Differential dissipativity analysis of reaction-diffusion systems*

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Abstract

This note shows how classical tools from linear control theory can be leveraged to provide a global analysis of nonlinear reaction-diffusion models. The approach is differential in nature. It proceeds from classical tools of contraction analysis and recent extensions to differential dissipativity.

Keywords: Differential analysis, reaction-diffusion systems, dominance theory, spatial homogeneity.

1. Introduction

Reaction-diffusion equations are broadly used for modeling the spatio-temporal evolution of processes appearing in many fields of science such as propagation of electrical activity on cells in cellular biology [16]; reactions between substances on active media in chemistry [18]; transport phenomena in semiconductor devices in electronics [26]; and combustion processes and heat propagation in physics [31], to name a few. They have attracted recent interest in control, most notably [1] and [2], because the close link between reaction-diffusion systems and synchronization models under diffusive coupling: the linear diffusion term in reaction-diffusion partial differential equations is the continuum limit of the diffusive (or incrementally passive) interconnection network of agents sharing the same reaction dynamics. In that sense, the results in [1] and [2] are infinite dimensional generalizations of classical finite dimensional results pertaining to synchronization [24, 27, 29, 32].

Our contribution in the present note is to further emphasize the potential of classical tools from linear control theory in the analysis and design of reaction-diffusion systems. Our observation is twofold. First, we model reaction diffusion systems as the interconnection of a linear spatially and time-invariant (LTSI) model with a static nonlinearity. This natural decomposition calls for a dissipativity analysis of the interconnection, with complementary input-output dissipation inequalities imposed on the LTSI model and on the static nonlinearity, respectively. Second, we study this interconnection differentially along arbitrary solutions, thereby studying a nonlinear model through a family on linearized systems.

The proposed approach is largely inspired from [1] and [2], which analyze spatial homogeneity via contraction theory. The purely differential approach in the present paper is thought to offer further potential especially in situations where the attractor is difficult to characterize explicitely. In this note, we illustrate the benefits of a differential approach in two distinct ways: (i) we use the classical KYP lemma to complement existing state-space analysis results with a frequency-domain analysis; and (ii) we use recent results of differential dissipativity theory [12] to characterize the attractor of two classical reaction-diffusion models: Nagumo model of bistability [21], and Fitzugh-Nagumo model of oscillation [11].

Some notation

Let $L_n^2(\Omega)$ denote the Hilbert space of square integrable functions mapping $\Omega \subset \mathbb{R}$ to \mathbb{R}^n with the conventional inner product $\langle x,y\rangle_{L_n^2(\Omega)}=\int_\Omega x(\theta)^\top y(\theta)d\theta$ and norm denoted by $\|\cdot\|_{L_n^2(\Omega)}$. When clear from the context, we will drop the subindex. For vectors ξ,ψ in \mathbb{R}^n , the inner product is denoted as $\xi^\top \psi$ and the associated norm as $|\cdot|$. The set $\mathbb{C}_+:=\{a+jb\in\mathbb{C}|a\geq 0\}$ denotes the set of complex numbers with non-negative real part, whereas \mathbb{R}_+ denotes the set of non-negative real numbers. A symmetric, positive (semi-) definite matrix Π is denoted as $(\Pi \succeq 0)$ $\Pi \succ 0$, whereas, I_n represents the identity matrix of dimension n.

2. Reaction-diffusion systems

The family of distributed systems under consideration has the form

$$\frac{\partial x}{\partial t}(\theta, t) = D\Delta x(\theta, t) + Ax(\theta, t) - B\varphi(Cx(\theta, t))$$

where $x(\theta, t) \in \mathbb{R}^n$ denotes the state of the system at position $\theta \in \Omega \subset \mathbb{R}$ and time $t \geq 0$. The nonlinear function $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ represents a static nonlinearity and its properties are stated below. Spatial diffusion is modeled via the

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matrix $D \in \mathbb{R}^{n \times n}$ which is symmetric and positive definite, and the Laplace operator $\Delta : \mathrm{Dom}(\Delta) \subset L^2_n(\Omega) \to L^2_n(\Omega)$ with domain specified below, whereas the matrices A, B and C are constant and of the appropriate dimensions.

Here, we regard reaction-diffusion systems as the feedback interconnection of a linear system and a static nonlinearity:

$$\Sigma : \begin{cases} \frac{\partial x}{\partial t}(\theta, t) = D\Delta x(\theta, t) + Ax(\theta, t) + Bu(\theta, t) \\ y(\theta, t) = Cx(\theta, t) \end{cases}$$
 (1a)

$$u(\theta, t) = -\varphi(y(\theta, t)) \tag{1b}$$

where $u(\theta,t) \in \mathbb{R}^m$, $y(\theta,t) \in \mathbb{R}^m$ are the distributed input and output, respectively. For simplicity we consider the spatial domain $\Omega \subset \mathbb{R}$ as the boundary of the unit circle S^1 . Thus, $\theta \in [0,2\pi]$ and we have the following periodic boundary conditions

$$x(0,t) = x(2\pi, t) \tag{2a}$$

$$\frac{\partial x}{\partial \theta}(0,t) = \frac{\partial x}{\partial \theta}(2\pi,t) \tag{2b}$$

Such models on the circle find application in computational biology for modeling directional sensing in living cells and systems with symmetries see, e.g., [8, 22, 30]. Additionally, a compact domain simplifies some of the technical assumptions. For instance, it guarantees a discrete frequency spectrum for the Laplace operator with domain

$$Dom(\Delta) = \{x(\cdot, t) \in H^2(\Omega; \mathbb{R}^n) | (2) \text{ holds} \}.$$
 (3)

Here $H^2(\Omega; \mathbb{R}^n) = H^2(\Omega) \times \cdots \times H^2(\Omega)$ denotes the Sobolev space of functions in $L_n^2(\Omega)$ such that the *i*-th component $x_i(\cdot,t)$ is differentiable (in the generalized sense) with derivatives in $L_2(\Omega)$.

The static nonlinearity $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is assumed to be continuously differentiable (i.e., $\varphi \in C^1(\mathbb{R}^n)$) and satisfies the following standing assumption.

Assumption 2.1.

- 1. There exists $0 < M < \infty$ such that $\eta^{\top} \varphi(\eta) < 0$ for all $\eta \in \mathbb{R}^m$ with $\|\eta\| > M$.
- 2. The function $\varphi: \mathbb{R}^m \to \mathbb{R}^m$ satisfies the differential dissipation inequality

$$\begin{bmatrix} I_m \\ -J_{\varphi}(\eta) \end{bmatrix}^{\top} \begin{bmatrix} Q & L \\ L^{\top} & R \end{bmatrix} \begin{bmatrix} I_m \\ -J_{\varphi}(\eta) \end{bmatrix} \leq 0 \tag{4}$$

for all $\eta \in \mathbb{R}^m$, where $J_{\varphi}(\eta) \in \mathbb{R}^{m \times m}$ denotes the Jacobian matrix of φ at η , the matrices $Q, L, R \in \mathbb{R}^{m \times m}$ are constant and $R = R^{\top} \succ 0$.

The dissipation inequality (4) is a classical differential sector condition. In the scalar case (m = 1), it reduces to

$$(J_{\omega}(\eta) - K_1)^{\top} (J_{\omega}(\eta) - K_2) \leq 0 \tag{5}$$

with $Q = \frac{1}{2} \left(K_1^{\top} K_2 + K_2^{\top} K_1 \right)$, $L = \frac{1}{2} (K_1 + K_2)^{\top}$ and $R = I_m$. Condition (5) then expresses that the slope of φ at any point lies in the interval $[K_1, K_2]$, whenever $K_1 < K_2$. See Figure 1 for an illustration.

The reader will note that model (1) reduces to

$$\frac{\partial x}{\partial t}(\theta, t) = D\Delta x(\theta, t) - \varphi(x(\theta, t)) \tag{6}$$

in the special case defined by m = n, A = 0, and $B = C = I_n$. This latter form is the classical form of a reaction-diffusion system in the literature [25].

Remark 2.2. Assumption 2.1 ensures that the system (1)-(2) admits a unique (classical) solution for any initial condition $x(\theta,0) = x_0(\theta)$ which is defined in the whole time interval $t \in [0,+\infty)$, given as

$$x(\theta,t) = T(t)x_0(\theta) + \int_0^t T(t-\tau)F(x(\theta,\tau))d\tau$$

where $T(t): L_n^2(\Omega) \to L_n^2(\Omega)$ is the C_0 -semigroup generated by the operator $D\Delta$ and $F(x(\theta,t)) = Ax(\theta,t) + B\varphi(Cx(\theta,t))$. See e.g., [23, Theorems 1.4 and 1.5, Chapter 6].

3. Differential analysis of reaction diffusion systems

Differential analysis consists in analyzing the properties of infinitesimal variations $\delta x(\theta,t)$ around an arbitrary solution $x(\theta,t)$ of (1)-(2) as is made in [9, 28] for the case of finite-dimensional systems.

Namely, let $\phi(\theta, t, x_0)$ denote the solution of the reaction-diffusion system (1)-(2) at position θ and time t with initial condition $x(\theta, 0) = x_0(\theta)$. Let $x^1(\theta, 0)$ and $x^2(\theta, 0)$ be two given initial conditions and let $\gamma: S^1 \times [0, 1] \to \mathbb{R}^n$ be a smooth curve such that $\gamma(\cdot, 0) = x^1(\cdot, 0)$, $\gamma(\cdot, 1) = x^2(\cdot, 0)$ and $\gamma(0, s) = \gamma(2\pi, s)$ for all $s \in [0, 1]$. In addition, let $\psi(\theta, t, s) = \phi(\theta, t, \gamma_s)$, where $\gamma_s(\cdot) = \gamma(\cdot, s)$, i.e., $\psi(\theta, t, s)$ is the solution of (1)-(2) at position θ and time t with initial condition $\psi(\theta, 0, s) = \gamma_s(\theta) = \gamma(\theta, s)$, $s \in [0, 1]$. It follows that

$$\begin{split} \frac{\partial}{\partial t} \left(\frac{\partial \psi}{\partial s}(\theta,t,s) \right) &= \frac{\partial}{\partial s} \left(\frac{\partial \psi}{\partial t}(\theta,t,s) \right) \\ &= \frac{\partial}{\partial s} \left(D\Delta \psi(\theta,t,s) + A\psi(\theta,t,s) \right) \\ &\quad - B\varphi(C\psi(\theta,t,s))) \\ &= D\Delta \frac{\partial \psi}{\partial s}(\theta,t,s) + A \frac{\partial \psi}{\partial s}(\theta,t,s) \\ &\quad - BJ_{\varphi}(C\psi(\theta,t,s))C \frac{\partial \psi}{\partial s}(\theta,t,s) \end{split}$$

Defining $\delta x(\theta,t,s):=\frac{\partial \psi}{\partial s}(\theta,t,s)$, leads to the variational equation

$$\frac{\partial \delta x}{\partial t}(\theta, t, s) = D\Delta \delta x(\theta, t, s) + A\delta x(\theta, t, s) - BJ_{\omega}(C\psi(\theta, t, s))C\delta x(\theta, t, s)$$

Equivalently, in Lur'e form

$$\delta \Sigma : \begin{cases} \frac{\partial \delta x}{\partial t}(\theta, t) = D\Delta \delta x(\theta, t) + A\delta x(\theta, t) + B\delta u(\theta, t) \\ \delta y(\theta, t) = C\delta x(\theta, t) \end{cases}$$
(7a)

$$\delta u(\theta, t) = -J_{\varphi}(y(\theta, t))\delta y(\theta, t) \tag{7b}$$

with boundary conditions

$$\delta x(0,t) = \delta x(2\pi, t) \tag{8a}$$

$$\frac{\partial \delta x}{\partial \theta}(0, t) = \frac{\partial \delta x}{\partial \theta}(2\pi, t) \tag{8b}$$

The variational system is linear. It is the interconnection of the same LTSI model (1) with a time-varying output feedback gain evaluated along an arbitrary solution $x(\theta,t)$. In the following subsections we focus on the analysis of the differential model (7)-(8). We analyze spatial and temporal variations separately.

3.1. Differential inhomogeneous dynamics

The spatial infinitesimal variation of the solution $x(\theta,t)$ at time t is

$$\lim_{\nu \to 0} \frac{x(\theta + \nu, t) - x(\theta, t)}{\nu} = \frac{\partial x}{\partial \theta}(\theta, t)$$

Note that $\frac{\partial x}{\partial \theta}$ is an infinitesimal variation for the family of curves $\gamma(\theta, s) = \gamma(\theta + s)$. Thus, $\frac{\partial x}{\partial \theta}$ satisfies (7)-(8). In addition, it follows from (2a) and the fundamental theorem of calculus that $\frac{\partial x}{\partial \theta}$ satisfies the integral constraint

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial x}{\partial \theta}(\theta, t) d\theta = x(2\pi, t) - x(0, t) = 0, \qquad (9)$$

which means that $\frac{\partial x}{\partial \theta}$ is a zero mean solution of (7)-(8). Let us consider the bounded linear operator $T: L_n^2(\Omega) \to L_n^2(\Omega)$ mapping

$$\delta x \mapsto \int_{\Omega} \delta x(\theta, t) d\theta =: \overline{\delta x}$$
 (10)

where $\overline{\delta x}: \mathbb{R}_+ \to \mathbb{R}^n$ is a function dependent on time but no longer dependent on the spatial variable. More generally, any variation δx admits the decomposition

$$\delta x = \overline{\delta x} + \delta \xi$$

with $\delta \xi \in \mathcal{N}(T)$ and $\overline{\delta x} \in \mathcal{R}(T)$, where $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the null and range space of T, respectively. It follows from (9) and the uniqueness of the splitting that $\delta \xi = \frac{\partial x}{\partial \theta} \in \mathcal{N}(T)$. Motivated by the discussion above, we label the dynamics associated to $\overline{\delta x} = T \delta x$ as the spatially homogeneous dynamics, whereas we refer to the dynamics associated to $\delta \xi \in \mathcal{N}(T)$ as the spatially inhomogeneous dynamics.

In the sequel we analyze separately the exponential contraction of $\delta \xi$ and the convergence of $\overline{\delta x}$ leading to spatially homogeneous motions. The definition of spatial homogeneity was introduced in [1], which we recall in the following lines.

Definition 3.1 (Spatial homogeneity [1]). The system (1)-(2) is spatially homogeneous with rate $\mu > 0$ if for any given initial condition

$$\|\frac{\partial x}{\partial \theta}(\cdot, t)\|_{L_n^2(\Omega)} \le M e^{-\mu t} \|\frac{\partial x}{\partial \theta}(\cdot, 0)\|_{L_n^2(\Omega)}, \tag{11}$$

where M > 0.

Definition 3.1 states that for any initial condition the spatial mismatch between any two trajectories, that are infinitesimally close, decays exponentially to zero, that is, all trajectories converge to each other in the spatial domain, enforcing an homogeneous motion in space. Spatial homogeneity is thus equivalent to contraction of the spatially inhomogeneous dynamics.

Proposition 3.2. System (1)-(2) is spatially homogeneous with rate $\mu \geq 0$, if and only if, the origin of the system (7)-(8)-(9) is uniformly exponentially stable with the same rate $\mu \geq 0$.

Proof. The proof is a direct consequence of Definition 3.1 and the fact that $\delta \xi = \frac{\partial x}{\partial \theta}$ is the solution to (7)-(8)-(9).

Conditions guaranteeing the exponential homogeneity of (1)-(2) have been studied extensively, see e.g., [1, 2, 6, 15]. The dissipativity formulation of those conditions is as follows. Let $\sigma: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the quadratic differential supply rate

$$\sigma(\delta y(\theta, t), \delta u(\theta, t)) := \begin{bmatrix} \delta y(\theta, t) \\ \delta u(\theta, t) \end{bmatrix}^{\top} \begin{bmatrix} Q & L \\ L^{\top} & R \end{bmatrix} \begin{bmatrix} \delta y(\theta, t) \\ \delta u(\theta, t) \end{bmatrix}$$
(12)

where the matrices Q, L and R are constant and $R = R^{\uparrow} \succ 0$

Definition 3.3. The LTSI system (1a)-(2) is uniformly differential dissipative with rate $\mu \geq 0$ and with respect to the supply rate (12), if there exists a matrix $\Pi = \Pi^{\top} \succ 0$ such that the following inequality holds for all admissible δu with $(\delta x, \delta y)$ satisfying (7a)-(8)

$$\int_{\Omega} \begin{bmatrix} \frac{\partial}{\partial t} \delta x \\ \delta x \end{bmatrix}^{\top} \begin{bmatrix} 0 & \Pi \\ \Pi & 2\mu\Pi + \varepsilon I_n \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial t} \delta x \\ \delta x \end{bmatrix} d\theta \leq \int_{\Omega} \sigma(\delta y, \delta u) d\theta \quad (13)$$

Additionally, if (13) holds in an invariant subspace $\mathcal{V} \subset L_2^n(\Omega)$ of δx then we say that the system is uniformly differential dissipative in \mathcal{V} . The property is strict if $\varepsilon > 0$ in (13).

Henceforth, dissipativity is always assumed with respect to the supply rate (12). With those definitions in place, the dissipativity analysis of spatial homogeneity of (1)-(2) is an infinite-dimensional version of the classical circle criterion.

Theorem 3.4. Let φ satisfy the dissipation inequality (4). If the LTSI system (1a)-(2) is uniformly differential dissipative with rate $\mu \geq 0$ in $\mathcal{N}(T)$, then the closed-loop system (1)-(2) is spatially homogeneous with the same rate μ .

Proof. From Proposition 3.2 it follows that spatial homogeneity of (1)-(2) is equivalent to exponential stability of the gradient dynamics given by (7)-(9). Let $\delta \xi = \frac{\partial x}{\partial \theta} \in \mathcal{N}(T)$. By hypothesis, there exists $\Pi = \Pi^{\top} \succ 0$ such that (13) holds for $\delta x = \delta \xi$. Now, let $S(\delta \xi) = \langle \delta \xi(\cdot,t), \Pi \delta \xi(\cdot,t) \rangle_{L^2(\Omega)}$, then (13) is equivalent to

$$\frac{d}{dt}S(\delta\xi) \le \int_{\Omega} \sigma(\delta y, \delta u) d\theta - 2\mu S(\delta\xi) - \varepsilon \|\delta\xi(\cdot, t)\|_{L_{n}^{2}(\Omega)}$$

Using $\delta u(\theta, t) = -J_{\varphi}(y(\theta, t))\delta y(\theta, t)$ together with the sector bound (4) leads to

$$\frac{d}{dt}S(\delta\xi) \le -2\mu S(\delta\xi) - \varepsilon \|\delta\xi(\cdot,t)\|_{L_n^2(\Omega)} \tag{14}$$

Now, multiplying both sides of (14) by $e^{2\mu t}$ and integrating from $\tau = 0$ up to $\tau = t$, yields

$$\|\delta\xi(\cdot,t)\|_{L^2_n(\Omega)} \leq \sqrt{\frac{\lambda_{\max}(\Pi)}{\lambda_{\min}(\Pi)}} e^{-\mu t} \|\delta\xi(\cdot,0)\|_{L^2_n(\Omega)}$$

where $\lambda_{\max}(\Pi)$ and $\lambda_{\min}(\Pi)$ denote the maximum and minimum eigenvalue of Π . Therefore, $\delta \xi = \frac{\partial x}{\partial \theta}$ goes exponentially to the zero with rate $\mu \geq 0$.

The following theorem provides a numerical test for uniform differential dissipativity of the LTSI system (1a)-(2). The result is a reformulation of [2, Theorem 1] for reaction-diffusion systems with periodic spatial domain S^1 .

Theorem 3.5. Let $\Pi \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix such that

$$\Pi D + D^{\top} \Pi \succeq 0 \tag{15}$$

and

$$\begin{bmatrix} \Theta_{1,1} & \Pi B - C^{\top} L \\ B^{\top} \Pi - L^{\top} C & -R \end{bmatrix} \preceq 0 \tag{16}$$

where $\Theta_{1,1} = (A - \lambda_2 D)^{\top} \Pi + \Pi(A - \lambda_2 D) + 2\mu \Pi - C^{\top} Q C + \varepsilon I_n$. Then the inhomogeneous dynamics of (1)-(2) is uniformly differential dissipative with rate $\mu \geq 0$, where λ_q is the q-th eigenvalue of the opertor Δ with domain (3), respectively.

Proof. The reader is addressed to [2] for a detailed proof of this fact.

For the differential spatial dynamics (9) implies that $\mathcal{N}(T) = \text{span}\{\nu_1\}$, where ν_1 is the eigenvector associated to the eigenvalue λ_1 . Therefore, uniform differential dissipativity of the inhomogenous dynamics is tested by solving (15)-(16) with $\lambda_2 = 1$.

3.2. Differential homogeneous dynamics

The gradient dynamics describes the time evolution of fluctuations in space. The complementary dynamics associated to $\overline{\delta x} := T\delta x$, describes the homogeneous behavior of the differential dynamics. Thus, the differential dynamics constrained to $\mathcal{R}(T)$ is constant in space and therefore can be described by an ODE. Such intuition is formalized in the following theorem.

Theorem 3.6. The dynamics of (7)-(8) projected into the subspace $\mathcal{R}(T)$ reduces to

$$\begin{cases} \frac{d}{dt}\overline{\delta x}(t) = A\overline{\delta x}(t) + B\overline{\delta u}(t) \\ \overline{\delta y}(t) = C\overline{\delta x}(t) \end{cases}$$
(17a)

$$\overline{\delta u}(t) = -\left(\int_{\Omega} J_{\varphi}(y(\theta, t)) d\theta\right) \overline{\delta y}(t)$$
 (17b)

Proof. Applying the projection T, previously defined in (10), on both sides of (7a) yields

$$\begin{cases} \frac{d}{dt}\overline{\delta x}(t) = A\overline{\delta x}(t) + B\overline{\delta u}(t) \\ \overline{\delta y}(t) = C\overline{\delta x}(t) \end{cases}$$
(18)

where $\overline{\delta x}$ denotes the average in space of the differential variable δx , introduced in (10), similar for $\overline{\delta y}$, and $\overline{\delta u}$. We recall that $\delta x = \overline{\delta x} + \delta \xi$, where $\delta \xi = \frac{\partial x}{\partial \theta} \in \mathcal{N}(T)$. Hence, $\overline{\delta u}(t)$ obeys,

$$\begin{split} \overline{\delta u}(t) &= -\int_{\Omega} \left(J_{\varphi}(y(\theta,t)) \delta y(\theta,t) \right) d\theta \\ &= -\int_{\Omega} \left(\frac{\partial}{\partial \theta} \varphi(y(\theta,t)) + J_{\varphi}(y(\theta,t)) \overline{\delta y}(t) \right) d\theta \\ &= -\left(\int_{\Omega} J_{\varphi}(y(\theta,t)) d\theta \right) \overline{\delta y}(t) \end{split}$$

where we used the fact that $\delta y = C\overline{\delta x} + C\delta \xi$ in the second equation, and the fundamental theorem of calculus together with (2) to obtain the last equation.

The above result agrees with the traditional approach of [6, 15] in which spatial homogeneity reduces a PDE into an ODE. Thus, (17) describes the differential dynamics of the homogeneous behavior which we identify as the differential temporal dynamics.

We now illustrate the use of differential dissipativity analysis to study non-equilibrium asymptotic behaviors of the homogeneous dynamics. We make use of a recent development of the theory in [12, 19, 20]. We recall that the inertia of a symmetric matrix $P \in \mathbb{R}^{n \times n}$ is the triple (ν, ζ, π) where each entry denotes the number of negative, zero and positive eigenvalues, respectively.

Note that for the homogeneous dynamics, the associated supply rate is the same as (12) but with functions that are independent of the spatial variable. Namely, it follows from (4) that

$$0 \ge \langle \overline{\delta y}, Q \overline{\delta y} \rangle_{L_{n}^{2}(\Omega)} + \langle (J_{\varphi} \circ y) \overline{\delta y}, R (J_{\varphi} \circ y) \overline{\delta y} \rangle_{L_{n}^{2}(\Omega)} - \langle \overline{\delta y}, L (J_{\varphi} \circ y) \overline{\delta y} \rangle_{L_{n}^{2}(\Omega)} - \langle (J_{\varphi} \circ y) \overline{\delta y}, L^{\top} \overline{\delta y} \rangle_{L_{n}^{2}(\Omega)} \ge \begin{bmatrix} \overline{\delta y}(t) \\ \overline{\delta u}(t) \end{bmatrix}^{\top} \begin{bmatrix} Q & L \\ L^{\top} & R \end{bmatrix} \begin{bmatrix} \overline{\delta y}(t) \\ \overline{\delta u}(t) \end{bmatrix} =: \bar{\sigma}(\overline{\delta y}, \overline{\delta u})$$
(19)

where we applied Jensen's inequality to the quadratic term $\langle \eta, R \eta \rangle_{L^2(\Omega)}$ to obtain the last inequality.

Associated to the variational dynamics (17) we consider the family of lumped systems

$$\begin{cases} \frac{d}{dt}\bar{x}(t) = A\bar{x}(t) + B\bar{u}(t) \\ \bar{y}(t) = C\bar{x}(t) \end{cases}$$
 (20a)

$$\begin{bmatrix} \overline{\delta y}(t) \\ \overline{\delta u}(t) \end{bmatrix}^{\top} \begin{bmatrix} Q & L \\ L^{\top} & R \end{bmatrix} \begin{bmatrix} \overline{\delta y}(t) \\ \overline{\delta u}(t) \end{bmatrix} \le 0$$
 (20b)

where \bar{u} is defined implicitly through its variational representation δu .

Definition 3.7. The linear system (20a) is p-dissipative with rate $\lambda \geq 0$ and with respect to the supply rate $\bar{\sigma}$, if there exists a symmetric matrix $P = P^{\top}$ with inertia (p, 0, n-p) such that for all admissible $\overline{\delta u}$ and all $(\overline{\delta x}, \overline{\delta y})$ satisfying (17) the following holds

$$\begin{bmatrix} \frac{d}{dt}\overline{\delta x} \\ \overline{\delta x} \end{bmatrix}^{\top} \begin{bmatrix} 0 & P \\ P & 2\lambda P + \varepsilon I_n \end{bmatrix} \begin{bmatrix} \frac{d}{dt}\overline{\delta x} \\ \overline{\delta x} \end{bmatrix} \leq \overline{\sigma}(\overline{\delta y}, \overline{\delta u})$$
 (21)

The property is strict if $\varepsilon > 0$.

Analogous to the inhomogeneous case above, in what follows, we consider only quadratic supply rates of the form (19). The following theorem, taken from [19] and repeated here for completeness, provides useful information for characterizing the homogeneous part of the asymptotic behavior.

Theorem 3.8. Let φ satisfy the dissipation inequality (4). If the system (20a) is strictly p-dissipative with rate $\lambda \geq 0$. Then the homogeneous dynamics of the closed-loop (1)-(2) is p-dominant. In particular, each bounded solution asymptotically converges to an equilibrium for p=1 and to a simple limit set (equilibrium, closed orbit, or connected arc of equilibria) for p=2.

Proof. The homogeneous differential dynamics of the closed-loop (1)-(2) is given by (17), which is a lumped Lur'e system, Theorem 3.6. The result thus follows from [19, Theorem 4.2].

It follows from Definition 3.7 that the homogeneous dynamics associated to (1)-(2) is strictly p-dissipative with rate $\lambda \geq 0$ if there exist $\varepsilon > 0$ and a matrix $P = P^{\top}$ with inertia (p,0,n-p) satisfying

$$\begin{bmatrix} \hat{\Theta}_{1,1} & PB - C^{\top}L \\ B^{\top}P - L^{\top}C & -R \end{bmatrix} \leq 0$$
 (22)

where
$$\hat{\Theta}_{1,1} = A^{\top}P + PA + 2\lambda P - C^{\top}QC + \varepsilon I_n$$

In this way, the differential model (7)-(8) contains all the information needed for the study of the global behavior of (1)-(2).

Example 3.9. We illustrate the above analysis with an application to the Nagumo model describing the spatiotemporal dynamics of a bistable transmission line [21],

$$\frac{\partial x}{\partial t}(\theta, t) = D\Delta x(\theta, t) + Ax(\theta, t) - \varphi(x(\theta, t)) \tag{23}$$

where $x(\theta, t) \in \mathbb{R}$, D > 0, and $\varphi : \mathbb{R} \to \mathbb{R}$ is an "N-shape" function as the one shown in Figure 1. Thus, φ satisfies (5) for some $K_1 < 0 < K_2$. The boundary conditions are the same as in (2).

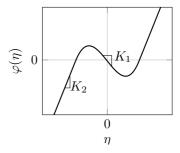


Figure 1: "N-shape" nonlinear function in the differential sector $[K_1,K_2]$.

In this example, condition (15) reduces to $\Pi > 0$ and by using Schur's complement formula it follows that (16) is equivalent to the condition,

$$\Pi^{2} + 2\left(A + \mu - D - \frac{K_{1} + K_{2}}{2}\right)\Pi + \frac{(K_{1} - K_{2})^{2}}{4} < 0$$

Straightforward computations reveal that uniform differential dissipativity of the inhomogeneous dynamics with rate at least μ is guaranteed whenever

$$D > A + \mu - K_1 \tag{24}$$

which implies spatial homogeneity of the closed-loop (1)-(2) according to Theorem 3.4. Now, the complementary dynamics in $\mathcal{R}(T)$ is given by (17), whose dissipativity property is verified by (22), which in this case reduces into

$$P^{2} + 2\left(A + \lambda - \frac{K_{1} + K_{2}}{2}\right)P + \frac{(K_{1} - K_{2})^{2}}{4} < 0 \quad (25)$$

It is easy to verify that if $K_1 > A$ then (25) admits a positive solution P > 0, that is, the homogeneous dynamics is 0-dissipative with rate $0 < \lambda < K_1 - A$. In such case, there is a unique equilibrium for (1)-(2) that is globally asymptotically stable, that is, the complete spatio-temporal behavior goes towards the unique equilibrium. On the other hand, if $A > K_1$, then (25) admits a negative solution P < 0, that is, the homogeneous dynamics is 1-dissipative

with positive rates λ satisfying $\lambda > K_2 - A$. Further, from a conventional local stability analysis one gets that the origin of the dynamics in $\mathcal{R}(T)$ is unstable whenever $A > K_1$. Thus, when condition (24) and $A > K_1$ hold, then the PDE (23) will have a homogeneous bistable behavior. Figure 2 shows the spatio-temporal evolution of the system to two different initial conditions with the following parameters A = 0, D = 1.1, $K_1 = -1$, and $K_2 = 1$.

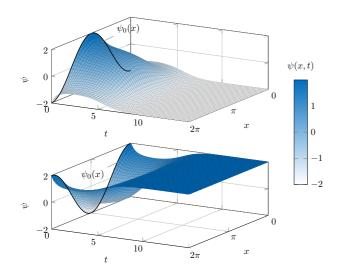


Figure 2: Spatio-temporal evolution of trajectories of Nagumo's equation (23) to two different initial conditions showing both, the spatial homogeneity of solutions and the bistable nature of the transmission line.

4. Analysis in the frequency domain

The linear system (1)-(2) is both space and time invariant (LTSI): solutions shifted in time and in space satisfy the same equation [5].

Spatial and temporal invariance properties of linear systems allow for insightful frequency domain analysis. In this section, we briefly illustrate the frequency-domain interpretation of the results of the previous sections.

4.1. Differential inhomogeneous dynamics

Spatial invariance allows to analyze a linear PDE as a family of ODEs parametrized by the spatial frequency ζ [4, 5, 10, 13]. By taking the Fourier transform of (7a) with respect to the spatial variable θ , we transform the PDE (7a) into the family of linear systems

$$\begin{cases} \frac{d}{dt}\delta x_{\zeta}(t) = (A - \zeta^{2}D)\delta x_{\zeta}(t) + B\delta u_{\zeta}(t) \\ \delta y_{\zeta}(t) = C\delta x_{\zeta}(t) \end{cases}$$
(26)

where, for the case of $\Omega = S^1$, $\zeta \in \mathbb{Z}$ (the dual group to S^1). Notice that each $\delta x_{\zeta}(t)$, $\zeta \in \mathbb{Z}$, is a coefficient on the Fourier series expansion of $\delta x(\cdot,t)$, [10].

The splitting between spatial and temporal differential dynamics in the previous section has an obvious interpretation in the frequency domain: (26) reduces to the differential temporal dynamics for the uniform spatial mode, that is $\zeta = 0$, whereas the differential spatial dynamics correspond to all other modes $\zeta \in \mathbb{Z} \setminus \{0\}$.

The following theorem provides sufficient conditions that guarantee the spatial homogeneity of the closed-loop (1)-(2) via the family of ODEs (26).

Theorem 4.1. Suppose that for each $\zeta \in \mathbb{Z} \setminus \{0\}$, the linear system (26) is 0-dissipative with rate $\mu \geq 0$ and with the same storage function $S(\delta x_{\zeta}) = \delta x_{\zeta}^{\top} \Pi \delta x_{\zeta}$. Then the closed-loop (1)-(2) is spatially homogeneous with the same rate μ .

Proof. The hypothesis on the family of systems (26) is equivalent to the existence of a matrix $\Pi = \Pi^{\top} \succ 0$ satisfying the following family of parametrized LMIs

$$\Theta_{\zeta}(\Pi) := \begin{bmatrix} \Theta_{1,1}(\zeta) & \Pi B - C^{\top} L \\ B^{\top} \Pi - L^{\top} C & -R \end{bmatrix} \preceq 0$$
 (27)

were $\Theta_{1,1}(\zeta) = (A - \zeta^2 D)^\top \Pi + \Pi(A - \zeta^2 D) + 2\mu \Pi - C^\top Q C + \varepsilon I_n$. The rest of the proof consists in showing that (27) is equivalent to conditions (15)-(16). To that end, let $\tau = \frac{1}{\zeta^2} \in (0,1]$. It then follows that condition (27) holds for all $\zeta \in \mathbb{Z} \setminus \{0\}$ if and only if

$$\begin{bmatrix} \tilde{\Theta}_{1,1}(\tau) & \tau \left(\Pi B - C^{\top} L \right) \\ \tau \left(B^{\top} \Pi - L^{\top} C \right) & -\tau R \end{bmatrix} \preceq 0 \qquad (28)$$

holds for all $\tau \in (0,1]$, where $\tilde{\Theta}_{1,1,1}(\tau) = (\tau A - D)^{\top}\Pi + \Pi(\tau A - D) - \tau(C^{\top}QC + 2\mu\Pi + \varepsilon I_n)$. Now, let us assume first that (28) holds. Thus, setting $\tau = 1$ in (28) implies (16). Next, a necessary condition for (28) to hold is

$$-D^{\top}\Pi - \Pi D + \tau (A^{\top}\Pi + \Pi A - C^{\top}QC + 2\mu\Pi + \varepsilon I_n) \leq 0$$

for all $\tau \in (0,1]$. Such condition is possible only if (15) holds. The converse statement follows directly by noting that (28) is contained in the convex combination of conditions (15)-(16). Hence (27) implies uniform differential dissipativity of (1a)-(2) with rate μ on $\mathcal{N}(T)$ and the conclusion follows from Theorem 3.4.

Remark 4.2. The LMI (27) has the interpretation of a dissipativity analysis of the family of systems (26) in feedback interconnection with a family of ζ -parametrized timevarying gains $\delta u_{\zeta} = -\tilde{J}_{\zeta}(t)\tilde{y}_{\zeta}$ satisfying

$$\begin{bmatrix} I_m \\ -\tilde{J}_{\zeta}(t) \end{bmatrix}^{\top} \begin{bmatrix} Q & L \\ L^{\top} & R \end{bmatrix} \begin{bmatrix} I_m \\ -\tilde{J}_{\zeta}(t) \end{bmatrix} \leq 0.$$
 (29)

For each value of ζ , the storage $S(\delta x_{\zeta}) = \delta x_{\zeta}^{\top} \Pi \delta x_{\zeta}$, where $\Pi = \Pi \succeq 0$ satisfies

$$\frac{d}{dt}S(\delta x_{\zeta}) = \begin{bmatrix} \delta x_{\zeta} \\ \delta u_{\zeta} \end{bmatrix}^{\top} \begin{bmatrix} (A - \zeta^{2}D)^{\top}\Pi + \Pi(A - \zeta^{2}D) & \Pi B \\ B^{\top}\Pi & 0 \end{bmatrix} \begin{bmatrix} \delta x_{\zeta} \\ \delta u_{\zeta} \end{bmatrix}$$

and the application of the S-procedure yields (27) as a sufficient condition for the uniform exponential stability of the family of closed-loops. It is worth stressing that in general $\tilde{J}_{\zeta}(t)\delta y_{\zeta}(t)$ is not the spatial Fourier transform of the term $J_{\varphi}(y(\theta,t))\delta y(\theta,t)$ in (7b).

In the previous subsection we analyzed spatial homogenity via the LMIs (15)-(16). The analysis in the spatial frequency domain in this section provides an alternative: because the Fourier transform is an isometry between $L_n^2(\Omega)$ and $l_n^2(\mathbb{Z})$, it is sufficient to show that the dynamics of each Fourier coefficient, given by (26), converges exponentially to zero with rate at least μ for each $\zeta \in \mathbb{Z} \setminus \{0\}$. Therefore, it is enough to verify the stability of (26) subject to the quadratic constraint $\sigma(\delta y_{\zeta}, \delta u_{\zeta}) \leq 0$. That is, to verify only the individual dissipativity properties of each Fourier coefficient. To this end, let us introduce the family of transfer functions associated to (26) as

$$G_{\zeta}(s) = C \left(sI - (A - \zeta^2 D) \right)^{-1} B \tag{30}$$

where $s \in \mathbb{C}$ and $\zeta \in \mathbb{Z} \setminus \{0\}$. In the SISO case, graphical tests (circle criterion) can be derived. Let $\mathcal{D}(K_1, K_2)$ be the disk in the complex plane given by the set

$$\mathcal{D}(K_1, K_2) := \left\{ x + jy \in \mathbb{C} \middle| \left(x + \frac{K_1 + K_2^2}{2K_1 K_2^2} \right) + y^2 \right.$$

$$\leq \left(\frac{K_2 - K_1}{2K_1 K_2} \right)^2 \right\}$$
 (31)

Theorem 4.3. Let $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be such that it satisfies the differential sector condition (5). If for each $\zeta \in \mathbb{Z} \setminus \{0\}$ there exists $\mu \geq 0$ such that

- 1. $G_{\zeta}(s-\mu)$ has no poles on the closure of \mathbb{C}_+ ;
- 2. one of the following conditions is satisfied
 - (a) $0 < K_1 < K_2$ and the Nyquist plot of $G(s \mu)$ lies outside the disk $\mathcal{D}(K_1, K_2)$.
 - (b) $K_1 < 0 < K_2$ and the Nyquist plot of $G(s \mu)$ lies inside the disk $\mathcal{D}(K_1, K_2)$.
 - (c) $K_1 < K_2 < 0$ and the Nyquist plot of $G(s \mu)$ lies outside the disk $\mathcal{D}(K_1, K_2)$.

Then the closed-loop (1)-(2) is spatially homogeneous with rate μ .

Proof. The proof is the same as in the standard circle criterion, see e.g., [14, 17].

Remark 4.4. It is worth to stress that in Theorem (4.3) we have disregarded the cases in which the Nyquist plot make encirclements of the disk $\mathcal{D}(K_1, K_2)$. This is because the Riemann-Lebesgue lemma [7, p. 36] states that $\delta x_{\zeta} \rightarrow 0$ as $|\zeta| \rightarrow +\infty$. Therefore, the family of Nyquist plots cannot make encirclements of any given disk.

4.2. Differential homogeneous dynamics

The second part of the analysis concerns the asymptotic behavior of the model (17), which is finite dimensional. In such case the frequency domain approach is explored in [19], where sufficient conditions are guaranteed.

The analysis is now centered around the feedback interconnection of (26) with $\zeta = 0$ and a nonlinear term $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ satisfying (5). For the sake of completeness we state the main result for the case of SISO systems, whose proof can be found in [19].

Theorem 4.5 (Extended circle criterion). Consider the closed-loop system (20). Let $G_0(s)$ be the transfer function associated to (20a) and let $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy the differential sector condition (5). Then the homogeneous dynamics is p-dominant with rate $\lambda > 0$ if

- 1. $G_0(s-\lambda)$ has q poles on the interior of \mathbb{C}_+ and no poles on the $j\omega$ -axis;
- 2. The Nyquist plot of $G_0(s \lambda)$ makes E = p q clockwise encirclements of the point $-1/K_1$;
- 3. one of the following conditions is satisfied
 - (a) $0 < K_1 < K_2$ and the Nyquist plot of $G(s \lambda)$ lies outside the disk $\mathcal{D}(K_1, K_2)$.
 - (b) $K_1 < 0 < K_2$ and the Nyquist plot of $G(s \lambda)$ lies inside the disk $\mathcal{D}(K_1, K_2)$.
 - (c) $K_1 < K_2 < 0$ and the Nyquist plot of $G(s \lambda)$ lies outside the disk $\mathcal{D}(K_1, K_2)$.

Theorem 4.3 gives us a sufficient condition for spatial homogeneity of reaction-diffusion systems, whereas Theorem 4.5 gives us a sufficient condition for the type of homogeneous motion.

Example 4.6. We apply our approach to the FitzHugh-Nagumo equation

$$\begin{cases} \frac{\partial x_1}{\partial t}(\theta, t) = D_{1,1} \Delta x_1(\theta, t) - x_2(\theta, t) + u(\theta, t) \\ \varepsilon \frac{\partial x_2}{\partial t}(\theta, t) = D_{2,2} \Delta x_2(\theta, t) + a x_1(\theta, t) - b x_2(\theta, t) \\ y(\theta, t) = x_1(\theta, t) \\ u(\theta, t) = -\varphi(y(\theta, t)) \end{cases}$$
(32)

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a nonlinear "N-shape" function in the differential sector $[K_1, K_2]$, as the one shown in Figure 1. We first focus on the analysis of spatial homogeneity. The family of transfer functions $G_{\zeta}(s)$ has the form

$$G_{\zeta}(s) = \frac{s + \frac{1}{\varepsilon}(b + \zeta^2 D_{2,2,})}{\left(s + \zeta^2 D_{1,1}\right)\left(s + \frac{1}{\varepsilon}(b + \zeta^2 D_{2,2})\right) + \frac{a}{\varepsilon}}$$
(33)

Now, we check the conditions stated in Theorem 4.3. Thus, if

$$\mu < \min \left\{ D_{1,1}, \frac{1}{\varepsilon} (b + D_{2,2}) \right\}$$

then condition 1 holds for all $\zeta \in \mathbb{Z} \setminus \{0\}$.

Setting the parameters as, $D_{1,1}=0.5,\ D_{2,2}=0.02,\ \varepsilon=0.1,\ a=0.1,\ b=0.05,\ K_1=-1.0$ and $K_2=1.0,\ we$

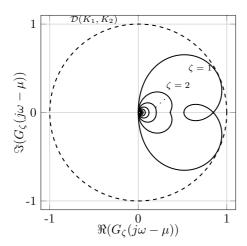


Figure 3: Family of Nyquist plots of (33) for $\zeta \in \mathbb{Z} \setminus \{0\}$ and with parameters $D_{1,1}=0.5,\ D_{2,2}=0.02,\ \varepsilon=0.1,\ a=0.1,\ b=0.05$ and $\mu=0.01$.

now look for the values of μ for which condition 2-(b) also holds. Thus, setting $\mu = 0.01$, we get the family of Nyquist plots depicted in Figure 3.

From Figure 3 it follows that, with our choice of parameters, we can expect a rate of convergence of the synchronization error of at least $\mu=0.01$. With that information on the rate μ , we verify a solution to the LMI conditions (15)-(16) and we get a positive definite solution Π as

$$\Pi = \begin{bmatrix} 1.16451 & -0.61023 \\ -0.61023 & 1.1594 \end{bmatrix}$$

which confirms the spatial homogeneity of the FitzHugh-Nagumo equation with the selected parameters.

The following step consists in retrieving the type of synchronized motion. To that end, we make use of the extension of the circle criterion on Theorem 4.5. The transfer function of interest is

$$G_0(s) = \frac{s + \frac{b}{\varepsilon}}{s^2 + \frac{b}{\varepsilon}s + \frac{a}{\varepsilon}}$$

whose poles are at $\left\{\frac{-b\pm\sqrt{b^2-4a\varepsilon}}{2\varepsilon}\right\}$. Now, we proceed to verify assumptions 1-3 in Theorem 4.5. First, with our choice of parameters, we have that $b^2-4a\varepsilon=-0.0375<0$. It follows that for any $\lambda>\frac{b}{2\varepsilon}=0.25$, assumption 1 is satisfied with q=2. Selecting $\lambda=0.8$, we get the Nyquist diagram of Figure 4 from which, we verify condition 2, (with E=0 and therefore p=2), and 3-(b). Hence, the homogeneous dynamcis is 2-dominant.

Further analysis shows that the origin of the closed-loop (20) is the unique equilibrium point and it is unstable. Hence, a cohesive oscillatory behavior is expected. Figure 5 confirms the analysis.

It is noteworthy that the methods presented here also find application in the analysis of systems with compact spatial domain and Neumann boundary conditions, as stated

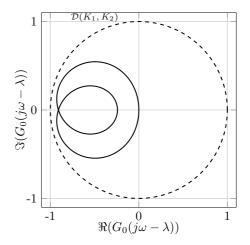


Figure 4: Nyquist plot of the transfer function $G_0(s - \lambda)$ associated to system (17a) with temporal rate $\lambda = 0.8$.

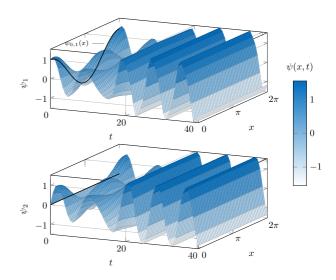


Figure 5: Spatio-temporal evolution of state trajectories of FitzHugh-Nagumo model (32) showing an homogeneous oscillatory behavior.

in [3]. Roughly speaking, let $F(x) = D\Delta x + Ax - B\varphi(Cx)$, then F is equivariant if

$$T_{\vartheta} \circ F = F \circ T_{\vartheta}$$
, and $R \circ F = F \circ R$

where $T_{\vartheta}x(\theta,t) = x(\theta+\vartheta,t)$ and $Rx(\theta,t) = x(-\theta,t)$. Thus, for equivariant equations, any solution satisfying Neumann boundary conditions can be extended, by reflection around the origin, to a solution of the periodic problem. Additionally, the even part of solutions to the periodic problem are also solutions to the Neumann problem, see [3] for details. Thereby, our approach also extends to the analysis of reaction-diffusion systems with Neumann constraints.

5. Conclusions

We illustrated the potential of differential dissipativity for the analysis of nonlinear reaction-diffusion systems.

The differential dynamics naturally decompose into two components, the differential inhomogeneous dynamics and the differential homogeneous dynamics. We illustrated sufficient conditions for spatial homogeneity, that is, contraction of the differential inhomogeneous dynamics, and for p-differential dissipativity of the differential homogeneous dynamics. Future work will explore the same framework to analyze asymptotic spatiotemporal behaviors that are homogeneous neither in space nor in time. Such behaviors include traveling waves and spatiotemporal patterns.

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