{ij}(k)$, from agent j . Namely, $\lambda_j(k) + \zeta_{ij}(k)$ and $z_j(k) + \epsilon_{ij}(k)$ are the values received by agent i from agent j at time k , which are not separable. Moreover, agent i also has the local noisy gradient observation $\nabla f_i(x) + v_i(k)$ and noisy resource observation $d_i + \delta_i(k)$.

The SA-based distributed recursive algorithm for agent i is given as follows:

SA – based Distributed Resource Allocation Algorithm

$$\begin{aligned} x_i(k+1) &= P_{\Omega_i}(x_i(k) + \alpha_k(-(\nabla f_i(x_i(k)) + v_i(k)) + \lambda_i(k))), \\ \lambda_i(k+1) &= \lambda_i(k) + \alpha_k((d_i + \delta_i(k)) - x_i(k) \\ &\quad - \sum_{j=1}^n a_{ij}(k)(\lambda_j(k) - (\lambda_j(k) + \zeta_{ij}(k))) \\ &\quad - \sum_{j=1}^n a_{ij}(k)(z_j(k) - (z_j(k) + \epsilon_{ij}(k))), \\ z_i(k+1) &= z_i(k) + \alpha_k \sum_{j=1}^n a_{ij}(k)(\lambda_j(k) - (\lambda_j(k) + \zeta_{ij}(k))), \end{aligned} \quad (3)$$

where the step-size $\{\alpha_k\}$ satisfies

$$\alpha_k > 0, \quad \sum_{k=1}^{\infty} \alpha_k = \infty, \quad \sum_{k=1}^{\infty} \alpha_k^2 < \infty. \quad (4)$$

Obviously, the algorithm (3) is a **fully distributed** one since each agent only uses its local noisy observations and the noisy information received from its neighbors, and only performs local projection with its local set Ω_i .

Since the local objective functions $f_i(x_i)$ is convex and continuously differentiable, the KKT condition of (2) is

$$\begin{aligned} \mathbf{0}_m &\in \nabla f_i(x_i^*) - \lambda^* + N_{\Omega_i}(x_i^*), \quad i = 1, \dots, n \\ \sum_{i \in \mathcal{N}} x_i^* &= \sum_{i \in \mathcal{N}} d_i \quad x_i^* \in \Omega_i, \end{aligned} \quad (5)$$

Algorithm (3) is developed by combining the ODE methods for KKT condition (5) and the ODE methods for stochastic approximation. In some sense, λ_i in (3) is the local ‘‘copy’’ of Lagrangian multiplier for λ^* in (5), and z_i in (3) is given for the consensus of λ_i to reach the same λ^* .

The communication noises $\epsilon_{ij}(k)$, $\zeta_{ij}(k)$ can be used to model information sharing uncertainties due to quantization errors (see [13]) or communication channel fading (see [22] and [24]). Additionally, noises can be actively added to achieve differential privacy protection as done in [25].

Define the σ -algebra at time k as:

$$\begin{aligned} \mathcal{F}_k &= \sigma\{\epsilon_{ij}(t), \zeta_{ij}(t), \delta_i(t), v_i(t), L(t), 0 \leq t \leq k, \\ &\quad i, j = 1, \dots, N, X(0), \Lambda(0), Z(0)\}. \end{aligned} \quad (6)$$

Define $\mathcal{F}'_k = \sigma\{\mathcal{F}_k, L(k+1)\}$. The following assumptions imposed on $\epsilon_{ij}(k)$, $\zeta_{ij}(k)$, $\delta_i(k)$, $v_i(k)$ were also adopted in the existing SA and distributed optimization works (see [8][9][10][24]).

Assumption 4. For any $i \in \mathcal{N}$, $\{\delta_i(k)\}$ is an i.i.d. sequence with zero mean and bounded second moments $\sigma_{i,\delta}^2 = E[\|\delta_i(k)\|^2]$.

Assumption 5. (i) The communication noises have conditional zero mean, i.e., $E[\zeta_{ij}(k)|\mathcal{F}'_{k-1}] = \mathbf{0}$ and $E[\epsilon_{ij}(k)|\mathcal{F}'_{k-1}] = \mathbf{0}$.

(ii) There is a uniform bound on conditional variances of the communication noise, i.e., there exists a constant $\mu > 0$ such that for any $i, j \in \mathcal{N}$ and any $k \geq 0$, $E[\|\zeta_{ij}(k)\|^2|\mathcal{F}'_{k-1}] \leq \mu^2$ and $E[\|\epsilon_{ij}(k)\|^2|\mathcal{F}'_{k-1}] \leq \mu^2$.

(iii) There exists a positive constant c such that for any $i \in \mathcal{N}$ and any $k \geq 0$,

$$E[v_i(k)|\mathcal{F}_{k-1}] = 0, \quad E[\|v_i(k)\|^2|\mathcal{F}_{k-1}] \leq c(1 + \|x_i(k)\|^2).$$

(iv) For all $i \in \mathcal{N}$, the sequences $\{L(k)\}$ and $\{\delta_i(k)\}$ are mutually independent. The sequences $\{L(k)\}$ and $\{\delta_i(k)\}_{i \in \mathcal{N}}$ are independent of \mathcal{F}_{k-1} .

3. Convergence Analysis

In this section, we employ the ODE method for SA algorithm to give the convergence analysis for algorithm (3). It is shown with the following outline. Theorem 3.1 shows that the equilibrium point of the underlying ODE contains the optimal solution to problem (2), while Lemma 3.2 shows the convergence of the underlying ODE. Then Lemma 3.3 investigates properties of the extended noise sequences, and Lemma 3.4 shows that the iteration sequence generated by (3) are bounded. Finally, Theorem 3.5 shows that the estimates generated by (3) converge to the optimal resource allocation with probability one.

Set $\zeta_i(k) = \sum_{j=1}^n a_{ij}(k)\zeta_{ij}(k)$ and $\epsilon_i(k) = \sum_{j=1}^n a_{ij}(k)\epsilon_{ij}(k)$, and

$$\begin{aligned} X(k) &= \text{col}\{x_1(k), \dots, x_n(k)\}, & \Lambda(k) &= \text{col}\{\lambda_1(k), \dots, \lambda_n(k)\}, \\ Z(k) &= \text{col}\{z_1(k), \dots, z_n(k)\}, & \delta(k) &= \text{col}\{\delta_1(k), \dots, \delta_n(k)\}, \\ v(k) &= \text{col}\{v_1(k), \dots, v_n(k)\}, & D &= \text{col}\{d_1, \dots, d_n\}, \\ \zeta(k) &= \text{col}\{\zeta_1(k), \dots, \zeta_n(k)\}, & \epsilon(k) &= \text{col}\{\epsilon_1(k), \dots, \epsilon_n(k)\}, \\ \nabla f(X(k)) &= \text{col}\{\nabla f_1(x_1(k)), \dots, \nabla f_n(x_n(k))\}. \end{aligned}$$

Then the recursive algorithm (3) can be rewritten in the compact form as follows:

$$\begin{aligned} X(k+1) &= P_\Omega(X(k) + \alpha_k(-\nabla f(X(k)) + \Lambda(k) - v(k))), \\ \Lambda(k+1) &= \Lambda(k) + \alpha_k(-(L(k) \otimes I_m)(\Lambda(k) + Z(k)) \\ &\quad + D - X(k) + \delta(k) + \zeta(k) + \epsilon(k)), \\ Z(k+1) &= Z(k) + \alpha_k((L(k) \otimes I_m)\Lambda(k) - \zeta(k)), \end{aligned} \quad (7)$$

where $\Omega = \prod_{i=1}^n \Omega_i$ denotes the Cartesian product of Ω_i .

Denote by $e_1(k) = ((\bar{L} - L(k)) \otimes I_m)(\Lambda_k + Z_k)$, $e_2(k) = \zeta(k) + \delta(k) + \epsilon(k)$, and by $e_3(k) = ((L(k) - \bar{L}) \otimes I_m)\Lambda(k) - \zeta(k)$. We then have

$$\begin{aligned} X(k+1) &= P_\Omega(X(k) + \alpha_k(-\nabla f(X(k)) + \Lambda(k) - v(k))), \\ \Lambda(k+1) &= \Lambda(k) + \alpha_k(-(\bar{L} \otimes I_m)(\Lambda(k) + Z(k)) \\ &\quad + D - X(k) + e_1(k) + e_2(k)), \\ Z(k+1) &= Z(k) + \alpha_k((\bar{L} \otimes I_m)\Lambda(k) + e_3(k)). \end{aligned} \quad (8)$$

By setting $S(k) = \text{col}\{X(k), \Lambda(k), Z(k)\}$, we can regard the algorithm (7) as an SA algorithm with the following form:

$$S(k+1) = P_\Phi(S(k) + \alpha_k(J(S(k)) + \xi(k))), \quad (9)$$

where

$$\begin{aligned} J(S) &= \begin{pmatrix} -\nabla f(X) + \Lambda \\ -(\bar{L} \otimes I_m)(\Lambda + Z) + D - X \\ (\bar{L} \otimes I_m)\Lambda \end{pmatrix}, \\ \xi(k) &= \begin{pmatrix} -v(k) \\ e_1(k) + e_2(k) \\ e_3(k) \end{pmatrix}, \quad \Phi = \Omega \times \mathbf{R}^{mn} \times \mathbf{R}^{mn}. \end{aligned} \quad (10)$$

The convergence proof of (3) relies on the ODE method for SA (referring to [27] and [28]). Define the following continuous-time projected dynamics as the underlying ODE of (3)

$$\dot{S} = J(S) + z, S(0) = \text{col}\{X(0), \Lambda(0), Z(0)\}, \quad (11)$$

with $z \in -N_\Phi(S)$ being the minimum force to keep the solution of (11) in Φ , and $J(S)$ is defined by (10).

Theorem 3.1. *Under Assumptions 1, 2, and 3, (11) has at least one equilibrium point. Furthermore, suppose $S^* = \text{col}\{X^*, \Lambda^*, Z^*\}$ is an equilibrium point of (11), then S^* has X^* as the optimal solution to problem (2).*

Proof: Because problem (2) is assumed to be solvable, there exist optimal solution X^* and $\lambda^* \in \mathbf{R}^m$ such that (5) can be satisfied. Then take $\Lambda = \mathbf{1}_n \otimes \lambda^*$, $\bar{L} \otimes I_m \Lambda^* = \mathbf{0}$. By $(\mathbf{1}_n^T \otimes I_m)X^* = (\mathbf{1}_n^T \otimes I_m)D$ (that is $\sum_{i \in \mathcal{N}} x_i^* = \sum_{i \in \mathcal{N}} d_i$), we have $D - X \in \ker\{\mathbf{1}_n^T \otimes I_m\}$. Notice that $\ker\{\mathbf{1}_n^T \otimes I_m\}$ and $\text{range}\{\mathbf{1}_n \otimes I_m\}$ form an orthogonal decomposition of \mathbf{R}^{mn} by

the fundamental theorem of linear algebra. Combined with $\ker(\bar{L} \otimes I_m) = \text{range}\{\mathbf{1}_n \otimes I_m\}$ due to Assumption 3, we have $D - X \in \ker(\bar{L} \otimes I_m)^\perp$. Therefore, $D - X \in \text{range}(\bar{L} \otimes I_m)$, that is there exists Z^* such that $-\bar{L} \otimes I_m Z^* + D - X^* = \mathbf{0}$. Hence, combined with (5), $S^* = \text{col}\{X^*, \Lambda^*, Z^*\}$ is an equilibrium point of (11).

On the other hand, when $S^* = \text{col}\{X^*, \Lambda^*, Z^*\}$ is an equilibrium point of (11), it satisfies:

$$\begin{aligned} -\nabla f_i(x_i^*) + \lambda_i^* &\in N_{\Omega_i}(x_i^*), x_i^* \in \Omega_i \\ (\bar{L} \otimes I_m)(\Lambda^* + Z^*) - (D - X^*) &= \mathbf{0} \\ (\bar{L} \otimes I_m)\Lambda^* &= \mathbf{0} \end{aligned} \quad (12)$$

Since \bar{L} is the weighted Laplacian of an undirected connected graph by Assumption 3, it follows from $(\bar{L} \otimes I_m)\Lambda^* = \mathbf{0}_{mn}$ that $\Lambda^* = \mathbf{1}_n \otimes \lambda^*$ for some $\lambda^* \in \mathbf{R}^m$. As a result, $\mathbf{0}_m \in \nabla f_i(x_i^*) - \lambda^* + N_{\Omega_i}(x_i^*)$. Furthermore, $(\bar{L} \otimes I_m)\Lambda^* + (\bar{L} \otimes I_m)Z^* - (D - X^*) = \mathbf{0}_{mn}$ implies that $(\bar{L} \otimes I_m)Z^* = D - X^*$. Then by noticing $\mathbf{1}_n^T \bar{L} = \mathbf{0}_n^T$ we derive $\sum_{i \in \mathcal{N}} d_i = \sum_{i \in \mathcal{N}} x_i^*$. Moreover, $x_i^* \in \Omega_i$ due to the viability of ODE (11).

Thus, any equilibrium point S^* of (11) satisfies the KKT condition (5), and hence, X^* is the optimal solution to problem (2). ■

Lemma 3.2 shows that (11) converges to its equilibrium point S^* .

Lemma 3.2. *Under Assumptions 1, 2 and 3, the trajectories of (11) are bounded and converge to its equilibrium point for any finite initial points.*

Proof: Take a Lyapunov function $V(S) = \frac{1}{2}\|S - S^*\|$, where S^* is an equilibrium point of (11). Take $n_\Omega(X^*) \in N_\Omega(X^*)$ such that $\nabla f(X^*) - \Lambda^* + n_\Omega(X^*) = \mathbf{0}$, then

$$\begin{aligned} \frac{dV}{dt} &= (S - S^*)^T (J(S) + z) \\ &\leq (X - X^*)^T (-\nabla f(X) + \Lambda + \nabla f(X^*) - \Lambda^* + n_\Omega(X^*)) \\ &\quad + (\Lambda - \Lambda^*)^T (-\bar{L} \otimes I_m(\Lambda + Z) + (D - X) \\ &\quad + \bar{L} \otimes I_m(\Lambda^* + Z^*) - (D - X^*)) + (Z - Z^*)^T \bar{L} \otimes I_m(\Lambda - \Lambda^*) \\ &\leq -(X - X^*)^T (\nabla f(X) - \nabla f(X^*)) + (X - X^*)^T n_\Omega(X^*) \\ &\quad - (\Lambda - \Lambda^*)^T \bar{L}(\Lambda - \Lambda^*) \leq 0 \end{aligned} \quad (13)$$

Hence, any equilibrium point of (11) is Lyapunov stable, and given finite initial point $S(0)$, the trajectories of (11) are bounded and belong to the compact forward invariant set $I_S = \{S | V(S) \leq V(0)\}$.

Denote E as the set within I_S such that $\dot{V} = 0$. Then we can show that the maximal invariance set in E can only be $\{S | \dot{S} = 0\}$. With the strict convexity of f_i , $X = X^*$ must hold within set E . Furthermore, $\Lambda - \Lambda^* \in \ker\{\bar{L} \otimes I_m\}$ by (13) and Assumption 3. Therefore, $\dot{Z} = \mathbf{0}_n$ and $Z = Z^*$ within set E . Moreover, $\dot{\Lambda} = -L \otimes I_n Z^* + D - X^*$, and $\dot{\Lambda}$ must be $\mathbf{0}$; otherwise Λ will go to infinity, which contradicts the boundedness of the trajectories. Hence, $\Lambda^* = \mathbf{1}_n \otimes \lambda^*$. Therefore, all the trajectories of (3) converge to the points in the maximal invariance set $\{S | \dot{S} = 0\}$. Recalling the Lyapunov stability of S^* and the LaSalle invariance principle, the dynamics (11) converges to its equilibrium point S^* , which leads to the conclusion. ■

3.1. Extended noise property

By definition of \mathcal{F}_k given in (6), $S(k)$ is adapted to \mathcal{F}_{k-1} according to (3). The extended noise sequence $\{\xi(k)\}$ is state-dependent, and its properties are shown in Lemma 3.3.

Lemma 3.3. *Suppose Assumptions 3, 4 and 5 hold. Then*

$$E[\xi(k)|\mathcal{F}_{k-1}] = 0, \quad E[\|\xi(k)\|^2|\mathcal{F}_{k-1}] \leq c_1\|S(k)\|^2 + c_2 \quad a.s. \quad (14)$$

for some finite constants c_1, c_2 .

Proof: By Assumption 5 (iii),

$$\begin{aligned} E[v(k)|\mathcal{F}_{k-1}] &= 0, \\ E[\|v(k)\|^2|\mathcal{F}_{k-1}] &= \sum_{i=1}^n E[\|v_i(k)\|^2|\mathcal{F}_{k-1}] \\ &\leq nc + c\|X(k)\|^2. \end{aligned} \quad (15)$$

Since $a_{ij}(k)$ is adapted to \mathcal{F}'_{k-1} , by Assumption 5 (i) we obtain

$$\begin{aligned} E[\zeta_i(k)|\mathcal{F}'_{k-1}] &= \sum_{j=1}^n E[a_{ij}(k)\zeta_{ij}(k)|\mathcal{F}'_{k-1}] \\ &= a_{ij}(k) \sum_{j=1}^n E[\zeta_{ij}(k)|\mathcal{F}'_{k-1}] = 0, \end{aligned}$$

and hence $E[\zeta(k)|\mathcal{F}'_{k-1}] = 0$. By noting that $\mathcal{F}_k \subset \mathcal{F}'_k$ we derive

$$E[\zeta(k)|\mathcal{F}_{k-1}] = E[E[\zeta(k)|\mathcal{F}'_{k-1}]|\mathcal{F}_{k-1}] = 0.$$

Similarly, it follows that $E[\epsilon(k)|\mathcal{F}'_{k-1}] = 0$ and $E[\epsilon(k)|\mathcal{F}_{k-1}] = 0$.

Since $L(k)$ is independent of $\delta_i(k)$ and \mathcal{F}_{k-1} by Assumption 5 (iv), we obtain $E[\delta_i(k)|\mathcal{F}'_{k-1}] = E[\delta_i(k)|\mathcal{F}_{k-1}, L(k)] = E[\delta_i(k)|\mathcal{F}_{k-1}]$. Then by Assumptions 4 and 4 (iv), we have that, for any $i \in \mathcal{N}$ $E[\delta_i(k)|\mathcal{F}'_{k-1}] = E[\delta_i(k)|\mathcal{F}_{k-1}] = E[\delta_i(k)] = 0$. Thus,

$$E[e_2(k)|\mathcal{F}'_{k-1}] = E[\zeta(k)|\mathcal{F}'_{k-1}] + E[\delta(k)|\mathcal{F}'_{k-1}] + E[\epsilon(k)|\mathcal{F}'_{k-1}] = 0, \quad (16)$$

which implies that $E[e_2(k)|\mathcal{F}_{k-1}] = E[E[e_2(k)|\mathcal{F}'_{k-1}]|\mathcal{F}_{k-1}] = 0$.

Note that $\Lambda(k)$ and $Z(k)$ are adapted to \mathcal{F}_{k-1} , while $L(k)$ is independent of \mathcal{F}_{k-1} . Then, by Assumption 3

$$\begin{aligned} E[e_1(k)|\mathcal{F}_{k-1}] &= E[\{(\bar{L} - L(k)) \otimes I_m\}(\Lambda(k) + Z(k))|\mathcal{F}_{k-1}] \\ &= E[\{(\bar{L} - L(k)) \otimes I_m\}|\mathcal{F}_{k-1}](\Lambda(k) + Z(k)) \\ &= E[\{(\bar{L} - L(k)) \otimes I_m\}](\Lambda(k) + Z(k)) = 0, \\ E[e_3(k)|\mathcal{F}_{k-1}] &= E[\{(\bar{L} - L(k)) \otimes I_m\}\Lambda(k)|\mathcal{F}_{k-1}] \\ &= E[\{(\bar{L} - L(k)) \otimes I_m\}]\Lambda(k) = 0. \end{aligned} \quad (17)$$

Consequently, we conclude that $E[\xi(k)|\mathcal{F}_{k-1}] = 0$.

Since $e_1(k)$ is adapted to \mathcal{F}'_{k-1} and $\mathcal{F}_k \subset \mathcal{F}'_k$, it follows from (16) that

$$\begin{aligned} E[e_1(k)^T e_2(k)|\mathcal{F}_{k-1}] &= E[E[e_1(k)^T e_2(k)|\mathcal{F}'_{k-1}]|\mathcal{F}_{k-1}] \\ &= E[e_1(k)^T E[e_2(k)|\mathcal{F}'_{k-1}]|\mathcal{F}_{k-1}] = 0. \end{aligned} \quad (18)$$

Since $\Lambda(k)$ and $L(k)$ are adapted to \mathcal{F}'_{k-1} , by $E[\zeta(k)|\mathcal{F}'_{k-1}] = 0$ and $\mathcal{F}_k \subset \mathcal{F}'_k$, we get

$$\begin{aligned} E[\{(L(k) - \bar{L}) \otimes I_m\}\Lambda(k)^T \zeta(k)|\mathcal{F}_{k-1}] \\ &= E[E[\{(L(k) - \bar{L}) \otimes I_m\}\Lambda(k)^T \zeta(k)|\mathcal{F}'_{k-1}]|\mathcal{F}_{k-1}] \\ &= E[\{(L(k) - \bar{L}) \otimes I_m\}\Lambda(k)^T E[\zeta(k)|\mathcal{F}'_{k-1}]|\mathcal{F}_{k-1}] = 0. \end{aligned} \quad (19)$$

By the conditional Hölder inequality

$$E[\|X^T Y\|] \leq (E[\|X\|^2|\mathcal{F}])^{\frac{1}{2}} (E[\|Y\|^2|\mathcal{F}])^{\frac{1}{2}}, \quad (20)$$

from Assumption 5 (ii) we see that

$$\begin{aligned} E[\zeta_{ij}(k)^T \zeta_{ip}(k)|\mathcal{F}'_{k-1}] &\leq E[\|\zeta_{ij}(k)\|^T \zeta_{ip}(k)|\mathcal{F}'_{k-1}] \\ &\leq (E[\|\zeta_{ij}(k)\|^2|\mathcal{F}'_{k-1}])^{\frac{1}{2}} (E[\|\zeta_{ip}(k)\|^2|\mathcal{F}'_{k-1}])^{\frac{1}{2}} \leq \mu^2. \end{aligned}$$

Then, since $A(k)$ is adapted to \mathcal{F}'_{k-1} , we have

$$\begin{aligned} E[\|\zeta_i(k)\|^2|\mathcal{F}'_{k-1}] &= E[\sum_{j,p=1}^n a_{ij}(k)a_{ip}(k)\zeta_{ij}(k)^T \zeta_{ip}(k)|\mathcal{F}'_{k-1}] \\ &= \sum_{j,p=1}^n a_{ij}(k)a_{ip}(k)E[\zeta_{ij}(k)^T \zeta_{ip}(k)|\mathcal{F}'_{k-1}] \\ &\leq \sum_{j,p=1}^n \mu^2 = n^2 \mu^2. \end{aligned}$$

Similarly, $E[\|\epsilon_i(k)\|^2|\mathcal{F}'_{k-1}] \leq n^2 \mu^2 \forall i \in \mathcal{N}$. From Assumption 5 (iv), it is clear that $\delta_i(k)$ is independent of \mathcal{F}'_{k-1} , and hence, by Assumption 4 $E[\|\delta(k)\|^2|\mathcal{F}'_{k-1}] = E[\|\delta(k)\|^2] = \sum_{i=1}^n E[\|\delta_i(k)\|^2]$. In summary,

$$\begin{aligned} E[\|\zeta(k)\|^2|\mathcal{F}'_{k-1}] &\leq n^3 \mu^2, \quad E[\|\epsilon(k)\|^2|\mathcal{F}'_{k-1}] \leq n^3 \mu^2, \\ E[\|\delta(k)\|^2|\mathcal{F}'_{k-1}] &= \sum_{i=1}^n \sigma_{i,\delta} \triangleq \sigma_\delta. \end{aligned} \quad (21)$$

Then

$$\begin{aligned} E[\|e_2(k)\|^2|\mathcal{F}'_{k-1}] \\ &\leq 3(E[\|\epsilon(k)\|^2|\mathcal{F}'_{k-1}] + E[\|\zeta(k)\|^2|\mathcal{F}'_{k-1}] + E[\|\delta(k)\|^2|\mathcal{F}'_{k-1}]) \\ &\leq 3(2n^3 \mu^2 + \sigma_\delta^2) \triangleq C_1, \end{aligned}$$

and hence, by $\mathcal{F}_k \subset \mathcal{F}'_k$, we get

$$E[\|e_2(k)\|^2|\mathcal{F}_{k-1}] = E[E[\|e_2(k)\|^2|\mathcal{F}'_{k-1}]|\mathcal{F}_{k-1}] \leq C_1. \quad (22)$$

Because $\Lambda(k)$ and $Z(k)$ are adapted to \mathcal{F}_{k-1} , and $L(k)$ is independent of \mathcal{F}_{k-1} by Assumption 4 (iv), we have

$$\begin{aligned} E[\|e_1(k)\|^2|\mathcal{F}_{k-1}] &= E[\|(\bar{L} - L(k)) \otimes I_m\}(\Lambda(k) + Z(k))\|^2|\mathcal{F}_{k-1}] \\ &\leq C_2 \|\Lambda(k) + Z(k)\|^2, \end{aligned}$$

where $C_2 = E[\|L(k) - \bar{L}\|^2]$ is finite. It, along with (18) (22), yields

$$\begin{aligned} E[\|e_1(k) + e_2(k)\|^2|\mathcal{F}_{k-1}] &= E[\|e_1(k)\|^2|\mathcal{F}_{k-1}] \\ &+ E[\|e_2(k)\|^2|\mathcal{F}_{k-1}] + 2E[e_1(k)^T e_2(k)|\mathcal{F}_{k-1}] \\ &\leq C_2 \|\Lambda(k) + Z(k)\|^2 + C_1. \end{aligned} \quad (23)$$

By (19) (21) and $\mathcal{F}_k \subset \mathcal{F}'_k$, we derive

$$\begin{aligned} E[\|e_3(k)\|^2|\mathcal{F}_{k-1}] &= E[\|((L(k) - \bar{L}) \otimes I_m)\Lambda(k) - \zeta(k)\|^2|\mathcal{F}_{k-1}] \\ &= E[\|((L(k) - \bar{L}) \otimes I_m)\Lambda(k)\|^2|\mathcal{F}_{k-1}] \\ &\quad - 2E[\{((L(k) - \bar{L}) \otimes I_m)\Lambda(k)\}^T \zeta(k)|\mathcal{F}_{k-1}] \\ &\quad + E[\|\zeta(k)\|^2|\mathcal{F}_{k-1}] \leq C_2 \|\Lambda(k)\|^2 + n^3 \mu^2. \end{aligned} \quad (24)$$

In summary, from (15), (23), and (24), we obtain

$$\begin{aligned} E[\|\xi(k)\|^2|\mathcal{F}_{k-1}] &= E[\|v(k)\|^2|\mathcal{F}_{k-1}] + E[\|e_3(k)\|^2|\mathcal{F}_{k-1}] \\ &\quad + E[\|e_1(k) + e_2(k)\|^2|\mathcal{F}_{k-1}] \\ &\leq nc + c\|X(k)\|^2 + C_2 \|\Lambda(k)\|^2 + n^3 \mu^2 \\ &\quad + C_2 \|\Lambda(k) + Z(k)\|^2 + C_1 \\ &\leq c_1 \|S(k)\|^2 + c_2 \end{aligned}$$

for some positive constants c_1, c_2 . ■

3.2. Stability

The following result is about the boundedness of the iterations before showing its convergence.

Lemma 3.4. *Under Assumptions 1-4, $\{S(k)\}$ generated by the distributed algorithm (3) is bounded with probability one given any finite initial value $S(0)$.*

Proof: Denote by S^* as an equilibrium point of (11), i.e., $J(S^*) \in N_\Phi(S^*)$. Then, by Assumption 1 and the KKT condition (5), S^* is a finite value. Take $v(S) = \|S - S^*\|^2$ as a Lyapunov function. Then from (9) and the non-expansive property of the projection operator (1) we derive

$$\begin{aligned} v(S(k+1)) &= \|S(k+1) - S^*\|^2 \\ &\leq \|S(k) - S^* + \alpha_k(J(S(k)) + \xi(k))\|^2 \\ &\leq \|S(k) - S^*\|^2 + 2\alpha_k(S(k) - S^*)^T(J(S(k)) + \xi(k)) \\ &\quad + \alpha_k^2(\|J(S(k))\|^2 + 2\xi(k)^T J(S(k)) + \|\xi(k)\|^2). \end{aligned}$$

Since $S(k)$ is adapted to \mathcal{F}_{k-1} , by recalling $E[\xi(k)|\mathcal{F}_{k-1}] = 0$ from Lemma 3.3 we obtain

$$\begin{aligned} E[v(S(k+1))|\mathcal{F}_{k-1}] &\leq v(S(k)) + 2\alpha_k(S(k) - S^*)^T J(S(k)) \\ &\quad + \alpha_k^2(\|J(S(k))\|^2 + E[\|\xi(k)\|^2|\mathcal{F}_{k-1}]). \end{aligned}$$

Similar to the proof of Lemma (3.2), $(S(k) - S^*)^T J(S(k)) \leq 0$. Then by Lemma 3.3, we get

$$E[v(S(k+1))|\mathcal{F}_{k-1}] \leq v(S(k)) + \alpha_k^2(\|J(S(k))\|^2 + c_1\|S(k)\|^2 + c_2). \quad (25)$$

From Assumption 1 and taking $n_\Omega(X^*) \in N_\Omega(X^*)$ such that $\nabla f(X^*) - \Lambda^* + n_\Omega(X^*) = \mathbf{0}$, we have

$$\begin{aligned} \|J(S(k))\|^2 &= \|-\nabla f(X(k)) + \Lambda(k) + \nabla f(X^*) - \Lambda^* + n_\Omega(X^*)\|^2 \\ &\quad + \|(\bar{L} \otimes I_m)(Z(k) - Z^*) + (\Lambda(k) - \Lambda^*)\|^2 \\ &\quad + \|X(k) - X^*\|^2 + \|(\bar{L} \otimes I_m)(\Lambda(k) - \Lambda^*)\|^2 \\ &\leq 3(\|\nabla f(X(k)) - \nabla f(X^*)\|^2 + \|\Lambda(k) - \Lambda^*\|^2 \\ &\quad + \|n_\Omega(X^*)\|^2 + \|(\bar{L} \otimes I_m)(\Lambda(k) - \Lambda^*)\|^2 + \|X(k) - X^*\|^2 \\ &\quad + \|(\bar{L} \otimes I_m)(Z(k) - Z^*)\|^2) + \|(\bar{L} \otimes I_m)(\Lambda(k) - \Lambda^*)\|^2 \\ &\leq (3l_c^2 + 3)\|X(k) - X^*\|^2 + 3c_4\|Z(k) - Z^*\|^2 \\ &\quad + (3 + 4c_4)\|(\Lambda(k) - \Lambda^*)\|^2 + c_n \\ &\leq (3 + 3l_c^2 + 4c_4)\|S(k) - S^*\|^2 + c_n = c_5v(S(k)) + c_n, \end{aligned} \quad (26)$$

where $c_4 = \|\bar{L}\|$ and $c_n = \|\nabla f(X^*) - \Lambda^*\|^2$. Note that

$$\|S(k)\|^2 \leq 2(\|S(k) - S^*\|^2 + \|S^*\|^2) = 2(v(S(k)) + \|S^*\|^2).$$

Incorporated with (25) and (26), it yields

$$\begin{aligned} E[v(S(k+1))|\mathcal{F}_{k-1}] &\leq v(S(k)) \\ &\quad + \alpha_k^2(c_5v(S(k)) + c_n + 2c_1v(S(k)) + 2c_1\|S^*\|^2 + c_2) \\ &\triangleq (1 + c_6\alpha_k^2)v(S(k)) + c_7\alpha_k^2, \end{aligned} \quad (27)$$

where $c_6 = 2c_1 + c_5, c_7 = 2c_1\|S^*\|^2 + c_2 + c_n$.

Since $\{\alpha_k\}$ satisfies (4), with probability one $\lim_{k \rightarrow \infty} v(S(k))$ exists and is finite by Lemma 5.2 in Appendix. Therefore, $\{S(k)\}$ is bounded with probability one. ■

3.3. Convergence

The following result gives the main convergence result for the SA-based distributed algorithm (3).

Theorem 3.5. *Suppose Assumptions (1)-(5) hold. Let sequences $\{x_i(k)\}$, $\{\lambda_i(k)\}$, $\{z_i(k)\}$ be produced by (3) given any finite initial values $x_i(0)$, $\lambda_i(0)$, $z_i(0)$. Then*

$$\lim_{k \rightarrow \infty} x_i(k) = x_i^* \text{ a.s.},$$

where $X^* = \text{col}\{x_1^*, \dots, x_n^*\}$ is the optimal resource allocation to problem (2).

Proof: Note that θ_k , Y_k , $g(\theta)$ and Φ for (30) correspond to $\theta_k = S(k)$, $Y_k = J(S(k)) + \xi(k)$, $g(\theta) = J(S)$ and $\Phi = \Omega \times \mathbf{R}^{mn} \times \mathbf{R}^{mn}$ for (9). Then we can apply Theorem 5.1 in Appendix to prove the conclusion, and it suffices to check conditions C1-C4 given in Appendix.

Since $S(k)$ is adapted to \mathcal{F}_{k-1} , by (25) we drive

$$E[\|Y_k\|^2|\mathcal{F}_{k-1}] \leq \|J(S(k))\|^2 + c_1\|S(k)\|^2 + c_2 \text{ a.s.}$$

Then by Lemma 3.4, (26) and Assumption 1 we conclude that C1 hold. From (10) and Lemma (3.3) it is easily seen that C2 holds. By definition of $J(S)$ given by (10) and Assumption 1 we know that C3 holds. Since $\{S(k)\}$ is bounded with probability one from Lemma 3.4, we then have C4.

As a result, C1-C4 hold. Since $\Phi = \Omega \times \mathbf{R}^{mn} \times \mathbf{R}^{mn}$, with Assumption 2 it is easily seen that Φ satisfies the similar conditions as Ω_i . Then, by Theorem 5.1, $S(k)$ converge with probability one to the invariant set of (11). Thus, by Lemma 3.2, $X(k)$ converges with probability one to the optimal solution X^* . ■

4. Demand Response Management and Simulations

In this section, we apply the RA optimization model (2) and algorithm (3) to distributed multi-period demand response management in power systems (see [3] and [30]).

Suppose that a group of load aggregators (with index $\mathcal{N} = \{1, \dots, n\}$) need to decide the load demand in the following T periods $P_i^d \in \mathbf{R}^T$, in order to meet the generation scheduling $P_i^g \in \mathbf{R}^T$ and minimize the disutilities. P_i^g is usually decided by other decision processes based on the generator unit commitment or real-time generation prediction of renewables, which is fixed and assumed to be only informed or observable to agent i . Aggregator i formulates its local objective function $f_i(P_i^d)$ to consider the costs or disutilities due to demand response P_i^d . Moreover, $P_i^d \in \Omega_i$ specifies the local response constraints, which considers the lower and upper bounds in each period, the total demand in the following T periods, ramping constraints, and other local specifications. Hence, the multi-period demand response management problem is formulated as:

$$\begin{aligned} \min_{P_i^d \in \mathbf{R}^T, i \in \mathcal{N}} & \sum_{i \in \mathcal{N}} f_i(P_i^d) \\ \text{subject to} & \sum_{i \in \mathcal{N}} P_i^d = \sum_{i \in \mathcal{N}} P_i^g, \quad P_i^d \in \Omega_i \end{aligned} \quad (28)$$

In many practical cases, P_i^g can only be observed indirectly through local measurements of wind speed, or solar radiation,

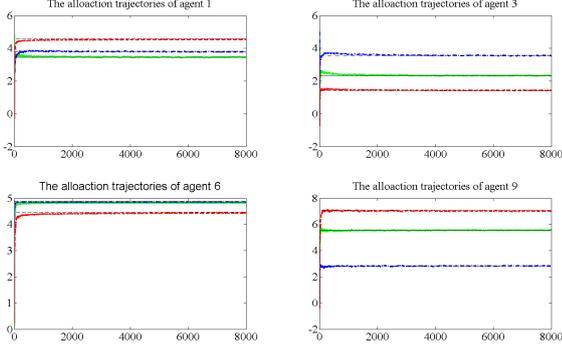


Figure 1: The averaging trajectories of some agents' allocation variables

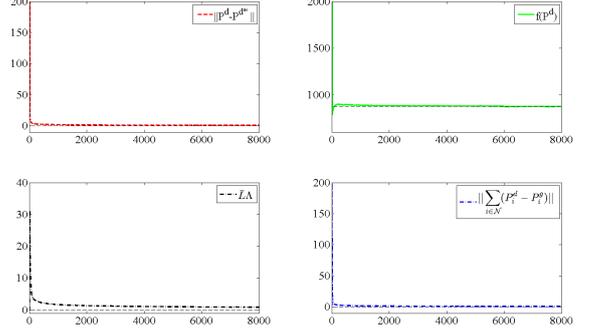


Figure 2: The averaging trajectories of some performance indexes.

or local frequency deviation, and hence, suffers from various observation noises. In addition, $f_i(P_i^d)$ should take full consideration of user's demand requirements, (dis)utility, satisfactory levels, and payoffs, and hence, is influenced by various external factors, such as temperature, electricity price, and renewable generations. Therefore, the gradient observation of $f_i(P_i^d)$ may also be noisy. The aggregators may share information through wireless communication networks with switching topologies and noisy channels. As a result, algorithm (3) can be applied to handle the above challenges for problem (28). Compared with previous works [3] and [30], the proposed model here considers the demand response in multi-periods and local load response feasibility constraints, and the algorithm can handle various observations and communication uncertainties, which may be more practical in many cases.

In what follows, we give a numerical experiment to illustrate the algorithm performance.

Example 4.1. Consider the following three-period demand response management problem:

$$\begin{aligned} \min_{P_i^d \in \mathbf{R}^3, i \in \mathcal{N}} \quad & \sum_{i \in \mathcal{N}} E_{\Psi_i, \theta_i} [P_i^{dT} (Q_i + \Psi_i) P_i^d + (c_i + \theta_i)^T P_i^d] \\ \text{s.t.} \quad & \sum_{i \in \mathcal{N}} P_i^d = \sum_{i \in \mathcal{N}} P_i^g \\ & R_i P_i^d \leq l_i, R_i \in \mathbf{R}^{12 \times 3}, l_i \in \mathbf{R}^{12 \times 1}, i \in \mathcal{N}, \end{aligned} \quad (29)$$

where $R_i P_i^d \leq l_i$ is the compact form of the following local load response feasibility constraints: $[l_i]_{11} \leq \mathbf{1}^T P_i^d \leq [l_i]_{21}$, $[l_i]_{31} \leq [P_i^d]_{11} - [P_i^d]_{21} \leq [l_i]_{41}$, $[l_i]_{51} \leq [P_i^d]_{21} - [P_i^d]_{31} \leq [l_i]_{61}$, $[l_i]_{71} \leq [P_i^d]_{11} \leq [l_i]_{81}$, $[l_i]_{91} \leq [P_i^d]_{21} \leq [l_i]_{10,1}$ and $[l_i]_{11,1} \leq [P_i^d]_{31} \leq [l_i]_{12,1}$.

The basic simulation experiment settings are given as follows. The number of agents is set to be 10. Q_i and c_i are randomly generated symmetric positive definite matrices and random vectors, respectively. Each P_i^g and l_i are also randomly generated vector that can ensure Assumptions 1 and 2.

Consider a graph set \mathcal{G}_s containing 30 graphs, each of which is generated according to the random graph model $G(10, P)$, where P is the probability of occurrence for any possible edge. The probability P is randomly and uniformly drawn from $[0.05, 0.1]$ for each graph in \mathcal{G}_s . Select a graph set \mathcal{G}_s with its union graph being connected. At time k , a graph is randomly drawn from the graph set \mathcal{G}_s according to the uniform distribution.

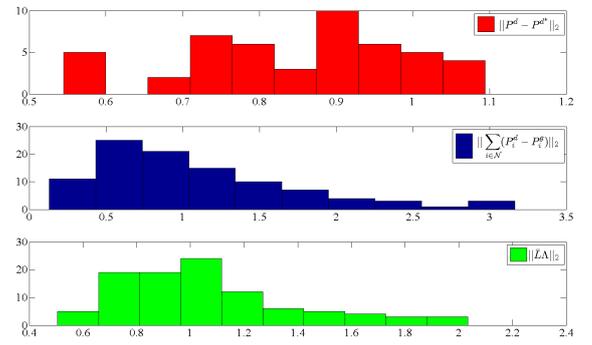


Figure 3: The histogram of some performance indexes at iteration time $k = 8000$.

For $i \in \mathcal{N}$, $[\Psi_i]_{ij}$, $[\theta_i]_j$ are i.i.d. random variables satisfying the Gaussian distribution $N(0, 0.5)$ with zero mean and variance 0.5. Let both the generation observation noise δ_i and communication noise ζ_{ij} , ϵ_{ij} be i.i.d. random vectors satisfying the Gaussian distribution $N(\mathbf{0}, I_3)$ with zero mean vector and covariance matrix I_3 . Hence, Assumptions 4 and 5 are satisfied. The stepsize α_k in (3) is set as $\alpha_k = \frac{1}{(k+1)^{0.6}}$.

Experiment 1: Given a randomly generated graph set \mathcal{G}_s and a randomly generated setting for problem (29), we apply algorithm (3) to generate 200 independent sample paths with iteration length of 8000. Figure 1 shows the averaging trajectories of some agents' allocation variables, and illustrates how the agents find the optimal allocation. Moreover, Figure 2 shows the averaging trajectories of some algorithm performance indexes, including the distance to optimal solution $\|P^d - P^{d*}\|$, function value $f(P^d)$, $\|L\Lambda\|$, and $\|\sum_{i \in \mathcal{N}} (P_i^d - P_i^g)\|$.

Experiment 2: Let us randomly generate a graph set \mathcal{G}_s and a setting for problem (29) at each round of this simulation, and employ algorithm (3) to generate one sample path of this setting with iteration length of 8000. We repeat the procedure for 100 rounds, and use Figure 3 to show the histogram of some performance indexes at iteration time 8000. It illustrates that algorithm (3) can almost surely find the optimal allocation for different problem settings with only one sample path.

5. Conclusions

In this paper, an SA-based distributed algorithm was proposed to solve a class of RA optimization problems under various uncertainties. The gradient and resource observation noises were taken into consideration, and the communication network was assumed with randomly switching topologies and noisy communication channels. The algorithm was proved to converge to the optimal solution with probability one by resorting to the ODE method for SA algorithm, which may demonstrate great potentials of SA algorithm and ODE methods for distributed decision problems over network systems under noisy data observations.

Appendix

Here is the convergence result for the constrained stochastic approximation. Consider

$$\theta_{k+1} = P_{\Phi}\{\theta_k + \alpha_k Y_k\}, \quad (30)$$

where $\Phi \in \mathbf{R}^m$ is a convex constraint set. Next follows the conditions for its convergence analysis.

C1: $\sup_k E[\|Y_k\|^2] < \infty$.

C2: There is a measurable function $g(\cdot)$ such that

$$E_k[Y_k] = E[Y_k|\theta_0, Y_i, i < k] = g(\theta_k).$$

C3: $g(\cdot)$ is continuous.

C4: θ_k is bounded with probability one.

Theorem 5.1. [27, Theorem 5.2.1 and Theorem 5.2.3] *Let C1-C4, and (4) hold for algorithm (30). If Φ satisfies the same condition as that imposed on Ω_i in Assumption 2, then with probability one θ_k converges to the invariant set of the following projected ODE in Φ :*

$$\dot{\theta} = g(\theta) + z,$$

where $z \in -N_{\Phi}(\theta)$ is the minimum force to keep the trajectories of the projected ODE in Φ .

The following lemma shows convergence properties for non-negative super-martingales.

Lemma 5.2 (Robbins-Siegmund). ([29]) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$ be a sequence of σ -algebra of \mathcal{F} . Let $\{d_k\}$ and $\{w_k\}$ be nonnegative \mathcal{F}_k -measurable random variables such that*

$$E[d_{k+1}|\mathcal{F}_k] \leq (1 + \alpha_k)d_k + w_k,$$

where $\alpha_k \geq 0$ are deterministic scalars with $\sum_{k=1}^{\infty} \alpha_k < \infty$. If $\sum_{k=1}^{\infty} w_k < \infty$, then $\{d_k\}$ converges with probability one to some finite random variable.

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