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Stabilisation with general decay rate by delay feedback control for nonlinear neutral stochastic functional differential equations with infinite delay

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Abstract

This paper considers a class of neutral-type stochastic functional differential equations with infinite delay (IDNSFDEs) and highly nonlinear coefficients (i.e., coefficients do not satisfy the linear growth condition). These systems are often unstable. Main aim of this paper is to design a delay feedback control to make them become stable with general decay rate. This general decay stability contains the exponential stability and the polynomial stability. Finally, to illustrate our results more clearly, as examples, this paper also introduces unstable scalar neutral-type stochastic integro-differential equations and discusses their exponential and polynomial stabilisation by delay feedback controls, respectively.

Keywords. Neutral stochastic functional differential equations, Infinite delay, Delay feedback control, General decay rate.

1 Introduction

It is well recognized that the functional differential equations can describe systems whose changes depend not only on the present state but also on some of their past states. It has also been well recognized that such systems provide more realistic models for many applications in non-instant transmission phenomena, for example, high velocity fields in wind funnel experiments, or other memory processes, or biological applications such as species' growth or incubating time on disease models among many others; see, for example, [1–3]. However, in many real world applications such as science and engineering, specially in the chemical engineering and the theory of aero-elasticity, the future development of the system depends not only on the present and the past states, but

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also their derivatives with delays, these systems are often described by neutral-type functional differential equations [4]. Moreover, when a system has different delay intervals while a unified model is required (such as the pantograph equation in physics) or a complete influence of the whole past of the state, it is necessary to consider systems with infinite delay [5,6]. Theory of functional differential equations with infinite delay and its applications were developed in the 1970s and 1980s; see [7] and references therein. Since many real world applications are inherently random, neutral stochastic functional differential equations with finite delay or infinite delay have received growing attention; see, for instance, [8–17].

This paper will consider the IDNSFDE

$$d[x(t) - D(x_t,t)] = f(x_t,t)dt + g(x_t,t)dB(t), \tag{1.1}$$

where $D : BC((-\infty,0];\mathbb{R}^n)\times\mathbb{R}_+ \to \mathbb{R}^n$ is a continuous functional, $f : BC((-\infty,0];\mathbb{R}^n)\times\mathbb{R}_+ \to \mathbb{R}^n$ and $g : BC((-\infty,0];\mathbb{R}^n)\times\mathbb{R}_+ \to \mathbb{R}^{n\times m}$ are Borel measurable. This system may be unstable (the instability of stochastic systems could be verified by applying the known criteria [18, Theorem 3.5, P123] or numerical simulations). To make this system become stable, this paper introduces a feedback control $u(x(t-\tau),t)$ such that the controlled system

$$d[x(t) - D(x_t,t)] = [f(x_t,t) + u(x(t-\tau),t)]dt + g(x_t,t)dB(t) \tag{1.2}$$

is stable, where $\tau > 0$ is a constant that stands for the time lag between the time when the observation of the state is made and the time when the feedback control reaches the system, the control function $u : \mathbb{R}^n\times\mathbb{R}_+ \to \mathbb{R}^n$ is Borel measurable.

Since Mao and his co-authors [19] examined the stabilisation problem for stochastic systems by delay feedback controls, theory of stabilisation has been developed very quickly; for example, [9,20,21] for the stabilisation by delay feedback controls, [22–24] for the stabilisation by discrete-time feedback control. However, for the stabilisation problem of neutral stochastic differential equations by delay feedback controls, the aforementioned papers only concern neutral stochastic functional differential equations with finite delay (FDNSFDEs); for instance, [21] for a class of linear and nonlinear FDNSFDEs, [9] for a class of nonlinear FDNSFDEs. It is therefore interesting to consider the stabilisation problem for a class of IDNSFEDs by delay feedback controls.

Moreover, stability is one of the important issues for stochastic systems disturbed by uncertainties and delays. According to the convergence speeds, stability includes exponential stability [25,26], and polynomial stability [13,14,27] and so on. Existing results mainly concern the exponential stabilization [20,26]. However, some systems, such as the following Eq. (1.4), can only be stabilised with a polynomial rate. In other words, these systems cannot be stabilised by using the traditional stabilisation method with exponential speed. This paper hopes to introduce a delay feedback control to stabilize IDNSFDEs with a general decay rate.

The key contribution of this paper are highlighted below:

- introducing the neutral term;
- considering infinite delay;

- examining a more general decay rate.

In addition, in order to clarify our results clearly, as examples, we illustrate that our designed controller can achieve not only the exponential stabilisation of the following scalar IDNSFDE with exponential kernel function (the example in the final section shows that it is unstable):

$$d\Big[x(t)-\int_{-\infty}^{0}x(t+\theta)e^{10\theta}d\theta\Big]=f(x_t,t)dt+g(x_t,t)dB(t), \tag{1.3}$$

where

$$\begin{cases} f(x_t,t)=-5x^3(t)+0.5x(t)\int_{-\infty}^{0}x(t+\theta)e^{\theta}d\theta+x(t)\\ g(x_t,t)=0.5|x(t)|^{3/2}+0.5\int_{-\infty}^{0}|x(t+\theta)|^{3/2}e^{\theta}d\theta, \end{cases}$$

but also the polynomial stabilisation of the following scalar IDNSFDE with polynomial kernel function (the example in the final section shows that it is unstable):

$$d\Big[x(t)-\int_{-\infty}^{0}x(t+\theta)(1-\theta)^{-11}d\theta\Big]=f(x_t,t)dt+g(x_t,t)dB(t), \tag{1.4}$$

where

$$\begin{cases} f(x_t,t)=-5x^3(t)+5x(t)\int_{-\infty}^{0}x(t+\theta)(1-\theta)^{-11}d\theta+x(t)\\ g(x_t,t)=0.5|x(t)|^{3/2}+5\int_{-\infty}^{0}|x(t+\theta)|^{3/2}(1-\theta)^{-11}d\theta. \end{cases}$$

2 Notation and standing hypotheses

Let us introduce some notations and assumptions that will be used. Let $\{\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing, right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t)=(B_1(t),...,B_m(t))^T$ be an m–dimensional Brownian motion defined on this probability space. If $x(t)$ is an $\mathbb{R}^n$–valued stochastic process on $t\in\mathbb{R}$, define $x_t=x_t(\theta)=\{x(t+\theta):-\infty<\theta\leq 0\}$ for $t\geq 0$ and let $\tilde{x}(t)=x(t)-D(x_t,t)$.

Let $|x|$ be the Euclidean norm in $\mathbb{R}^n$. Denote by $C((-\infty,0];\mathbb{R}^n)$ the family of continuous functions from $(-\infty,0]$ to $\mathbb{R}^n$. Similarly, denote by $BC((-\infty,0];\mathbb{R}^n)$ the family of bounded continuous functions from $(-\infty,0]$ to $\mathbb{R}^n$ with the norm $\|\phi\|=\sup_{\theta\leq 0}|\phi(\theta)|<\infty$, which forms a Banach space. Let $L^p((-\infty,0];\mathbb{R}^n)$ denote all functions $h:(-\infty,0]\to\mathbb{R}^n$ such that $\int_{-\infty}^{0}|h(s)|^p ds<\infty$. If A is a vector or a matrix, its transpose is denoted by A^T. For a matrix A, denote its trace norm by $|A|=\sqrt{\mathrm{trace}(A^TA)}$ and operator norm by $\|A\|=\sup_{|v|=1}|Av|$. Let $\mathbb{R}_+=[0,\infty)$ and $\tau>0$. If both a and b are real numbers, then $a\wedge b:=\min\{a,b\}$, $a\vee b:=\max\{a,b\}$ and $a_+:=a\vee 0$. Throughout the paper, C denotes a generic positive constant, whose value may change for different usage. Similarly, denote by $C(\alpha)$ the generic positive constant depending on parameter α.

Let us first introduce the general decay function (ψ–type function) and stability with the general decay rate (ψ–type stability), also see [13].

Definition 2.1. *The function $\psi : \mathbb{R} \to (0, \infty)$ is said to be the ψ–type function if this function satisfies the following conditions:*

(1) *it is continuous and nondecreasing in $\mathbb{R}$ and differentiable in $\mathbb{R}_+$;*

(2) *$\psi(0) = 1$ and $\psi(\infty) = \infty$;*

(3) *$\phi = \sup_{t\geq 0}[\psi'(t)/\psi(t)] < \infty$;*

(4) *for any $t, s \geq 0, \psi(t) \leq \psi(s)\psi(t-s)$.*

It is obvious that functions $\psi(t) = e^{\alpha t}, \psi(t) = (1+t_+)^\alpha$ for any $\alpha > 0$ are ψ–type functions since they satisfy the above four conditions.

Definition 2.2. *System (1.2) is said to be ψ–type stable in δ–th moment if there exists a pair of positive constants q and δ such that for any initial data $\varphi \in BC((-\infty, 0]; \mathbb{R}^n) \cap L^\delta((-\infty, 0]; \mathbb{R}^n)$,*

$$\limsup_{t\to\infty} \frac{\ln \mathbb{E}|x(t)|^\delta}{\ln \psi(t)} \leq -q.$$

When $\delta = 2$, it is said to be ψ–type stable in mean square.

Definition 2.3. *System (1.2) is said to be almost surely ψ–type stable if there exists a pair of positive constants q and δ such that for any initial data $\varphi \in BC((-\infty, 0]; \mathbb{R}^n) \cap L^\delta((-\infty, 0]; \mathbb{R}^n)$,*

$$\limsup_{t\to\infty} \frac{\ln |x(t)|}{\ln \psi(t)} \leq -q \qquad a.s..$$

Clearly, the ψ–type stability implies the exponential stability and polynomial stability when $\psi(t) = e^{\alpha t}$ and $\psi(t) = (1+t_+)^\alpha$ for any $\alpha > 0$, respectively.

Remark 2.1. Compared to the traditional stability, the ψ–type stability is a class of more general stability, which includes the ordinary exponential stability and polynomial stability. In the example behind, it can be observed that by the ψ–type stability, we can not only obtain the exponential stability of the system (4.4), but also the polynomial stability of the system (4.6). In other words, the different stability for different system can be examined simultaneously.

Lemma 2.2. *Let $\phi \in BC((-\infty, 0]; \mathbb{R}^n) \cap L^p((-\infty, 0]; \mathbb{R}^n)$ for some $p > 0$. Then for any $q > p$, $\phi \in BC((-\infty, 0]; \mathbb{R}^n) \cap L^q((-\infty, 0]; \mathbb{R}^n)$.*

This lemma can be found in [14]. Denote by $\mathcal{M}_0$ the set of probability measures on $(-\infty, 0]$. For a given ψ which satisfies Definition 2.1, let us further define $\mathcal{M}_\epsilon$ for each $\epsilon > 0$, the subset of $\mathcal{M}_0$, by

$$\mathcal{M}_\epsilon = \left\{ \mu \in \mathcal{M}_0;\ \mu_\epsilon \ := \int_{-\infty}^0 \psi^\epsilon(-\theta)\mu(d\theta) < \infty \right\}.$$

Lemma 2.3. *Fix $\epsilon_0 > 0$. If $\mu \in \mathcal{M}_{\epsilon_0}$, then for any $\epsilon \in (0, \epsilon_0)$, μ_ϵ is continuously nondecreasing and satisfies $\mu_{\epsilon_0} \geq \mu_\epsilon \geq \mu_0 = 1$.*

This lemma can be found in [13]. Let us impose the following assumptions:

Assumption 2.1. *Assume for each integer $h \geq 1$, there exists a positive constant k_h such that*

$$|f(\xi,t)-f(\eta,t)| \vee |g(\xi,t)-g(\eta,t)| \leq k_h\|\xi-\eta\| \tag{2.1}$$

for all $\xi,\eta \in BC((-\infty,0];\mathbb{R}^n)$ with $\|\xi\| \vee \|\eta\| \leq h$ and all $t \in \mathbb{R}_+$. Assume there exist positive constants K, $q_1 > 1$, $q_2 \geq 1$ and probability measures $\mu_j \in \mathcal{M}_{\epsilon_0}$ $(j=1,2)$ for some given $\epsilon_0 > 0$ such that for $(\xi,t) \in BC((-\infty,0];\mathbb{R}^n) \times \mathbb{R}_+$,

$$\begin{cases} |f(\xi,t)| \leq K\Big(\int_{-\infty}^0 |\xi(\theta)|^{q_1}\mu_1(d\theta) + |\xi(0)|^{q_1} + \int_{-\infty}^0 |\xi(\theta)|\mu_1(d\theta) + |\xi(0)|\Big), \\ |g(\xi,t)| \leq K\Big(\int_{-\infty}^0 |\xi(\theta)|^{q_2}\mu_2(d\theta) + |\xi(0)|^{q_2} + \int_{-\infty}^0 |\xi(\theta)|\mu_2(d\theta) + |\xi(0)|\Big). \end{cases} \tag{2.2}$$

This assumption implies that $f(0,t) = g(0,t) = 0$.

Assumption 2.2. *For ϵ_0 given in Assumption 2.1, and there exists $L_* \in (0,1)$ such that*

$$\frac{\{(q_1-1)\big[(1-L_*)^{-q_1}+L_*\big]+2\}\frac{L_*}{2}}{(q_1+1)(1-L_*)^{(q_1-2)_+}(1-L_*^{(q_1-1)\wedge 1})} + \frac{L_*}{2} \leq 1,$$

then there exist a probability measure $\mu_3 \in \mathcal{M}_{\epsilon_0}$ and a constant $L \in (0,L^*)$ such that

$$|D(\xi,t)-D(\eta,t)| \leq L\int_{-\infty}^0 |\xi(\theta)-\eta(\theta)|\mu_3(d\theta) \tag{2.3}$$

for $(\xi,\eta,t) \in BC((-\infty,0];\mathbb{R}^n) \times BC((-\infty,0];\mathbb{R}^n) \times \mathbb{R}_+$. For the purpose of stability, assume that $D(0,t) = 0$.

These two assumptions imply that Eq. (1.1) admits a trivial solution.

Assumption 2.3. *For μ_j $(j=1,2,3)$ and L in Assumptions 2.1, 2.2, there exist positive constants β_j $(1 \leq j \leq 4)$, k_j $(1 \leq j \leq 4)$ and $p > 2$ with $(q_1+1) \vee (2q_2-q_1+1) \leq p$ and*

$$[(q_1-1)\big((1-L)^{2-p-q_1}+L\big)+p](\beta_1+\beta_2+\beta_3) < (p+q_1-1)(1-L)^{(q_1-2)_+}(1-L^{(q_1-1)\wedge 1})\beta_4 \tag{2.4}$$

such that for $(\xi,t) \in BC((-\infty,0];\mathbb{R}^n) \times \mathbb{R}_+$,

$$\begin{aligned} &[\xi(0)-D(\xi,t)]^T f(\xi,t) + \frac{q_1}{2}|g(\xi,t)|^2 \\ &\leq \sum_{j=1}^3 \beta_j \int_{-\infty}^0 |\xi(\theta)|^p \mu_j(d\theta) - \beta_4|\xi(0)|^p + \sum_{j=1}^3 k_j \int_{-\infty}^0 |\xi(\theta)|^2\mu_j(d\theta) + k_4|\xi(0)|^2. \end{aligned} \tag{2.5}$$

Remark 2.4. It is easily observed from (2.4) and (2.5) that $\beta_1+\beta_2+\beta_3 < \beta_4$ and

$$\begin{aligned} &[\xi(0)-D(\xi,t)]^T f(\xi,t) + \frac{1}{2}|g(\xi,t)|^2 \\ &\leq \sum_{j=1}^3 \beta_j \int_{-\infty}^0 |\xi(\theta)|^p \mu_j(d\theta) - \beta_4|\xi(0)|^p + \sum_{j=1}^3 k_j \int_{-\infty}^0 |\xi(\theta)|^2\mu_j(d\theta) + k_4|\xi(0)|^2. \end{aligned}$$

By using a standard argument as in the proofs of [26, Theorem 2.5] and [13, Theorem 3.2], we can prove that under Assumptions 2.1–2.3, Eq. (1.1) has a unique global solution for any given initial data $\varphi \in BC((-\infty,0];\mathbb{R}^n) \cap L^2((-\infty,0];\mathbb{R}^n)$.

Before presenting the following Theorem 2.6, we prepare a lemma first.

Lemma 2.5. Let Assumption 2.2 holds. Then for any $(\xi, t) \in BC((-\infty, 0]; \mathbb{R}^n) \times \mathbb{R}_+$,

$$\begin{cases} |\xi(0) - D(\xi,t)|^q \geq (1-L)^{(q-1)_+}\Big(|\xi(0)|^q - L^{q\wedge 1}\Big|\int_{-\infty}^0 \xi(\theta)\mu_3(d\theta)\Big|^q\Big), & q > 0, \\ |\xi(0) - D(\xi,t)|^q \leq (1-L)^{1-q}|\xi(0)|^q + L\int_{-\infty}^0 |\xi(\theta)|^q\mu_3(d\theta), & q > 1. \end{cases}$$

Proof. Recall the elementary inequality: for $q > 1$, $\varrho > 0$, $a, b \in \mathbb{R}$, $|a+b|^q \leq (1+\varrho)^{q-1}(|a|^q + \varrho^{1-q}|b|^q)$ (see [18, Lemma 4.1, page211]). Setting $\varrho = L/(1-L)$ gives

$$|a+b|^q \leq (1-L)^{1-q}|a|^q + L^{1-q}|b|^q. \tag{2.6}$$

Applying the Hölder inequality gives

$$\begin{aligned} |\xi(0) - D(\xi,t)|^q &\leq \Big[|\xi(0)| + L\int_{-\infty}^0 |\xi(\theta)|\mu_3(d\theta)\Big]^q \\ &\leq (1-L)^{1-q}|\xi(0)|^q + L\int_{-\infty}^0 |\xi(\theta)|^q\mu_3(d\theta). \end{aligned} \tag{2.7}$$

While for $0 < q \leq 1$,

$$|a+b|^q \leq |a|^q + |b|^q. \tag{2.8}$$

Combining (2.6) and (2.8) yields that for any $q > 0$,

$$|a+b|^q \leq (1-L)^{-(q-1)_+}|a|^q + L^{-(q-1)_+}|b|^q,$$

which implies for any $q > 0$,

$$\begin{aligned} |\xi(0)|^q &= |\xi(0) - D(\xi,t) + D(\xi,t)|^q \\ &\leq (1-L)^{-(q-1)_+}|\xi(0) - D(\xi,t)|^q + L^{q\wedge 1}\Big|\int_{-\infty}^0 \xi(\theta)\mu_3(d\theta)\Big|^q. \end{aligned}$$

Hence,

$$|\xi(0) - D(\xi,t)|^q \geq (1-L)^{(q-1)_+}\Big(|\xi(0)|^q - L^{q\wedge 1}\Big|\int_{-\infty}^0 \xi(\theta)\mu_3(d\theta)\Big|^q\Big). \tag{2.9}$$

Then the desired assertion follows from (2.7) and (2.9) immediately. □

Theorem 2.6. Let Assumptions 2.1–2.3 hold. For any given initial data $\varphi \in BC((-\infty, 0]; \mathbb{R}^n) \cap L^{q_1+1}((-\infty, 0]; \mathbb{R}^n)$, the solution of Eq. (1.1) satisfies $\sup_{0\leq t\leq T}\mathbb{E}|x(t)|^{q_1+1} < \infty$ for all $T > 0$.

By applying the Itô formula and Lemma 2.5, we first reveals $\sup_{0\leq t\leq T}\mathbb{E}|\tilde{x}(t)|^{q_1+1} \leq C$. Then the desired result follows easily. To keep the flow of the presentation, this proof is deferred to part A1 in Section Appendix.

3 Stabilisation with general decay rate

The system (1.1) may be unstable. The main aim of this section is to design a delay feedback control $u(x(t-\tau),t)$ by which the controlled system (1.2) is stable with general decay rate. Before presenting the main results, we give the following assumptions on the control function.

Assumption 3.1. *There exists a positive number $\kappa > 0$ such that*

$$|u(x,t) - u(y,t)| \le \kappa |x-y|$$

for all $x, y \in \mathbb{R}^n$ and all $t \ge 0$. Moreover, assume that $u(0,t) \equiv 0$.

Under Assumptions 2.1–2.3 and 3.1, the controlled system (1.2) admits a unique global solution and the solution satisfies $\sup_{0\le t\le T} \mathbb{E}|x(t)|^{q_1+1} < \infty$ for all $T > 0$, which can be proved by the same methods as Theorem 2.6, so the proof is omitted.

Assumption 3.2. *For L, μ_j $(1 \le j \le 3)$ and β_j $(1 \le j \le 4)$ given in Assumptions 2.1–2.3, there exist positive constants α_j, $\hat{\alpha}_j$, $\hat{\beta}_j$ $(1 \le j \le 4)$ with $\alpha_4 \ge 2\beta_4$, $\alpha_j \le 2\beta_j$ $(j = 1, 2, 3)$ and*

$$\begin{cases} \Big((q_1-1)\big[(1-L)^{-q_1} + L\big] + 2\Big)(\hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_3) < (q_1+1)(1-L)^{(q_1-2)_+}(1 - L^{(q_1-1)\wedge 1})\hat{\beta}_4, \\ \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 < \hat{\alpha}_4. \end{cases} \tag{3.1}$$

such that for $(\xi, t) \in BC((-\infty,0];\mathbb{R}^n) \times \mathbb{R}_+$,

$$\begin{aligned} &2[\xi(0) - D(\xi,t)]^T [f(\xi,t) + u(\xi(0),t)] + |g(\xi,t)|^2 \\ &\quad \le \sum_{j=1}^{3} \alpha_j \int_{-\infty}^{0} |\xi(\theta)|^p \mu_j(d\theta) - \alpha_4 |\xi(0)|^p + \sum_{j=1}^{3} \hat{\alpha}_j \int_{-\infty}^{0} |\xi(\theta)|^2 \mu_j(d\theta) - \hat{\alpha}_4 |\xi(0)|^2 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &[\xi(0) - D(\xi,t)]^T [f(\xi,t) + u(\xi(0),t)] + \frac{q_1}{2} |g(\xi,t)|^2 \\ &\quad \le \sum_{j=1}^{3} \beta_j \int_{-\infty}^{0} |\xi(\theta)|^p \mu_j(d\theta) - \beta_4 |\xi(0)|^p + \sum_{j=1}^{3} \hat{\beta}_j \int_{-\infty}^{0} |\xi(\theta)|^2 \mu_j(d\theta) - \hat{\beta}_4 |\xi(0)|^2. \end{aligned} \tag{3.3}$$

It should be pointed out that there is a rich class of control functions which satisfy Assumptions 3.1 and 3.2 under Assumptions 2.1– 2.3. For example, choose a symmetric matrix A such that $\|A\| = 1$ and

$$\lambda_m := \lambda_{\min}(A) > \lambda_* := \frac{\{(q_1-1)\big[(1-L_*)^{-q_1} + L_*\big] + 2\}\frac{L_*}{2}}{(q_1+1)(1-L_*)^{(q_1-2)_+}(1 - L_*^{(q_1-1)\wedge 1})} + \frac{L_*}{2}.$$

Design $u(x,t) = -\kappa A x$ for $\kappa > 0$. Then $u(x,t)$ satisfies Assumption 3.1 clearly. We will now show that it satisfies Assumption 3.2 provided κ is sufficiently large. In fact, for any $(\xi, t) \in BC((-\infty,0];\mathbb{R}^n) \times \mathbb{R}_+$,

$$\big(\xi(0) - D(\xi,t)\big)^T u(\xi(0),t) \le -\Big(\kappa\lambda_m - \frac{L\kappa}{2}\Big)|\xi(0)|^2 + \frac{L\kappa}{2} \int_{-\infty}^{0} |\xi(\theta)|^2 \mu_3(d\theta).$$

It then follows from (2.5) that

$$\begin{aligned}&2[\xi(0)-D(\xi,t)]^T[f(\xi,t)+u(\xi(0),t)]+|g(\xi,t)|^2\\&\quad\le 2\sum_{j=1}^{3}\beta_j\int_{-\infty}^{0}|\xi(\theta)|^p\mu_j(d\theta)-2\beta_4|\xi(0)|^p+2\sum_{j=1}^{2}k_j\int_{-\infty}^{0}|\xi(\theta)|^2\mu_j(d\theta)\\&\qquad+(2k_3+L\kappa)\int_{-\infty}^{0}|\xi(\theta)|^2\mu_3(d\theta)-(2\kappa\lambda_m-L\kappa-2k_4)|\xi(0)|^2\end{aligned}$$

and

$$\begin{aligned}&[\xi(0)-D(\xi,t)]^T[f(\xi,t)+u(\xi(0),t)]+\frac{q_1}{2}|g(\xi,t)|^2\\&\quad\le \sum_{j=1}^{3}\beta_j\int_{-\infty}^{0}|\xi(\theta)|^p\mu_j(d\theta)-\beta_4|\xi(0)|^p+\sum_{j=1}^{2}k_j\int_{-\infty}^{0}|\xi(\theta)|^2\mu_j(d\theta)\\&\qquad+\Big(k_3+\frac{L\kappa}{2}\Big)\int_{-\infty}^{0}|\xi(\theta)|^2\mu_3(d\theta)-\Big(\kappa\lambda_m-\frac{L\kappa}{2}-k_4\Big)|\xi(0)|^2\end{aligned}$$

hold for $(\xi,t)\in BC((-\infty,0];\mathbb{R}^n)\times\mathbb{R}_+$. Since $L\in(0,L^*)$ and $\lambda_m>\lambda_*$, we can always choose κ large enough such that

$$\begin{cases}\Big((q_1-1)\big[(1-L)^{-q_1}+L\big]+2\Big)\big(k_1+k_2+k_3\big)+(q_1+1)(1-L)^{(q_1-2)_+}(1-L^{(q_1-1)\wedge 1})k_4\\ <\Big\{(q_1+1)(1-L)^{(q_1-2)_+}(1-L^{(q_1-1)\wedge 1})\big(\lambda_m-\frac{L}{2}\big)-\Big((q_1-1)\big[(1-L)^{-q_1}+L\big]+2\Big)\frac{L}{2}\Big\}\kappa,\\ k_1+k_2+k_3+k_4<(\lambda_m-L)\kappa,\end{cases}$$

which is equivalent to

$$\begin{cases}\Big((q_1-1)\big[(1-L)^{-q_1}+L\big]+2\Big)\big(k_1+k_2+k_3+\frac{L\kappa}{2}\big)\\ \qquad<(q_1+1)(1-L)^{(q_1-2)_+}(1-L^{(q_1-1)\wedge 1})\big(\kappa\lambda_m-\frac{L\kappa}{2}-k_4\big),\\ 2k_1+2k_2+(2k_3+L\kappa)<2\kappa\lambda_m-L\kappa-2k_4.\end{cases}\tag{3.4}$$

This shows that the control function u satisfies Assumption 3.2.

Define $L_1V: BC((-\infty,0];\mathbb{R}^n)\times\mathbb{R}_+\to\mathbb{R}$ by

$$\begin{aligned}L_1V(\xi,t)&=2[\xi(0)-D(\xi,t)]^T\big[f(\xi,t)+u(\xi(0),t)\big]+|g(\xi,t)|^2+(q_1+1)|\xi(0)-D(\xi,t)|^{q_1-1}\\&\quad\times\Big\{[\xi(0)-D(\xi,t)]^T[f(\xi,t)+u(\xi(0),t)]+\frac{q_1}{2}|g(\xi,t)|^2\Big\}.\end{aligned}\tag{3.5}$$

The following lemma will plays an important role in this paper.

Lemma 3.1. *Let Assumptions 2.1–2.3 and 3.1–3.2 hold. Then there exist positive constants ρ_1–ρ_6, γ_j, $\hat\gamma_j$, $\bar\gamma_j$, $\tilde\gamma_j$ $(j=1,2,3)$ and $\epsilon^*\in(0,\epsilon_0]$ such that*

$$\begin{aligned}&L_1V(\xi,t)+\rho_1\big(2|\xi(0)-D(\xi,t)|+(q_1+1)|\xi(0)-D(\xi,t)|^{q_1}\big)^2+\rho_2|f(\xi,t)|^2+\rho_3|g(\xi,t)|^2\\&\quad\le-\rho_4|\xi(0)|^2-\rho_5|\xi(0)|^{q_1+1}-\rho_6|\xi(0)|^{p+q_1-1}+\sum_{j=1}^{3}\gamma_jH_{j,\epsilon^*}^{p+q_1-1}+\sum_{j=1}^{3}\hat\gamma_jH_{j,\epsilon^*}^{q_1+1}\end{aligned}$$

$$+\sum_{j=1}^{3}\bar{\gamma}_j H^p_{j,\epsilon^*}+\sum_{j=1}^{3}\tilde{\gamma}_j H^2_{j,\epsilon^*},$$

where $H^{\alpha}_{j,\epsilon^*}=\int_{-\infty}^{0}|\xi(\theta)|^{\alpha}\mu_j(d\theta)-\mu_{j\epsilon^*}|\xi(0)|^{\alpha}$.

This proof is deferred to part A2 in Section Appendix. In the following, we will prove the controlled system (1.2) is ψ–type stable with the help of Lemma 3.1. Define $\mathcal{L}V: BC((-\infty,0];\mathbb{R}^n)\times\mathbb{R}_+\to\mathbb{R}$ by

$$\begin{aligned}\mathcal{L}V(\xi,t) \;=\;& 2[\xi(0)-D(\xi,t)]^T[f(\xi,t)+u(\xi(-\tau),t)]+|g(\xi,t)|^2\\&+(q_1+1)|\xi(0)-D(\xi,t)|^{q_1-1}[\xi(0)-D(\xi,t)]^T[f(\xi,t)+u(\xi(-\tau),t)]\\&+\frac{q_1+1}{2}|\xi(0)-D(\xi,t)|^{q_1-1}|g(\xi,t)|^2\\&+\frac{(q_1+1)(q_1-1)}{2}|\xi(0)-D(\xi,t)|^{q_1-3}\big|[\xi(0)-D(\xi,t)]^Tg(\xi,t)\big|^2. \end{aligned}\tag{3.6}$$

Recalling the definition of L_1V and $q_1>1$, it is obvious that

$$\begin{aligned}\mathcal{L}V(\xi,t) \;\le\;& 2[\xi(0)-D(\xi,t)]^T[f(\xi,t)+u(\xi(-\tau),t)]+|g(\xi,t)|^2\\&+(q_1+1)|\xi(0)-D(\xi,t)|^{q_1-1}\Big\{[\xi(0)-D(\xi,t)]^T[f(\xi,t)+u(\xi(-\tau),t)]+\frac{q_1}{2}|g(\xi,t)|^2\Big\}\\=\;& L_1V(\xi,t)+\big\{2+(q_1+1)|\xi(0)-D(\xi,t)|^{q_1-1}\big\}\\&\times[\xi(0)-D(\xi,t)]^T[u(\xi(-\tau),t)-u(\xi(0),t)].\end{aligned}\tag{3.7}$$

Theorem 3.2. *Let conditions in Lemma 3.1 hold. For ρ_1 – ρ_4 and ϵ^* in Lemma 3.1, if $\tau>0$ satisfies*

$$\tau<\frac{(1-L)\sqrt{\rho_1\rho_4}}{2\kappa^2}\wedge\frac{(1-L)}{4\sqrt{2}\kappa}\quad\text{and}\quad\tau\le\frac{(1-L)\sqrt{\rho_1\rho_2}}{\sqrt{2}\kappa}\wedge\frac{(1-L)^2\rho_1\rho_3}{\kappa^2},\tag{3.8}$$

then there exists $q^*\in(0,\epsilon^*]$ such that for any initial data $\varphi\in BC((-\infty,0];\mathbb{R}^n)\cap L^2((-\infty,0];\mathbb{R}^n)$, the solution of the controlled system (1.2) satisfies

$$\mathbb{E}\int_0^{\infty}\psi^{q^*}(s)|x(s)|^{\hat{p}}ds<C,\qquad\forall\hat{p}\in[2,p+q_1-1]\tag{3.9}$$

and

$$\limsup_{t\to\infty}\frac{\ln\mathbb{E}|x(t)|^{\hat{p}}}{\ln\psi(t)}\le -q^*,\qquad\forall\hat{p}\in[2,q_1+1].\tag{3.10}$$

The proof follows the idea in the proof of Theorem 3.6 in [26] by using the method of the Lyapunov functionals. It is deferred to part A3 in Section Appendix.

Theorem 3.3. *Let conditions of Theorem 3.2 hold. Then the controlled system (1.2) is almost surely ψ–type stable.*

The key is to $\mathbb{E}\big(\sup_{0\le t<\infty}\psi^{q^*}(t)|x(t)|^2\big)<\infty$ by the Itô formula and inequality (3.9). Then the desired result then follows easily. We defer the proof to part A4 in Section Appendix.

Remark 3.4. The main aim of this paper is to examine the ψ–type stability, which shows the general convergence speed. Compared to the conventional exponential stabilisation in [20, 26], the results in Theorems 3.2 and 3.3 are more general results. Theorem 3.2 shows that the neutral term has effect on the bound of τ by the parameter L. The smaller L will lead to the bigger τ. In this paper, the stability conditions are imposed on the non-delay control in Assumption 3.2. To obtain the stability results by the delay feedback control, in Lemma 3.1, ρ_1, ρ_2 and ρ_3 represent stability costs. The results in Theorem 3.2 also show that the bigger ρ_1, ρ_2 and ρ_3 will lead to the bigger τ.

Remark 3.5. By virtue of the continuous semimartingale convergence theory, [13] established the almost sure ψ–type stability of IDNSFDEs. In this paper, suppose that the given IDNSFDE is unstable, and we are required to design a delay feedback control in the drift part so that the given equation become ψ–type stable in moment and almost surely ψ–type stable. In this paper, since the stability conditions are imposed on the non-delay control in Assumption 3.2, the continuous semimartingale convergence theorem cannot be applied to obtain the almost sure ψ–type stability of the delay controlled system, directly. To overcome this difficulty, new techniques are developed to prove (3.9), which makes the almost sure ψ–type stabilisation possible.

Remark 3.6. The results of this paper are neither simple generalisations of the results in [9] from finite delay to infinite delay, nor simply a case of adding a neutral item in [26]. In [9], only the moment stabilisation is proved. This paper also examines the almost sure stabilisation possible due to (3.9). Moreover, this paper also reveals the rate at which the solution tends to zero. In addition, exponential stabilisation is only considered in [26]. Due to the coexistence of neutral term and infinite delay, the method of Theorem 3.8 in [26] cannot be applied to obtain the almost sure ψ–type stability. In order to overcome this difficulty, more delicate estimates and more complex computations are needed.

4 Example

Many real phenomenon can be modeled by neutral functional or delay differential equations, for instance, [28] examined a coupled oscillator-pendulum system. In particular, [29] discussed the collision problem in electrodynamics by the neutral delay differential equation

$$\dot{x}(t) = f_1(x(t), x(\delta(t))) + f_2(x(t), x(\delta(t)))\dot{x}(\delta(t)),$$

where $\delta(t) \leq t$. These systems can be generalized to the following neutral functional differential equation

$$d[x(t) - D(x_t, t)] = f(x_t, t)dt.$$

Taking into account stochastic perturbations leads to the neutral stochastic functional differential equation

$$d[x(t) - D(x_t, t)] = f(x_t, t)dt + g(x_t, t)dB(t). \tag{4.1}$$

In the following, as examples, two special cases will be discussed.

Case 1 : exponential kernel function. Let $\mu_1(d\theta) = \mu_2(d\theta) = e^{\theta}d\theta$, $\mu_3(d\theta) = 10e^{10\theta}d\theta$ for $\theta \in (-\infty, 0]$ and

$$\begin{aligned} f(x_t,t) &= -5x^3(t) + 0.5x(t)\int_{-\infty}^{0} x(t+\theta)\mu_1(d\theta) + x(t) \\ g(x_t,t) &= 0.5|x(t)|^{3/2} + 0.5\int_{-\infty}^{0} |x(t+\theta)|^{3/2}\mu_2(d\theta) \\ D(x_t,t) &= 0.1\int_{-\infty}^{0} x(t+\theta)\mu_3(d\theta). \end{aligned}$$

In this case, Eq. (4.1) can be rewritten as Eq. (1.3). Choosing $\epsilon_0 = 0.9$ and $\psi(t) = e^t$, it is easy to see that

$$\mu_{1\epsilon_0} = \int_{-\infty}^{0} e^{-\epsilon_0\theta}de^{\theta} = 10,$$

which implies that $\mu_1 \in M_{0.9}$. Similarly, $\mu_2, \mu_3 \in M_{0.9}$. Clearly, Assumption 2.1 holds with $q_1 = 3$, $q_2 = 1.5$, $K = 5$ and Assumption 2.2 holds with $L = 0.1$. It can be found Assumption 2.3 holds with $p = 4$, $\beta_1 = 0.0125$, $\beta_2 = 0.375$, $\beta_3 = 0.1375$, $\beta_4 = 4$, $k_1 = 0.25$, $k_2 = 0.375$, $k_3 = 0.05$, $k_4 = 1.45$. Let the initial value

$$\varphi(t) = \begin{cases} 5e^{0.01t} - 5e^{-1}, & \text{if} \quad t \in (-100, 0], \\ 0, & \text{if} \quad t \in (-\infty, -100]. \end{cases} \tag{4.2}$$

By Remark 2.4 and Theorem 2.6, Eq. (1.3) has a unique solution with $\sup_{0\le t\le T}\mathbb{E}|x(t)|^4 < \infty$ for any $T > 0$. The simulation (Figure 1(a)) shows the system (1.3) is not stable.

Let us now introduce a delay feedback control to stabilize the system (1.3). We take the control function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ as follows:

$$u(x,t) = -6x. \tag{4.3}$$

Obviously, Assumption 3.1 is satisfied with $\kappa = 6$ and the controlled system

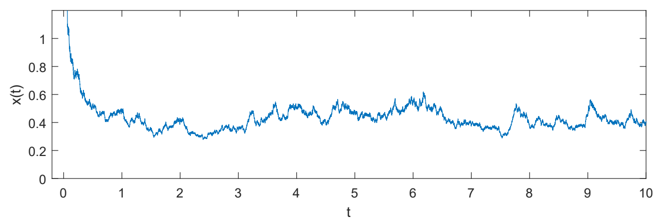
$$d\Big[x(t) - \int_{-\infty}^{0} x(t+\theta)e^{10\theta}d\theta\Big] = [f(x_t,t) + u(x(t-\tau),t)]dt + g(x_t,t)dB(t) \tag{4.4}$$

has a unique global solution. Moreover, using the Cauchy inequality and the Young inequality gives

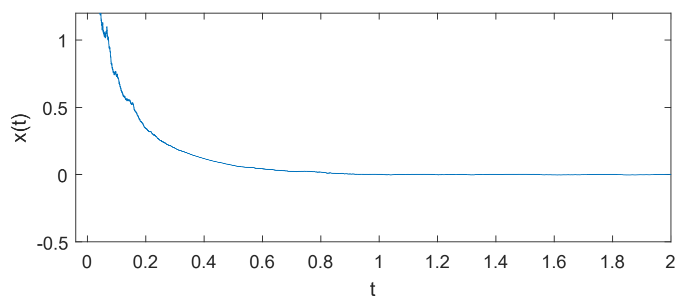
$$\begin{aligned} &[x(t) - D(x_t,t)]^T[f(x_t,t) + u(x(t),t)] + 1.5|g(x_t,t)|^2 \\ &\quad\le x^T(t)[f(x_t,t) + u(x(t),t)] + |D(x_t,t)||f(x_t,t)| + |D(x_t,t)||u(x(t),t)| + 1.5|g(x_t,t)|^2 \\ &\quad\le -4|x(t)|^4 + 0.0125\int_{-\infty}^{0}|x(t+\theta)|^4\mu_1(d\theta) + 0.375\int_{-\infty}^{0}|x(t+\theta)|^4\mu_2(d\theta) \\ &\qquad +0.1375\int_{-\infty}^{0}|x(t+\theta)|^4\mu_3(d\theta) - 4.25|x(t)|^2 + 0.25\int_{-\infty}^{0}|x(t+\theta)|^2\mu_1(d\theta) \\ &\qquad +0.375\int_{-\infty}^{0}|x(t+\theta)|^2\mu_2(d\theta) + 0.35\int_{-\infty}^{0}|x(t+\theta)|^2\mu_3(d\theta) \end{aligned}$$

and

$$2[x(t) - D(x_t,t)]^T[f(x_t,t) + u(x(t),t)] + |g(x_t,t)|^2$$



(a) The computer simulation of the sample path of the solution to Eq. (1.3) using the Euler-Maruyama method with step size 10^{-4}.



(b) The computer simulation of the sample path of the solution to Eq. (4.4) using the Euler-Maruyama method with step size 10^{-4}.

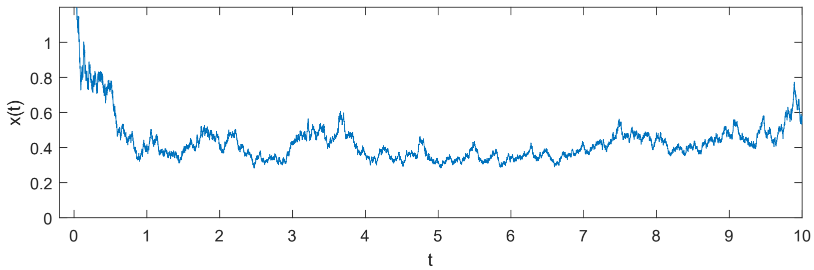
Figure 1: with exponential kernel functions

$$
\begin{aligned}
&\leq -8.5|x(t)|^4 + 0.025\int_{-\infty}^{0}|x(t+\theta)|^4\mu_1(d\theta) + 0.25\int_{-\infty}^{0}|x(t+\theta)|^4\mu_2(d\theta)\\
&\quad +0.275\int_{-\infty}^{0}|x(t+\theta)|^4\mu_3(d\theta) - 9|x(t)|^2 + 0.5\int_{-\infty}^{0}|x(t+\theta)|^2\mu_1(d\theta)\\
&\quad +0.25\int_{-\infty}^{0}|x(t+\theta)|^2\mu_2(d\theta) + 0.7\int_{-\infty}^{0}|x(t+\theta)|^2\mu_3(d\theta).
\end{aligned}
$$

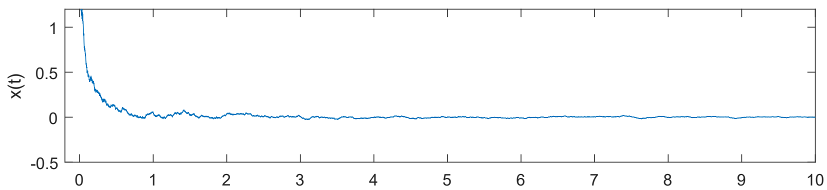
Hence, Assumption 3.2 holds with $\hat{\beta}_1 = 0.25$, $\hat{\beta}_2 = 0.375$, $\hat{\beta}_3 = 0.35$, $\hat{\beta}_4 = 4.25$, $\alpha_1 = 0.025$, $\alpha_2 = 0.25$, $\alpha_3 = 0.275$, $\alpha_4 = 8.5$, $\hat{\alpha}_1 = 0.5$, $\hat{\alpha}_2 = 0.25$, $\hat{\alpha}_3 = 0.7$ and $\hat{\alpha}_4 = 9$. Let $\epsilon^* = 0.1$, $V(\tilde{x}(t)) = |\tilde{x}(t)|^2 + |\tilde{x}(t)|^4$, $\rho_1 = 0.1$, $\rho_2 = 0.05$ and $\rho_3 = 8$. We can check that

$$
\begin{aligned}
&L_1V(x_t,t) + \rho_1\big(2|\tilde{x}(t)| + (q_1+1)|\tilde{x}(t)|^{q_1}\big)^2 + \rho_2|f(x_t,t)|^2 + \rho_3|g(x_t,t)|^2\\
&\quad \leq 0.0404\tilde{H}_1^6 + 1.2111\tilde{H}_2^6 + 1.3229\tilde{H}_3^6 + 0.6306\tilde{H}_1^4 + 3.1583\tilde{H}_2^4 + 2.0828\tilde{H}_3^4 + 0.5188\tilde{H}_1^2\\
&\qquad + 2.2500\tilde{H}_2^2 + 0.7080\tilde{H}_3^2 - 1.0894|x(t)|^6 - 14.7581|x(t)|^4 - 4.4207|x(t)|^2,
\end{aligned}
$$

where $\tilde{H}_j^\alpha = \int_{-\infty}^0 |x(t+\theta)|^\alpha\mu_j(d\theta) - \mu_{j\epsilon^*}|x(t)|^\alpha$. This implies Lemma 3.1 holds with $\gamma_1 = 0.0404$, $\gamma_2 = 1.2111$, $\gamma_3 = 1.3229$, $\bar{\gamma}_1 + \hat{\gamma}_1 = 0.6036$, $\bar{\gamma}_2 + \hat{\gamma}_2 = 3.1583$, $\bar{\gamma}_3 + \hat{\gamma}_3 = 2.0828$, $\tilde{\gamma}_1 = 0.5188$, $\tilde{\gamma}_2 = 2.2500$, $\tilde{\gamma}_3 = 1.0894$, $\rho_4 = 4.4207$, $\rho_5 = 14.7581$ and $\rho_6 = 1.0894$. Accordingly, condition (3.8) implies $\tau \leq 0.0075$. Theorem 3.3 shows the controlled system (4.4) is almost surely exponentially stable provided $\tau \leq 0.0075$. When we choose $\tau = 0.007$ and initial data (4.2), the simulation can support our theoretical results, which is shown in Figure 1(b).



(a) The computer simulation of the sample path of the solution to Eq. (1.4) using the Euler-Maruyama method with step size 10^{-4}.



(b) The computer simulation of the sample path of the solution to Eq. (4.6) using the Euler-Maruyama method with step size 10^{-4} .

Figure 2: with polynomial kernel functions

Case 2 : polynomial kernel function. Let $\mu_1(d\theta) = \mu_2(d\theta) = (1-\theta)^{-2}d\theta$, $\mu_3(d\theta) = 10(1-\theta)^{-11}d\theta$ for $\theta \in (-\infty, 0]$ and

$$\begin{aligned} f(x_t,t) &= -5x^3(t) + 0.5x(t)\int_{-\infty}^{0} x(t+\theta)\mu_1(d\theta) + x(t) \\ g(x_t,t) &= 0.5|x(t)|^{3/2} + 0.5\int_{-\infty}^{0} |x(t+\theta)|^{3/2}\mu_2(d\theta) \\ D(x_t,t) &= 0.1\int_{-\infty}^{0} x(t+\theta)\mu_3(d\theta). \end{aligned}$$

In this case, Eq. (4.1) can be rewritten as Eq. (1.4). Choosing $\epsilon_0 = 0.9$ and $\psi(t) = 1+t$, it is easy to see that

$$\mu_{1\epsilon_0} = \int_{-\infty}^{0} (1-\theta)^{\epsilon_0}(1-\theta)^{-2}d\theta = 10,$$

which implies that $\mu_1 \in M_{0.9}$. Similarly, $\mu_2, \mu_3 \in M_{0.9}$. Let the initial value

$$\varphi = \begin{cases} 2(1-0.01t), & \text{if} \quad t \in (-100, 0], \\ 0, & \text{if} \quad t \in (-\infty, -100]. \end{cases} \tag{4.5}$$

The computer simulation (Figure 2(a)) shows Eq. (1.4) is not stable.

Letting $\epsilon^* = 0.1$, $V(\tilde{x}(t)) = |\tilde{x}(t)|^2 + |\tilde{x}(t)|^4$ and $\rho_1 = 0.1$, $\rho_2 = 0.05$, $\rho_3 = 8$, by a similar argument as Case 1, it is easy to show from Theorem 3.2 that the controlled system

$$d\Big[x(t) - \int_{-\infty}^{0} x(t+\theta)(1-\theta)^{-11}d\theta\Big] = [f(x_t,t) + u(x(t-\tau),t)]dt + g(x_t,t)dB(t) \tag{4.6}$$

with control function (4.3) is almost surely polynomially stable provided $\tau \leq 0.0075$. The simulation with $\tau = 0.007$ and initial data (4.5) can support our theoretical results, which is shown in Figure 2(b).

Remark 4.1. The above examples shows the exponential stabilisation and polynomial stabilisation simultaneously. We also see from the example that the control function (4.3) can stabilize not only Eq. (1.3) with a exponential rate and Eq. (1.4) with a polynomial rate. In other words, the controlled system (4.4) converges to zero exponentially while the controlled system (4.6) converges to zero polynomially. Accordingly, it can be observed from Figure 1(b) and Figure 2(b) that the controlled system (4.4) tends to zero much faster than the controlled system (4.6). These two examples also show that some systems can only be stabilised with the lower decay rate, for example, Eq. (1.4) can only be stabilised under a polynomial rate. In other words, this class of the system cannot be stabilised by using the traditional stabilisation method with exponential speed.

5 Conclusion

In this paper, we consider a class of IDNSFDEs with highly nonlinear coefficients. These systems are often unstable. Under suitable conditions, we prove that a delay feedback control can be designed to stabilize these systems with general convergence rate. More precisely, we consider IDNSFDE system (1.1) under Assumptions 2.1–2.3. Introducing a delay feedback control function satisfying Assumptions 3.1 and 3.2, we prove that the corresponding controlled system (1.2) can be ψ–type stable in moment as well as in probability one. The use of our theory depends on the design of the control function $u(x,t)$. We does not only show that there is a rich class of control functions which satisfy Assumptions 3.1 and 3.2 but also explicitly explain how they can be designed.

Our Theorem 3.2 shows that for each control function satisfying Assumptions 3.1 and 3.2, there is a pair of positive numbers τ and q^* for (3.10) to hold. Note that q^* is the indicator of the control gain while τ is the tolerable delay in the feedback control. It is therefore more desired to have both of them as larger as possible. However, our Lemma 3.1 could only show the existence of positive numbers $\rho_1 - \rho_4$ and ϵ^* but we still do not know how to determine them wisely so that we can have the optimal τ and q^*. It is even more challenged to find a better control function among the rich class so that we could have larger τ and q^*. We will tackle these open problems in the future.

Appendix: Technical complements

A1. Proof of Theorem 2.6.

By the Itô formula,

$$d|x(t) - D(x_t,t)|^{q_1+1} = LV(x_t,t)dt + d\bar{M}(t), \tag{A1.1}$$

where $\bar{M}(t)$ is a local martingale with $\bar{M}(0) = 0$, while $LV(x_t,t)$ is defined by

$$LV(x_t,t) \quad = \quad (q_1+1)|x(t) - D(x_t,t)|^{q_1-1}[x(t) - D(x_t,t)]^T f(x_t,t)$$

$$+\frac{q_1+1}{2}|x(t)-D(x_t,t)|^{q_1-1}|g(x_t,t)|^2$$
$$+\frac{(q_1+1)(q_1-1)}{2}|x(t)-D(x_t,t)|^{q_1-3}\big|[x(t)-D(x_t,t)]^T g(x_t,t)\big|^2.$$

It is obvious that

$$LV(x_t,t)\le (q_1+1)|x(t)-D(x_t,t)|^{q_1-1}\Big\{[x(t)-D(x_t,t)]^T f(x_t,t)+\frac{q_1}{2}|g(x_t,t)|^2\Big\}. \tag{A1.2}$$

Substituting (2.5) into (A1.2) yields

$$\begin{aligned}
LV(x_t,t) \quad\le\quad & (q_1+1)|x(t)-D(x_t,t)|^{q_1-1}\Big\{\sum_{j=1}^{3}\beta_j\int_{-\infty}^{0}|x(t+\theta)|^p\mu_j(d\theta)-\beta_4|x(t)|^p\\
&+\sum_{j=1}^{3}k_j\int_{-\infty}^{0}|x(t+\theta)|^2\mu_j(d\theta)+k_4|x(t)|^2\Big\}\\
= \quad & \sum_{j=1}^{3}(q_1+1)\beta_j|x(t)-D(x_t,t)|^{q_1-1}\int_{-\infty}^{0}|x(t+\theta)|^p\mu_j(d\theta)\\
&-(q_1+1)\beta_4|x(t)-D(x_t,t)|^{q_1-1}|x(t)|^p\\
&+\sum_{j=1}^{3}(q_1+1)k_j|x(t)-D(x_t,t)|^{q_1-1}\int_{-\infty}^{0}|x(t+\theta)|^2\mu_j(d\theta)\\
&+(q_1+1)k_4|x(t)-D(x_t,t)|^{q_1-1}|x(t)|^2\\
=: \quad & E_1-E_2+E_3+E_4.
\end{aligned} \tag{A1.3}$$

The Young inequality and the Hölder inequality yield

$$\begin{aligned}
&|x(t)-D(x_t,t)|^{q_1-1}\int_{-\infty}^{0}|x(t+\theta)|^p\mu_j(d\theta)\\
&\le\frac{q_1-1}{p+q_1-1}|x(t)-D(x_t,t)|^{p+q_1-1}+\frac{p}{p+q_1-1}\int_{-\infty}^{0}|x(t+\theta)|^{p+q_1-1}\mu_j(d\theta)
\end{aligned} \tag{A1.4}$$

and

$$\begin{aligned}
&|x(t)|^p\Big|\int_{-\infty}^{0}x(t+\theta)\mu_3(d\theta)\Big|^{q_1-1}\\
&\le\frac{p}{p+q_1-1}|x(t)|^{p+q_1-1}+\frac{q_1-1}{p+q_1-1}\Big|\int_{-\infty}^{0}x(t+\theta)\mu_j(d\theta)\Big|^{p+q_1-1}\\
&\le\frac{p}{p+q_1-1}|x(t)|^{p+q_1-1}+\frac{q_1-1}{p+q_1-1}\int_{-\infty}^{0}|x(t+\theta)|^{p+q_1-1}\mu_j(d\theta).
\end{aligned} \tag{A1.5}$$

It then follows from (A1.4) and Lemma 2.5 that

$$\begin{aligned}
E_1 \quad\le\quad & Q_1(\beta_1+\beta_2+\beta_3)|x(t)-D(x_t,t)|^{p+q_1-1}+Q_2\sum_{j=1}^{3}\beta_j\int_{-\infty}^{0}|x(t+\theta)|^{p+q_1-1}\mu_j(d\theta)\\
\le\quad & Q_1(1-L)^{2-p-q_1}(\beta_1+\beta_2+\beta_3)|x(t)|^{p+q_1-1}+Q_1L(\beta_1+\beta_2+\beta_3)\int_{-\infty}^{0}|x(t+\theta)|^{p+q_1-1}\mu_3(d\theta)
\end{aligned}$$

$$+Q_2\sum_{j=1}^{3}\beta_j\int_{-\infty}^{0}|x(t+\theta)|^{p+q_1-1}\mu_j(d\theta),$$

where

$$Q_1=\frac{(q_1^2-1)}{p+q_1-1}\ \text{ and }\ Q_2=\frac{p(q_1+1)}{p+q_1-1}.$$

In light of Lemma 2.5 and (A1.5), we arrive at

$$\begin{aligned}-E_2 &\leq -(q_1+1)(1-L)^{(q_1-2)_+}\beta_4\Big[|x(t)|^{q_1-1}-L^{(q_1-1)\wedge 1}\Big|\int_{-\infty}^{0}x(t+\theta)\mu_3(d\theta)\Big|^{q_1-1}\mu_3(d\theta)\Big]|x(t)|^{p}\\ &\leq -(q_1+1)(1-L)^{(q_1-2)_+}\beta_4|x(t)|^{p+q_1-1}+Q_2(1-L)^{(q_1-2)_+}L^{(q_1-1)\wedge 1}\beta_4|x(t)|^{p+q_1-1}\\ &\quad +Q_1(1-L)^{(q_1-2)_+}L^{(q_1-1)\wedge 1}\beta_4\int_{-\infty}^{0}|x(t+\theta)|^{p+q_1-1}\mu_3(d\theta).\end{aligned}$$

Similar to estimation of E_1, we derive

$$E_3+E_4\leq C\sum_{j=1}^{3}\int_{-\infty}^{0}|x(t+\theta)|^{q_1+1}\mu_j(d\theta)+C|x(t)|^{q_1+1}.$$

Substituting E_1, E_2, E_3+E_4 into (A1.3) gives

$$\begin{aligned}LV(x_t,t) &\leq \sum_{j=1}^{3}C_j\int_{-\infty}^{0}|x(t+\theta)|^{p+q_1-1}\mu_j(d\theta)+C\sum_{j=1}^{3}\int_{-\infty}^{0}|x(t+\theta)|^{q_1+1}\mu_j(d\theta)\\ &\quad -Q_3|x(t)|^{p+q_1-1}+C|x(t)|^{q_1+1}\\ &\leq \sum_{j=1}^{3}C_j\Big[\int_{-\infty}^{0}|x(t+\theta)|^{p+q_1-1}\mu_j(d\theta)-|x(t)|^{p+q_1-1}\Big]\\ &\quad +C\sum_{j=1}^{3}\Big[\int_{-\infty}^{0}|x(t+\theta)|^{q_1+1}\mu_j(d\theta)-|x(t)|^{q_1+1}\Big]+Q_4,\end{aligned}\tag{A1.6}$$

where

$$\begin{aligned}&C_1=Q_2\beta_1,\quad C_2=Q_2\beta_2,\quad C_3=Q_2\beta_3+Q_1L(\beta_1+\beta_2+\beta_3)+Q_1(1-L)^{(q_1-2)_+}L^{(q_1-1)\wedge 1}\beta_4,\\ &Q_3=(q_1+1)(1-L)^{(q_1-2)_+}\beta_4-Q_2(1-L)^{(q_1-2)_+}L^{(q_1-1)\wedge 1}\beta_4-Q_1(1-L)^{2-p-q_1}(\beta_1+\beta_2+\beta_3),\\ &Q_4=\sup_{u\geq 0}\{-(Q_3-C_1-C_2-C_3)u^{p+q_1-1}+Cu^{q_1+1}\}.\end{aligned}$$

According to (2.4), we have

$$Q_3-C_1-C_2-C_3>0.$$

which implies $Q_4<\infty$. Let k_0 be a sufficiently large integer such that $\|\varphi\|<k_0$. For each integer $k>k_0$, we define the stopping time

$$\sigma_k=\inf\{t\geq 0:|x(t)|\geq k\}.\tag{A1.7}$$

It then follows (A1.1) that

$$\mathbb{E}|\tilde{x}(t\wedge\sigma_k)|^{q_1+1}\ \leq\ |\tilde{x}(0)|^{q_1+1}+\mathbb{E}\int_{0}^{t\wedge\sigma_k}\Big\{\sum_{j=1}^{3}C_j\Big[\int_{-\infty}^{0}|x(s+\theta)|^{p+q_1-1}\mu_j(d\theta)-|x(s)|^{p+q_1-1}\Big]$$

$$+C\sum_{j=1}^{3}\Big[\int_{-\infty}^{0}|x(s+\theta)|^{q_1+1}\mu_j(d\theta)-|x(s)|^{q_1+1}\Big]+Q_4\Big\}ds. \tag{A1.8}$$

Note that $\varphi\in BC((-\infty,0];\mathbb{R}^n)\cap L^{q_1+1}((-\infty,0];\mathbb{R}^n)$. Lemma 2.2 shows $\varphi\in BC((-\infty,0];\mathbb{R}^n)\cap L^{p+q_1-1}((-\infty,0];\mathbb{R}^n)$. Using the Fubini theorem and a substitution technique gives

$$\begin{aligned}
&\int_0^{t\wedge\sigma_k}\Big[\int_{-\infty}^{0}|x(s+\theta)|^{p+q_1-1}\mu_j(d\theta)-|x(s)|^{p+q_1-1}\Big]ds\\
&=\int_{-\infty}^{0}\mu_j(d\theta)\int_{\theta}^{t\wedge\sigma_k+\theta}|x(s)|^{p+q_1-1}ds-\int_0^{t\wedge\sigma_k}|x(s)|^{p+q_1-1}ds\\
&\le\int_{-\infty}^{t\wedge\sigma_k}|x(s)|^{p+q_1-1}ds-\int_0^{t\wedge\sigma_k}|x(s)|^{p+q_1-1}ds\\
&=\int_{-\infty}^{0}|\varphi(s)|^{p+q_1-1}ds<C.
\end{aligned}$$

Similarly,

$$\int_0^{t\wedge\sigma_k}\Big[\int_{-\infty}^{0}|x(s+\theta)|^{q_1+1}\mu_j(d\theta)-|x(s)|^{q_1+1}\Big]ds<C.$$

It then follows from (A1.8) that

$$\sup_{0\le t\le T}\mathbb{E}|\tilde{x}(t\wedge\sigma_k)|^{q_1+1}<C(T)<\infty.$$

Note that $\sigma_k\to\infty$ as $k\to\infty$. Letting $k\to\infty$ yields

$$\sup_{0\le t\le T}\mathbb{E}|\tilde{x}(t)|^{q_1+1}<C(T).$$

Moreover, applying (2.6) gives

$$\begin{aligned}
&\sup_{-\infty\le t\le T}\mathbb{E}|x(t)|^{q_1+1}\\
&\le(\|\varphi\|^{q_1+1})\vee\Big[\sup_{0\le t\le T}\mathbb{E}|x(t)-D(x_t,t)+D(x_t,t)|^{q_1+1}\Big]\\
&\le(\|\varphi\|^{q_1+1})\vee\Big[(1-L)^{-q_1}\sup_{0\le t\le T}\mathbb{E}|\tilde{x}(t)|^{q_1+1}+L\int_{-\infty}^{0}(\sup_{0\le t\le T}\mathbb{E}|x(t+\theta)|^{q_1+1})\mu_3(d\theta)\Big]\\
&\le\|\varphi\|^{q_1+1}+(1-L)^{-q_1}\sup_{0\le t\le T}\mathbb{E}|\tilde{x}(t)|^{q_1+1}+L\int_{-\infty}^{0}(\sup_{-\infty\le t\le T}\mathbb{E}|x(t)|^{q_1+1})\mu_3(d\theta)\\
&=\|\varphi\|^{q_1+1}+(1-L)^{-q_1}\sup_{0\le t\le T}\mathbb{E}|\tilde{x}(t)|^{q_1+1}+L\sup_{-\infty\le t\le T}\mathbb{E}|x(t)|^{q_1+1},
\end{aligned}$$

which implies

$$\sup_{-\infty\le t\le T}\mathbb{E}|x(t)|^{q_1+1}\le(1-L)^{-1}\|\varphi\|^{q_1+1}+(1-L)^{-(q_1+1)}\sup_{0\le t\le T}\mathbb{E}|\tilde{x}(t)|^{q_1+1}.$$

Noting that $\varphi\in BC((-\infty,0];\mathbb{R}^n)$ and $\sup_{0\le t\le T}\mathbb{E}|\tilde{x}(t)|^{q_1+1}<C(T)$, we have $\sup_{0\le t\le T}\mathbb{E}|x(t)|^{q_1+1}<C(T)<\infty$. The desired conclusion follows immediately.

A2. Proof of Lemma 3.1.

Let

$$
\begin{aligned}
W(\xi,t) &= L_1V(\xi,t)+\rho_1\big(2|\xi(0)-D(\xi,t)|+(q_1+1)|\xi(0)-D(\xi,t)|^{q_1}\big)^2 \\
&\quad +\rho_2|f(\xi,t)|^2+\rho_3|g(\xi,t)|^2. \qquad\qquad (A2.1)
\end{aligned}
$$

Hence it suffices to prove

$$
\begin{aligned}
W(\xi,t) &\le -\rho_4|\xi(0)|^2-\rho_5|\xi(0)|^{q_1+1}-\rho_6|\xi(0)|^{p+q_1-1}+\sum_{j=1}^{3}\gamma_j H_{j,\epsilon^*}^{p+q_1-1}+\sum_{j=1}^{3}\hat{\gamma}_j H_{j,\epsilon^*}^{q_1+1} \\
&\quad +\sum_{j=1}^{3}\bar{\gamma}_j H_{j,\epsilon^*}^{p}+\sum_{j=1}^{3}\tilde{\gamma}_j H_{j,\epsilon^*}^{2}. \qquad\qquad (A2.2)
\end{aligned}
$$

We divide the following proof into two steps.

Step 1: Estimation of L_1V. Applying (3.2), (3.3) and (3.5) gives

$$
\begin{aligned}
L_1V(\xi,t) &= \sum_{j=1}^{3}\alpha_j\int_{-\infty}^{0}|\xi(\theta)|^p\mu_j(d\theta)-\alpha_4|\xi(0)|^p+\sum_{j=1}^{3}\hat{\alpha}_j\int_{-\infty}^{0}|\xi(\theta)|^2\mu_j(d\theta)-\hat{\alpha}_4|\xi(0)|^2 \\
&\quad +(q_1+1)|\xi(0)-D(\xi,t)|^{q_1-1}\Big\{\sum_{j=1}^{3}\beta_j\int_{-\infty}^{0}|\xi(\theta)|^p\mu_j(d\theta)-\beta_4|\xi(0)|^p \\
&\quad +\sum_{j=1}^{3}\hat{\beta}_j\int_{-\infty}^{0}|\xi(\theta)|^2\mu_j(d\theta)-\hat{\beta}_4|\xi(0)|^2\Big\}. \qquad\qquad (A2.3)
\end{aligned}
$$

Similar to the estimation of (A1.6), we have for any $\epsilon\in(0,\epsilon_0]$,

$$
\begin{aligned}
&(q_1+1)|\xi(0)-D(\xi,t)|^{q_1-1}\Big\{\sum_{j=1}^{3}\beta_j\int_{-\infty}^{0}|\xi(\theta)|^p\mu_j(d\theta)-\beta_4|\xi(0)|^p \\
&\qquad\qquad +\sum_{j=1}^{3}\hat{\beta}_j\int_{-\infty}^{0}|\xi(\theta)|^2\mu_j(d\theta)-\hat{\beta}_4|\xi(0)|^2\Big\} \\
&\le \sum_{j=1}^{3}C_j\int_{-\infty}^{0}|\xi(\theta)|^{p+q_1-1}\mu_j(d\theta)+\sum_{j=1}^{3}\hat{C}_j\int_{-\infty}^{0}|\xi(\theta)|^{q_1+1}\mu_j(d\theta)-Q_3|\xi(0)|^{p+q_1-1}-\hat{Q}_3|\xi(0)|^{q_1+1} \\
&= \sum_{j=1}^{3}C_jH_{j,\epsilon}^{p+q_1-1}+\sum_{j=1}^{3}\hat{C}_jH_{j,\epsilon}^{q_1+1}-Q_{11}^{\epsilon}|\xi(0)|^{p+q_1-1}-Q_{22}^{\epsilon}|\xi(0)|^{q_1+1},
\end{aligned}
$$

where Q_3 and C_j $(j=1,2,3)$ come from the proof of Theorem 2.6 and

$$
\begin{aligned}
&\hat{C}_1=2\hat{\beta}_1,\quad \hat{C}_2=2\hat{\beta}_2,\\
&\hat{C}_3=2\hat{\beta}_3+(q_1-1)L(\hat{\beta}_1+\hat{\beta}_2+\hat{\beta}_3)+(q_1-1)(1-L)^{(q_1-2)_+}L^{(q_1-1)\wedge 1}\hat{\beta}_4,\\
&\hat{Q}_3=(q_1+1)(1-L)^{(q_1-2)_+}\hat{\beta}_4-2(1-L)^{(q_1-2)_+}L^{(q_1-1)\wedge 1}\hat{\beta}_4-(q_1-1)(1-L)^{-q_1}(\hat{\beta}_1+\hat{\beta}_2+\hat{\beta}_3),\\
&Q_{11}^{\epsilon}=Q_3-C_1\mu_{1\epsilon}-C_2\mu_{2\epsilon}-C_3\mu_{3\epsilon},
\end{aligned}
$$

$$Q_{22}^{\epsilon} = \hat{Q}_3 - \hat{C}_1\mu_{1\epsilon} - \hat{C}_2\mu_{2\epsilon} - \hat{C}_3\mu_{3\epsilon}.$$

It then follows from (A2.3) that for any $\epsilon \in (0, \epsilon_0]$,

$$\begin{aligned} L_1V(\xi,t) \quad \leq \quad & \sum_{j=1}^{3} C_j H_{j,\epsilon}^{p+q_1-1} + \sum_{j=1}^{3} \hat{C}_j H_{j,\epsilon}^{q_1+1} + \sum_{j=1}^{3} \alpha_j H_{j,\epsilon}^{p} + \sum_{j=1}^{3} \hat{\alpha}_j H_{j,\epsilon}^{2} \\ & -Q_{11}^{\epsilon}|\xi(0)|^{p+q_1-1} - Q_{22}^{\epsilon}|\xi(0)|^{q_1+1} - Q_{33}^{\epsilon}|\xi(0)|^{p} - Q_{44}^{\epsilon}|\xi(0)|^2, \end{aligned} \tag{A2.4}$$

where $Q_{33}^{\epsilon} = \alpha_4 - \alpha_1\mu_{1\epsilon} - \alpha_2\mu_{2\epsilon} - \alpha_3\mu_{3\epsilon}$ and $Q_{44}^{\epsilon} = \hat{\alpha}_4 - \hat{\alpha}_1\mu_{1\epsilon} - \hat{\alpha}_2\mu_{2\epsilon} - \hat{\alpha}_3\mu_{3\epsilon}$.

Step 2: Estimation of $W(\xi,t)$. Employing Lemma 2.5 gives

$$\begin{aligned} & \rho_1(2|\xi(0) - D(\xi,t)| + (q_1+1)|\xi(0) - D(\xi,t)|^{q_1})^2 \\ & \quad \leq 8\rho_1|\xi(0) - D(\xi,t)|^2 + 2\rho_1(q_1+1)^2|\xi(0) - D(\xi,t)|^{2q_1} \\ & \quad \leq 8\rho_1(1-L)^{-1}|\xi(0)|^2 + 8\rho_1 L\int_{-\infty}^{0}|\xi(\theta)|^2\mu_3(d\theta) \\ & \qquad +2\rho_1(q_1+1)^2(1-L)^{1-2q_1}|\xi(0)|^{2q_1} + 2\rho_1(q_1+1)^2 L\int_{-\infty}^{0}|\xi(\theta)|^{2q_1}\mu_3(d\theta), \end{aligned}$$

and from (2.2),

$$\begin{aligned} & \rho_2|f(\xi,t)|^2 + \rho_3|g(\xi,t)|^2 \\ & \quad \leq 4\rho_2K^2\Big[\int_{-\infty}^{0}|\xi(\theta)|^{2q_1}\mu_1(d\theta) + |\xi(0)|^{2q_1} + \int_{-\infty}^{0}|\xi(\theta)|^2\mu_1(d\theta) + |\xi(0)|^2\Big] \\ & \qquad +4\rho_3K^2\Big[\int_{-\infty}^{0}|\xi(\theta)|^{2q_2}\mu_2(d\theta) + |\xi(0)|^{2q_2} + \int_{-\infty}^{0}|\xi(\theta)|^2\mu_2(d\theta) + |\xi(0)|^2\Big]. \end{aligned}$$

Recalling that $(q_1+1) \vee (2q_2 - q_1 + 1) \leq p$ and $\big(|\xi(\theta)|^{2q_1} \vee |\xi(\theta)|^{2q_2}\big) \leq |\xi(\theta)|^2 + |\xi(\theta)|^{p+q_1-1}$, we have

$$\begin{aligned} & \rho_1(2|\xi(0) - D(\xi,t)| + (q_1+1)|\xi(0) - D(\xi,t)|^{q_1})^2 + \rho_2|f(\xi,t)|^2 + \rho_3|g(\xi,t)|^2 \\ & \quad \leq 4\rho_2K^2\int_{-\infty}^{0}|\xi(\theta)|^{p+q_1-1}\mu_1(d\theta) + 4\rho_3K^2\int_{-\infty}^{0}|\xi(\theta)|^{p+q_1-1}\mu_2(d\theta) \\ & \qquad +2\rho_1(q_1+1)^2L\int_{-\infty}^{0}|\xi(\theta)|^{p+q_1-1}\mu_3(d\theta) \\ & \qquad +8\rho_2K^2\int_{-\infty}^{0}|\xi(\theta)|^2\mu_1(d\theta) + 8\rho_3K^2\int_{-\infty}^{0}|\xi(\theta)|^2\mu_2(d\theta) \\ & \qquad +(8\rho_1L + 2\rho_1(q_1+1)^2L)\int_{-\infty}^{0}|\xi(\theta)|^2\mu_3(d\theta) \\ & \qquad +(2\rho_1(q_1+1)^2(1-L)^{1-2q_1} + 4\rho_2K^2 + 4\rho_3K^2)|\xi(0)|^{p+q_1-1} \\ & \qquad +(8\rho_1(1-L)^{-1} + 2\rho_1(q_1+1)^2(1-L)^{1-2q_1} + 8\rho_2K^2 + 8\rho_3K^2)|\xi(0)|^2. \end{aligned}$$

Substituting this and (A2.4) into (A2.1) yields that for any $\epsilon \in (0, \epsilon_0]$,

$$W(\xi,t) \quad \leq \quad (C_1 + 4\rho_2K^2)H_{1,\epsilon}^{p+q_1-1} + (C_2 + 4\rho_3K^2)H_{2,\epsilon}^{p+q_1-1} + (C_3 + 2\rho_1(q_1+1)^2L)H_{3,\epsilon}^{p+q_1-1}$$

$$
\begin{aligned}
&+\sum_{j=1}^{3}\hat{C}_j H_{j,\epsilon}^{q_1+1}+\sum_{j=1}^{3}\alpha_j H_{j,\epsilon}^{p}+(\hat{\alpha}_1+8\rho_2K^2)H_{1,\epsilon}^2+(\hat{\alpha}_2+8\rho_3K^2)H_{2,\epsilon}^2\\
&+\Big(\hat{\alpha}_3+8\rho_1L+2\rho_1(q_1+1)^2L\Big)H_{3,\epsilon}^2\\
&-\big[Q_{11}^{\epsilon}-Q_{55}^{\epsilon}\big]|\xi(0)|^{p+q_1-1}-Q_{22}^{\epsilon}|\xi(0)|^{q_1+1}-Q_{33}^{\epsilon}|\xi(0)|^{p}-\big[Q_{44}^{\epsilon}-Q_{66}^{\epsilon}\big]|\xi(0)|^2. \quad \text{(A2.5)}
\end{aligned}
$$

where

$$
\begin{aligned}
Q_{55}^{\epsilon}&=4\rho_2K^2\mu_{1\epsilon}+4\rho_3K^2\mu_{2\epsilon}+2\rho_1(q_1+1)^2L\mu_{3\epsilon}+2\rho_1(q_1+1)^2(1-L)^{1-2q_1}+4\rho_2K^2+4\rho_3K^2,\\
Q_{66}^{\epsilon}&=8\rho_2K^2\mu_{1\epsilon}+8\rho_3K^2\mu_{2\epsilon}+\big(8\rho_1L+2\rho_1(q_1+1)^2L\big)\mu_{3\epsilon}+8\rho_1(1-L)^{-1}\\
&\quad+2\rho_1(q_1+1)^2(1-L)^{1-2q_1}+8\rho_2K^2+8\rho_3K^2.
\end{aligned}
$$

By (2.4) and (3.1), we have

$$
Q_3-C_1-C_2-C_3>0,\quad \hat{Q}_3-\hat{C}_1-\hat{C}_2-\hat{C}_3>0\ \text{ and }\ \alpha_4-\alpha_1-\alpha_2-\alpha_3>0.
$$

Recalling $\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_3<\hat{\alpha}_4$, by Lemma 2.3, there exists $\bar{\epsilon}\in(0,\epsilon_0]$ sufficiently small such that for any $\epsilon\in(0,\bar{\epsilon}]$,

$$
\begin{aligned}
Q_{11}^{\epsilon}&=Q_3-C_1\mu_{1\epsilon}-C_2\mu_{2\epsilon}-C_3\mu_{3\epsilon}>0,\\
Q_{22}^{\epsilon}&=\hat{Q}_3-\hat{C}_1\mu_{1\epsilon}-\hat{C}_2\mu_{2\epsilon}-\hat{C}_3\mu_{3\epsilon}>0,\\
Q_{33}^{\epsilon}&=\alpha_4-\alpha_1\mu_{1\epsilon}-\alpha_2\mu_{2\epsilon}-\alpha_3\mu_{3\epsilon}>0,\\
Q_{44}^{\epsilon}&=\hat{\alpha}_4-\hat{\alpha}_1\mu_{1\epsilon}-\hat{\alpha}_2\mu_{2\epsilon}-\hat{\alpha}_3\mu_{3\epsilon}>0.
\end{aligned}
$$

We can choose appropriate positive constants ρ_j $(1\le j\le 3)$ sufficiently small such that

$$
\begin{cases}
8\rho_2K^2+8\rho_3K^2+2\rho_1(q_1+1)^2L+2\rho_1(q_1+1)^2(1-L)^{1-2q_1}<0.5Q_{11}^{\epsilon},\\
16\rho_2K^2+16\rho_3K^2+\big(8\rho_1L+2\rho_1(q_1+1)^2L\big)+8\rho_1(1-L)^{-1}+2\rho_1(q_1+1)^2(1-L)^{1-2q_1}<0.5Q_{44}^{\epsilon},
\end{cases}
$$

then there exists $\tilde{\epsilon}\in(0,\epsilon_0]$ sufficiently small such that $Q_{55}^{\epsilon}\le 0.5Q_{11}^{\epsilon}$ and $Q_{66}^{\epsilon}\le 0.5Q_{44}^{\epsilon}$ for any $\epsilon\in(0,\tilde{\epsilon}]$. It then follows from (A2.5) that there exists $\epsilon^*\in(0,\bar{\epsilon}\wedge\tilde{\epsilon}]$ such that

$$
\begin{aligned}
W(\xi,t)\ \le\ &(C_1+4\rho_2K^2)H_{1,\epsilon^*}^{p+q_1-1}+(C_2+4\rho_3K^2)H_{2,\epsilon^*}^{p+q_1-1}+(C_3+2\rho_1(q_1+1)^2L)H_{3,\epsilon^*}^{p+q_1-1}\\
&+\sum_{j=1}^{3}\hat{C}_jH_{j,\epsilon^*}^{q_1+1}+\sum_{j=1}^{3}\alpha_jH_{j,\epsilon^*}^{p}+(\hat{\alpha}_1+8\rho_2K^2)H_{1,\epsilon^*}^2+(\hat{\alpha}_2+8\rho_3K^2)H_{2,\epsilon^*}^2\\
&+\Big(\hat{\alpha}_3+8\rho_1L+2\rho_1(q_1+1)^2L\Big)H_{3,\epsilon^*}^2-0.5Q_{11}^{\epsilon^*}|\xi(0)|^{p+q_1-1}-Q_{22}^{\epsilon^*}|\xi(0)|^{q_1+1}\\
&-0.5Q_{44}^{\epsilon^*}|\xi(0)|^2.
\end{aligned}
$$

This implies (A2.2) holds with $\gamma_1=C_1+4\rho_2K^2$, $\gamma_2=C_2+4\rho_2K^2$, $\gamma_3=C_3+2\rho_1(q_1+1)^2L$, $\hat{\gamma}_j=\hat{C}_j(j=1,2,3)$, $\bar{\gamma}_j=\alpha_j$ for $j=1,2,3$, $\tilde{\gamma}_1=\hat{\alpha}_1+8\rho_2K^2$, $\tilde{\gamma}_2=\hat{\alpha}_2+8\rho_3K^2$, $\tilde{\gamma}_3=\hat{\alpha}_3+8\rho_1L+2\rho_1(q_1+1)^2L$, $\rho_4=0.5Q_{44}^{\epsilon^*}$, $\rho_5=Q_{22}^{\epsilon^*}$ and $\rho_6=0.5Q_{11}^{\epsilon^*}$. This completes the proof.

A3. Proof of Theorem 3.2.

Let $V(\tilde{x}(t)) = |\tilde{x}(t)|^2 + |\tilde{x}(t)|^{q_1+1}$ and

$$U(x_t, t) = V(\tilde{x}(t)) + \lambda \int_{-\tau}^{0} \int_{t+s}^{t} J(v)dvds, \quad t \geq 0, \tag{A3.1}$$

where $J(t) = [\tau|f(x_t, t) + u(x(t-\tau), t)|^2 + |g(x_t, t)|^2]$ and λ is a positive constant to be determined later. Let us also define

$$\begin{aligned} \mathbb{L}U(x_t, t) \quad = \quad & L_1V(x_t, t) + \big\{2 + (q_1+1)|\tilde{x}(t)|^{q_1-1}\big\}[\tilde{x}(t)]^T[u(x(t-\tau), t) - u(x(t), t)] \\ & +\lambda\tau J(t) - \lambda \int_{t-\tau}^{t} J(v)dv. \end{aligned} \tag{A3.2}$$

In the following, let us divide this proof into three steps.

Step 1 : Estimation of $\mathbb{L}U(x_t, t)$. Applying the Itô formula to $V(\tilde{x}(t))$ yields

$$dV(\tilde{x}(t)) = \mathcal{L}V(x_t, t)dt + dM(t), \tag{A3.3}$$

where $\mathcal{L}V$ comes from (3.6) and M(t) is a continuous local martingale with $M(0) = 0$. On the other hand,

$$d\Big(\lambda \int_{-\tau}^{0} \int_{t+s}^{t} J(v)dvds\Big) = \Big(\lambda\tau J(t) - \lambda \int_{t-\tau}^{t} J(v)dv\Big)dt. \tag{A3.4}$$

It then follows from (A3.1)–(A3.4) that

$$dU(x_t, t) = \Big(\mathcal{L}V(x_t, t) + \lambda\tau J(t) - \lambda \int_{t-\tau}^{t} J(v)dv\Big)dt + dM(t),$$

that is, $U(x_t, t)$ is an Itô process as claimed. It then follows from (3.7) and (A3.2) that

$$dU(x_t, t) \leq \mathbb{L}U(x_t, t)dt + dM(t).$$

Recalling $|u(x(t), t)| \leq \kappa|x(t)|$ and $x^Ty \leq |x||y|$, we get

$$\begin{aligned} & [2 + (q_1+1)|\tilde{x}(t)|^{q_1-1}][\tilde{x}(t)]^T[u(x(t-\tau), t) - u(x(t), t)] \\ & \quad \leq \big(2|\tilde{x}(t)| + (q_1+1)|\tilde{x}(t)|^{q_1}\big)\kappa|x(t-\tau) - x(t)| \\ & \quad \leq \rho_1\big(2|\tilde{x}(t)| + (q_1+1)|\tilde{x}(t)|^{q_1}\big)^2 + \frac{\kappa^2}{4\rho_1}|x(t) - x(t-\tau)|^2. \end{aligned} \tag{A3.5}$$

Let $\lambda = \kappa^2/\rho_1(1-L)^2$. From (3.8), $2\lambda\tau^2 \leq \rho_2$ and $\lambda\tau \leq \rho_3$. It then follows from Lemma 3.1 that

$$\begin{aligned} \mathbb{L}U(x_t, t) \quad \leq \quad & L_1V(x_t, t) + \rho_1\big(2|\tilde{x}(t)| + (q_1+1)|\tilde{x}(t)|^{q_1}\big)^2 + \frac{\kappa^2}{4\rho_1}|x(t) - x(t-\tau)|^2 \\ & +\rho_2|f(x_t, t)|^2 + 2\lambda\tau^2\kappa^2|x(t-\tau)|^2 + \rho_3|g(x_t, t)|^2 - \lambda \int_{t-\tau}^{t} J(v)dv \\ \leq \quad & -\Big(\rho_4 - \frac{4\tau^2\kappa^4}{\rho_1(1-L)^2}\Big)|x(t)|^2 - \rho_5|x(t)|^{q_1+1} - \rho_6|x(t)|^{p+q_1-1} \\ & + \sum_{j=1}^{3} \gamma_j \bar{H}_j^{p+q_1-1}(t) + \sum_{j=1}^{3} \hat{\gamma}_j \bar{H}_j^{q_1+1}(t) + \sum_{j=1}^{3} \bar{\gamma}_j \bar{H}_j^{p}(t) + \sum_{j=1}^{3} \tilde{\gamma}_j \bar{H}_j^{2}(t) \end{aligned}$$

$$+\Big(\frac{\kappa^2}{4\rho_1}+\frac{4\tau^2\kappa^4}{\rho_1(1-L)^2}\Big)|x(t)-x(t-\tau)|^2-\frac{\kappa^2}{\rho_1(1-L)^2}\int_{t-\tau}^{t}J(v)dv, \quad \text{(A3.6)}$$

where $\bar{H}_j^\alpha(t)=\int_{-\infty}^0|x(t+\theta)|^\alpha\mu_j(d\theta)-\mu_{j\epsilon^*}|x(t)|^\alpha$.

Step 2 : Estimation of $\mathbb{E}\psi^q(t\wedge\sigma_k)U(x_{t\wedge\sigma_k},t\wedge\sigma_k)$. For σ_k given in (A1.7) and any $q\in(0,\epsilon^*]$, applying the Itô formula yields

$$\mathbb{E}\psi^q(t\wedge\sigma_k)U(x_{t\wedge\sigma_k},t\wedge\sigma_k)\le U(x_0,0)+\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)[q\phi U(x_s,s)+\mathbb{L}U(x_s,s)]ds,$$

where $\phi=\sup_{s\ge0}[\psi'(s)/\psi(s)]<\infty$. Note that (3.8) shows $\rho_4>\frac{4\tau^2\kappa^4}{\rho_1(1-L)^2}$. Let $a=(\rho_4-\frac{4\tau^2\kappa^4}{\rho_1(1-L)^2})\wedge\rho_5$. We deduce from the definition of U and (A3.6) that

$$\begin{aligned}
&\mathbb{E}\psi^q(t\wedge\sigma_k)U(x_{t\wedge\sigma_k},t\wedge\sigma_k)\\
&\le U(x_0,0)+\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\Big[-a(|x(s)|^2+|x(s)|^{q_1+1})-\rho_6|x(s)|^{p+q_1-1}\\
&\quad+q\phi(|\tilde{x}(s)|^2+|\tilde{x}(s)|^{q_1+1})\Big]ds+R_1+R_2+R_3-R_4,
\end{aligned}\quad \text{(A3.7)}$$

where

$$\begin{aligned}
R_1&=q\phi\frac{\kappa^2}{\rho_1(1-L)^2}\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\int_{-\tau}^0\int_{s+l}^sJ(v)dvdlds\\
R_2&=\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\Big\{\sum_{j=1}^3\gamma_j\bar{H}_j^{p+q_1-1}(s)+\sum_{j=1}^3\hat{\gamma}_j\bar{H}_j^{q_1+1}(s)+\sum_{j=1}^3\bar{\gamma}_j\bar{H}_j^{p}(s)+\sum_{j=1}^3\tilde{\gamma}_j\bar{H}_j^{2}(s)\Big\}ds\\
R_3&=\Big(\frac{\kappa^2}{4\rho_1}+\frac{4\tau^2\kappa^4}{\rho_1(1-L)^2}\Big)\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)|x(s)-x(s-\tau)|^2ds\\
R_4&=\frac{\kappa^2}{\rho_1(1-L)^2}\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\int_{s-\tau}^sJ(v)dvds.
\end{aligned}$$

It is obvious that

$$R_1\le q\phi\frac{\kappa^2}{\rho_1(1-L)^2}\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\tau\int_{s-\tau}^sJ(v)dvds=q\phi\tau R_4. \quad \text{(A3.8)}$$

Note that $\varphi\in BC((-\infty,0];\mathbb{R}^n)\cap L^2((-\infty,0];\mathbb{R}^n)$. Lemma 2.2 shows $\varphi\in BC((-\infty,0];\mathbb{R}^n)\cap L^{p+q_1-1}((-\infty,0];\mathbb{R}^n)$. Using the Fubini theorem and a substitution technique gives

$$\begin{aligned}
&\int_0^{t\wedge\sigma_k}\psi^q(s)\bar{H}_j^{p+q_1-1}(s)ds\\
&=\int_0^{t\wedge\sigma_k}\psi^q(s)\Big[\int_{-\infty}^0|x(s+\theta)|^{p+q_1-1}\mu_j(d\theta)-\mu_{j\epsilon^*}|x(s)|^{p+q_1-1}\Big]ds\\
&=\int_{-\infty}^0\mu_j(d\theta)\int_0^{t\wedge\sigma_k}\psi^q(-\theta)\psi^q(s+\theta)|x(s+\theta)|^{p+q_1-1}ds-\mu_{j\epsilon^*}\int_0^{t\wedge\sigma_k}\psi^q(s)|x(s)|^{p+q_1-1}ds\\
&\le\int_{-\infty}^0\psi^q(-\theta)\mu_j(d\theta)\int_\theta^{t\wedge\sigma_k+\theta}\psi^q(s)|x(s)|^{p+q_1-1}ds-\mu_{j\epsilon^*}\int_0^{t\wedge\sigma_k}\psi^q(s)|x(s)|^{p+q_1-1}ds\\
&\le\mu_{jq}\int_{-\infty}^{t\wedge\sigma_k}\psi^q(s)|x(s)|^{p+q_1-1}ds-\mu_{j\epsilon^*}\int_0^{t\wedge\sigma_k}\psi^q(s)|x(s)|^{p+q_1-1}ds
\end{aligned}$$

$$\leq \mu_{j\epsilon^*}\int_{-\infty}^{0}\psi^q(s)|x(s)|^{p+q_1-1}ds \leq \mu_{j\epsilon^*}\int_{-\infty}^{0}|\varphi(s)|^{p+q_1-1}ds < C.$$

Similarly,

$$\int_0^{t\wedge\sigma_k}\psi^q(s)\bar{H}_j^{q_1+1}(s)ds < C,\quad \int_0^{t\wedge\sigma_k}\psi^q(s)\bar{H}_j^{p}(s)ds < C,\quad \int_0^{t\wedge\sigma_k}\psi^q(s)\bar{H}_j^{2}(s)ds < C,$$

which implies

$$R_2 < C. \tag{A3.9}$$

Moreover, applying Lemma 2.5 and the Fubini theorem yields

$$\begin{aligned}|x(s)-x(s-\tau)|^2 \quad &\leq \quad \frac{1}{1-L}\Big|\int_{s-\tau}^{s}[f(x_v,v)+u(x(v-\tau),v)]dv+\int_{s-\tau}^{s}g(x_v,v)dB(v)\Big|^2\\ &\quad +L\int_{-\infty}^{0}|x(s+\theta)-x(s-\tau+\theta)|^2\mu_3(d\theta)\end{aligned}$$

and

$$\begin{aligned}&\int_0^{t\wedge\sigma_k}\psi^q(s)\int_{-\infty}^{0}|x(s+\theta)-x(s-\tau+\theta)|^2\mu_3(d\theta)ds\\ &=\int_{-\infty}^{0}\int_{\theta}^{t\wedge\sigma_k+\theta}\psi^q(s-\theta)|x(s)-x(s-\tau)|^2ds\mu_3(d\theta)\\ &\leq\int_{-\infty}^{0}\psi^q(-\theta)\mu_3(d\theta)\int_{-\infty}^{0}\psi^q(s)|x(s)-x(s-\tau)|^2ds\\ &\quad+\int_{-\infty}^{0}\psi^q(-\theta)\mu_3(d\theta)\int_0^{t\wedge\sigma_k}\psi^q(s)|x(s)-x(s-\tau)|^2ds\\ &\leq C+\mu_{3q}\int_0^{t\wedge\sigma_k}\psi^q(s)|x(s)-x(s-\tau)|^2ds.\end{aligned}$$

Noting that $\mu_{30}=1$ and $L\in(0,L^*)$, by Lemma 2.3, we always can choose $q^{(1)}\in(0,\epsilon^*]$ sufficiently small such that $L\mu_{3q^{(1)}}<1$. Using the Hölder inequality and the Fubini theorem yields that for any $q\in(0,q^{(1)}]$,

$$R_3\leq C+\frac{2}{(1-L)(1-L\mu_{3q})}\Big(\frac{\kappa^2}{4\rho_1}+\frac{4\tau^2\kappa^4}{\rho_1(1-L)^2}\Big)\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\int_{s-\tau}^{s}J(v)dvds.$$

By (3.8), $\kappa\tau<(1-L)/4\sqrt{2}$. Hence,

$$\frac{2}{(1-L)^2}\Big(\frac{\kappa^2}{4\rho_1}+\frac{4\tau^2\kappa^4}{\rho_1(1-L)^2}\Big)<\frac{3\kappa^2}{4\rho_1(1-L)^2}.$$

Then there exists $q^{(2)}\in(0,q^{(1)}]$ sufficiently small such that for any $q\in(0,q^{(2)}]$,

$$\frac{2}{(1-L)(1-L\mu_{3q})}\Big(\frac{\kappa^2}{4\rho_1}+\frac{4\tau^2\kappa^4}{\rho_1(1-L)^2}\Big)\leq\frac{3\kappa^2}{4\rho_1(1-L)^2}.$$

It then follows that for any $q\in(0,q^{(2)}]$,

$$R_3\leq C+\frac{3\kappa^2}{4\rho_1(1-L)^2}\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\int_{s-\tau}^{s}J(v)dvds=C+\frac{3}{4}R_4. \tag{A3.10}$$

Inserting (A3.8)–(A3.10) into (A3.7) gives that for any $q \in (0, q^{(2)}]$,

$$\begin{aligned}
&\mathbb{E}\psi^q(t\wedge\sigma_k)U(x_{t\wedge\sigma_k}, t\wedge\sigma_k)\\
&\le C - \Big(\frac{1}{4} - q\phi\tau\Big)\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\int_{s-\tau}^{s}J(v)dvds + \mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\Big[-a(|x(s)|^2 + |x(s)|^{q_1+1})\\
&\quad -\rho_6|x(s)|^{p+q_1-1} + q\phi(|\tilde{x}(s)|^2 + |\tilde{x}(s)|^{q_1+1})\Big]ds.
\end{aligned}$$

Choosing $q^{(3)} \in (0, \epsilon^*]$ sufficiently small such that $q^{(3)}\phi\tau \le \frac{1}{4}$, then for any $q \in (0, q^{(2)} \wedge q^{(3)}]$,

$$\begin{aligned}
\mathbb{E}\psi^q(t\wedge\sigma_k)U(x_{t\wedge\sigma_k}, t\wedge\sigma_k) \;\le\;& C + \mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\Big[-a(|x(s)|^2 + |x(s)|^{q_1+1}) - \rho_6|x(s)|^{p+q_1-1}\\
&+q\phi(|\tilde{x}(s)|^2 + |\tilde{x}(s)|^{q_1+1})\Big]ds. \qquad \text{(A3.11)}
\end{aligned}$$

Step 3 : proof of (3.9) **and** (3.10)**.** Let $h(t,q) = \psi^q(t)(|x(t)|^2 + |x(t)|^{q_1+1})$ and $\tilde{h}(t,q) = \psi^q(t)(|\tilde{x}(t)|^2 + |\tilde{x}(t)|^{q_1+1})$. We derive from (A3.11) that for any $q \in (0, q^{(2)} \wedge q^{(3)}]$,

$$\mathbb{E}\tilde{h}(t\wedge\sigma_k, q) \le C - a\mathbb{E}\int_0^{t\wedge\sigma_k}h(s,q)ds + q\phi\mathbb{E}\int_0^{t\wedge\sigma_k}\tilde{h}(s,q)ds - \rho_6\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)|x(s)|^{p+q_1-1}ds.$$

By Lemma 2.5, we have

$$\begin{aligned}
|\tilde{x}(s)|^2 \;&\le\; (1-L)^{-1}|x(s)|^2 + L\int_{-\infty}^{0}|x(s+\theta)|^2\mu_3(d\theta)\\
|\tilde{x}(s)|^{q_1+1} \;&\le\; (1-L)^{-q_1}|x(s)|^{q_1+1} + L\int_{-\infty}^{0}|x(s+\theta)|^{q_1+1}\mu_3(d\theta), \qquad \text{(A3.12)}
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_0^{t\wedge\sigma_k}\tilde{h}(s,q)ds = \int_0^{t\wedge\sigma_k}\psi^q(s)(|\tilde{x}(s)|^2 + |\tilde{x}(s)|^{q_1+1})ds\\
&\quad\le \int_0^{t\wedge\sigma_k}\psi^q(s)\Big\{(1-L)^{-q_1}(|x(s)|^2 + |x(s)|^{q_1+1})\\
&\qquad +L\int_{-\infty}^{0}(|x(s+\theta)|^2 + |x(s+\theta)|^{q_1+1})\mu_3(d\theta)\Big\}ds\\
&\quad= (1-L)^{-q_1}\int_0^{t\wedge\sigma_k}h(s,q)ds + L\int_0^{t\wedge\sigma_k}\int_{-\infty}^{0}\psi^q(s)(|x(s+\theta)|^2 + |x(s+\theta)|^{q_1+1})\mu_3(d\theta)ds\\
&\quad\le (1-L)^{-q_1}\int_0^{t\wedge\sigma_k}h(s,q)ds + L\int_0^{t\wedge\sigma_k}\int_{-\infty}^{0}\psi^q(-\theta)h(s+\theta,q)\mu_3(d\theta)ds. \qquad \text{(A3.13)}
\end{aligned}$$

Therefore, for any $q \in (0, q^{(2)} \wedge q^{(3)}]$,

$$\begin{aligned}
\mathbb{E}\tilde{h}(t\wedge\sigma_k, q) \;\le\;& C - (a - q\phi(1-L)^{-q_1})\mathbb{E}\int_0^{t\wedge\sigma_k}h(s,q)ds\\
&+q\phi L\mathbb{E}\int_0^{t\wedge\sigma_k}\int_{-\infty}^{0}\psi^q(-\theta)h(s+\theta,q)\mu_3(d\theta)ds\\
&-\rho_6\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)|x(s)|^{p+q_1-1}ds. \qquad \text{(A3.14)}
\end{aligned}$$

However,

$$\begin{aligned}
&L\mathbb{E}\int_0^{t\wedge\sigma_k}\int_{-\infty}^0 \psi^q(-\theta)h(s+\theta,q)\mu_3(d\theta)ds\\
&\le L\mathbb{E}\int_{-\infty}^0 \psi^q(-\theta)\mu_3(d\theta)\int_\theta^{t\wedge\sigma_k+\theta} h(s,q)ds\\
&\le L_q\int_{-\infty}^0 \psi^q(s)[|x(s)|^2+|x(s)|^{q_1+1}]ds + L_q\mathbb{E}\int_0^{t\wedge\sigma_k} h(s,q)ds\\
&\le L_q\int_{-\infty}^0 [|\varphi(s)|^2+|\varphi(s)|^{q_1+1}]ds + L_q\mathbb{E}\int_0^{t\wedge\sigma_k} h(s,q)ds,
\end{aligned} \tag{A3.15}$$

where $L_q = L\int_{-\infty}^0 \psi^q(-\theta)\mu_3(d\theta)$. Since $\varphi \in BC((-\infty,0];\mathbb{R}^n)\cap L^2((-\infty,0];\mathbb{R}^n)$, by Lemma 2.2, $\int_{-\infty}^0[|\varphi(s)|^2+|\varphi(s)|^{q_1+1}]ds < \infty$. Noting that $L\in(0,1)$ and $\int_{-\infty}^0 \psi^q(-\theta)\mu_3(d\theta)\to 1$ as $q\to 0$, we can always choose $q^{(4)}\in(0,\epsilon^*]$ sufficiently small such that $L_q\in(0,1)$ for any $q\in(0,q^{(4)}]$. Therefore, from (A3.14) we see that for any $q\in(0,q^{(2)}\wedge q^{(3)}\wedge q^{(4)}]$,

$$\mathbb{E}\tilde{h}(t\wedge\sigma_k,q)\le C-\big(a-q\phi(1-L)^{-q_1}-q\phi L_q\big)\mathbb{E}\int_0^{t\wedge\sigma_k} h(s,q)ds-\rho_6\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)|x(s)|^{p+q_1-1}ds.$$

Choosing $q^{(5)}\in(0,\epsilon^*]$ sufficiently small such that $q^{(5)}\phi(1-L)^{-q_1}+q^{(5)}\phi L_{q^{(5)}}<a$, we get that for any $q\in(0,q^{(2)}\wedge q^{(3)}\wedge q^{(4)}\wedge q^{(5)}]$,

$$\mathbb{E}\tilde{h}(t\wedge\sigma_k,q)\le C-\Big[\big(a-q\phi(1-L)^{-q_1}-q\phi L_q\big)\wedge\rho_6\Big]\mathbb{E}\int_0^{t\wedge\sigma_k}\psi^q(s)\big[|x(s)|^2+|x(s)|^{q_1+1}+|x(s)|^{p+q_1-1}\big]ds.$$

Noting that $p>2$ and $|x(s)|^{q_1+1}\le|x(s)|^2+|x(s)|^{p+q_1-1}$, then there exists $q^*\in(0,q^{(2)}\wedge q^{(3)}\wedge q^{(4)}\wedge q^{(5)}]$ such that

$$\mathbb{E}\tilde{h}(t\wedge\sigma_k,q^*)\le C,\quad \mathbb{E}\int_0^{t\wedge\sigma_k}\psi^{q^*}(s)\big[|x(s)|^2+|x(s)|^{p+q_1-1}\big]ds\le C.$$

Letting $k\to\infty$, and then $t\to\infty$ gives

$$\limsup_{t\to\infty}\mathbb{E}\tilde{h}(t,q^*)\le C,\quad \mathbb{E}\int_0^{\infty}\psi^{q^*}(s)\big[|x(s)|^2+|x(s)|^{p+q_1-1}\big]ds\le C. \tag{A3.16}$$

Similar method to (A3.13) can give

$$\mathbb{E}h(t,q^*)=\mathbb{E}\psi^{q^*}(t)(|x(t)|^2+|x(t)|^{q_1+1})\le(1-L)^{-q_1}\mathbb{E}\tilde{h}(t,q^*)+L\mathbb{E}\int_{-\infty}^0\psi^{q^*}(-\theta)h(t+\theta,q^*)\mu_3(d\theta).$$

Hence,

$$\begin{aligned}
&\sup_{-\infty\le s\le t}\mathbb{E}h(s,q^*)\\
&\quad\le(\|\varphi\|^2+\|\varphi\|^{q_1+1})\vee\Big((1-L)^{-q_1}\sup_{0\le s\le t}\mathbb{E}\tilde{h}(s,q^*)+L\int_{-\infty}^0\psi^{q^*}(-\theta)\big(\sup_{0\le s\le t}\mathbb{E}h(s+\theta,q^*)\big)\mu_3(d\theta)\Big)\\
&\quad\le\|\varphi\|^2+\|\varphi\|^{q_1+1}+(1-L)^{-q_1}\sup_{0\le s\le t}\mathbb{E}\tilde{h}(s,q^*)+L\int_{-\infty}^0\psi^{q^*}(-\theta)\big(\sup_{-\infty\le s\le t}\mathbb{E}h(s,q^*)\big)\mu_3(d\theta)
\end{aligned}$$

$$= \|\varphi\|^2 + \|\varphi\|^{q_1+1} + (1-L)^{-q_1} \sup_{0\le s\le t} \mathbb{E}\tilde{h}(s,q^*) + L_{q^*} \sup_{-\infty\le s\le t} \mathbb{E}h(s,q^*), \tag{A3.17}$$

which implies

$$\sup_{-\infty\le s\le t} \mathbb{E}h(s,q^*) \le (1-L_{q^*})^{-1}(\|\varphi\|^2 + \|\varphi\|^{q_1+1}) + (1-L_{q^*})^{-1}(1-L)^{-q_1} \sup_{0\le s\le t} \mathbb{E}\tilde{h}(s,q^*). \tag{A3.18}$$

This, together with (A3.16), leads to $\limsup_{t\to\infty} \mathbb{E}h(t,q^*) \le C$. This implies $\mathbb{E}[|x(t)|^2 + |x(t)|^{q_1+1}] \le C\psi^{-q^*}(t)$ for all $t\ge 0$. Therefore,

$$\limsup_{t\to\infty} \frac{\ln \mathbb{E}[|x(t)|^2 + |x(t)|^{q_1+1}]}{\ln \psi(t)} \le -q^*. \tag{A3.19}$$

Noting that $|x(t)|^{\hat{p}} \le |x(t)|^2 + |x(t)|^{p+q_1-1}$ for $\hat{p}\in[2,p+q_1-1]$ and $|x(t)|^{\hat{p}} \le |x(t)|^2 + |x(t)|^{q_1+1}$ for $\hat{p}\in[2,q_1+1]$, the required assertion (3.9) and (3.10) follows from (A3.16) and (A3.19) immediately.

A4. Proof of Theorem 3.3.

Let us divide this proof into two steps.

Step 1 : Estimation of $\mathbb{E}\Big(\sup_{0\le t<\infty} \psi^{q^*}(t)|x(t)|^2\Big)$. For q^* given in Theorem 3.2 and any $T>0$, applying the Itô formula yields

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T\wedge\sigma_k} \psi^{q^*}(t)|\tilde{x}(t)|^2\Big)\\
&\le |\tilde{x}(0)|^2 + \mathbb{E}\sup_{0\le t\le T\wedge\sigma_k} \int_0^t \psi^{q^*}(s)\Big\{q^*\phi|\tilde{x}(s)|^2 + 2\tilde{x}'(s)[f(x_s,s) + u(x(s-\tau),s)]\\
&\quad + |g(x_s,s)|^2\Big\}ds + \mathbb{E}\Big(\sup_{0\le t\le T\wedge\sigma_k} \Gamma_t\Big)\\
&\le |\tilde{x}(0)|^2 + \mathbb{E}\int_0^{T\wedge\sigma_k} \psi^{q^*}(s)\Big\{q^*\phi|\tilde{x}(s)|^2 + 2|\tilde{x}(s)||f(x_s,s)| + 2|\tilde{x}(s)||u(x(s-\tau),s)|\\
&\quad + |g(x_s,s)|^2\Big\}ds + \mathbb{E}\Big(\sup_{0\le t\le T\wedge\sigma_k} \Gamma_t\Big),
\end{aligned} \tag{A4.1}$$

where σ_k is given in (A1.7), $\phi = \sup_{s\ge 0}[\psi'(s)/\psi(s)] < \infty$ and $\Gamma_t = 2\int_0^t \psi^{q^*}(s)\tilde{x}^T(s)g(x_s,s)dB(s)$. The Burkholder-David-Gundy inequality gives

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le T\wedge\sigma_k} \Gamma_t\Big) &\le 12\mathbb{E}\Big(\int_0^{T\wedge\sigma_k} \psi^{2q^*}(s)|\tilde{x}^T(s)g(x_s,s)|^2 ds\Big)^{\frac{1}{2}}\\
&\le 12\mathbb{E}\Big[\Big(\sup_{0\le s\le T\wedge\sigma_k} \psi^{q^*}(s)|\tilde{x}(s)|^2\Big)^{\frac{1}{2}}\Big(\int_0^{T\wedge\sigma_k} \psi^{q^*}(s)|g(x_s,s)|^2 ds\Big)^{\frac{1}{2}}\Big]\\
&\le \frac{1}{2}\mathbb{E}\Big(\sup_{0\le s\le T\wedge\sigma_k} \psi^{q^*}(s)|\tilde{x}(s)|^2\Big) + C\mathbb{E}\int_0^{T\wedge\sigma_k} \psi^{q^*}(s)|g(x_s,s)|^2 ds.
\end{aligned}$$

Substituting this into (A4.1) and using (A3.12) yields

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T\wedge\sigma_k} \psi^{q^*}(t)|\tilde{x}(t)|^2\Big)\\
&\le |\tilde{x}(0)|^2 + C\mathbb{E}\int_0^{T\wedge\sigma_k} \psi^{q^*}(s)\Big\{|\tilde{x}(s)|^2 + |\tilde{x}(s)||f(x_s,s)| + |\tilde{x}(s)||u(x(s-\tau),s)|
\end{aligned}$$

$$
\begin{aligned}
&+|g(x_s,s)|^2\Big\}ds\\
&\le |\tilde{x}(0)|^2+C\mathbb{E}\int_0^{T\wedge\sigma_k}\psi^{q^*}(s)\Big\{|x(s)|^2+\int_0^\infty |x(s+\theta)|^2\mu_3(d\theta)\\
&+|\tilde{x}(s)||f(x_s,s)|+|\tilde{x}(s)||u(x(s-\tau),s)|+|g(x_s,s)|^2\Big\}ds. \qquad \text{(A4.2)}
\end{aligned}
$$

Note that $(q_1+1)\vee(2q_2)\le p+q_1-1$ and $\big(|x(s)|^{q_1+1}\vee|x(s)|^{2q_2}\big)\le |x(s)|^2+|x(s)|^{p+q_1-1}$. By Assumption 2.1, we infer from the Young inequality and (A3.12) that

$$
\begin{aligned}
|\tilde{x}(s)||f(x_s,s)| \;\le\; & C\Big(\int_0^\infty |x(s+\theta)|^{q_1+1}\mu_1(d\theta)+|x(s)|^{q_1+1}+\int_0^\infty |x(s+\theta)|^2\mu_1(d\theta)\\
&+|x(s)|^2+|\tilde{x}(s)|^{q_1+1}+|\tilde{x}(s)|^2\Big)\\
\le\; & C\Big(\int_0^\infty |x(s+\theta)|^{p+q_1-1}\mu_1(d\theta)+|x(s)|^{p+q_1-1}+\int_0^\infty |x(s+\theta)|^2\mu_1(d\theta)\\
&+|x(s)|^2+\int_0^\infty |x(s+\theta)|^{p+q_1-1}\mu_3(d\theta)+\int_0^\infty |x(s+\theta)|^2\mu_3(d\theta)\Big) \qquad \text{(A4.3)}
\end{aligned}
$$

and

$$
\begin{aligned}
|g(x_s,s)|^2 \;\le\; & C\Big(\int_0^\infty |x(s+\theta)|^{2q_2}\mu_2(d\theta)+|x(s)|^{2q_2}+\int_0^\infty |x(s+\theta)|^2\mu_2(d\theta)\\
&+|x(s)|^2+|\tilde{x}(s)|^{2q_2}+|\tilde{x}(s)|^2\Big)\\
\le\; & C\Big(\int_0^\infty |x(s+\theta)|^{p+q_1-1}\mu_2(d\theta)+|x(s)|^{p+q_1-1}+\int_0^\infty |x(s+\theta)|^2\mu_2(d\theta)\\
&+|x(s)|^2+|\tilde{x}(s)|^{p+q_1-1}+|\tilde{x}(s)|^2\Big)\\
\le\; & C\Big(\int_0^\infty |x(s+\theta)|^{p+q_1-1}\mu_2(d\theta)+|x(s)|^{p+q_1-1}+\int_0^\infty |x(s+\theta)|^2\mu_2(d\theta)\\
&+|x(s)|^2+\int_0^\infty |x(s+\theta)|^{p+q_1-1}\mu_3(d\theta)+\int_0^\infty |x(s+\theta)|^2\mu_3(d\theta)\Big). \qquad \text{(A4.4)}
\end{aligned}
$$

Similarly, it follows from Assumption 3.1, the Young inequality and (A3.12) that

$$
\begin{aligned}
|\tilde{x}(s)||u(x(s-\tau),s)| \;\le\; & C(|\tilde{x}(s)|^2+|x(s-\tau)|^2)\\
\le\; & C\Big(|x(s)|^2+\int_0^\infty |x(s+\theta)|^2\mu_3(d\theta)+|x(s-\tau)|^2\Big). \qquad \text{(A4.5)}
\end{aligned}
$$

Substituting (A4.3)–(A4.5) into (A4.2) and using (3.9) yields

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T\wedge\sigma_k}\psi^{q^*}(t)|\tilde{x}(t)|^2\Big)\\
&\le C+C\mathbb{E}\int_0^{T\wedge\sigma_k}\psi^{q^*}(s)\Big\{\sum_{j=1}^3\int_0^\infty \big[|x(s+\theta)|^{p+q_1-1}+|x(s+\theta)|^2\big]\mu_j(d\theta)\\
&\qquad +|x(s-\tau)|^2\Big\}ds. \qquad \text{(A4.6)}
\end{aligned}
$$

Note that $\varphi\in BC((-\infty,0];\mathbb{R}^n)\cap L^2((-\infty,0];\mathbb{R}^n)$. Similar method to (A3.15) can give

$$
C\mathbb{E}\int_0^{T\wedge\sigma_k}\psi^{q^*}(s)\Big\{\sum_{j=1}^3\int_0^\infty \big[|x(s+\theta)|^{p+q_1-1}+|x(s+\theta)|^2\big]\mu_j(d\theta)\Big\}ds
$$

$$\leq C + C\mathbb{E}\int_0^{T\wedge\sigma_k} \psi^{q^*}(s)[|x(s)|^{p+q_1-1} + |x(s)|^2]ds.$$

Moreover,

$$\begin{aligned} C\mathbb{E}\int_0^{T\wedge\sigma_k} \psi^{q^*}(s)|x(s-\tau)|^2 ds &\leq C\mathbb{E}\int_0^{T\wedge\sigma_k} \psi^{q^*}(\tau)\psi^{q^*}(s-\tau)|x(s-\tau)|^2 ds \\ &\leq C + C\mathbb{E}\int_0^{T\wedge\sigma_k} \psi^{q^*}(s)|x(s)|^2 ds. \end{aligned}$$

Therefore, it follows from (3.9) and (A4.6) that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T\wedge\sigma_k} \psi^{q^*}(t)|\tilde{x}(t)|^2\Big) \leq C + C\mathbb{E}\int_0^{\infty} \psi^{q^*}(s)[|x(s)|^{p+q_1-1} + |x(s)|^2]ds \leq C.$$

Similar the way to derive (A3.17) and (A3.18) can give

$$\mathbb{E}\Big(\sup_{0\leq t\leq T\wedge\sigma_k} \psi^{q^*}(t)|x(t)|^2\Big) \leq C.$$

Letting $k\to\infty$, and then $T\to\infty$ gives

$$\mathbb{E}\Big(\sup_{0\leq t<\infty} \psi^{q^*}(t)|x(t)|^2\Big) \leq C. \tag{A4.7}$$

Step 2 : Almost sure stability. It follows from (A4.7) that

$$\sup_{0\leq t<\infty} \psi^{q^*}(t)|x(t)|^2 < \infty \quad a.s..$$

Therefore,

$$\limsup_{t\to\infty} \frac{q^*\ln\psi(t) + 2\ln|x(t)|}{\ln\psi(t)} = 0 \quad a.s.,$$

which implies

$$\limsup_{t\to\infty} \frac{\ln|x(t)|}{\ln\psi(t)} = -q^*/2 \quad a.s..$$

That is, the controlled system (1.2) is ψ–type stable. This completes the proof.

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