



Nonzero-sum risk-sensitive stochastic differential games: A multi-parameter eigenvalue problem approach[☆]

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ABSTRACT

We study nonzero-sum stochastic differential games with risk-sensitive ergodic cost criterion. Under certain conditions, using multi parameter eigenvalue approach, we establish the existence of a Nash equilibrium in the space of stationary Markov strategies. We achieve our results by studying the relevant systems of coupled Hamilton–Jacobi–Bellman (HJB) equations. Exploiting the stochastic representation of the principal eigenfunctions we completely characterize Nash equilibrium points in the space of stationary Markov strategies. The complete characterization of Nash equilibrium points is established under an additive structural assumption on the running cost and the drift term.

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1. Introduction

We study non zero-sum risk-sensitive stochastic differential games in a multi parameter eigenvalue problem framework. In the literature of stochastic differential games, one usually considers the expectation of the integral of costs ([1–3] etc.). This is the so called risk-neutral situation where the players (i.e., the decision makers or controllers) ignore the risk. If the players are risk-sensitive (i.e., risk-averse or risk-seeking), then one of the most appropriate cost criteria is the expectation of the exponential of the integral of costs as it leads to certainty equivalence [4]. Since the cost criterion is the expectation of the exponential of the integral costs, it is multiplicative as opposed to the additive nature of the cost criterion in the expectation of the integral costs case. Due to this, the analysis of the risk-sensitive case is significantly different from its risk-neutral counterpart. To our knowledge, the risk-sensitive criterion was first introduced by Bellman [5]; see [6] and the references therein. Though this criterion has been studied extensively for stochastic optimal control problems [7–21], the corresponding literature in the context of stochastic differential games is rather limited. Some exceptions are [22–25]. Basar [22] proves the existence of a Nash equilibrium for finite horizon nonzero-sum risk sensitive games.

El-Karoui and Hamadene [25] study risk-sensitive control, zero-sum and nonzero-sum game problems. They prove the existence of an optimal control, a saddle-point and a Nash equilibrium point for relevant cases. In [25], authors use Pontryagin's minimum principle to characterize the optimality condition and the adjoint problem leads to some special backward stochastic differential equations. Basu and Ghosh [23] study infinite horizon risk-sensitive zero-sum stochastic differential games and establish the existence of saddle points which are mini-max selectors of the associated Hamilton–Jacobi–Isaacs (HJI) equation. In a recent work Biswas and Saha [24] consider risk-sensitive zero-sum stochastic differential games for controlled diffusion process in \mathbb{R}^d . Under fairly general conditions on the drift and the diffusion coefficients (e.g., the coefficients are locally Lipschitz continuous and have some global growth condition), they study the ergodic cost criterion. They completely characterize saddle point equilibria in the space of stationary Markov strategies, under the assumption that running cost function satisfies either small cost condition or dominated by some inf-compact function.

In the framework of reflecting diffusions Ghosh and Pradhan [26] (in bounded domain), [27] (in orthonant) have studied similar nonzero-sum game problem for risk-sensitive ergodic cost criterion. They studied the game problems by studying the associated system of coupled HJB equations. In the reflecting diffusion setup, the associated coupled systems are semi-linear elliptic pdes with some oblique boundary conditions. The authors used the principal eigenvalue approach to completely characterize all possible Nash equilibria in the space of stationary Markov strategies. Due to the presence of these nontrivial boundary conditions,

[☆] This paper is dedicated to the memory of Ari Arapostathis.

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in order to establish the existence of principal eigenpair to associated coupled HJB equations, the authors in [27] crucially used the fact that the drift term, diffusion matrix are uniformly bounded and the running cost function satisfies certain small cost condition. Together with the ergodic cost criterion, in [27] the authors studied the game problem for discounted cost criterion as well.

For controlled diffusion models, similar game problem under discounted cost criterion is studied in [28]. By studying the associated system of coupled HJB equations, which is in this case is a coupled system of semi-linear parabolic pdes, they have established the existence of Nash equilibrium points in the class of eventually stationary Markov strategies. The uniform boundedness assumptions on the diffusion coefficients and the running cost functions play important role in the analysis of this game problem.

In this paper, we address the existence of Nash equilibria for stochastic differential games where the state of the system is governed by a controlled diffusion processes in the whole space \mathbb{R}^d . We consider the risk-sensitive ergodic cost evaluation criterion. We analyze this game problem by analyzing the associated system of coupled HJB equation, which is a system of coupled semi-linear elliptic pdes in \mathbb{R}^d . Compared to [26–28], under a relatively weaker set of assumptions on diffusion coefficients (e.g., the drift term and diffusion matrix are locally Lipschitz continuous and have some global growth condition) (see Assumption 1), using principal eigenvalue approach we establish the existence of a Nash equilibrium in the space of stationary Markov strategies. Also, in this present study, we are allowing our running cost function to be unbounded as well (see Assumption 2(ii)).

In order to establish the existence of principal eigenpair of the associated coupled system of Hamilton–Jacobi–Bellman (HJB) equation, we first study the corresponding Dirichlet eigenvalue problem on smooth bounded domains in \mathbb{R}^d . Applying a version of non-linear Krein–Rutman theorem we show that principal eigenpair exists for Dirichlet eigenvalue problem. Then increasing these domains to \mathbb{R}^d and employing Fan’s fixed point theorem [29], we establish the existence of principal eigenpair to the associated coupled system of HJB equation in the whole space \mathbb{R}^d , which lead to the existence of a Nash equilibrium. Furthermore, exploiting the stochastic representation of the principal eigenfunctions we completely characterize all possible Nash equilibria in the space of stationary Markov strategies. Thus, the main results of this article can be roughly described as follows.

- *Existence and uniqueness of solution to the coupled HJB equation:* Using Principal eigenvalue approach, we establish the existence and uniqueness of solution to the associated coupled HJB equation in an appropriate function space.
- *Characterization of Nash equilibrium:* Using Fan’s fixed point theorem we first establish the existence of Nash equilibrium in the space of stationary Markov strategies. Then utilizing the stochastic representation of the principal eigenfunctions we completely characterize all possible Nash equilibria in the space of stationary Markov strategies.

The rest of this paper is organized as follows. Section 2 deals with the problem description. In Section 3 we discuss the principal eigenvalue problem for controlled diffusion operators on smooth bounded domains. Section 4 is devoted to study the eigenvalue problem for controlled diffusion operator in whole space \mathbb{R}^d . The complete characterization of Nash equilibrium in the space of stationary Markov strategies is presented in Section 5.

2. Problem description

For the sake of notational simplicity we treat two player case. Let $U_i, i = 1, 2$ be compact metric spaces and $V_i = \mathcal{P}(U_i)$, the space of probability measures on the compact metric space U_i with the topology of weak convergence. Let $\bar{b} = (\bar{b}_1, \dots, \bar{b}_d) : \mathbb{R}^d \times U_1 \times U_2 \rightarrow \mathbb{R}^d, \bar{r}_i : \mathbb{R}^d \times U_1 \times U_2 \rightarrow [0, \infty), i = 1, 2, \sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ be given functions satisfying Assumption 1 (to be described below).

Define $b = (b_1, \dots, b_d) : \mathbb{R}^d \times V_1 \times V_2 \rightarrow \mathbb{R}^d, r_i : \mathbb{R}^d \times V_1 \times V_2 \rightarrow [0, \infty)$ by

$$b_k(x, v_1, v_2) = \int_{U_2} \int_{U_1} \bar{b}_k(x, u_1, u_2) v_1(du_1) v_2(du_2),$$

$$r_i(x, v_1, v_2) = \int_{U_2} \int_{U_1} \bar{r}_i(x, u_1, u_2) v_1(du_1) v_2(du_2), x \in \mathbb{R}^d, \\ v_1 \in V_1, v_2 \in V_2, k = 1, \dots, d, i = 1, 2.$$

We consider a nonzero-sum stochastic differential game whose state is evolving according to a controlled diffusion process given by the solution of the following stochastic differential equation (s.d.e.)

$$dX(t) = b(X(t), v_1(t), v_2(t))dt + \sigma(X(t))dW(t), \quad (2.1)$$

where $W(\cdot)$ is an \mathbb{R}^d -valued standard Wiener process, $v_i(\cdot)$ is a V_i -valued process which is a non-anticipative functional of the state process $X(\cdot)$, i.e., $v_i(t) = f_i(t, X([0, t]))$ where $X([0, t])(s) = X(s \wedge t)$ for all $s \in [0, \infty)$ and $f_i : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \rightarrow V_i$. Such a strategy is called an admissible strategy. For $i = 1, 2, \mathcal{A}_i$ denotes the space of all admissible strategies of Player i . In order to ensure the existence of a solution to Eq. (2.1) and the existence of Nash equilibrium (to be describe in (2.6)), we impose following conditions on the drift term \bar{b} , and the dispersion matrix σ .

Assumption 1.

- (i) *Local Lipschitz continuity:* The function $\sigma = [\sigma^{ij}] : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ and $\bar{b} : \mathbb{R}^d \times U_1 \times U_2 \rightarrow \mathbb{R}^d$ are locally Lipschitz continuous in x (uniformly with respect to the rest), i.e., for each $R \geq 0$, there exists a constant $C_R > 0$ depending on $R > 0$, such that

$$|\bar{b}(x, u_1, u_2) - \bar{b}(y, u_1, u_2)|^2 + \|\sigma(x) - \sigma(y)\|^2 \leq C_R |x - y|^2$$

for all $x, y \in B_R$ ($:= \{x \in \mathbb{R}^d : |x| < R\}$), $i = 1, 2$ and $(u_1, u_2) \in U_1 \times U_2$, where $\|\sigma\| := \sqrt{\text{tr}(\sigma\sigma^T)}$ and $\bar{b} = (\bar{b}_1, \dots, \bar{b}_d)^T$. Also, we assume that b, r_i are jointly continuous in (x, u_1, u_2) for $i = 1, 2$.

- (ii) *Affine growth condition:* \bar{b} and σ satisfy a global growth condition of the form

$$\sup_{u_1 \in U_1, u_2 \in U_2} \langle \bar{b}(x, u_1, u_2), x \rangle + \|\sigma(x)\|^2 \leq C_0(1 + |x|^2) \quad \forall x \in \mathbb{R}^d,$$

for some constant $C_0 > 0$.

- (iii) *Nondegeneracy:* For each $R > 0$, it holds that

$$\sum_{i,j=1}^d a^{ij}(x) z_i z_j \geq C_R^{-1} |z|^2 \quad \forall x \in B_R,$$

and for all $z = (z_1, \dots, z_d)^T \in \mathbb{R}^d$, where $a = [a^{ij}] := \frac{1}{2} \sigma \sigma^T$.

Also, we assume that the running cost functions $\bar{r}_i : \mathbb{R}^d \times U_1 \times U_2 \rightarrow \mathbb{R}_+$ $i = 1, 2$ are jointly continuous in (x, u_1, u_2) and locally Lipschitz continuous in x (uniformly with respect to the rest), i.e., for all $R \geq 0$ and $x, y \in B_R$ there exists a constant $C_R > 0$ depending on $R > 0$, such that

$$|\bar{r}_i(x, u_1, u_2) - \bar{r}_i(y, u_1, u_2)|^2 \leq C_R |x - y|^2 \quad \text{for all } (u_1, u_2) \in U_1 \times U_2.$$

It is well known that, under [Assumption 1](#), for any $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ and initial condition $X(0) = x$, the s.d.e. (2.1) admits a unique weak solution which is a strong Markov process (see [30, Theorem 2.2.11, p.42]). For the stochastic differential game, the controlled diffusion given by (2.1) has the following interpretation: The i th player controls the state dynamics, i.e., the controlled diffusion given above, through the choice of her/his strategy v_i . The function \bar{r}_i represents the running cost function of Player i . If the strategy v_i has the form $v_i(t) = \bar{v}_i(t, X(t))$, $t \geq 0$ for some $\bar{v}_i : [0, \infty) \times \mathbb{R}^d \rightarrow V_i$, then v_i or by an abuse of notation \bar{v}_i is called a Markov strategy for Player i . Let $\mathcal{M}_i = \{v_i : [0, \infty) \times \mathbb{R}^d \rightarrow V_i \mid v_i \text{ is measurable}\}$ be the set of all Markov strategies for Player i . Under a pair of Markov strategies the s.d.e. (2.1) admits a unique strong solution which is a strong Markov process (see [30, Theorem 2.2.12, p.45]). If v_i does not have explicit dependence on t , i.e., $\bar{v}_i(t, x) = \bar{v}_i(x)$, $x \in \mathbb{R}^d$, $t \geq 0$, it is said to be a stationary Markov strategy for Player i . The set of all stationary Markov strategies for Player i is denoted by \mathcal{S}_i , $i = 1, 2$. We topologize \mathcal{S}_i , $i = 1, 2$, using a metrizable weak* topology on $L^\infty(\mathbb{R}^d; \mathcal{M}_s(U_i))$, where $\mathcal{M}_s(U_i)$ denotes the space of all signed measures on U_i with weak* topology. Since \mathcal{S}_i is a subset of the unit ball of $L^\infty(\mathbb{R}^d; \mathcal{M}_s(U_i))$, it is compact under the above weak* topology. One also has the following characterization of the topology given by the following convergence criterion: For $i = 1, 2$, $v_i^n \rightarrow v_i$ in \mathcal{S}_i as $n \rightarrow \infty$ if and only if

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f(x) \int_{U_i} g(x, u_i) v_i^n(x)(du_i) dx = \int_{\mathbb{R}^d} f(x) \int_{U_i} g(x, u_i) v_i(x)(du_i) dx, \quad (2.2)$$

for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $g \in C_b(\mathbb{R}^d \times U_i)$; see [30, p. 57], for details.

For $v_i \in V_i$, $i = 1, 2$, let $\mathcal{L}^{v_1, v_2} : C^2(\mathbb{R}^d) \rightarrow C(\mathbb{R}^d)$, be given by

$$\mathcal{L}^{v_1, v_2} f(x) = a^{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + b_i(x, v_1, v_2) \frac{\partial f(x)}{\partial x_i}, \quad f \in C^2(\mathbb{R}^d), \quad (2.3)$$

where Einstein summation convention is used. Further, let

$$\mathcal{G}_1^{v_2} f = \inf_{v_1 \in V_1} [\mathcal{L}_1^{v_1, v_2} f + r_1(x, v_1, v_2(x))f], \quad v_2 \in \mathcal{S}_2, \quad (2.4)$$

$$\mathcal{G}_2^{v_1} f = \inf_{v_2 \in V_2} [\mathcal{L}_2^{v_1, v_2} f + r_2(x, v_1(x), v_2)f], \quad v_1 \in \mathcal{S}_1, f \in C^2(\mathbb{R}^d),$$

where for $f \in C^2(\mathbb{R}^d)$,

$$\mathcal{L}_1^{v_1, v_2} f(x) = \mathcal{L}^{v_1, v_2(x)} f(x) \quad \forall v_1 \in V_1, v_2 \in \mathcal{S}_2$$

and

$$\mathcal{L}_2^{v_1, v_2} f(x) = \mathcal{L}^{v_1(x), v_2} f(x) \quad \forall v_1 \in \mathcal{S}_1, v_2 \in V_2.$$

For $(v_1, v_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, it is easy to see that

$$\begin{aligned} \mathcal{L}_1^{v_1(x), v_2} f(x) &= \mathcal{L}_2^{v_1, v_2(x)} f(x) = \mathcal{L}^{v_1(x), v_2(x)} f(x) = a^{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \\ &\quad + b_i(x, v_1(x), v_2(x)) \frac{\partial f(x)}{\partial x_i}. \end{aligned}$$

The analysis of our game problem will be based on the analysis of the eigenvalue problems of the above defined operators.

2.1. Ergodic cost criterion

Given the running cost functions $r_i : \mathbb{R}^d \times V_1 \times V_2 \rightarrow \mathbb{R}_+$, $i = 1, 2$, for any $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, the associated risk-sensitive ergodic cost of Player i is defined by

$$\rho_i(x, v_1, v_2) = \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^T r_i(X(t), v_1(t), v_2(t)) dt} \right], \quad i = 1, 2. \quad (2.5)$$

The definition of a Nash equilibrium is standard, i.e., $(v_1^*, v_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ is a Nash equilibrium among the class of admissible strategies if

$$\begin{aligned} \rho_1(x, v_1^*, v_2^*) &\leq \rho_1(x, v_1, v_2^*), \quad \text{for all } v_1 \in \mathcal{A}_1, \\ \rho_2(x, v_1^*, v_2^*) &\leq \rho_2(x, v_1^*, v_2), \quad \text{for all } v_2 \in \mathcal{A}_2, \quad \text{for all } x \in \mathbb{R}^d. \end{aligned} \quad (2.6)$$

We assume that our running cost functions r_i , $i = 1, 2$ satisfy [Assumption 1](#)(i). Now for each $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, define

$$\begin{aligned} \lambda_1(x, v_2) &= \inf_{v_1' \in \mathcal{A}_1} \rho_1(x, v_1', v_2), \quad \lambda_1(v_2) = \inf_{x \in \mathbb{R}^d} \lambda_1(x, v_2), \\ \Lambda_1(x, v_2) &= \inf_{v_1' \in \mathcal{S}_1} \rho_1(x, v_1', v_2), \quad \Lambda_1(v_2) = \inf_{x \in \mathbb{R}^d} \Lambda_1(x, v_2), \\ \lambda_2(x, v_1) &= \inf_{v_2' \in \mathcal{A}_2} \rho_2(x, v_1, v_2'), \quad \lambda_2(v_1) = \inf_{x \in \mathbb{R}^d} \lambda_2(x, v_1), \\ \Lambda_2(x, v_1) &= \inf_{v_2' \in \mathcal{S}_2} \rho_2(x, v_1, v_2'), \quad \Lambda_2(v_1) = \inf_{x \in \mathbb{R}^d} \Lambda_2(x, v_1). \end{aligned} \quad (2.7)$$

Now we outline our program for establishing the existence of a Nash equilibrium. We analyze our game problem by analyzing the corresponding system of coupled Hamilton–Jacobi–Bellman (HJB) equations. Suppose that one of the players, say Player 2 announces his strategy $v_2 \in \mathcal{S}_2$ in advance, then Player 1 tries to minimize associated cost $\rho_1(x, v_1, v_2)$ (see, Eq. (2.5)) over all $v_1 \in \mathcal{A}_1$, which is a (stochastic) optimal control problem for Player 1. Such an optimal control problem has been studied in [7,13,14] and it is shown that one can characterize the optimal value and optimal controls by analyzing the corresponding HJB equation given by

$$\lambda_1 \psi_1(x) = \mathcal{G}_1^{v_2} \psi_1(x) \quad \text{with} \quad \psi_1(0) = 1. \quad (2.8)$$

It is well known that (see [7]) the principal eigenvalue of the HJB equation is the optimal value $\lambda_1(v_2)$ and any minimizing selector of (2.8) (which is same as the minimizing selector of (2.4)), i.e., any $v_1^* \in \mathcal{S}_1$ which satisfies

$$\lambda_1(v_2) \psi_1(x) = \mathcal{G}_1^{v_2} \psi_1(x) = \mathcal{L}_1^{v_1^*, v_2} \psi_1 + r_1(x, v_1^*(x), v_2(x)) \psi_1,$$

is an optimal control for Player 1. In particular, $v_1^* \in \mathcal{S}_1$ is an optimal response for Player 1 corresponding to the announced strategy v_2 of Player 2. Note that v_1^* depends on v_2 and the map $v_2 \in \mathcal{S}_2 \rightarrow$ the optimal responses of Player 1

may be multi-valued. Analogous result holds for Player 2 if Player 1 announces his strategy $v_1 \in \mathcal{S}_1$ in advance. From the above discussion, it is easy to see that for any given pair of strategies $(v_1, v_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, one can construct a set of pairs of optimal responses $\{(v_1^*, v_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2\}$ from their corresponding HJB equations. Clearly any fixed point of this multi-valued map is a Nash equilibrium. The above discussion leads to the following program for finding a pair of Nash equilibrium strategies for ergodic cost criterion. Suppose that there exist a pair of stationary strategies $(v_1^*, v_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$, a pair of scalars (λ_1, λ_2) and a pair of functions (ψ_1, ψ_2) in an appropriate function space satisfying the following coupled HJB equations

$$\begin{aligned} \lambda_1 \psi_1 &= \mathcal{G}_1^{v_2^*} \psi_1 = \mathcal{L}_1^{v_1^*, v_2^*} \psi_1 + r_1(x, v_1^*(x), v_2^*(x)) \psi_1 \\ \lambda_2 \psi_2 &= \mathcal{G}_2^{v_1^*} \psi_2 = \mathcal{L}_2^{v_1^*, v_2^*} \psi_2 + r_2(x, v_1^*(x), v_2^*(x)) \psi_2, \end{aligned}$$

then (v_1^*, v_2^*) will be a pair of Nash equilibrium. The above discussion leads us to study the principal eigenvalues associated with the above coupled equations in the subsequent sections.

3. Dirichlet eigenvalue problem for controlled diffusion operators

In this section, we discuss the principal eigenvalue problem associated with the nonlinear operators $\mathcal{G}_i^{v_j}$ on smooth bounded

domains $D \subset \mathbb{R}^d$. The generalized principal eigenvalue of the semi-linear operator $\mathcal{G}_i^{v_j}$ with Dirichlet boundary condition on D is defined by

$$\lambda_i^+(v_j, D) = \inf\{\lambda \in \mathbb{R} \mid \text{for some } \varphi \in W^{2,p}(D) \cap C(\bar{D}), p > d, \varphi > 0, \mathcal{G}_i^{v_j} \varphi \leq \lambda \varphi \text{ in } D\}, \quad (3.1)$$

for $i \neq j$, $i, j = 1, 2$. Now we prove the existence of the principal eigenvalues of a certain parametric family of semi-linear elliptic pdes.

Theorem 3.1. *Suppose that Assumption 1 holds. Let $v_j \in S_j$ and D be a bounded smooth domain in \mathbb{R}^d . Then there exists (unique up to a scalar multiplication) $\psi_D \in W^{2,p}(D) \cap C(\bar{D})$, $p > d$, $\psi_D > 0$ such that*

$$\mathcal{G}_i^{v_j} \psi_D = \lambda_i^+(v_j, D) \psi_D, \quad (3.2)$$

$$\psi_D = 0 \text{ on } \partial D, \quad i, j = 1, 2 \text{ with } i \neq j.$$

Proof. We take $i = 1, j = 2$. Suppose $r_1 \leq 0$ (this will be dropped shortly). For $\phi \in C_0^1(D) := C_0(D) \cap C^1(D)$, $f \in L^p(D)$, let

$$\Gamma_1(\phi, f)(x) = - \inf_{v_1 \in V_1} \{b_1(x, v_1, v_2(x)) \frac{\partial \phi(x)}{\partial x_i} + r_1(x, v_1, v_2(x)) \phi(x)\} + f(x),$$

and consider

$$a_{ij}(x) \frac{\partial^2 \hat{\phi}(x)}{\partial x_i \partial x_j} = \Gamma_1(\phi, f)(x), \quad \text{with } \hat{\phi} = 0 \text{ on } \partial D. \quad (3.3)$$

Then by [31, Theorem 9.15, p.241], [31, Theorem 9.14, p.240], there exists a unique solution $\hat{\phi} \in W^{2,p}(D) \cap C(\bar{D})$, $p > d$, satisfying

$$\|\hat{\phi}\|_{W^{2,p}(D)} \leq \kappa_1 (\|\hat{\phi}\|_\infty + \|\Gamma_1(\phi, f)\|_{L^p(D)}), \quad (3.4)$$

for some positive constant $\kappa_1 = \kappa_1(p, D)$ which is independent of $\hat{\phi}, \phi, f$. From [31, Theorem 9.1, p.220], we deduce that

$$\|\hat{\phi}\|_\infty \leq \kappa_2 \|\Gamma_1(\phi, f)\|_{L^p(D)},$$

for some constant $\kappa_2 > 0$. Hence, from (3.4), we obtain

$$\|\hat{\phi}\|_{W^{2,p}(D)} \leq \kappa_3 \|\Gamma_1(\phi, f)\|_{L^p(D)} \quad (3.5)$$

for some positive constant κ_3 . Now consider an operator \mathfrak{T} mapping $\phi \in C_0^1(D)$ to the corresponding solution $\hat{\phi}$ of (3.3), i.e., $\mathfrak{T}(\phi) = \hat{\phi}$. Since the embedding $W^{2,p}(D) \hookrightarrow C^{1,\alpha}(D)$ for $p > d$ and $\alpha \in (0, 1 - \frac{d}{p})$ is compact, the operator \mathfrak{T} is compact and continuous. Now we want to show that the following space of functions

$$\{\phi \in C_0^1(D) : \phi = v \mathfrak{T}(\phi) \text{ for some } v \in [0, 1]\},$$

is bounded in $C_0^1(D)$. Suppose that there exists a sequence (ϕ_n, v_n) with $\|\phi_n\|_{C_0^1(D)} \rightarrow \infty$ and $v_n \rightarrow v \in [0, 1]$ as $n \rightarrow \infty$. Scaling ϕ_n appropriately we assume that $\|\phi_n\|_{C_0^1(D)} = 1$. Hence, in view of the estimate (3.5), extracting a suitable subsequence, there exists a nontrivial $\tilde{\phi}$ satisfying

$$a_{ij}(x) \frac{\partial^2 \tilde{\phi}(x)}{\partial x_i \partial x_j} = -v \inf_{v_1 \in V_1} \{b_1(x, v_1, v_2(x)) \frac{\partial \tilde{\phi}(x)}{\partial x_i} + r_1(x, v_1, v_2(x)) \tilde{\phi}(x)\},$$

with $\tilde{\phi} = 0$ on ∂D . This is a contradiction to the strong maximum principle [31, Theorem 9.6, p.225]. This implies that the above space is bounded. Hence, by the Leray-Schauder fixed point theorem [31, Theorem 11.3, p.280], it follows that \mathfrak{T} admits a fixed point $\varphi \in W^{2,p}(D) \cap C(\bar{D})$, i.e., we have

$$\mathcal{G}_1^{v_2} \varphi(x) = f(x), \quad \text{with } \varphi = 0 \text{ on } \partial D.$$

Also, by the strong maximum principle [31, Theorem 9.6] it is clear that φ satisfying the equation is unique.

Let $\mathfrak{X} = C_0(D)$ and \mathfrak{C} the cone of non-negative functions in \mathfrak{X} . Now define an operator $\hat{\mathfrak{T}}$ which maps $f \in \mathfrak{X}$ to corresponding solution $\varphi \in W^{2,p}(D) \cap C(\bar{D})$ satisfying

$$\mathcal{G}_1^{v_2} \varphi(x) = -f(x), \quad \text{with } \varphi = 0 \text{ on } \partial D.$$

From the above discussion it is easy to see that the operator $\hat{\mathfrak{T}}$ is well defined. Thus, combining [31, Theorem 9.1] and [31, Theorem 9.14], we deduce that

$$\|\varphi\|_{W^{2,p}(D)} \leq \kappa_1 \sup_D |\varphi|, \quad (3.6)$$

for some positive constant κ_1 . From (3.6), it is clear that $\hat{\mathfrak{T}}$ is compact and continuous. Also, from the definition one can see that $\hat{\mathfrak{T}}$ is 1-homogeneous (i.e., $\hat{\mathfrak{T}}(\tilde{\lambda}f) = \tilde{\lambda} \hat{\mathfrak{T}}(f)$ for all $\tilde{\lambda} \geq 0$). Suppose $\hat{\mathfrak{T}}(f_k) = \varphi_k$, $k = 1, 2$, with $f_1 \leq f_2$. Thus, we have $\mathcal{G}_1^{v_2} \varphi_1(x) \geq \mathcal{G}_1^{v_2} \varphi_2(x)$. Since $\mathcal{G}_1^{v_2}$ is concave, it follows that $\mathcal{G}_1^{v_2}(\varphi_2 - \varphi_1)(x) \leq 0$. Hence, applying [32, Theorem 3.1] we obtain $\varphi_2 \geq \varphi_1$ and if $f_1 < f_2$ (i.e., $f_1 \leq f_2$ and $f_1 \neq f_2$) then we have $\varphi_2 > \varphi_1$ (see [32, Lemma 3.1]). This implies that $\hat{\mathfrak{T}}$ is order preserving. Let $\phi \in \mathfrak{C}$ be nontrivial nonnegative function with compact support, hence from the above discussion we deduce that $\hat{\mathfrak{T}}(\phi) > 0$. Thus, one can choose $\kappa_2 > 0$ such that $\kappa_2 \hat{\mathfrak{T}}(\phi) - \phi > 0$ in D . Therefore, by Krein-Rutman theorem (see Theorem A.1), we conclude that there exists $(\hat{\lambda}, \psi_D) \in \mathbb{R}_+ \times W^{2,p}(D) \cap C(\bar{D})$ with $\psi_D > 0$ satisfying

$$\mathcal{G}_1^{v_2} \psi_D = \hat{\lambda} \psi_D \text{ in } D, \quad \text{and } \psi_D = 0 \text{ on } \partial D. \quad (3.7)$$

Where ψ_D is unique up to scalar multiplication. Now, $r_1 \geq 0$ (which is the case by our assumption), since r_1 is bounded in \bar{D} replacing r_1 by $(r_1 - \|r_1\|_{\infty, D})$, following the above arguments there exists $(\lambda_D, \psi_D) \in \mathbb{R} \times W^{2,p}(D) \cap C(\bar{D})$ with $\psi_D > 0$ satisfying (3.7).

Next, we show that

$$\lambda_D = \lambda_1^+(v_2, D).$$

Clearly,

$$\lambda_D \geq \lambda_1^+(v_2, D). \quad (3.8)$$

Suppose $\lambda_1^+(v_2, D) < \lambda_D$. Then for each $\varepsilon > 0$, there exists $\varepsilon' \leq \varepsilon$ and $\varphi' \in W^{2,p}(D) \cap C(\bar{D})$, $\varphi' > 0$ such that

$$\mathcal{G}_1^{v_2} \varphi' \leq (\lambda_1^+(v_2, D) + \varepsilon') \varphi'. \quad (3.9)$$

Choose $\varepsilon' > 0$ small enough such that $\lambda_1^+(v_2, D) + \varepsilon' < \lambda_D$. Also, we have

$$\mathcal{G}_1^{v_2} \psi_D - (\lambda_1^+(v_2, D) + \varepsilon') \psi_D > \mathcal{G}_1^{v_2} \psi_D - \lambda_D \psi_D = 0. \quad (3.10)$$

Hence by Theorem A.3, it follows that $\psi_D = t \varphi'$ for some $t > 0$. This gives a contradiction. Therefore we get $\lambda_D = \lambda_1^+(v_2, D)$. This completes the proof. \square

4. Eigenvalue problem for controlled diffusion operators in \mathbb{R}^d

In this section we explore the existence of principal eigenvalue of the controlled diffusion operator $\mathcal{G}_i^{v_j}$, $v_j \in \mathcal{A}_j$ in the whole space \mathbb{R}^d and establish their relations with the risk-sensitive ergodic optimal control problem. The generalized principal eigenvalue of $\mathcal{G}_i^{v_j}$ in the whole space is defined by

$$\lambda_i^+(v_j) = \inf\{\lambda \in \mathbb{R} \mid \text{for some } \varphi \in W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d), p > d, \varphi > 0, \mathcal{G}_i^{v_j} \varphi \leq \lambda \varphi \text{ a.e.}\}. \quad (4.1)$$

In order to study our game problem we enforce following Foster-Lyapunov condition on the dynamics.

Assumption 2.

(i) **In bounded cost case:** There exist $\mathcal{V} \in C^2(\mathbb{R}^d)$ with $\inf_{\mathbb{R}^d} \mathcal{V} \geq 1$, constants $\delta, \tilde{\alpha} > 0$ and a compact set \tilde{K} such that

$$\sup_{u_i \in U_i, i=1,2} \mathcal{L}^{u_1, u_2} \mathcal{V} \leq \tilde{\alpha} I_{\tilde{K}} - \delta \mathcal{V}. \tag{4.2}$$

and $\max_{i=1,2} \|r_i\|_\infty < \delta$.
Or,

(ii) **In unbounded cost case:** There exist $\mathcal{V} \in C^2(\mathbb{R}^d)$ with $\inf_{\mathbb{R}^d} \mathcal{V} \geq 1$, an inf-compact positive $\ell \in C(\mathbb{R}^d)$ (i.e., the sublevel sets $\{\ell \leq \kappa\}$ are compact, or empty, in \mathbb{R}^d for each $\kappa \in \mathbb{R}$), a constant $\tilde{\alpha} > 0$ and a compact set \tilde{K} such that

$$\sup_{u_i \in U_i, i=1,2} \mathcal{L}^{u_1, u_2} \mathcal{V} \leq \tilde{\alpha} I_{\tilde{K}} - \ell \mathcal{V}, \tag{4.3}$$

and for $i = 1, 2$

$$\ell(x) - \sup_{u_i \in U_i, i=1,2} \bar{r}_i(x, u_1, u_2) \text{ is inf-compact.} \tag{4.4}$$

As noted in [7,9], if a and b are bounded, it might not be possible to find an unbounded function ℓ which satisfies (4.3). In view of this, we are assuming (4.2).

For $i \neq j$, it is easy to see that under Assumption 2(i)

$$\sup_{v_1 \in \mathcal{A}_1} \sup_{v_2 \in \mathcal{A}_2} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^T r_1(X(t), v_1(t), v_2(t)) dt} \right] \leq \|r_i\|_\infty < \infty.$$

Also, under Assumption 2(ii), applying Itô–Krylov formula, it follows that

$$\sup_{v_1 \in \mathcal{A}_1} \sup_{v_2 \in \mathcal{A}_2} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^T \ell(X(t)) dt} \right] \leq \frac{\tilde{\alpha}}{\min_{\tilde{K}} \mathcal{V}}.$$

From (4.4), it is clear that $\sup_{u_k \in U_k, k=1,2} \bar{r}_i(\cdot, u_1, u_2) \leq \kappa_1 + \ell(\cdot)$, for some positive constant κ_1 . Therefore, we obtain

$$\sup_{v_1 \in \mathcal{A}_1} \sup_{v_2 \in \mathcal{A}_2} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^T r_i(X(t), v_1(t), v_2(t)) dt} \right] \leq \kappa_1 + \frac{\tilde{\alpha}}{\min_{\tilde{K}} \mathcal{V}}. \tag{4.5}$$

Now we proceed to prove the existence of the principal eigenpair to certain semi-linear elliptic pdes in the whole space \mathbb{R}^d .

Theorem 4.1. *Let Assumptions 1 and 2 hold. Suppose $v_j \in S_j$, then there exists a unique $\psi \in W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, $p > d$, $\psi > 0$ such that*

$$\mathcal{G}_i^{v_j} \psi = \lambda_i^+(v_j) \psi \text{ with } \psi(0) = 1. \tag{4.6}$$

Moreover $\lambda_i^+(v_j)$ is simple and satisfies

$$\lambda_i^+(v_j) \leq \lambda_i(v_j), \text{ for } i \neq j, i, j = 1, 2. \tag{4.7}$$

Proof. Take $i = 1, j = 2$. Let $D = B_n$, $n \geq 1$, denote the open ball centered at the origin with radius n . From Theorem 3.1, there exists a (unique) $\psi_n \in W^{2,p}(B_n) \cap C(\bar{B}_n)$, $\psi_n > 0$ in B_n with $\psi_n(0) = 1$ satisfying

$$\begin{aligned} \mathcal{G}_1^{v_2} \psi_n &= \lambda_n \psi_n \\ \psi_n &= 0 \text{ on } \partial B_n, \end{aligned} \tag{4.8}$$

where $\lambda_n = \lambda_1^+(v_2, B_n)$. Choose $v_1 \in \mathcal{A}_1$, since $\psi_n = 0$ on ∂B_n applying Ito–Dynkin’s formula we obtain

$$\begin{aligned} \psi_n(x) &\leq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^T (r_1(X(t), v_1(t), v_2(X(t))) - \lambda_n) dt} \psi_n(X(T)) \mathbb{1}_{\{T \leq \tau\}} \right] \\ &\leq \|\psi_n\|_{\infty, B_n} \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^T (r_1(X(t), v_1(t), v_2(X(t))) - \lambda_n) dt} \right] \\ &\text{for all } (T, x) \in \mathbb{R}_+ \times B_n, \end{aligned}$$

where τ is the first exit time of the process $X(t)$ from B_n and $\|\psi_n\|_{\infty, B_n} := \sup_{x \in B_n} \psi_n(x)$. Thus, taking logarithm on both sides of the inequality, dividing by T and letting $T \rightarrow \infty$, it follows that

$$\lambda_n \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^T r_1(X(t), v_1(t), v_2(X(t))) dt} \right] < \infty. \tag{4.9}$$

Since λ_n is nondecreasing in n (see, (3.1)), it follows that $\lim_n \lambda_n = \lambda$ exists.

Now using Harnack inequality (see [31, Corollary 8.21, p.199]) and the interior estimates [31, Theorem 9.11, p.235], we get for each bounded domain D , there exists n_0 such that

$$\sup_{n \geq n_0} \|\psi_n\|_{W^{2,p}(D)} < \infty. \tag{4.10}$$

Hence, by a standard diagonalization procedure and Banach–Alaoglu theorem, we can extract a subsequence $\{\psi_{n_k}\}$ such that for some $\psi \in W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, $p \geq 2$

$$\begin{cases} \psi_{n_k} \rightarrow \psi \text{ in } W_{loc}^{2,p}(\mathbb{R}^d) \text{ (weakly)} \\ \psi_{n_k} \rightarrow \psi \text{ in } C^{1,\alpha}(K) \text{ (strongly) for all compact set } K \subset \mathbb{R}^d, \end{cases} \tag{4.11}$$

where $0 < \alpha < 1 - \frac{d}{p}$. Now multiplying both sides of (4.8) by a test function $\varphi \in C_c^\infty(\mathbb{R}^d)$, integrating, and then letting $n \rightarrow \infty$, we deduce that $\psi \in W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, $p \geq 2$ satisfies

$$\mathcal{G}_1^{v_2} \psi = \lambda \psi \text{ in } \mathbb{R}^d. \tag{4.12}$$

From (4.9), it follows that

$$\lambda \leq \lambda_1(v_2).$$

Since for each $n \in \mathbb{N}$ we have $\psi_n > 0$ it is clear that $\psi \geq 0$ in \mathbb{R}^d and since $\psi_n(0) = 1$ for all n , we have $\psi(0) = 1$. Thus, applying Harnack’s inequality we deduce that $\psi > 0$ in \mathbb{R}^d .

Next from the definition of the generalized principal eigenvalue, it is immediate that

$$\lambda \geq \lambda_1^+(v_2). \tag{4.13}$$

Also from the definition of the generalized principle eigenvalue (see Eq. (3.1)), it follows that

$$\lambda_n = \lambda_1^+(v_2, B_n) \leq \lambda_1^+(v_2). \tag{4.14}$$

Thus, combining (4.13) and (4.14) we get

$$\lambda = \lambda_1^+(v_2).$$

Next we show that any eigenvalue of $\mathcal{G}_1^{v_2}$ corresponding to a positive eigenfunction in the class $W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ is simple. This, in particular, would imply the simplicity of the generalized principal eigenvalue $\lambda_1^+(v_2)$.

Let $\psi_k \in W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, $k = 1, 2$ be positive eigenfunctions corresponding to an eigenvalue λ (in particular, we are interested in $\lambda = \lambda_1^+(v_2)$) satisfying $\psi_k(0) = 1$. Let $t_0 > 0$ be such that $\psi_1 - t_0 \psi_2 \geq 0$ in \bar{B}_R .

Let v_1 be a minimizing selector of $\mathcal{G}_1^{v_2} \psi_1$. Thus

$$\begin{aligned} \mathcal{L}_1^{v_1, v_2} \psi_1 + r_1(x, v_1(x), v_2(x)) \psi_1 &= \mathcal{G}_1^{v_2} \psi_1 = \lambda \psi_1 \\ \mathcal{L}_1^{v_1, v_2} \psi_2 + r_1(x, v_1(x), v_2(x)) \psi_2 &\geq \mathcal{G}_1^{v_2} \psi_2 = \lambda \psi_2. \end{aligned}$$

This gives us the following inequality

$$\mathcal{L}_1^{v_1, v_2} (\psi_1 - t_0 \psi_2) + r_1(x, v_1(x), v_2(x)) (\psi_1 - t_0 \psi_2) \leq \lambda (\psi_1 - t_0 \psi_2).$$

Since $\psi_1 - t_0 \psi_2 \geq 0$ in \bar{B}_R , it follows that

$$\mathcal{L}_1^{v_1, v_2} (\psi_1 - t_0 \psi_2) - (r_1(x, v_1(x), v_2(x)) - \lambda)^- (\psi_1 - t_0 \psi_2) \leq 0 \text{ in } B_R.$$

Hence using the maximum principle [31, Theorem 9.6], we have $\psi_1 - t_0\psi_2 = 0$ in B_R and since $\psi_1(0) = \psi_2(0) = 1$, we get $t_0 = 1$ and hence $\psi_1 = \psi_2$ in B_R . Since the choice of $R > 0$ arbitrary (by choosing large $R > 0$), it follows that $\psi_1 = \psi_2$ in \mathbb{R}^d . This completes the proof. \square

We denote the eigenfunction corresponding to $\lambda_i^+(v_j)$ satisfying $\psi(0) = 1$ by $\psi_i(v_j)$. Next theorem proves that the eigenfunction $\psi_i(v_j)$ corresponding to the principal eigenvalue $\lambda_i^+(v_j)$ admits certain stochastic representation. This result plays crucial role in obtaining complete characterization of Nash equilibrium in the space of stationary Markov strategies.

Theorem 4.2. *Let Assumptions 1, 2 hold. Then, for $v_j \in S_j$, the eigenfunction $\psi_i(v_j)$ corresponding to principal eigenvalue $\lambda_i^+(v_j)$ satisfies*

$$\psi_i(v_j)(x) = \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_i^+(v_j)) dt} \psi_i(v_j)(X(\check{\tau}_r)) \right], \quad r > 0, \quad (4.15)$$

where $\check{\tau}_r$ is the hitting time of $X(t)$ to B_r and $v_i \in S_i$ is a minimizing selector of $\mathcal{G}_i^{v_j} \psi_i(v_j)$, for $i, j = 1, 2$ with $i \neq j$.

Proof. Take $i = 1, j = 2$. Let $(\hat{\lambda}_n, \hat{\psi}_n)$ denote the generalized principal eigenpair of the Dirichlet eigenvalue problem of $\mathcal{L}_1^{v_1, v_2} + r_1(x, v_1(x), v_2(x))$ in B_n with $\hat{\psi}_n(0) = 1$. Using the monotonicity of $\hat{\lambda}_n$ with respect to the running cost, the following representation holds (see, [8, Lemma 2.10 (i)])

$$\hat{\psi}_n(x) = \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \hat{\lambda}_n) dt} \hat{\psi}_n(X(\check{\tau}_r)) I_{\{\check{\tau}_r < \tau_n\}} \right], \quad (4.16)$$

where $\tau_n = \tau(B_n)$, the exit time from B_n . Also as in the proof of Theorem 4.1, it follows that $\hat{\lambda}_n \uparrow \lambda_1^+(v_1, v_2)$, the generalized principal eigenvalue of $\mathcal{L}_1^{v_1, v_2} + r_1(x, v_1(x), v_2(x))$. Again using Harnack's inequality and the standard approximation argument (as in Theorem 4.1), it follows that there exists $\psi \in W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, $\psi > 0$ satisfying

$$\mathcal{L}_1^{v_1, v_2} \psi + r_1(x, v_1(x), v_2(x))\psi = \lambda_1^+(v_1, v_2)\psi. \quad (4.17)$$

Consider

$$\begin{aligned} & \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \hat{\lambda}_n) dt} \hat{\psi}_n(X(\check{\tau}_r)) I_{\{\check{\tau}_r < \tau_n\}} \right] \\ & \leq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \hat{\lambda}_n) dt} \psi(X(\check{\tau}_r)) I_{\{\check{\tau}_r < \infty\}} \right] \\ & + \sup_{\partial B_r} |\hat{\psi}_n - \psi| \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \hat{\lambda}_n) dt} I_{\{\check{\tau}_r < \tau_n\}} \right]. \end{aligned} \quad (4.18)$$

Using the monotone convergence theorem, the first term in the r.h.s. of (4.18) converges to

$$\mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_1, v_2)) dt} \psi(X(\check{\tau}_r)) I_{\{\check{\tau}_r < \infty\}} \right].$$

The second term

$$\begin{aligned} & \sup_{\partial B_r} |\hat{\psi}_n - \psi| \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \hat{\lambda}_n) dt} I_{\{\check{\tau}_r < \tau_n\}} \right] \\ & \leq \frac{\sup_{\partial B_r} |\hat{\psi}_n - \psi|}{\inf_{\partial B_r} \hat{\psi}_n} \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \hat{\lambda}_n) dt} \hat{\psi}_n(X(\check{\tau}_r)) I_{\{\check{\tau}_r < \tau_n\}} \right] \\ & = \frac{\sup_{\partial B_r} |\hat{\psi}_n - \psi|}{\inf_{\partial B_r} \hat{\psi}_n} \hat{\psi}_n(x) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

In the above, we have used the fact that $\hat{\psi}_n - \psi \rightarrow 0$ uniformly over compact sets and $\inf_{\partial B_r} \hat{\psi}_n > 0$ (by Harnack's inequality). Hence, we get

$$\psi(x) \leq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_1, v_2)) dt} \psi(X(\check{\tau}_r)) \right].$$

(4.19)

Since

$$\mathcal{G}_1^{v_2} \hat{\psi}_n \leq \mathcal{L}_1^{v_1, v_2} \hat{\psi}_n + r_1(x, v_1(x), v_2(x)) \hat{\psi}_n = \hat{\lambda}_n \hat{\psi}_n,$$

it follows that

$$\lambda_1^+(v_2, B_n) \leq \hat{\lambda}_n, n \geq 1.$$

Therefore

$$\lambda_1^+(v_2) \leq \lambda_1^+(v_1, v_2).$$

Since $v_1 \in S_1$ is a minimizing selector of $\mathcal{G}_1^{v_2} \psi_1(v_2)$, we have

$$\mathcal{L}_1^{v_1, v_2} \psi_1(v_2) + r_1(x, v_1(x), v_2(x)) \psi_1(v_2) = \lambda_1^+(v_2) \psi_1(v_2). \quad (4.20)$$

Using Ito-Krylov formula, for fixed $T > 0, x \in B_r^c \cap B_n, r > 0$ and n large enough, we have

$$\begin{aligned} \psi_1(v_2)(x) &= \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r \wedge T \wedge \tau_n} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \right. \\ & \quad \left. \times \psi_1(v_2)(X(\check{\tau}_r \wedge T \wedge \tau_n)) \right]. \end{aligned}$$

Letting $n \rightarrow \infty$, and $T \rightarrow \infty$ and using Fatou's lemma, it follows that

$$\begin{aligned} \psi_1(v_2)(x) &\geq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \psi_1(v_2)(X(\check{\tau}_r)) \right] \\ &\geq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_1, v_2)) dt} \psi_1(v_2)(X(\check{\tau}_r)) \right]. \end{aligned} \quad (4.21)$$

Hence for each $t > 0$,

$$\begin{aligned} \psi_1(v_2)(x) - t\psi(x) &\geq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_1, v_2)) dt} \right. \\ & \quad \left. \times (\psi_1(v_2)(X(\check{\tau}_r)) - t\psi(X(\check{\tau}_r))) \right]. \end{aligned} \quad (4.22)$$

Thus

$$\psi_1(v_2)(x) - t\psi(x) \geq 0 \text{ in } \bar{B}_r \text{ implies that } \psi_1(v_2)(x) - t\psi(x) \geq 0 \text{ in } \mathbb{R}^d.$$

Now choose $t > 0$ such that $\psi_1(v_2)(x) - t\psi(x) \geq 0$ in \bar{B}_r and attains its minimum value 0 in \bar{B}_r . Hence $\psi_1(v_2)(x) - t\psi(x) \geq 0$ in \mathbb{R}^d and attains its minimum in \mathbb{R}^d . Now using $\lambda_1^+(v_2) \leq \lambda_1^+(v_1, v_2)$, it is easy to verify that

$$\mathcal{L}_1^{v_1, v_2} (\psi_1(v_2) - t\psi) - (r_1(x, v_1(x), v_2(x)) - \lambda_1^+(v_2)) (\psi_1(v_2) - t\psi) \leq 0.$$

Hence using the strong maximum principle [31, Theorem 9.6], we get $\psi_1(v_2) = t\psi$. Since $\psi_1(v_2)(0) = \psi(0) = 1$, we have $t = 1$. Therefore, it follows that $\lambda_1^+(v_2) = \lambda_1^+(v_1, v_2)$ and $\hat{\psi}_n \rightarrow \psi_1(v_2)$ in $W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$.

Thus we have $\hat{\lambda}_n \uparrow \lambda_1^+(v_2)$ and along a subsequence $\hat{\psi}_n \rightarrow \psi_1(v_2)$ in $W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. Now combining (4.19) and (4.21) we get the required representation. This completes the proof of the theorem. \square

Remark 4.1. From the proof Theorem 4.2, we conclude that $\lambda_1^+(v_2) = \lambda_1^+(v_1, v_2)$ for any minimizing selector $v_1 \in S_1$ of the HJB equation $\mathcal{G}_1^{v_2} \psi_1(v_2) = \lambda_1^+(v_2) \psi_1(v_2)$, where $\lambda_1^+(v_1, v_2)$ is the generalized principle eigenvalue of $\mathcal{L}^{v_1, v_2} + r_1(x, v_1(x), v_2(x))$. Similarly, $\lambda_2^+(v_1) = \lambda_2^+(v_1, v_2)$ for any minimizing selector $v_2 \in S_2$ of the HJB equation $\mathcal{G}_2^{v_1} \psi_2(v_1) = \lambda_2^+(v_1) \psi_2(v_1)$, where $\lambda_2^+(v_1, v_2)$ is the generalized principle eigenvalue of $\mathcal{L}^{v_1, v_2} + r_2(x, v_1(x), v_2(x))$.

Now we claim that $\lambda_1^+(v_2), \lambda_2^+(v_1) \geq 0$. If not, suppose that $\lambda_1^+(v_2) < 0$. Then from (4.15), we deduce that $\psi_1(v_2)(x) \geq \min_{B_r} \psi_1(v_2)$ for all $x \in B_r^c$. Applying Itô-Krylov formula and Fatou's lemma, from (4.6) it is follows that

$$\begin{aligned} \psi_1(v_2)(x) &\geq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^T (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \psi_1(v_2)(X(T)) \right] \\ &\geq \min_{B_r} \psi_1(v_2) \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^T (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \right]. \end{aligned}$$

Taking logarithm of both sides, dividing by T and letting $T \rightarrow \infty$, we get

$$\lambda_1^+(v_2) \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}^{v_1, v_2} \left[e^{\int_0^T r_1(X(t), v_1(X(t)), v_2(X(t))) dt} \right] \geq 0. \tag{4.23}$$

This is a contradiction. Thus, $\lambda_1^+(v_2) \geq 0$. Similarly $\lambda_2^+(v_1) \geq 0$.

Now we show that the map $v_j \mapsto (\lambda_i^+(v_j), \psi_i(v_j))$ is continuous in the topology of \mathcal{S}_j for $i, j = 1, 2$. This result is useful in establishing the u.s.c. of a certain set-valued map (to be introduced soon), which in turn, will ensure the existence of a Nash equilibrium.

Theorem 4.3. *Let Assumptions 1 and 2 hold. Then the map $v_j \mapsto (\lambda_i^+(v_j), \psi_i(v_j))$ from \mathcal{S}_j to $\mathbb{R} \times W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ is continuous for $i, j = 1, 2$ with $i \neq j$.*

Proof. Take $i = 1, j = 2$. Let $v_2^n \rightarrow v_2$ in the topology of stationary Markov strategies. From the above observation and (4.5), we get

$$0 \leq \lambda_1^+(v_2^n) \leq \max\{\kappa_1 + \frac{\tilde{\alpha}}{\min_{\tilde{K}} \mathcal{V}}, \|r_1\|_\infty\}.$$

Now using Harnack inequality, see [31, Corollary 8.21, p.199], and the interior estimates [31, Theorem 9.11, p.235], we get for each bounded domain D , there exists n_0 such that

$$\sup_{n \geq n_0} \|\psi_1(v_2^n)\|_{W^{2,p}(D)} < \infty. \tag{4.24}$$

Hence, by a standard approximation procedure involving Sobolev imbedding (as in Theorem 4.1), we obtain the existence of $\psi \in W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, $p > d$, $\psi > 0$ and a limit point λ of $\lambda_1^+(v_2^n)$ satisfying

$$\mathcal{G}_1^{v_2} \psi = \lambda \psi \text{ in } \mathbb{R}^d. \tag{4.25}$$

Clearly

$$\lambda \geq \lambda_1^+(v_2).$$

Next we prove the reverse inequality. From Assumption 2, we deduce that there exist a compact set $\mathcal{B}(\supset \tilde{K})$ and a constant $\theta \in (0, 1)$ such that for all large $n \in \mathbb{N}$

- under Assumption 2(i): $(\sup_{u_i \in U_i, i=1,2} r_1(x, u_1, u_2) - \lambda_1^+(v_2^n)) < \theta \delta$ for all $x \in \mathcal{B}^c$
- under Assumption 2(ii): $(\sup_{u_i \in U_i, i=1,2} r_1(x, u_1, u_2) - \lambda_1^+(v_2^n)) < \theta \ell(x)$ for all $x \in \mathcal{B}^c$.

Let $r_0 > 0$ be such that $\mathcal{B} \subset B_{r_0}$. Applying Itô-Krylov formula and Fatou's lemma, from (4.2) and (4.3), for any $(v_1, v_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ we deduce that

$$\mathbb{E}_x^{v_1, v_2} \left[e^{\delta \check{\tau}_0} \mathcal{V}(X(\check{\tau}_0)) \right] \leq \mathcal{V}(x) \text{ and} \tag{4.26}$$

$$\mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_0} \ell(X(t)) dt} \mathcal{V}(X(\check{\tau}_0)) \right] \leq \mathcal{V}(x) \quad \forall x \in B_{r_0}^c.$$

Thus, from Theorem 4.2, for any minimizing selector v_1^n of $\mathcal{G}_1^{v_2^n}$ $\psi_1(v_2^n) = \lambda_1^+(v_2^n) \psi_1(v_2^n)$, and $x \in B_{r_0}^c$, it follows that

$$\begin{aligned} \psi_1(v_2^n)(x) &= \mathbb{E}_x^{v_1^n, v_2^n} \left[e^{\int_0^{\check{\tau}_0} (r_1(X(t), v_1^n(X(t)), v_2^n(X(t))) - \lambda_1^+(v_2^n)) dt} \right. \\ &\quad \left. \psi_1(v_2^n)(X(\check{\tau}_0)) \right] \\ &\leq \frac{\sup_{B_{r_0}} \psi_1(v_2^n)}{\inf_{B_{r_0}} \mathcal{V}^\theta} \mathbb{E}_x^{v_1^n, v_2^n} \left[e^{\theta \check{\tau}_0 \delta} \mathcal{V}^\theta(X(\check{\tau}_0)) \right] \end{aligned}$$

$$\begin{aligned} &\leq \frac{\sup_{B_{r_0}} \psi_1(v_2^n)}{\inf_{B_{r_0}} \mathcal{V}^\theta} \left(\mathbb{E}_x^{v_1^n, v_2^n} \left[e^{\check{\tau}_0 \delta} \mathcal{V}(X(\check{\tau}_0)) \right] \right)^\theta \\ &\quad \text{(by Jensen's inequality)} \\ &\leq \hat{\kappa}_2 \mathcal{V}^\theta(x) \text{ (by (4.26)),} \end{aligned} \tag{4.27}$$

where one can choose the constant $\hat{\kappa}_2 > 0$ independent of n (by Harnack's inequality). This implies that $\psi \leq \hat{\kappa}_2 \mathcal{V}^\theta$ (in the above calculations replacing δ by ℓ , it is easy to see that same estimate holds true under Assumption 2(ii)). Now for any minimizing selector v_1 of (4.6), applying Itô-Krylov formula from (4.25) for some $T > 0$ we deduce that

$$\psi(x) \leq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_0 \wedge T} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda) dt} \psi(X(\check{\tau}_0 \wedge T)) \right].$$

In view of (4.26), since $\psi \leq \hat{\kappa}_2 \mathcal{V}^\theta$, by the dominated convergence theorem letting $T \rightarrow \infty$, we get

$$\begin{aligned} \psi(x) &\leq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_0} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda) dt} \psi(X(\check{\tau}_0)) \right] \\ &\leq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_0} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \psi(X(\check{\tau}_0)) \right]. \end{aligned} \tag{4.28}$$

Thus, from (4.15) (for $i = 1, j = 2$) and (4.28), we have

$$\begin{aligned} (\psi_1(v_2) - \psi)(x) &\geq \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^{\check{\tau}_0} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \right. \\ &\quad \left. \times (\psi_1(v_2) - \psi)(X(\check{\tau}_0)) \right]. \end{aligned} \tag{4.29}$$

Let $\tilde{\kappa}_2 = \sup_{B_{r_0}} \frac{\psi_1(v_2)}{\psi}$. Hence (4.29) implies that $(\psi_1(v_2) - \tilde{\kappa}_2 \psi) \geq 0$ in \mathbb{R}^d , and for some $x_1 \in B_{r_0}$ we have $(\psi_1(v_2) - \tilde{\kappa}_2 \psi)(x_1) = 0$. Since $\lambda \geq \lambda_1^+(v_2)$, (4.20) and (4.25) give us

$$\mathcal{L}_1^{v_1, v_2} (\psi_1(v_2) - \tilde{\kappa}_2 \psi) - (r_1(x, v_1(x), v_2(x)) - \lambda_1^+(v_2)) - (\psi_1(v_2) - \tilde{\kappa}_2 \psi) \leq 0.$$

Thus, by the strong maximum principle [31, Theorem 9.6], we obtain $\psi_1(v_2) = \tilde{\kappa}_2 \psi$. But, we have $\psi_1(v_2)(0) = \psi(0) = 1$, this gives $\tilde{\kappa}_2 = 1$. Therefore, we deduce that $\psi_1(v_2) = \psi$ and $\lambda_1^+(v_2) \geq \lambda$. This, in particular, implies that $\lambda_1^+(v_2) = \lambda$. This proves the continuity of the map $v_2 \mapsto (\lambda_1^+(v_2), \psi_1(v_2))$ and the continuity of the other maps follows by analogous arguments. \square

Remark 4.2. For any $v \in \mathcal{S}_1$, by Itô-Krylov formula, from (4.6) we deduce that

$$\begin{aligned} \psi_1(v_2)(x) &\leq \mathbb{E}_x^{v, v_2} \left[e^{\int_0^{\check{\tau}_r \wedge \tau_n} (r_1(X(t), v(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \right. \\ &\quad \left. \times \psi_1(v_2)(X(\check{\tau}_r \wedge \tau_n)) \right] \\ &= \mathbb{E}_x^{v, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \right. \\ &\quad \left. \times \psi_1(v_2)(X(\check{\tau}_r)) \mathbb{1}_{\{\check{\tau}_r \leq \tau_n\}} \right] \\ &\quad + \mathbb{E}_x^{v, v_2} \left[e^{\int_0^{\tau_n} (r_1(X(t), v(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \right. \\ &\quad \left. \times \psi_1(v_2)(X(\tau_n)) \mathbb{1}_{\{\check{\tau}_r \geq \tau_n\}} \right]. \end{aligned} \tag{4.30}$$

Since $\psi_1(v_2) \leq \hat{\kappa}_2 \mathcal{V}^\theta$ for some $\theta \in (0, 1)$ (see Theorem 4.3, Eq. (4.27)), by mimicking the arguments as in the proof of [33, Theorem 3.2], it is easy to see that

$$\lim_{n \rightarrow \infty} \mathbb{E}_x^{v, v_2} \left[e^{\int_0^{\tau_n} (r_1(X(t), v(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \psi_1(v_2)(X(\tau_n)) \mathbb{1}_{\{\tau_n \leq \check{\tau}_r\}} \right] = 0.$$

Thus, by monotone convergence theorem letting $n \rightarrow \infty$, from (4.30) we conclude that

$$\psi_1(v_2)(x) \leq \mathbb{E}_x^{v, v_2} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), v(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \psi_1(v_2)(X(\check{\tau}_r)) \right]. \tag{4.31}$$

Next we show that for each $v_j \in S_j$ the generalized principal eigenvalue $\lambda_i^+(v_j)$ is the optimal ergodic cost of Player i , i.e., $\lambda_i^+(v_j) = \lambda_i(v_j)$, for $i, j = 1, 2$ with $i \neq j$.

Theorem 4.4. *Suppose that Assumptions 1 and 2 hold. Then for $i, j = 1, 2$ with $i \neq j$ we have*

$$\lambda_i^+(v_j) = \lambda_i(v_j).$$

Proof. From Theorem 4.1, we have $\lambda_i^+(v_j) \leq \lambda_i(v_j)$. Now to prove the reverse inequality, we approximate the running costs in the following way:

- When the cost is bounded: let $\{\phi_{i,n}\}$ be a sequence of test functions such that $\phi_{i,n} = 1$ in B_n and $\phi_{i,n} = 0$ in B_{n+1}^c . Since $\|r_i\|_\infty < \delta$, it is possible to choose constants $\hat{\delta}_i > 0$ small enough such that $\|r_i\|_\infty + \hat{\delta}_i < \delta$. For $(x, u_1, u_2) \in \mathbb{R}^d \times U_1 \times U_2$, set

$$r_{i,n}(x, u_1, u_2) = \phi_{i,n}(x)r_i(x, u_1, u_2) + (1 - \phi_{i,n}(x))(\|r_i\|_\infty + \hat{\delta}_i), \quad \forall n \in \mathbb{N}.$$

- When the cost is unbounded: For $(x, u_1, u_2) \in \mathbb{R}^d \times U_1 \times U_2$ we define

$$r_{i,n}(x, u_1, u_2) = r_i(x, u_1, u_2) + \frac{1}{2} (\ell(x) - r_i(x, u_1, u_2))^+ \mathbb{1}_{\{B_n^c\}}.$$

It is easy to see that for $r_{i,n}$ satisfies (4.4) for $i = 1, 2$.

Now from Theorem 4.1, for each $n \in \mathbb{N}$, there exists $(\lambda_{1,n}^+(v_2), \psi_{1,n}(v_2)) \in \mathbb{R} \times W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, $2 \leq p < \infty$, $\psi_{1,n}(v_2) > 0$, satisfying

$$\lambda_{1,n}^+(v_2)\psi_{1,n}(v_2)(x) = \inf_{v_1 \in V_1} [\mathcal{L}_1^{v_1, v_2} \psi_{1,n}(v_2) + r_{1,n}(x, v_1, v_2(x))\psi_{1,n}(v_2)], \quad \text{with } \psi_{1,n}(0) = 1, \quad (4.32)$$

and

$$\lambda_{1,n}^+(v_2) \leq \inf_{x \in \mathbb{R}^d} \inf_{v_1 \in \mathcal{A}_1} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}_x^{v_1, v_2} \left[e^{\int_0^T r_{1,n}(X(t), v_1(t), v_2(X(t))) dt} \right]. \quad (4.33)$$

It is clear from our construction that there exists a compact set \mathcal{K} containing \tilde{K} such that $\inf_{(u_1, u_2) \in U_1 \times U_2} r_{1,n}(x, u_1, u_2) - \lambda_{1,n}^+(v_2) \geq 0$ for all $x \in \mathcal{K}^c$. Under Assumption 2(i) one can take $\mathcal{K} = \bar{B}_{n+1}$ and under Assumption 2(ii) since $r_{1,n}$ is unbounded and it satisfies (4.4) one can suitably choose \mathcal{K} which satisfies the above inequality. Let

$$\tilde{\tau}(\mathcal{K}) = \inf\{t \geq 0 : X(t) \in \mathcal{K}\}.$$

Applying Itô-Krylov formula and Fatou's lemma, for any minimizing selector \hat{v}_1 of (4.32), it follows that

$$\begin{aligned} \psi_{1,n}(v_2)(x) &\geq E_x^{\hat{v}_1, v_2} \left[e^{\int_0^{\tilde{\tau}(\mathcal{K})} (r_{1,n}(X(t), \hat{v}_1(X(t)), v_2(X(t))) - \lambda_{1,n}^+(v_2)) dt} \right. \\ &\quad \left. \times \psi_{1,n}(v_2)(X(\tilde{\tau}(\mathcal{K}))) \right], \\ &\geq \inf_{\mathcal{K}} \psi_{1,n}(v_2), \quad \forall x \in \mathcal{K}^c. \end{aligned}$$

Thus, by another application of Itô-Krylov's formula and Fatou's lemma, we deduce that

$$\begin{aligned} \psi_{1,n}(v_2)(x) &\geq E_x^{\hat{v}_1, v_2} \left[e^{\int_0^T (r_{1,n}(X(t), \hat{v}_1(X(t)), v_2(X(t))) - \lambda_{1,n}^+(v_2)) dt} \psi_{1,n}(v_2)(X(T)) \right], \\ &\geq \inf_{\mathcal{K}} \psi_{1,n}(v_2) E_x^{\hat{v}_1, v_2} \left[e^{\int_0^T (r_{1,n}(X(t), \hat{v}_1(X(t)), v_2(X(t))) - \lambda_{1,n}^+(v_2)) dt} \right]. \end{aligned}$$

Taking logarithm on both sides, dividing by T and then letting $T \rightarrow \infty$, we get

$$\begin{aligned} \lambda_{1,n}^+(v_2) &\geq \limsup_{T \rightarrow \infty} \frac{1}{T} \log E_x^{\hat{v}_1, v_2} \left[e^{\int_0^T r_{1,n}(X(t), \hat{v}_1(X(t)), v_2(X(t))) dt} \right], \\ &\geq \limsup_{T \rightarrow \infty} \frac{1}{T} \log E_x^{\hat{v}_1, v_2} \left[e^{\int_0^T r_1(X(t), \hat{v}_1(X(t)), v_2(X(t))) dt} \right]. \quad (4.34) \end{aligned}$$

As in Theorem 4.1, using Harnack's inequality and Sobolev estimate from (4.32), one can clearly see that $\psi_{1,n}(v_2)$ is uniformly bounded in $W_{loc}^{2,p}(\mathbb{R}^d)$, $2 \leq p < \infty$. Thus, along a suitable subsequence $\{\psi_{1,n}(v_2)\}$ converges weakly in $W_{loc}^{2,p}(\mathbb{R}^d)$, $2 \leq p < \infty$, to some $\psi_{1,*}(v_2) \in W_{loc}^{2,p}(\mathbb{R}^d)$, $2 \leq p < \infty$, and strongly in $C_{loc}^{1,\hat{\alpha}}(\mathbb{R}^d)$, $\hat{\alpha} \in (0, 1)$. It is clear from (4.33) and (4.34), that $\{\lambda_{1,n}^+(v_2)\}$ is a bounded sequence. Thus, along a further subsequence it converges to a constant $\lambda_{1,*}(v_2)$. Now as in Theorem 4.1, letting $n \rightarrow \infty$ in (4.32), we get $(\lambda_{1,*}(v_2), \psi_{1,*}(v_2)) \in \mathbb{R} \times W_{loc}^{2,p}(\mathbb{R}^d)$, $2 \leq p < \infty$, satisfies

$$\begin{aligned} \lambda_{1,*}(v_2)\psi_{1,*}(v_2) &= \inf_{v_1 \in V_1} [\mathcal{L}_1^{v_1, v_2} \psi_{1,*}(v_2) + r_1(x, v_1, v_2(x))\psi_{1,*}(v_2)] \\ \psi_{1,*}(v_2)(0) &= 1. \quad (4.35) \end{aligned}$$

Following the argument as in Theorem 4.3 (see (4.27)), one can show that $\psi_{1,n}(v_2) \leq \hat{\kappa}_2 \mathcal{V}^\theta$, uniformly in n for some constant $\hat{\kappa}_2 > 0$ and $\theta \in (0, 1)$. This implies that, the limit $\psi_{1,*}(v_2) \leq \hat{\kappa}_2 \mathcal{V}^\theta$. Let $v_1 \in S_1$ be a minimizing selector of (4.6). Now, by the arguments as in Remark 4.2, for each large $n \in \mathbb{N}$, we have

$$\begin{aligned} \psi_{1,n}(v_2)(x) &\leq E_x^{v_1, v_2} \left[e^{\int_0^{\tilde{\tau}_r} (r_{1,n}(X(t), v_1(X(t)), v_2(X(t))) - \lambda_{1,n}^+(v_2)) dt} \psi_{1,n}(X(\tilde{\tau}_r)) \right], \\ \forall x &\in B_r^c, \quad (4.36) \end{aligned}$$

for some $r > 0$. Since $\psi_{1,n}(v_2) \leq \hat{\kappa}_2 \mathcal{V}^\theta$ (uniformly in n), in view of estimates as in (4.26), by the dominated convergence theorem letting $n \rightarrow \infty$ from (4.36) we deduce that

$$\psi_{1,*}(v_2)(x) \leq E_x^{v_1, v_2} \left[e^{\int_0^{\tilde{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_{1,*}(v_2)) dt} \psi_{1,*}(X(\tilde{\tau}_r)) \right], \quad (4.37)$$

for all $x \in B_r^c$.

From (4.34), it is easy to see that $\lambda_{1,*}(v_2) \geq \lambda_1(v_2)$. To complete the proof, we have to show that $\lambda_{1,*}(v_2) \geq \lambda_{1,*}(v_2)$. If not, let $\lambda_{1,*}(v_2) < \lambda_{1,*}(v_2)$. From Theorem 4.2, we have for $x \in B_r^c$

$$\begin{aligned} \psi_1(v_2)(x) &= E_x^{v_1, v_2} \left[e^{\int_0^{\tilde{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_1^+(v_2)) dt} \psi_1(v_2)(X(\tilde{\tau}_r)) \right] \\ &\geq E_x^{v_1, v_2} \left[e^{\int_0^{\tilde{\tau}_r} (r_1(X(t), v_1(X(t)), v_2(X(t))) - \lambda_{1,*}(v_2)) dt} \psi_1(v_2)(X(\tilde{\tau}_r)) \right]. \quad (4.38) \end{aligned}$$

From (4.37) and (4.38), it follows that

$$\begin{aligned} (\psi_1(v_2) - \psi_{1,*}(v_2))(x) &\geq E_x^{v_1, v_2} \left[e^{\int_0^{\tilde{\tau}_r} (r_1(X(t), \hat{v}_1(X(t)), \hat{v}_2(X(t))) - \lambda_{1,*}(v_2)) dt} \right. \\ &\quad \left. \times (\psi_1 - \psi_{1,*})(X(\tilde{\tau}_r)) \right]. \end{aligned}$$

This implies that $(\psi_1(v_2) - \psi_{1,*}(v_2))(x) \geq 0$ for all $x \in \mathbb{R}^d$, if it holds in B_r . Now multiplying $\psi_{1,*}(v_2)$ by a suitable positive constant (say, $\hat{k}_1 = \inf_{B_r} \frac{\psi_1(v_2)}{\psi_{1,*}(v_2)}$), we obtain that $(\psi_1(v_2) - \tilde{\psi}_{1,*}(v_2))(x) \geq 0$ in B_r and it attains its minimum value 0 in B_r , where $\tilde{\psi}_{1,*}(v_2) = \hat{k}_1 \psi_{1,*}(v_2)$. It is clear that $\tilde{\psi}_{1,*}(v_2)$ also satisfies (4.34). Thus, from (4.6) and (4.34) (for $\tilde{\psi}_{1,*}$), we obtain

$$\begin{aligned} \mathcal{L}_1^{v_1, v_2}(\psi_1(v_2) - \tilde{\psi}_{1,*}(v_2)) - (r_1(x, v_1(x), v_2(x)) - \lambda_{1,*}(v_2))(\psi_1 - \tilde{\psi}_{1,*}) \\ \leq -(r_1(x, \hat{v}_1(x), \hat{v}_2(x)) - \lambda_{1,*}(v_2))^+(\psi_1(v_2) - \tilde{\psi}_{1,*}(v_2)) \leq 0. \end{aligned}$$

Thus, by an application of the strong maximum principle as in [31, Theorem 9.6], we have $\psi_1(v_2) = \tilde{\psi}_{1,*}(v_2)$. Since $\psi_1(v_2)(0) = \psi_{1,*}(v_2)(0) = 1$, we obtain $\psi_1(v_2) = \psi_{1,*}(v_2)$. Hence, from (4.6) and (4.34), we deduce that

$$\lambda_{1,*}(v_2)\psi_{1,*}(v_2) \leq \lambda_1^+(v_2)\psi_{1,*}(v_2).$$

Since $\psi_{1,*}(v_2) > 0$, we conclude that $\lambda_1^+(v_2) \geq \lambda_{1,*}(v_2)$. This contradicts the fact that $\lambda_1^+(v_2) < \lambda_{1,*}(v_2)$. Therefore we obtain $\lambda_1^+(v_2) \geq \lambda_{1,*}(v_2)$. This completes the proof of the theorem. \square

Remark 4.3. By closely following the arguments as in the proof of Theorem 4.4, one can conclude that for any $(v_1, v_2) \in \mathcal{S}_1 \times \mathcal{S}_2$ the generalized principle eigenvalue $\lambda_i^+(v_1, v_2)$ of $\mathcal{L}^{v_1, v_2} + r_i(x, v_1(x), v_2(x))$, satisfies $\lambda_i^+(v_1, v_2) = \rho_i(x, v_1, v_2)$ for $i = 1, 2$ and $x \in \mathbb{R}^d$.

5. Existence of Nash equilibrium

In this section using Fan's fixed point theorem, we establish the existence of Nash equilibria in the space of stationary Markov strategies. Also, exploiting the stochastic representation of the principal eigenfunctions of the associated coupled HJB equation we completely characterize all possible Nash equilibria in the space of stationary Markov strategies.

Let $(v_1, v_2) \in \mathcal{S}_1 \times \mathcal{S}_2$. Define

$$N(v_1, v_2) = N_1(v_2) \times N_2(v_1), \quad (5.1)$$

where

$$N_1(v_2) = \left\{ v_1^* \in \mathcal{S}_1 \mid F_1(x, v_1^*(x), v_2(x)) = \inf_{v_1 \in V_1} F_1(x, v_1, v_2(x)) \text{ a.e. } x \right\},$$

$$F_1(x, v_1, v_2(x)) = \langle b(x, v_1, v_2(x)), \nabla \psi_1(v_2) \rangle + r_1(x, v_1, v_2(x))\psi_1(v_2), \quad x \in \mathbb{R}^d, \quad v_1 \in V_1, \quad v_2 \in \mathcal{S}_2$$

and

$$N_2(v_1) = \left\{ v_2^* \in \mathcal{S}_2 \mid F_2(x, v_1(x), v_2^*(x)) = \inf_{v_2 \in V_2} F_2(x, v_1(x), v_2) \text{ a.e. } x \right\},$$

where

$$F_2(x, v_1(x), v_2) = \langle b(x, v_1(x), v_2), \nabla \psi_2(v_1) \rangle + r_2(x, v_1(x), v_2)\psi_1(v_2), \quad x \in \mathbb{R}^d, \quad v_2 \in V_2, \quad v_1 \in \mathcal{S}_1.$$

By a standard measurable selection theorem (see, [34]), it is clear that $N_1(v_2)$ is nonempty. Also, it is easy to see that $N_1(v_2)$ is convex. Under the topology of \mathcal{S}_1 , one can show that $N_1(v_2)$ is closed in \mathcal{S}_1 , hence compact. Similarly, one can show that $N_2(v_1)$ is nonempty, compact, convex subset of \mathcal{S}_2 . Therefore $N(v_1, v_2)$ is nonempty, convex and compact subset of $\mathcal{S}_1 \times \mathcal{S}_2$. To establish the existence of a Nash equilibrium, we next prove the upper semi-continuity (u.s.c.) of the map $(v_1, v_2) \mapsto N(v_1, v_2)$ from $\mathcal{S}_1 \times \mathcal{S}_2 \rightarrow 2^{\mathcal{S}_1 \times \mathcal{S}_2}$. In order to do so we impose some additive structure on the drift of the state dynamics and the running cost function, which is known as ADAC (additive drift and additive cost) condition, given as follows.

Assumption 3. We assume that $\bar{b} : \mathbb{R}^d \times U_1 \times U_2 \rightarrow \mathbb{R}^d$ and $\bar{r}_i : \mathbb{R}^d \times U_1 \times U_2 \rightarrow \mathbb{R}_+$, $i = 1, 2$, admit the following additive structures given by

$$\bar{b}(x, u_1, u_2) = \bar{b}_1(x, u_1) + \bar{b}_2(x, u_2)$$

$$\bar{r}_i(x, u_1, u_2) = \bar{r}_{i,1}(x, u_1) + \bar{r}_{i,2}(x, u_2)$$

where $\bar{b}_1, \bar{b}_2, \bar{r}_{i,1}, \bar{r}_{i,2}$ satisfy the conditions in Assumption 1(i)-(ii).

Next lemma shows that our set valued map $(v_1, v_2) \mapsto N(v_1, v_2)$ is upper semi-continuous.

Lemma 5.1. Let Assumptions 1–3 hold. Then the map $(v_1, v_2) \mapsto N(v_1, v_2)$ from $\mathcal{S}_1 \times \mathcal{S}_2 \rightarrow 2^{\mathcal{S}_1 \times \mathcal{S}_2}$ is u.s.c.

Proof. Consider a sequence $\{(v_1^n, v_2^n)\}_n$ in $\mathcal{S}_1 \times \mathcal{S}_2$ such that $(v_1^n, v_2^n) \rightarrow (v_1, v_2) \in \mathcal{S}_1 \times \mathcal{S}_2$. Choose $\hat{v}_1^n \in N_1(v_2^n)$, $n \geq 1$. Since \mathcal{S}_1 is compact, there exists a subsequence (denoting by the same notation without any loss of generality) $\{\hat{v}_1^n\}$ such that $\hat{v}_1^n \rightarrow \hat{v}_1$ for some $\hat{v}_1 \in \mathcal{S}_1$. Then $(\hat{v}_1^n, v_2^n) \rightarrow (\hat{v}_1, v_2)$ in $\mathcal{S}_1 \times \mathcal{S}_2$. In view of Assumption 3, the continuity results as in Theorem 4.3 and the topology of \mathcal{S}_i , $i = 1, 2$, we deduce that

$$\langle b(x, \hat{v}_1^n(x), v_2^n(x)), \nabla \psi_1(v_2^n) \rangle + r_1(x, \hat{v}_1^n(x), v_2^n(x))\psi_1(v_2^n)$$

converges weakly in $L_{loc}^2(\mathbb{R}^d)$ to

$$\langle b(x, \hat{v}_1(x), v_2(x)), \nabla \psi_1(v_2) \rangle + r_1(x, \hat{v}_1(x), v_2(x))\psi_1(v_2).$$

Thus, by Banach–Saks theorem [35], there exists a subsequence of the former whose convex combinations converges strongly in $L_{loc}^2(\mathbb{R}^d)$ to the latter. Therefore, along a suitable subsequence of the convergent sequence of convex combinations (without any loss of generality denoting by the same notation), it follows that

$$\lim_{n \rightarrow \infty} F_1(x, \hat{v}_1^n(x), v_2^n(x)) = F_1(x, \hat{v}_1(x), v_2(x)), \quad \text{a.e. in } x. \quad (5.2)$$

By analogous arguments, for any fixed $\hat{v}_1 \in \mathcal{S}_1$, we have

$$\lim_{n \rightarrow \infty} F_1(x, \hat{v}_1(x), v_2^n(x)) = F_1(x, \hat{v}_1(x), v_2(x)), \quad \text{a.e. in } x. \quad (5.3)$$

Since $\hat{v}_1^n \in N_1(v_2^n)$, from the definition of the set $N_1(v_2^n)$ it is easy to see that

$$F_1(x, \hat{v}_1^n(x), v_2^n(x)) \geq F_1(x, \hat{v}_1^n(x), v_2^n(x)), \quad \text{for all } n \geq 1.$$

Thus, from (5.2) and (5.3), we obtain

$$F_1(x, \hat{v}_1(x), v_2(x)) \geq F_1(x, \hat{v}_1(x), v_2(x)), \quad \text{for any } \hat{v}_1 \in \mathcal{S}_1.$$

This implies that $\hat{v}_1 \in N_1(v_2)$. By similar argument, one can show that if $\hat{v}_2^n \in N_2(v_1^n)$ and $\hat{v}_2^n \rightarrow \hat{v}_2$ in \mathcal{S}_2 then $\hat{v}_2 \in N_2(v_1)$. This proves that the set valued map is u.s.c. \square

In view of the u.s.c. of the above set valued map, using Fan's fixed point theorem, we now establish the existence of Nash equilibrium in the space of stationary Markov strategies.

Theorem 5.1. Let Assumptions 1–3 hold. Then there exists $(v_1^*, v_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$ such that

$$\lambda_1^+(v_2^*) = \lambda_1^+(v_1^*, v_2^*) \quad \text{and} \quad \lambda_2^+(v_1^*) = \lambda_2^+(v_1^*, v_2^*).$$

In particular, we have $(v_1^*, v_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$ is a Nash equilibrium.

Proof. From Lemma 5.1, we know that the set valued map $(v_1, v_2) \mapsto N(v_1, v_2)$ from $\mathcal{S}_1 \times \mathcal{S}_2 \rightarrow 2^{\mathcal{S}_1 \times \mathcal{S}_2}$ is u.s.c. Thus, by Fan's fixed point theorem [29], there exists a fixed point $(v_1^*, v_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$, of the map $(v_1, v_2) \mapsto N(v_1, v_2)$, i.e., $(v_1^*, v_2^*) \in N(v_1^*, v_2^*)$. Therefore, it follows that $(\lambda_1^+(v_2^*), \psi_1(v_2^*)), (\lambda_2^+(v_1^*), \psi_2(v_1^*)) \in \mathbb{R}_+ \times W_{loc}^{2,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, $p \geq 2$, satisfy the following coupled HJB equations

$$\lambda_1^+(v_2^*)\psi_1(v_2^*)(x) = G_1^{v_2^*}\psi_1(v_2^*)(x) = \mathcal{L}^{v_1^*, v_2^*}\psi_1(v_2^*)(x) + r_1(x, v_1^*(x), v_2^*(x))\psi_1(v_2^*)(x), \quad (5.4)$$

and

$$\lambda_2^+(v_1^*)\psi_2(v_1^*)(x) = G_2^{v_1^*}\psi_2(v_1^*)(x) = \mathcal{L}^{v_1^*, v_2^*}\psi_2(v_1^*)(x) + r_2(x, v_1^*(x), v_2^*(x))\psi_2(v_1^*)(x). \quad (5.5)$$

From Remark 4.1 (also see Theorem 4.2), it is easy to see that

$$\lambda_1^+(v_2^*) = \lambda_1^+(v_1^*, v_2^*) \quad \text{and} \quad \lambda_2^+(v_1^*) = \lambda_2^+(v_1^*, v_2^*).$$

Therefore, in view of [Theorem 4.4](#) and [Remark 4.3](#), we conclude that

$$\rho_1(x, v_1^*, v_2^*) \leq \rho_1(x, v_1, v_2^*) \quad \text{and} \quad \rho_2(x, v_1^*, v_2^*) \leq \rho_2(x, v_1^*, v_2),$$

for all $v_1 \in \mathcal{A}_1, v_2 \in \mathcal{A}_2$ and $x \in \mathbb{R}^d$. This completes the proof of the theorem. \square

In the above theorem we have shown the existence of a Nash equilibrium in the space of stationary Markov strategies. Conversely, we now prove that if there exists a Nash equilibrium $(\bar{v}_1^*, \bar{v}_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$, then $(\bar{v}_1^*, \bar{v}_2^*)$ is a pair of minimizing selectors of the associated coupled HJB equation.

Theorem 5.2. *Suppose that [Assumptions 1–3](#) hold. Then, if $(\bar{v}_1^*, \bar{v}_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$ is a Nash equilibrium, i.e.,*

$$\rho_1(x, \bar{v}_1^*, \bar{v}_2^*) \leq \rho_1(x, \bar{v}_1, \bar{v}_2^*), \quad \forall \bar{v}_1 \in \mathcal{A}_1, x \in \mathbb{R}^d,$$

$$\rho_2(x, \bar{v}_1^*, \bar{v}_2^*) \leq \rho_2(x, \bar{v}_1^*, \bar{v}_2), \quad \forall \bar{v}_2 \in \mathcal{A}_2, x \in \mathbb{R}^d,$$

then $(\bar{v}_1^*, \bar{v}_2^*)$ is a pair of minimizing selector of the corresponding coupled HJB equation

$$\lambda_1^+(\bar{v}_2^*)\psi_1(\bar{v}_2^*)(x) = \mathcal{G}_1^{\bar{v}_1^*}\psi_1(\bar{v}_2^*)(x). \quad (5.6)$$

$$\lambda_2^+(\bar{v}_1^*)\psi_2(\bar{v}_1^*)(x) = \mathcal{G}_2^{\bar{v}_2^*}\psi_2(\bar{v}_1^*)(x). \quad (5.7)$$

Proof. By limiting arguments as in [Theorem 4.1](#), for the given pair $(\bar{v}_1^*, \bar{v}_2^*) \in \mathcal{S}_1 \times \mathcal{S}_2$, one can prove that there exists a principal eigenpair $(\lambda_1^+(\bar{v}_1^*, \bar{v}_2^*), \psi_1(\bar{v}_1^*, \bar{v}_2^*)) \in \mathbb{R}_+ \times W_{loc}^{2,p}(\mathbb{R}^d)$, $\infty > p \geq 2$, with $\psi_1(\bar{v}_1^*, \bar{v}_2^*) > 0$ satisfying the following

$$\begin{aligned} \lambda_1^+(\bar{v}_1^*, \bar{v}_2^*)\psi_1(\bar{v}_1^*, \bar{v}_2^*) &= \mathcal{L}^{\bar{v}_1^*, \bar{v}_2^*}\psi_1(\bar{v}_1^*, \bar{v}_2^*) \\ &\quad + r_1(x, \bar{v}_1^*(x), \bar{v}_2^*(x))\psi_1(\bar{v}_1^*, \bar{v}_2^*) \\ \psi_1^{\bar{v}_1^*, \bar{v}_2^*}(0) &= 1. \end{aligned} \quad (5.8)$$

From [Remark 4.3](#), we deduce that $\lambda_1^+(\bar{v}_1^*, \bar{v}_2^*) = \rho_1(x, \bar{v}_1^*, \bar{v}_2^*)$. By similar argument as in [Theorem 4.2](#), we have

$$\begin{aligned} \psi_1(\bar{v}_1^*, \bar{v}_2^*)(x) &= \mathbb{E}_x^{\bar{v}_1^*, \bar{v}_2^*} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), \bar{v}_1^*(X(t)), \bar{v}_2^*(X(t))) - \lambda_1^+(\bar{v}_1^*, \bar{v}_2^*)) dt} \right. \\ &\quad \left. \times \psi_1(\bar{v}_1^*, \bar{v}_2^*)(X(\check{\tau}_r)) \right], \end{aligned} \quad (5.9)$$

for some $r > 0$. In view of [Theorem 4.1](#), for given $\bar{v}_2^* \in \mathcal{S}_2$, there exists a principal eigenpair $(\lambda_1^+(\bar{v}_2^*), \psi_1(\bar{v}_2^*)) \in \mathbb{R}_+ \times W_{loc}^{2,p}(\mathbb{R}^d)$, $\psi_1(\bar{v}_2^*) > 0$, $\infty > p \geq 2$, satisfying

$$\lambda_1^+(\bar{v}_2^*)\psi_1(\bar{v}_2^*) = \mathcal{G}_1^{\bar{v}_2^*}\psi_1(\bar{v}_2^*) \quad \text{with} \quad \psi_1(\bar{v}_2^*)(0) = 1. \quad (5.10)$$

[Remark 4.1](#) implies that for any minimizing selector $\bar{v}_1^* \in \mathcal{S}_1$ of [\(5.10\)](#), $\lambda_1^+(\bar{v}_2^*) = \rho_1(x, \bar{v}_1^*, \bar{v}_2^*)$. From [\(5.10\)](#), it is easy to see that

$$\lambda_1^+(\bar{v}_2^*)\psi_1(\bar{v}_2^*) \leq \mathcal{L}^{\bar{v}_1^*, \bar{v}_2^*}\psi_1(\bar{v}_2^*) + r_1(x, \bar{v}_1^*(x), \bar{v}_2^*(x))\psi_1(\bar{v}_2^*). \quad (5.11)$$

By Itô-Krylov formula, as in [Theorem 4.1](#), we obtain $\lambda_1^+(\bar{v}_2^*) \leq \rho_1(x, \bar{v}_1^*, \bar{v}_2^*)$. But we already have $\rho_1(x, \bar{v}_1^*, \bar{v}_2^*) \leq \rho_1(x, \bar{v}_1, \bar{v}_2^*)$, $\forall \bar{v}_1 \in \mathcal{A}_1, x \in \mathbb{R}^d$. Therefore we get $\lambda_1^+(\bar{v}_2^*) = \rho_1(x, \bar{v}_1^*, \bar{v}_2^*) = \rho_1(x, \bar{v}_1^*, \bar{v}_2^*)$. Following the proof of [Theorem 4.2](#), we get

$$\psi_1(\bar{v}_1^*)(x) \leq E_x^{\bar{v}_1^*, \bar{v}_2^*} \left[e^{\int_0^{\check{\tau}_r} (r_1(X(t), \bar{v}_1^*(X(t)), \bar{v}_2^*(X(t))) - \lambda_1^+(\bar{v}_2^*)) dt} \psi_1(\bar{v}_1^*)(X(\check{\tau}_r)) \right].$$

Now applying the maximum principle as in [Theorem 4.3](#), one can deduce that $\psi_1(\bar{v}_2^*) = \psi_1(\bar{v}_1^*, \bar{v}_2^*)$. Thus, from [\(5.6\)](#) and [\(5.8\)](#), it follows that \bar{v}_1^* is a minimizing selector of [\(5.6\)](#). By similar arguments one can show that \bar{v}_2^* is a minimizing selector of [\(5.7\)](#). This completes the proof of the theorem. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix

In this section we state some important results which we have used in our proofs. First we recall a version of the nonlinear Krein–Rutman theorem from [\[36\]](#).

Theorem A.1. *Let \mathcal{C} be a nonempty closed cone in an ordered Banach space \mathfrak{X} satisfying $\mathfrak{X} = \mathcal{C} - \mathcal{C}$ (where $\mathcal{C} - \mathcal{C} := \{f - g : f, g \in \mathcal{C}\}$). Suppose that $\mathfrak{T} : \mathfrak{X} \rightarrow \mathfrak{X}$ is order-preserving, 1-homogeneous, completely continuous map and for some nonzero f , and $M > 0$, we have $f \leq M\mathfrak{T}f$. Then there exists $\lambda > 0$ and $\phi \neq 0$ in \mathcal{C} such that $\mathfrak{T}\phi = \lambda\phi$.*

Here \leq denotes the partial ordering in \mathfrak{X} with respect to the cone \mathcal{C} , i.e., $f \leq g$ if and only if $g - f \in \mathcal{C}$. Also, we recall that a map $\mathfrak{T} : \mathfrak{X} \rightarrow \mathfrak{X}$ is called completely continuous if it is continuous and compact. Now we state the Aleksandrov–Bakelman–Pucci (ABP) estimate for certain semi-linear differential operator.

Theorem A.2. *Let $v_j \in \mathcal{S}_j$ and $\bar{r}_i(x, u_1, u_2) \leq 0$ for all $(x, u_1, u_2) \in \mathbb{R}^d \times U_1 \times U_2$ and $i, j = 1, 2$ with $i \neq j$. Suppose that $\phi \in W_{loc}^{2,p}(D) \cap C(\bar{D})$, $p > d$, satisfies*

$$\mathcal{G}_i^{v_j}\phi \geq f(x) \quad \text{in } \{\phi > 0\} \cap D, \quad \text{with } \phi = 0 \text{ on } \partial D. \quad (\text{A.1})$$

Then the following inequality holds

$$\sup_D \phi^+ \leq \sup_{\partial D} \phi^+ + \bar{\kappa} \|f^-\|_{L^d(D)},$$

for some constant positive constant $\bar{\kappa}$.

Proof. Since \bar{b} is jointly continuous, $M := \sup_{x \in D, u_1 \in U_1, u_2 \in U_2} |\bar{b}(x, u_1, u_2)| < \infty$. From [\(A.1\)](#), we deduce that

$$a_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) + M |\nabla \phi(x)| \geq f(x) \quad \text{in } \{\phi > 0\} \cap D, \quad \text{with } \phi = 0 \text{ on } \partial D.$$

Therefore, the result follows from [\[37, Proposition 3.3\]](#). \square

We also need the following maximum principle for small domains, which follows from [Theorem A.2](#).

Lemma A.1. *Let $v_j \in \mathcal{S}_j$. Then there exists $\epsilon_0 > 0$ such that if $|D| \leq \epsilon_0$, then any $\phi \in W_{loc}^{2,p}(D) \cap C(\bar{D})$ satisfying*

$$\begin{aligned} \mathcal{G}_i^{v_j}\phi &\geq \lambda\phi, \quad \text{in } D, \\ \phi &\leq 0 \quad \text{on } \partial D \end{aligned}$$

for some $\lambda \in \mathbb{R}$, is nonpositive in D , where $i \neq j$ and $i, j = 1, 2$.

Proof. Take $-(\|c\|_\infty + |\lambda|)|\varphi| = f$ and $M = \sup_{D \times U_1 \times U_2} |b(x, u_1, u_2)|$. Since on $\{\phi > 0\}$, we have $f^- = (\|c\|_\infty + |\lambda|)\varphi^+$. Thus, from (A.1), we get

$$\sup_D \varphi^+ \leq \sup_{\partial D} \varphi^+ + \hat{K} \|\varphi^+\|_{L^d(D)},$$

for some constant $\hat{K}(> 0)$. Now for the choice $\epsilon_0 = (2\hat{K})^{-d}$, it follows that for $|D| \leq \epsilon_0$,

$$\sup_D \varphi^+ \leq \frac{1}{2} \sup_D \varphi^+,$$

which is possible only when $\sup_D \varphi^+ = 0$. Hence $\varphi \leq 0$ in D . This completes the proof. \square

In view of the above lemma we have the following results. This is useful in establishing simplicity of the generalized principal eigenvalue of smooth bounded domains D . The proof of the following theorem follows from [32, Theorem 4.1]

Theorem A.3. Let $v_j \in S_j$ and $\varphi, \psi \in W_{loc}^{2,p}(D) \cap C(\bar{D})$, $p \geq d$ satisfies for some $\lambda \in \mathbb{R}$

$$\mathcal{G}_i^{v_j} \psi \leq \lambda \psi, \quad \psi > 0 \text{ in } D,$$

$$\mathcal{G}_i^{v_j} \varphi \geq \lambda \varphi \text{ in } D,$$

$$\varphi \leq 0 \text{ on } \partial D, \quad \varphi(x_0) > 0,$$

for some $x_0 \in D$, then $\psi = t\varphi$ for some $t > 0$, where $i \neq j$ and $i, j = 1, 2$.

Proof. Choose a compact $C \subset D$ such that $|D \setminus C| \leq \epsilon_0$, where ϵ_0 is given by Lemma A.1. Then, following the proof of [32, Theorem 4.1] and using the small domain maximum principle as in Lemma A.1 the result follows. \square

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