A filtering problem with uncertainty in observation

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Abstract. This paper is concerned with a generalized Kalman-Bucy filtering model and corresponding robust problem under model uncertainty. We find that this robust problem is equivalent to considering an estimate problem under some sublinear operator. Therefore, we turn to obtaining the minimum mean square estimator under a sublinear operator. By Girsanov theorem and minimax theorem, we obtain the optimal estimator \hat{x}_t of the signal process x_t for given time $t \in [0, T]$.

Key words. sublinear operator, minimum mean square estimator, Kalman-Bucy filtering, uncertainty.

1 Introduction

Let $(\Omega, \mathcal{F}, {\mathcal{F}_t}_{0 \le t \le T}, P)$ be a complete filtered probability space equipped with a natural filtration $\mathcal{F}_t = \sigma{w(s), v(s); 0 \le s \le t}, \mathcal{F} = \mathcal{F}_T$, where $(w(\cdot), v(\cdot))$ is 2-dimensional standard Brownian motion defined on the space, T > 0 is a fixed real number. Suppose that the signal process (x_t) and the observation process (m_t) under probability measure P satisfy respectively

$$dx_t = (F_t x_t + f_t)dt + dw_t,$$

$$x(0) = x_0,$$

$$dm_t = (G_t x_t + g_t)dt + dv_t,$$

$$m(0) = 0$$

(1.1)

where the coefficients F_t , f_t , G_t , g_t are bounded, continuous functions in t and x_0 is a given constant. The classical Kalman-Bucy filtering problem is to find the optimal estimator \bar{x}_t such that

$$\min_{\zeta} E_P \|x_t - \zeta\|^2 = E_P \|x_t - \bar{x}_t\|^2.$$
(1.2)

In 1961, Kalman and Bucy [13] gave the fundamental results of the filtering problem which are the foundation of modern filtering theory (see Bensoussan [2], Liptser and Shiryaev [16] et al). Based on the

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filtering technique, stochastic optimal control problems with partial information (or observation) have been studied extensively. In the field of finance and insurance, for example, Bensoussan and Keppo [3] and Lakner [15] considered the optimal consumption and portfolio investment problems of an investor who is interested in maximizing his utilities from consumption and terminal wealth under partial information; Xiong and Zhou [24] considered the mean-variance portfolio selection problems under partial information. In the field of stochastic control, Duncan and Pasik-Dunan [5] and [6] considered respectively the optimal control for a partially observed linear stochastic system with an exponential quadratic cost and with fractional brownian motions; Tang [21] gave the maximum principle for partially observed optimal control problems of stochastic differential equations; Wang and Wu [23] studied the Kalman-Bucy filtering equation of a certain forwardbackward stochastic differential equation system and solved a partially observed linear quadratic optimal control problem, and so on. Some fundamental researches based on forward-backward stochastic differential equations are surveyed by Ma and Yong [17] and Zhang [26].

In 2002, Chen and Epstein [4] proposed a kind of model uncertainty for continuous-time models which is the so called drift ambiguity. Drift ambiguity models an agent's uncertainty about the drift of the underlying Brownian motion. Moreover, in 2013, Epstein and Ji proposed more general uncertainty models (see [7] and [8] for details). In this paper, we introduce the following drift ambiguity in [4] into model (1.1) and focus on a corresponding robust problem. Consider the generalized Kalman-Bucy filtering model under some probability measure $P^{\theta} \in \mathcal{P}$:

$$dx_t = (F_t x_t + f_t)dt + dw_t,$$

$$x(0) = x_0,$$

$$dm_t = (G_t x_t + g_t + \theta_t)dt + dv_t^{\theta},$$

$$m(0) = 0$$

(1.3)

where (w_t) and (v_t^{θ}) are Brownian motions under P^{θ} and the probability measure P^{θ} is regarded as an observer's evaluation criterion for the signal process. Here the probability measure set \mathcal{P} denotes all the evaluation criterions by observers and $\theta \in \Theta$ is called ambiguity parameter. Note that Ji, Li and Miao [12] adopt a similar formulation in order to solve a dynamic contract problem. Then, we naturally consider the following worst-case minimum mean square estimate of the signal process (x_t) :

$$\min_{\zeta} \sup_{P^{\theta} \in \mathcal{P}} E_{P^{\theta}} \| x_t - \zeta \|^2$$
(1.4)

which is to minimize the maximum expected loss over a range of possible models, an idea that goes back at least as far as Wald [22] in 1945. Allan and Cohen [1] studied this type of estimate problem under nonlinear expectations by a control approach. Recently, Ji, Kong and Sun [11] considered a different generalized Kalman-Bucy filtering model where the ambiguity parameters affect the evolution of signal process.

In fact, $\sup_{P^{\theta} \in \mathcal{P}} E_{P^{\theta}}[\cdot]$ can be regarded as a sublinear operator $\mathcal{E}(\cdot)$ and the problem (1.4) can be reformulated as a estimate problem under sublinear operator:

$$\min_{\zeta} \mathcal{E}(\|x_t - \zeta\|^2).$$

The related literatures about the estimate problem under sublinear operators include Sun and Ji [20], Ji, Kong and Sun [10]. Sun and Ji [20] introduced a new conditional nonlinear expectation for bounded random variables which is based on the minimum mean square estimator for sublinear operators. However, the boundedness assumption for random variables has great limitations. Therefore, Ji, Kong and Sun [10] deleted the boundedness assumption and generalized the corresponding results to the case in which the random variables fall in the space $L_{\mathcal{F}}^{2+\epsilon}(\Omega, P)$ where ϵ is a constant such that $\epsilon \in (0, 1)$.

Under some mild conditions, we prove that the optimal estimator \hat{x} and the optimal probability measure P^{θ^*} exist. It results that we only need to consider the classical Kalman-Bucy filtering problem under the probability measure P^{θ^*} . Moreover, in some special cases, the optimal estimator \hat{x} can be decomposed to two parts. One part is the optimal estimator of the signal process under the probability measure P and the other part contains the parameter θ^* (see Corollary 3.5 for details).

The remainder of the paper proceeds as follows. In section 2, after a brief recall of the Kalman-Bucy filter and the drift ambiguity, a generalized robust Kalman-Bucy filtering problem is introduced. In section 3, the main general results are given and we consider a special case to further explain our results.

2 Problem formulation

Let $w(\cdot)$ and $v(\cdot)$ be *n*-dimensional and *m*-dimensional independent Brownian motions defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$ where $\mathcal{F}_t = \sigma\{w(s), v(s); 0 \leq s \leq t\}$, $\mathcal{F} = \mathcal{F}_T$ and T > 0 be a fixed terminal time. The means of $w(\cdot)$ and $v(\cdot)$ are zero and the covariance matrices are $Q(\cdot)$ and $R(\cdot)$ respectively. The matrix $R(\cdot)$ is uniformly positive definite. Denote by \mathbb{R}^n the *n*-dimensional real Euclidean space and $\mathbb{R}^{n \times k}$ the set of $n \times k$ real matrices. Let $\langle \cdot, \cdot \rangle$ (resp. $\|\cdot\|$) denote the usual scalar product (resp. usual norm) of \mathbb{R}^n and $\mathbb{R}^{n \times k}$. The scalar product (resp. norm) of $M = (m_{ij}), N = (n_{ij}) \in \mathbb{R}^{n \times k}$ is denoted by $\langle M, N \rangle = tr\{MN^{\mathsf{T}}\}$ (resp. $\|M\| = \sqrt{\langle M, M \rangle}$), where the superscript T denotes the transpose of vectors or matrices. For a \mathbb{R}^n -valued vector $x = (x_1, \cdots, x_n)^{\mathsf{T}}, |x| := (|x_1|, \cdots, |x_n|)^{\mathsf{T}}$; for two \mathbb{R}^n -valued vectors xand $y, x \leq y$ means that $x_i \leq y_i$ for $i = 1, \dots, n$.

Through out this paper, 0 denotes the matrix/vector with appropriate dimension whose all entries are zero and ϵ is a constant such that $0 < \epsilon < 1$.

Suppose that the signal process $(x_t) \in \mathbb{R}^n$ and the observation process $(m_t) \in \mathbb{R}^m$ under probability measure P satisfy model (1.1) where $F_t \in \mathbb{R}^{n \times n}$, $G_t \in \mathbb{R}^{m \times n}$, $f_t \in \mathbb{R}^n$, $g_t \in \mathbb{R}^m$ are bounded, continuous functions in $t, x_0 \in \mathbb{R}^n$ is a given constant vector. Let the filtration $\mathcal{Z}_t = \sigma\{m(s); 0 \le s \le t\}$ be the set of observable events up to time t. By the Kalman-Bucy filtering theory (see Bensoussan [2], Kalman and Bucy [13] and Liptser and Shiryaev [16] et al), the optimal solution $\bar{x}_t = E_P(x_t|\mathcal{Z}_t)$ of problem (1.2) is governed by

$$\begin{cases} d\bar{x}_t = (F_t \bar{x}_t + f_t) dt + P_t G_t^{\mathsf{T}} R_t^{-1} dI_t, \\ \bar{x}(0) = x_0, \end{cases}$$
(2.1)

and the variance of estimate error $P_t = E_P[(x_t - \bar{x}_t)(x_t - \bar{x}_t)^{\mathsf{T}}]$ is governed by

$$\frac{dP_t}{dt} = F_t P_t + P_t F_t^{\mathsf{T}} - P_t G_t^{\mathsf{T}} R_t^{-1} G_t P_t + Q_t,$$

$$P(0) = 0$$
(2.2)

where $I_t = m_t - \int_0^t (G_s \bar{x}_s + g_s) ds$ is called innovation process under probability measure P which is a Wiener process adapted to $\{\mathcal{Z}_t\}$. Furthermore, the filtration $\mathcal{I}_t = \sigma\{I(s); 0 \leq s \leq t\}$ equals to \mathcal{Z}_t for any time $t \in [0, T]$.

Now we are ready to give the drift ambiguity model. For a fixed \mathbb{R}^m -valued nonnegative constant vector μ , denote by Θ the set of all the \mathbb{R}^m -valued progressively measurable processes (θ_t) with $|\theta_t| \leq \mu$. Define

$$\mathcal{P} = \{ P^{\theta} | \frac{dP^{\theta}}{dP} = f_T^{P^{\theta}} \text{ with } \theta \in \Theta \}$$
(2.3)

where

$$f_T^{P^{\theta}} = \frac{dP^{\theta}}{dP} = \exp(\int_0^T \theta_t^{\mathsf{T}} dv_t - \frac{1}{2} \int_0^T \|\theta_t\|^2 dt).$$

Due to the boundness of θ , the Novikov's condition holds (see Karatzas and Shreve [14]). Therefore, P^{θ} defined by (2.3) is a probability measure which is equivalent to the probability measure P and the processes (w_t) and (v_t^{θ}) where $v_t^{\theta} = v_t - \int_0^t \theta_s ds$ are Brownian motions under this probability measure P^{θ} by Girsanov theorem. Then, with this generalized model (1.3) under probability measure P^{θ} , we consider naturally the following robust problem:

$$\inf_{\zeta \in L^{2+\epsilon}_{Z_t}(\Omega, P, \mathbb{R}^n)} \sup_{P^{\theta} \in \mathcal{P}} E_{P^{\theta}} \| x_t - \zeta \|^2,$$
(2.4)

where $L^{2+\epsilon}_{\mathcal{Z}_t}(\Omega, P, \mathbb{R}^n)$ is the set of all the \mathbb{R}^n -valued $(2+\epsilon)$ integrable \mathcal{Z}_t -measurable random variables.

However, if we denote $\mathcal{E}(\cdot) = \sup_{P^{\theta} \in \mathcal{P}} E_{P^{\theta}}[\cdot]$ which can be regarded as a sublinear operator, then the above robust problem can be considered as an estimate problem of the signal process under this sublinear operator $\mathcal{E}(\cdot)$. In more details, given the observation information $\{\mathcal{Z}_t\}$, we intend to find the optimal estimator \hat{x}_t of the signal process (x_t) at time $t \in [0, T]$ such that

$$\mathcal{E}\|x_t - \hat{x}_t\|^2 = \inf_{\zeta \in \mathcal{K}_t} \mathcal{E}\|x_t - \zeta\|^2,$$
(2.5)

where

$$\mathcal{K}_t = \{ \zeta : \Omega \to \mathbb{R}^n; \ \zeta \in L^{2+\epsilon}_{\mathcal{Z}_t}(\Omega, P, \mathbb{R}^n) \}$$

Remark 2.1. The optimal solution \hat{x}_t of problem (2.5) is called minimum mean square estimator. It is also regarded as a minimax estimator in statistical decision theory. If the sublinear operator $\mathcal{E}(\cdot)$ degenerates to linear expectation operator, then \mathcal{P}^{θ} contains only one probability measure P. In this case, it is well known that the minimum mean square estimator \hat{x}_t is just the conditional expectation $E_P(x_t|\mathcal{Z}_t)$.

3 Main results

In this section, we study the minimum mean square estimator \hat{x}_t of problem (2.5) for some time $t \in [0, T]$. Without loss of generality, we only prove one dimensional case and the multidimensional case can be proved similarly. **Lemma 3.1** The set $\{\frac{dP^{\theta}}{dP} : P^{\theta} \in \mathcal{P}\} \subset L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, P)$ is $\sigma(L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, P), L^{1+\frac{\epsilon}{2}}(\Omega, \mathcal{F}, P))$ -compact and the set \mathcal{P} is convex.

Proof. By Lemma 1 in Girsanov [9] and the boundness of θ , the set $\{\frac{dP^{\theta}}{dP} : P^{\theta} \in \mathcal{P}\} \subset L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, P)$ space. According to Simons [19], Chapter 1, Theorem 4.1, the set $\{\frac{dP^{\theta}}{dP} : P^{\theta} \in \mathcal{P}\}$ is $\sigma(L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, P), L^{1+\frac{\epsilon}{2}}(\Omega, \mathcal{F}, P))$ -compact.

The set \mathcal{P} is convex which can be referred to Chen and Epstein [4]. Let θ_1 and θ_2 belong to the set Θ . $f^{P^{\theta_i}}$, i = 1, 2 denote the exponential martingales respectively with

$$f_t^{P^{\theta_i}} = \exp(\int_0^t \theta_{i,s} dv_s - \frac{1}{2} \int_0^t \theta_{i,s}^2 ds)$$

and

$$df_t^{P^{\theta_i}} = f_t^{P^{\theta_i}} \theta_{i,t} dv_t.$$

Let $0 \le \lambda_i \le 1$, i = 1, 2 be constants with $\lambda_1 + \lambda_2 = 1$ and

$$\theta_t^{\lambda} = \frac{\lambda_1 \theta_{1,t} f_t^{P^{\theta_1}} + \lambda_2 \theta_{2,t} f_t^{P^{\theta_2}}}{\lambda_1 f_t^{P^{\theta_1}} + \lambda_2 f_t^{P^{\theta_2}}}.$$

Since $f^{P^{\theta_i}} > 0$, i = 1, 2, the process (θ_t^{λ}) belongs to the set Θ , which implies that the set Θ is stochastically convex. Moreover, it is also easy to calculate that

$$d(\lambda_1 f_t^{P^{\theta_1}} + \lambda_2 f_t^{P^{\theta_2}}) = (\lambda_1 f_t^{P^{\theta_1}} + \lambda_2 f_t^{P^{\theta_2}}) \theta_t^{\lambda} dv_t.$$

Therefore, the set \mathcal{P} is convex.

Remark 3.1. By Lemma 3.1, Lemma 1 in [9] and Theorem 6.3 in Chapter 1 of [25], the signal process (x_t) is $(4+2\epsilon)$ integrable, the set $\{\frac{dP^{\theta}}{dP}: P^{\theta} \in \mathcal{P}\}$ is uniformly normed bounded in $L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, P)$ space and also $\sigma(L^{1+\frac{2}{\epsilon}}(\Omega, \mathcal{F}, P), L^{1+\frac{\epsilon}{2}}(\Omega, \mathcal{F}, P))$ -compact. Therefore, we can apply the results in Ji, Kong and Sun [10] to guarantee that the optimal solution of problem (2.5) exists.

By Lemma 3.1, we can apply the minimax theorem (see Theorem B.1.2 in Pham [18]) to problem (2.4) which leads to the following theorem.

Theorem 3.2 For a given $t \in [0,T]$, there exists a $\theta^* \in \Theta$ such that

$$\inf_{\zeta \in \mathcal{K}_t} \mathcal{E} \| x_t - \zeta \|^2 = \inf_{\zeta \in \mathcal{K}_t} \sup_{P^{\theta} \in \mathcal{P}} E_{P^{\theta}} \| x_t - \zeta \|^2 = \inf_{\zeta \in \mathcal{K}_t} E_{P^{\theta^*}} \| x_t - \zeta \|^2.$$

Proof. Denote $f_n = \frac{dP^{\theta_n}}{dP}$ and choose a sequence $\{f_n\}_{n\geq 1}$ such that

$$\lim_{n \to \infty} \inf_{\zeta \in \mathcal{K}_t} E_P[f_n(x_t - \zeta)^2] = \lim_{n \to \infty} \inf_{\zeta \in \mathcal{K}_t} E_{P^{\theta_n}}[(x_t - \zeta)^2] = \sup_{P^{\theta} \in \mathcal{P}} \inf_{\zeta \in \mathcal{K}_t} E_{P^{\theta}}[(x_t - \zeta)^2].$$

By Komlós theorem in Pham [18], we know that there exist a subsequence $\{f_{n_k}\}_{k\geq 1} \subset \{f_n\}_{n\geq 1}$ and $f^* \in L^1(\Omega, \mathcal{F}, P)$ space such that

$$\lim_{n \to \infty} \frac{1}{m} \sum_{k=1}^{m} f_{n_k} = f^*, \ P - a.s..$$

Let $g_m = \frac{1}{m} \sum_{k=1}^m f_{n_k}$. We have $g_m \xrightarrow{P-a.s.} f^*$ and

$$\sup_{P^{\theta}\in\mathcal{P}}\inf_{\zeta\in\mathcal{K}_{t}}E_{P^{\theta}}[(x_{t}-\zeta)^{2}] = \lim_{n\to\infty}\inf_{\zeta\in\mathcal{K}_{t}}E_{P^{\theta_{n}}}[(x_{t}-\zeta)^{2}] = \lim_{k\to\infty}\inf_{\zeta\in\mathcal{K}_{t}}E_{P^{\theta_{n_{k}}}}[(x_{t}-\zeta)^{2}]$$
$$= \lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m}\inf_{\zeta\in\mathcal{K}_{t}}E_{P^{\theta_{n_{k}}}}[(x_{t}-\zeta)^{2}] \leq \lim_{m\to\infty}\inf_{\zeta\in\mathcal{K}_{t}}\frac{1}{m}\sum_{k=1}^{m}E_{P^{\theta_{n_{k}}}}[(x_{t}-\zeta)^{2}]$$
$$= \lim_{m\to\infty}\inf_{\zeta\in\mathcal{K}_{t}}E_{P}[g_{m}(x_{t}-\zeta)^{2}].$$
(3.1)

By Lemma 1 in [9], for any given constants p > 1 and m, we have $E_P(g_m)^K \leq M$ where $K = (1 + \frac{2}{\epsilon})p$ and $M = \exp(\frac{K^2 - K}{2}\mu^2 T)$. Then, we have $\left\{|g_m|^{1+\frac{2}{\epsilon}}: m = 1, 2, \cdots\right\}$ is uniformly integrable. Therefore,

 $g_m \xrightarrow{L^{1+\frac{2}{\epsilon}}(\Omega,\mathcal{F},P)} f^* \text{ and } f^* \in L^{1+\frac{2}{\epsilon}}(\Omega,\mathcal{F},P).$ According to the convexity weak compactness of set $\{\frac{dP^{\theta}}{dP}: P^{\theta} \in \mathcal{P}\}$, there exists a θ^* such that $\frac{dP^{\theta^*}}{dP} = f^*$ and the following relations hold

$$\sup_{P^{\theta} \in \mathcal{P}} \inf_{\zeta \in \mathcal{K}_{t}} E_{P^{\theta}}[(x_{t} - \zeta)^{2}] \geq \inf_{\zeta \in \mathcal{K}_{t}} E_{P^{\theta^{*}}}[(x_{t} - \zeta)^{2}]$$
$$= \inf_{\zeta \in \mathcal{K}_{t}} E_{P}[f^{*}(x_{t} - \zeta)^{2}]$$
$$\geq \lim_{m \to \infty} \sup_{m \to \infty} \inf_{\zeta \in \mathcal{K}_{t}} E_{P}[g_{m}(x_{t} - \zeta)^{2}]$$
$$\geq \sup_{P^{\theta} \in \mathcal{P}} \inf_{\zeta \in \mathcal{K}_{t}} E_{P^{\theta}}[(x_{t} - \zeta)^{2}]$$

where the second $' \geq '$ is based on upper semi-continuous property. It follows that

$$\sup_{P^{\theta} \in \mathcal{P}} \inf_{\zeta \in \mathcal{K}_t} E_{P^{\theta}}[(x_t - \zeta)^2] = \inf_{\zeta \in \mathcal{K}_t} E_{P^{\theta^*}}[(x_t - \zeta)^2].$$

By minimax theorem, we obtain

$$\sup_{P^{\theta} \in \mathcal{P}} \inf_{\zeta \in \mathcal{K}_t} E_{P^{\theta}}[(x_t - \zeta)^2] = \inf_{\zeta \in \mathcal{K}_t} \sup_{P^{\theta} \in \mathcal{P}} E_{P^{\theta}}[(x_t - \zeta)^2].$$

It results that

$$\inf_{\zeta \in \mathcal{K}_t} \sup_{P^{\theta} \in \mathcal{P}} E_{P^{\theta}}[(x_t - \zeta)^2] = \inf_{\zeta \in \mathcal{K}_t} E_{P^{\theta^*}}[(x_t - \zeta)^2].$$

Once we find the optimal θ^* , model (1.3) and problem (2.4) can be expressed under the new probability measure P^{θ^*} correspondingly. In more details, for this filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, P^{\theta^*})$, the processes (x_t) and (m_t) satisfy respectively

$$dx_t = (F_t x_t + f_t)dt + dw_t,$$

$$x(0) = x_0,$$

$$dm_t = (G_t x_t + g_t + \theta_t^*)dt + dv_t^{\theta^*},$$

$$m(0) = 0$$

$$(3.2)$$

and problem (2.4) turns into the minimum mean square estimate problem under probability measure P^{θ^*} :

$$E_{P^{\theta^*}} \|x_t - \hat{x}_t\|^2 = \inf_{\zeta \in \mathcal{K}_t} E_{P^{\theta^*}} \|x_t - \zeta\|^2.$$
(3.3)

With the above theorem, we consider the following estimate problem which is a Kalman-Bucy filtering problem with the parameter θ^* :

$$E_{P^{\theta^*}} \| x_t - \hat{\zeta} \|^2 = \inf_{\zeta \in \bar{\mathcal{K}}_t} E_{P^{\theta^*}} \| x_t - \zeta \|^2$$
(3.4)

where $\bar{\mathcal{K}}_t = \{\zeta: \Omega \to \mathbb{R}^n; \ \zeta \in L^2_{\mathcal{Z}_t}(\Omega, P^{\theta^*}, \mathbb{R}^n)\}.$

The model (3.2) and problem (3.4) constitute a classical construction for a linear, partially observable system with a parameter θ^* . This estimate problem is to characterize the conditional distribution $P^{\theta^*}(x_t \in A|\mathcal{Z}_t)$, where A is a Borel set in \mathbb{R}^n . Then, we are in the realm of Kalman-Bucy filtering and it is well known(see [13] and [16]) that the conditional distribution is again Gaussian and conditional mean $\hat{x}_t = E_{P^{\theta^*}}(x_t|\mathcal{Z}_t)$ solves the following equation:

$$\begin{cases} d\hat{x}_t = (F_t \hat{x}_t + f_t) dt + (P_t G_t^{\mathsf{T}} + \widehat{x_t \theta_t^{*\mathsf{T}}} - \hat{x}_t \widehat{\theta_t^{*\mathsf{T}}}) R_t^{-1} d\hat{I}_t, \\ \hat{x}(0) = x_0 \end{cases}$$
(3.5)

where $\widehat{\theta_t^*} = E_{P^{\theta^*}}[\theta_t^*|\mathcal{Z}_t], \ \widehat{x_t\theta_t^{*\mathsf{T}}} = E_{P^{\theta^*}}[x_t\theta_t^{*\mathsf{T}}|\mathcal{Z}_t], \ \hat{I}_t = m_t - \int_0^t (G_s \hat{x}_s + g_s + \widehat{\theta_s^*}) ds$ is \mathcal{Z}_t -measurable Brownian motion and the variance of error equation $P_t = E_{P^{\theta^*}}[(x_t - \hat{x}_t)^2|\mathcal{Z}_t] = E_{P^{\theta^*}}[(x_t - \hat{x}_t)^2]$ satisfies

$$\begin{cases} \frac{dP_t}{dt} = F_t P_t + P_t F_t^{\mathsf{T}} - E_{P^{\theta^*}} \left[(P_t G_t^{\mathsf{T}} + \widehat{x_t \theta_t^{*\mathsf{T}}} - \hat{x}_t \widehat{\theta_t^{*}}) R_t^{-1} (G_t P_t + \widehat{\theta_t^{*} x_t^{\mathsf{T}}} - \widehat{\theta_t^{*}} \widehat{x^{\mathsf{T}}}_t) \right] + Q_t, \\ P(0) = 0. \end{cases}$$

$$(3.6)$$

So far, the optimal estimator of problem (3.4) has been obtained. Next, we expound that this solution \hat{x}_t is also the optimal estimator of problem (2.5) at time $t \in [0, T]$.

Theorem 3.3 Under the above assumptions, \hat{x}_t governed by equation (3.5) is also the optimal solution of problem (2.5) for any time $t \in [0, T]$.

Proof. Note that

$$\inf_{\zeta \in \mathcal{K}_t} \sup_{P^{\theta} \in \mathcal{P}} E_{P^{\theta}} (x_t - \zeta)^2 = \inf_{\zeta \in \mathcal{K}_t} E_{P^{\theta^*}} (x_t - \zeta)^2 \ge \inf_{\zeta \in \bar{\mathcal{K}}_t} E_{P^{\theta^*}} (x_t - \zeta)^2.$$
(3.7)

In addition, since F_t , G_t , f_t and g_t are bounded continuous functions in t and θ^* is bounded, it is easy to verify that \hat{x}_t is not only square integrable but also $(4 + 2\epsilon)$ integrable under probability measure P^{θ^*} by Theorem 6.3 in Chapter 1 of [25]. Then, the solution \hat{x}_t of equation (3.5) also belongs to \mathcal{K}_t . It yields that \hat{x}_t is the optimal solution of problem (2.5) at time $t \in [0,T]$.

Corollary 3.4 If the optimal θ_t^* adapted to subfiltration \mathcal{Z}_t , then the optimal estimator \hat{x}_t satisfies the following simpler equation.

$$\begin{cases} d\hat{x}_t = (F_t \hat{x}_t + f_t) dt + P_t G_t^{\mathsf{T}} R_t^{-1} d\hat{I}_{1,t}, \\ \hat{x}(0) = x_0 \end{cases}$$
(3.8)

where $\hat{I}_{1,t} = m_t - \int_0^t (G_s \hat{x}_s + g_s + \theta_s^*) ds$ is \mathcal{Z}_t -measurable Brownian motion and P_t reduces to equation (2.2).

Define $A(t,s) = P_s G_s R_s^{-1} \exp \int_s^t (F_r - P_r G_r^2 R_r^{-1}) dr$, which is the impulse response of the classical Kalman-Bucy filter. After some simple calculations, the optimal estimator \hat{x} can be decomposed to two parts. One part is the optimal estimator of the signal process under the probability measure P and the other part contains the parameter θ^* .

Corollary 3.5 If the optimal θ_t^* adapted to subfiltration Z_t , with equations (2.1) and (3.8), then the optimal estimator \hat{x}_t for any time $t \in [0,T]$ can be expressed as

$$\hat{x}_t = \bar{x}_t - \int_0^t A(t, s) \theta_s^* ds.$$
(3.9)

where \bar{x}_t is defined by equation (2.1).

Similar to Theorem 5.6 in Sun and Ji [20], we give a sufficient and necessary condition for the existence of the optimal estimator in the following corollary.

Corollary 3.6 For a given $t \in [0,T]$, \hat{x}_t is the optimal solution of problem (2.5) if and only if it is the solution of the following equation

$$\inf_{\zeta \in \mathcal{K}_t} \mathcal{E}[(x_t - \hat{x}_t)(x_t - \zeta)] = \mathcal{E}(x_t - \hat{x}_t)^2.$$
(3.10)

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