

# SPARSITY IN LONG-TIME CONTROL OF NEURAL ODES

CARLOS ESTEVE-YAGÜE AND BORJAN GESHKOVSKI

**ABSTRACT.** We consider the neural ODE and optimal control perspective of supervised learning, with  $\ell^1$ -control penalties, where rather than only minimizing a final cost (the *empirical risk*) for the state, we integrate this cost over the entire time horizon. We prove that any optimal control (for this cost) vanishes beyond some positive stopping time. When seen in the discrete-time context, this result entails an *ordered* sparsity pattern for the parameters of the associated residual neural network: ordered in the sense that these parameters are all 0 beyond a certain layer. Furthermore, we provide a polynomial stability estimate for the empirical risk with respect to the time horizon. This can be seen as a *turnpike property*, for nonsmooth dynamics and functionals with  $\ell^1$ -penalties, and without any smallness assumptions on the data, both of which are new in the literature.

## CONTENTS

1. Introduction	1
2. Main result	5
3. Proofs	13
4. Concluding remarks	21
References	22

**Keywords.** Deep Learning; Neural ODEs; Supervised Learning; Sparsity; Optimal control; Turnpike property, Stabilization.

**AMS Subject Classification.** 49J15; 49M15; 49J20; 49K20; 93C20; 49N05.

## 1. INTRODUCTION

1.1. **Motivation.** Sparsity is a highly desirable property in many machine learning and optimization tasks due to the inherent reduction of computational complexity. Typically induced by  $\ell^1$  penalties/regularizations, it has been used extensively for simplifying machine learning tasks by selecting, in an automatized manner, a strict subset of the available features to be used. This is exemplified by the well-known Lasso (least absolute shrinkage and selection operator, [40, 42]), which consists in minimizing a least squares cost function and an  $\ell^1$  parameter penalty for an affine parametric model  $y = wx + b$ . As the  $\ell^1$  penalty enforces a subset of the optimizable parameters  $(w, b)$  to become zero, the associated features may be discarded safely.

With such insights in mind, in this work we analyze supervised learning problems viewed from the lens of optimal control and neural ODEs, and demonstrate the appearance of sparsity patterns for global minimizers in the context of  $\ell^1$  control penalties. Rather than typical sparsity in which, at a given time  $t$ , all but few of the components

---

*Date:* September 9, 2022.

of a control  $u(t) \in \mathbb{R}^{d_u}$  are zero, we shall demonstrate a *ordered* or *temporal* sparsity: an optimal control  $u(t)$  concentrates all its amplitude within a subinterval  $[0, T^*]$  (wherein it may very well be additionally sparse), and vanishes beyond time  $t \geq T^*$ . (See Theorem 2.1)

We motivate our setting and main result in what follows, and refer the reader to Section 1.6 for a roadmap of the paper.

**1.2. Supervised learning.** To put the above discussion into context, we recall that *supervised learning* addresses the problem of predicting from labeled data, which consists in approximating an unknown function  $f : \mathcal{X} \rightarrow \mathcal{Y}$  from known samples

$$\left\{ x^{(i)}, y^{(i)} \right\}_{i \in [n]} \subset \mathcal{X} \times \mathcal{Y}.$$

Here and henceforth,  $[n] := \{1, \dots, n\}$  and  $\mathcal{X} \subset \mathbb{R}^d$ . Depending on the nature of the label space  $\mathcal{Y}$ , one distinguishes two types of supervised learning tasks: *classification*, when labels take values in a finite set of  $m \geq 2$  classes, e.g.  $\mathcal{Y} = [m]$ , and *regression*, when the labels take continuous values in  $\mathcal{Y} \subset \mathbb{R}^m$  with  $m \geq 1$ . To solve a supervised learning problem, one seeks to construct a map  $f_{\text{approx}} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{Y})$ , which, desirably, is such that for any  $x \in \mathcal{X}$  and for any Borel measurable  $A \subset \mathcal{Y}$ ,  $f_{\text{approx}}(x)(A) \simeq 1$  whenever  $f(x) \in A$ , and  $f_{\text{approx}}(x)(A) \simeq 0$  whenever  $f(x) \notin A$ ; here,  $\mathcal{P}(\mathcal{Y})$  denotes the space of probability measures on  $\mathcal{Y}$ . In other words, one looks for a map  $f_{\text{approx}}$  which approximates the map  $x \rightarrow \delta_{f(x)}$  where  $\delta_z$  denotes the Dirac measure centered at  $z$ . Ultimately, this translates to simultaneously interpolating the above dataset through  $f_{\text{approx}}$ , whilst ensuring generalization/extrapolation, namely reliable prediction on points in  $\mathcal{X}$  which are outside of said dataset ([47]).

**1.3. An optimal control perspective.** There are various ways in which one can construct such an approximation  $f_{\text{approx}}$ , with different degrees of empirical and theoretical guarantees. In this paper, following a recent trend started with the works [12, 28, 9], we shall focus on parametrizing  $f_{\text{approx}}$  by the flow of neural ODEs, such as

$$\begin{cases} \dot{\mathbf{x}}_i(t) = w(t)\sigma(\mathbf{x}_i(t)) + b(t) & \text{for } t \in (0, T), \\ \mathbf{x}_i(0) = x^{(i)} \in \mathbb{R}^d, \end{cases} \quad (1.1)$$

for  $i \in [n]$  and  $T > 0$ , with  $\sigma$  being a scalar, globally Lipschitz function defined componentwise in (1.1). The matrix  $w(t) \in \mathbb{R}^{d \times d}$  and vector  $b(t) \in \mathbb{R}^d$  play the role of controls (called *parameters* in machine learning jargon), which in practice are found by solving an empirical risk minimization problem of the form

$$\inf_{\substack{u=(w,b) \in \mathfrak{U} \\ \mathbf{x}_i \text{ solves (1.1)}}} \underbrace{\frac{1}{n} \sum_{i=1}^n \text{loss} \left( P\mathbf{x}_i(T), y^{(i)} \right)}_{:= \mathcal{E}(\mathbf{x}(T))} + \int_0^T \|u(t)\|_1 dt. \quad (1.2)$$

Here,  $\mathfrak{U}$  is an appropriate Banach subspace of  $L^1(0, T; \mathbb{R}^{d_u})$ ,  $P : \mathbb{R}^d \rightarrow \mathbb{R}^m$  is an affine map which we suppose to be given<sup>1</sup>, and which serves to match the states  $\mathbf{x}_i(T)$  with the labels  $y^{(i)}$  (typically of different dimensions), while

$$\text{loss}(\cdot, \cdot) : \mathbb{R}^m \times \mathcal{Y} \rightarrow \mathbb{R}_+$$

<sup>1</sup>In practice,  $P$  is either an optimizable variable, or its coefficients may be chosen at random. While we fix  $P$  for technical purposes, our numerical experiments indicate that the results presented in what follows persist when  $P$  is optimized as well.

is such that  $x \mapsto \text{loss}(x, y)$  is continuous for all  $y \in \mathcal{Y}$ ,  $\text{loss}(x, y) \neq 0$  whenever  $\mu(x) \neq \delta_y$ , and  $\text{loss}(x, y) \rightarrow 0$  when  $\mu(x) \rightarrow \delta_y$  in an appropriate sense of measures (e.g., for some Wasserstein distance, or for the Kullback-Leibler divergence). A prototypical example is given by the square of the euclidean distance (*least squares error*). But more tailored loss functions may be used, including positive and non-coercive ones, such as the *cross-entropy* loss commonly used for classification tasks

$$\text{loss}(x, y) := -\log \left( \frac{e^{xy}}{\sum_{j=1}^m e^{x_j}} \right) \quad \text{for } x \in \mathbb{R}^m, y \in [m]. \quad (1.3)$$

Once a solution  $u = (w, b)$  to (1.2) is found, one may construct the approximation  $f_{\text{approx}}$  by setting  $f_{\text{approx}}(x) = \mu(\mathbf{x}(T))$  for  $x \in \mathcal{X} \subset \mathbb{R}^d$ , where  $\mathbf{x}(T)$  solves (1.1) with  $\mathbf{x}(0) = x$  and control  $u$ . The choice of  $\mu : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{Y})$  depends on the loss function and task at hand; for the least squares error loss for instance, one sets  $\mu(x) := \delta_{Px}$ , while for the cross-entropy loss, one sets  $\mu := \text{softmax} \circ P$ , with  $\text{softmax}(z)_\ell = e^{z_\ell} / \sum_{j=1}^m e^{z_j}$  for  $\ell \in [m]$  and  $z \in \mathbb{R}^m$ , as in (1.3) (designating a smooth approximation of the argmax).

The above presentation thus leads one to note that, in the neural ODE setting, supervised learning is a particular optimal control problem, wherein one looks to find a single pair of controls  $u = (w, b)$ , which steer  $n$  trajectories of a nonlinear ODE such as (1.1), corresponding to  $n$  different initial data, to  $n$  different targets.

**1.4. The role of  $T$ .** Let us motivate our reason for considering the neural ODE and optimal control interpretation of supervised learning. In practice, one typically considers some discrete-time analog of (1.1), e.g. a forward Euler scheme of the form

$$\begin{cases} \mathbf{x}_i^{k+1} = \mathbf{x}_i^k + \Delta t \left( w^k \sigma(\mathbf{x}_i^k) + b^k \right) & \text{for } k \in \{0, \dots, n_t - 1\}, \\ \mathbf{x}_i^0 = x^{(i)}, \end{cases} \quad (1.4)$$

for  $i \in [n]$ , where  $n_t \geq 2$  and  $\Delta t = T/n_t$ . The scheme (1.4) is an example of a *residual neural network* (ResNet), a popular neural network architecture introduced in [29]. As shown in [29], such neural networks provide, empirically, remarkable interpolation *and* extrapolation performance when  $n_t$  is large (of the orders of hundreds). Here,  $n_t$  is referred to as the *depth* of the network (1.4) and each time-step  $k$  is called a *layer*. However, the theory supporting these empirical results is not completely mature ([47]).

We observe that when  $\Delta t > 0$  is fixed, the time horizon  $T$  can be used to estimate the depth  $n_t$ . This warrants the study of the behavior of optimal control problems for neural ODEs when  $T$  is increased. On another hand, for many problems in optimal control, tracking the control and the trajectory over the entire time interval yields quantitative stability estimates for both when  $T$  is large enough. This is for instance the case in turnpike theory for linear quadratic (LQ) problems ([23]). This setup is further motivated by empirical studies in machine learning literature, where a penalty of the state over each layer has been seen to yield better larger margin predictors, and thus better generalization, for specific classification tasks ([14]). Consequently, in this work, rather than (1.2), we are led to consider

$$\inf_{\substack{u=(w,b) \in \mathcal{U} \\ \mathbf{x}_i \text{ solves (1.1)}}} \int_0^T \mathcal{E}(\mathbf{x}(t)) dt + \int_0^T \|u(t)\|_1 dt, \quad (1.5)$$

where  $\mathcal{E}$  is defined in (1.2), and where we set  $\mathbf{x}(t) = \{\mathbf{x}_i(t)\}_{i \in [n]}$ . Our goal in this work is to provide a rather complete picture of the behavior of solutions to (1.5) and (1.1) as functions of  $T$ .

**1.5. Our contributions.** We can illustrate our findings through numerical experiments<sup>2</sup> before proceeding with theoretical setups and proofs. In Figure 1 (see Figure 2–Figure 4 for related illustrations), we depict a solution of (1.5) for a binary classification task ( $\mathcal{Y} := \{1, 2\}$ , with the data in Figure 3), with  $\sigma \equiv \tanh$ , using the cross-entropy loss defined in (1.3),  $T = 5$ ,  $\Delta t = 1/4$  (thus 20 ResNet layers) with a midpoint scheme, and  $n = 3000$ . We also impose the constraint  $\|u(t)\|_1 \leq M$  with  $M = 8$ , to avoid concentration near  $t = 0$ . (See Remark 1.)

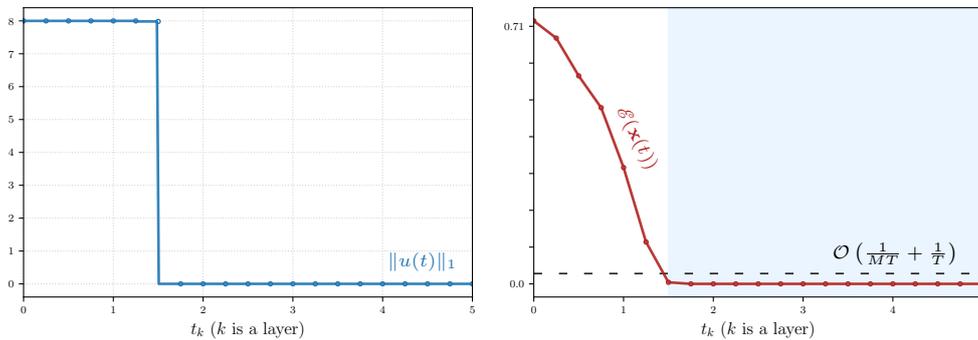


FIGURE 1. (Left) Optimal controls  $u(t)$  solving (1.5). (Right) The empirical risk  $\mathcal{E}(\mathbf{x}(t))$  of the optimal states  $\{\mathbf{x}_i(t)\}_{i \in [n]}$ . Both vanish beyond time  $T^* = 1.5$ , which corresponds to 7 layers.

- The numerics show that optimal controls  $u_T(t) = (w_T(t), b_T(t))$  concentrate within a subinterval  $[0, T^*]$ , and vanish beyond time  $T^*$  (the *ordered* sparsity pattern we had alluded to). The corresponding states  $\{\mathbf{x}_i(t)\}_{i \in [n]}$  are not only stationary for  $t \geq T^*$ , but actually in the regime in which  $\mathcal{E}(\mathbf{x}(t))$  is near 0, as desired.
- In practical terms, the ordered sparsity and stability results could then be used to discard unnecessary layers in the corresponding residual neural network (ResNet), without removing relevant information. They also provide a quantitative estimate of the number of layers needed to fit the data, whilst keeping the controls of user-prescribed amplitude (thus possibly helping in generalization). These estimates ensure and indicate that the time horizon (or number of layers) ought not to be large at all for the error to reach 0 (Figure 1).
- However, the presence of a minimal time  $T^*$  would mean that we still need several layers – namely a large enough  $T$  – before entering the stability regime, from which point on the empirical risk can be ensured to be small. This implies a trade-off in how large  $T$  should actually be. One should keep in mind that our numerical experiments are toy examples and do not convey possible difficulties encountered for various real-life datasets, which may be

<sup>2</sup>The PyTorch code may be found at <https://github.com/borjanG/dynamical.systems>.

significantly more complex. (This complexity can partially be seen through our upper bounds in Theorem 2.1, see Remark 4.)

- All in all, in computing terms, the pointwise stability estimate further indicates that a hybrid, model predictive control (MPC)-type strategy is warranted for an optimal choice of the stopping time (see [26, 15] for similar considerations). Our theoretical results provide further backbone for such ideas, which have been used in applied scenarios ([24]).

In the subsequent section, we shall mathematically formalize these results (Theorem 2.1) and provide rigorous proofs ensuring their validity in a wide array of functional settings.

**1.6. Outline.** The remainder of this work is structured as follows. In **Section 2**, we provide the functional setting and our main result (Theorem 2.1), which corroborates the numerical experiment presented just above. Further numerical visualizations of the same experiment may also be found therein. The proof of Theorem 2.1 may be found in **Section 3**. We conclude with a selection of open problems in **Section 4**.

## 2. MAIN RESULT

**2.1. Setup.** We henceforth suppose we are given a dataset

$$\left\{x^{(i)}, y^{(i)}\right\}_{i \in [n]} \subset \mathcal{X} \times \mathcal{Y} \quad (2.1)$$

with  $\mathcal{X} \subset \mathbb{R}^d$  and  $x^{(i)} \neq x^{(j)}$  for  $i \neq j$ . The label space  $\mathcal{Y}$  may either be a finite subset of  $\mathbb{N}$ , or a subset of  $\mathbb{R}^m$ . To have a more coherent presentation and simplify the technical details, we shall stack all of the trajectories  $\mathbf{x}_i(t)$  appearing in neural ODEs as (1.1), in order, into one single vector  $\mathbf{x}(t) \in \mathbb{R}^{dn}$ . Namely, we set

$$\mathbf{x}(t) := \begin{bmatrix} \mathbf{x}_1(t) \\ \vdots \\ \mathbf{x}_n(t) \end{bmatrix} \in \mathbb{R}^{d_x}, \quad \mathbf{x}^0 := \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(n)} \end{bmatrix} \in \mathbb{R}^{d_x}$$

for  $i \in [n]$  and  $t \geq 0$ , where  $d_x := dn$ , and consider stacked neural ODEs in the general form

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u(t)) & \text{for } t \in (0, T), \\ \mathbf{x}(0) = \mathbf{x}^0, \end{cases} \quad (2.2)$$

where  $u(t) := (w(t), b(t)) \in \mathbb{R}^{d^2+d}$ . As presented in (1.1), for the stacked system the nonlinearity  $\mathbf{f} : \mathbb{R}^{d_x} \times \mathbb{R}^{d_u} \rightarrow \mathbb{R}^{d_x}$  may take the form

$$\mathbf{f}(\mathbf{x}, u) = \begin{bmatrix} w & & \\ & \ddots & \\ & & w \end{bmatrix} \sigma(\mathbf{x}) + \begin{bmatrix} b \\ \vdots \\ b \end{bmatrix} \quad (2.3)$$

for  $\mathbf{x} \in \mathbb{R}^{d_x}$  and  $u = (w, b) \in \mathbb{R}^{d_u}$ , with  $d_u := d^2 + d$ . Once again,  $\sigma \in \text{Lip}(\mathbb{R})$  is defined componentwise, so that each component of  $\mathbf{f}$  coincides with the neural ODE given in (1.1). Permutations may also be considered, such as

$$\mathbf{f}(\mathbf{x}, u) = \sigma \left( \begin{bmatrix} w & & \\ & \ddots & \\ & & w \end{bmatrix} \mathbf{x} + \begin{bmatrix} b \\ \vdots \\ b \end{bmatrix} \right), \quad (2.4)$$

as in the original paper [12]. The key assumption we shall henceforth make regarding  $\mathbf{f}$  is the following.

**Assumption 1** (Homogeneous dynamics). *We suppose that  $\sigma \in \text{Lip}(\mathbb{R})$ . We suppose that  $\mathbf{f}$  is 1-homogeneous with respect to the controls  $u$ , in the sense that*

$$\mathbf{f}(\mathbf{x}, \alpha u) = \alpha \mathbf{f}(\mathbf{x}, u)$$

for all  $(\mathbf{x}, u) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_u}$  and for all  $\alpha > 0$ .

This is clearly the case for dynamics  $\mathbf{f}$  parametrized as in (2.3), whilst for (2.4), we shall moreover assume that  $\sigma$  is 1-homogeneous – a prototypical example is the ReLU  $\sigma(x) = \max\{x, 0\}$ , or more general variants such as  $\sigma(x) = \max\{ax, x\}$  for  $a \in [0, 1)$ . (Such homogeneity assumptions are not an oddity in theoretical contexts, see [10] for instance.) Now, as seen in (1.5), given  $T > 0$  we shall consider the following minimization problem

$$\inf_{\substack{u \in \mathfrak{U}_{\text{ad}, T} \\ \mathbf{x} \text{ solves (2.2)}}} \underbrace{\int_0^T \mathcal{E}(\mathbf{x}(t)) dt + \int_0^T \|u(t)\|_1 dt}_{:= \mathcal{J}_T(u)}, \quad (2.5)$$

where  $\mathcal{E}$  is defined in (1.2), and

$$\mathfrak{U}_{\text{ad}, T} := \left\{ u \in L^1(0, T; \mathbb{R}^{d_u}) : \|u(t)\|_1 \leq M \text{ a.e. in } (0, T) \right\}$$

for a fixed thresholding constant  $M > 0$ . Note that for such controls, (2.2) admits a unique solution  $\mathbf{x} \in C^0([0, T]; \mathbb{R}^{d_x})$  by the Cauchy-Lipschitz theorem. We postpone commenting the need of having an  $L^\infty$  constraint in  $\mathfrak{U}_{\text{ad}, T}$  to Remark 1. Before doing so, we make precise the exact assumptions we shall henceforth make regarding the loss function inducing the error  $\mathcal{E}$ , defined in (1.2), appearing in (2.5).

**Assumption 2** (The loss function). *We suppose that  $\text{loss}(\cdot, \cdot) : \mathbb{R}^m \times \mathcal{Y} \rightarrow \mathbb{R}_+$  appearing in (1.2) satisfies*

$$\text{loss}(\cdot, y) \in \text{Lip}_{\text{loc}}(\mathbb{R}^m; \mathbb{R}_+) \quad \text{and} \quad \inf_{x \in \mathbb{R}^m} \text{loss}(x, y) = 0$$

for all  $y \in \mathcal{Y}$ .

This assumption is generic among most losses considered in practice, including all those induced by a distance (e.g., least squares error) and the cross-entropy loss (1.3).

**2.2. Main result.** Throughout the paper, we will assume that the neural ODE can interpolate the dataset defined in (2.1), either in finite or in infinite time. This is an exact controllability assumption, as we shall suppose that there exist controls for which the corresponding stacked trajectory  $\mathbf{x}(t)$  makes  $\mathcal{E}(\mathbf{x}(\cdot))$  (defined in (1.2)) vanish in finite or in infinite time respectively.

**Definition 2.1** (Interpolation). *We say that*

- (i) (2.2) interpolates the dataset (2.1) in some time  $T > 0$  if there exists  $T > 0$  and  $u \in L^\infty(0, T; \mathbb{R}^{d_u})$  such that the solution  $\mathbf{x} \in C^0([0, T]; \mathbb{R}^{d_x})$  to (2.2) satisfies

$$\mathcal{E}(\mathbf{x}(T)) = 0.$$

- (ii) (2.2) asymptotically interpolates the dataset (2.1) if there exist  $T > 0$ , some function  $h \in C^\infty([T, +\infty); \mathbb{R}_+)$  satisfying

$$\dot{h} < 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} h(t) = 0,$$

and some  $u \in L^\infty(\mathbb{R}_+; \mathbb{R}^{d_u})$  such that the solution  $\mathbf{x} \in C^0(\mathbb{R}_+; \mathbb{R}^{d_x})$  to (2.2) set on  $\mathbb{R}_+$  satisfies

$$\mathcal{E}(\mathbf{x}(t)) \leq h(t)$$

for  $t \geq T$ .

These conditions actually hold for the dynamics  $\mathbf{f}$  and many of the errors  $\mathcal{E}$  we consider here – we postpone this discussion to Remark 2. We may now state our main result.

**Theorem 2.1.** *Suppose  $T > 0$  and  $M > 0$  are fixed. Let  $u_T \in \mathfrak{U}_{\text{ad}, T}$  be any (should it exist<sup>3</sup>) minimizer of (2.5). Let  $\mathbf{x}_T \in C^0([0, T]; \mathbb{R}^{d_x})$  denote the corresponding solution to (2.2). Then, there exists some time  $T^* \in (0, T]$  such that*

$$\begin{aligned} \|u_T(t)\|_1 &= M && \text{for a.e. } t \in (0, T^*), \\ \|u_T(t)\|_1 &= 0 && \text{for a.e. } t \in (T^*, T). \end{aligned} \quad (2.6)$$

Moreover,  $T^*$  is such that

$$\mathcal{E}(\mathbf{x}_T(T^*)) \leq \mathcal{E}(\mathbf{x}_T(t)) \quad \text{for } t \in [0, T], \quad (2.7)$$

and, furthermore,

- (i) *If system (2.2) interpolates the dataset in some time  $T_0 > 0$  as per Definition 2.1, then there exists a constant  $\mathfrak{C} > 0$  independent of both  $T$  and  $M$ , such that*

$$T^* \leq \mathfrak{C} \left( \frac{1}{M} + \frac{1}{M^2} \right)$$

and

$$\mathcal{E}(\mathbf{x}_T(T^*)) \leq \frac{\mathfrak{C}}{T} \left( \frac{1}{M} + 1 \right).$$

- (ii) *If system (2.2) asymptotically interpolates the dataset as per Definition 2.1, then there exists a constant  $\mathfrak{C}(M) > 0$  independent of  $T$  such that*

$$T^* \leq \frac{\mathfrak{C}(M)}{M} h^{-1} \left( \frac{1}{T} \right) + \frac{1}{M}$$

and

$$\mathcal{E}(\mathbf{x}_T(T^*)) \leq \frac{\mathfrak{C}(M)}{T} h^{-1} \left( \frac{1}{T} \right) + \frac{1}{T},$$

where  $h^{-1}$  denotes the inverse function of  $h$ .

---

<sup>3</sup>One can show that a minimizer exists when  $\mathbf{f}$  is as in (2.3) by means of the direct method in the calculus of variations. However, for  $\mathbf{f}$  as in (2.4), it's not clear if there is enough compactness to convert weak convergences into pointwise ones for passing to the limit inside  $\sigma$ .

*Sketch of the proof.* In the proof of the theorem, which may be found in Section 3, the stopping time  $T^* > 0$  is precisely defined as

$$T^* := \min \left\{ t \in [0, T]: \mathcal{E}(\mathbf{x}_T(t)) = \min_{s \in [0, T]} \mathcal{E}(\mathbf{x}_T(s)) \right\}.$$

This implies (2.7) by definition. One then shows that the temporal sparsity in equations (2.6) holds. This is done by a contradiction argument: one supposes that either of both conclusions doesn't hold, and in both cases, constructs auxiliary controls which are strict minimizers for  $\mathcal{J}_T$  defined in (2.5). This is quite transparent in the case in which  $\|u_T(t)\| \neq 0$  for  $t \geq T^*$ , in which case, one can simply use a zero extension of  $u_T(t)$  for  $t \geq T^*$  to conclude. On the other hand, if  $\|u_T(t)\| < M$  for  $t \in (0, T^*)$ , the construction is more delicate and technical, and makes crucial use of the scaling provided by the homogeneous dynamics, and the invariance of the  $L^1(0, T; \mathbb{R}^{d_u})$  by this scaling. The estimates on the stopping time  $T^*$  and on the error evaluated at the stopping time can then be obtained by making use of the interpolation assumptions and the mentioned scaling, for constructing suboptimal controls which can be estimated appropriately. In particular, our arguments do not rely on studying the first-order optimality system, and is specifically tailored to the particular ODEs in question. This allows us to avoid smallness assumptions on the data, and smoothness assumptions on the nonlinearity.

**2.3. Turnpike property.** The behavior displayed in Theorem 2.1 and Figure 1 – Figure 4 can, in some contexts, be seen as a novel manifestation of the *turnpike property* in optimal control: over long time horizons, the optimal pair  $(u_T(t), \mathbf{x}_T(t))$  should be "near" an optimal steady pair  $(\bar{u}, \bar{\mathbf{x}})$ , namely a solution to the problem

$$\inf_{\substack{(u, \mathbf{x}) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_x} \\ \mathbf{f}(\mathbf{x}, u) = 0}} \mathcal{E}(\mathbf{x}) + \|u\|_1. \quad (2.8)$$

(See [23].) Let us suppose that  $\text{loss}(x, y) = \|x - y\|_2^2$  (but the discussion remains true for any distance) and drop the subscript  $T$ , hence

$$\mathcal{E}(\mathbf{x}(t)) = \frac{1}{n} \sum_{i=1}^n \left\| P\mathbf{x}_i(t) - y^{(i)} \right\|_2^2.$$

Theorem 2.1 then implies that

$$\left\| P\mathbf{x}_i(t) - y^{(i)} \right\|_2^2 \leq \frac{C(M)}{T} \quad (2.9)$$

for all  $t \geq T^*$  and  $i \in [n]$ . Now note that  $\mathbf{f}(\bar{\mathbf{x}}, 0) = 0$  for any  $\bar{\mathbf{x}} \in \mathbb{R}^{d_x}$ . In particular, if  $P: \mathbb{R}^d \rightarrow \mathbb{R}^m$  is surjective, then taking  $\bar{\mathbf{x}}_i \in P^{-1}(\{y^{(i)}\})$  for  $i \in [n]$ , we see that there exists some  $\bar{\mathbf{x}} \in \mathbb{R}^{d_x}$ , with  $\bar{\mathbf{x}}_i \in P^{-1}(\{y^{(i)}\})$  such that  $(0, \bar{\mathbf{x}})$  is the unique solution to the steady problem (2.8). Now, on one hand, the sparsity in time result already ensures a finite-time turnpike property for the optimal controls  $u_T(t)$  to the steady correspondent  $\bar{u} \equiv 0$ . On the other hand, (2.9) can be seen as

$$\left\| P(\mathbf{x}_i(t) - \bar{\mathbf{x}}_i) \right\|_2^2 \leq \frac{C(M)}{T}$$

for all  $t \geq T^*$ ,  $i \in [n]$  and for some  $\bar{\mathbf{x}}_i \in P^{-1}(\{y^{(i)}\})$ . This is a turnpike property for (a projection of) the state  $\mathbf{x}(t)$ .

Actually, one can see that the above phenomenon is not bound to machine learning, and applies to more classical optimal control problems of the form

$$\inf_{\substack{u \in \mathfrak{U}_{\text{ad},T} \\ \mathbf{x} \text{ solves (2.11)}}} \int_0^T \|\mathbf{x}(t) - \bar{\mathbf{x}}\|_p^p + \int_0^T \|u(t)\|_1 dt, \quad (2.10)$$

where  $p \in [1, +\infty)$ ,  $\bar{\mathbf{x}} \in \mathbb{R}^{d_x}$  is fixed, and the underlying system is of *driftless control-affine* form

$$\begin{cases} \dot{\mathbf{x}}(t) = \sum_{j=1}^{d_u} u_j(t) f_j(\mathbf{x}(t)) & \text{in } (0, T), \\ \mathbf{x}(0) = \mathbf{x}^0, \end{cases} \quad (2.11)$$

with  $f_j : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_x}$  for  $j \in [d_u]$ . Then  $(\bar{u}, \bar{\mathbf{x}}) = (0, \bar{\mathbf{x}})$  is the optimal steady pair, namely the unique solution to

$$\inf_{\substack{(u, \mathbf{x}) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_x} \\ \sum_{j=1}^{d_u} u_j f_j(\mathbf{x}) = 0}} \|\mathbf{x} - \bar{\mathbf{x}}\|_p^p + \|u\|_1,$$

and we have the following corollary of Theorem 2.1.

**Corollary 2.1** (Turnpike property). *Suppose  $\mathbf{x}_0, \bar{\mathbf{x}} \in \mathbb{R}^{d_x}$  are given, and let  $T > 0$ ,  $M > 0$  be fixed. Suppose  $f_j \in \text{Lip}(\mathbb{R}^{d_x}; \mathbb{R}^{d_x})$  for  $j \in [d_u]$ . Let  $u_T \in \mathfrak{U}_{\text{ad},T}$  be any solution to (2.10). Let  $\mathbf{x}_T$  denote the corresponding solution to (2.11). Then there exists some time  $T^* \in (0, T]$  and some constant  $\mathfrak{C} > 0$  independent of both  $T$  and  $M$  such that*

$$\|u_T(t)\|_1 = M \mathbf{1}_{[0, T^*]}(t)$$

holds for a.e.  $t \in (0, T)$ , and

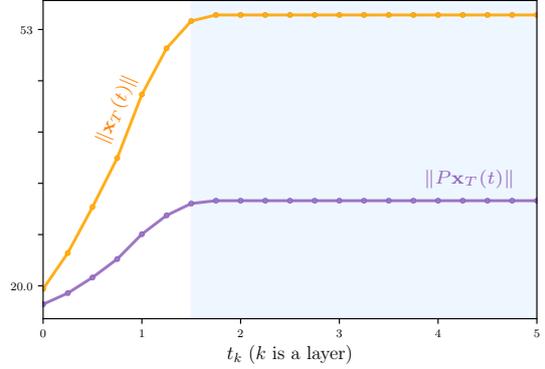
$$\|\mathbf{x}_T(t) - \bar{\mathbf{x}}\|_p^p \leq \frac{\mathfrak{C}}{T} \left( \frac{1}{M} + 1 \right).$$

holds for all  $t \in [T^*, T]$ .

Theorem 2.1 and Corollary 2.1 can then be seen as a new result in the turnpike literature: they provide a finite-time, exact turnpike for any optimal control  $u_T$  solving (2.10) (new on its own, due to the  $L^1$  penalty of the controls), and a polynomial turnpike for the corresponding optimal state  $\mathbf{x}_T(t)$  for  $t \in [T^*, T]$ , without any smallness assumptions on the initial data  $\mathbf{x}^0$ , on the target  $\bar{\mathbf{x}}$ , or smoothness assumptions on the dynamics  $f$ . The latter are deemed necessary for arguments which make use of the Pontryagin Maximum Principle and linearization ([43]). A final arc near  $t = T$  doesn't appear as the running cost is at its minimal value for  $t \in [T^*, T]$ . Another possible approach for proving turnpike would be through the avenue of dissipativity theory in the sense of Willems (see the recent survey [18]), but due to its non-smooth nature, showing that this problem fits in the dissipativity setting is not straightforward. Similar results have been obtained for  $L^2$  penalties in [15, 16] (see also [19, 13, 27, 17]).

It is gripping that in Figure 2, we actually see this phenomenon for the trajectories when  $\mathcal{E}$  is given by the cross-entropy loss (1.3). In this case,  $\mathcal{E}$  is not coercive:  $\mathcal{E}(\mathbf{x}(t))$  approaches 0 only if the margin  $\gamma(\mathbf{x}_T(T))$  defined in (2.12) goes to  $+\infty$ . Namely, every trajectory  $\mathbf{x}_i(T)$  for  $i \in [n]$  ought to grow to  $+\infty$  in an appropriate direction in  $\mathbb{R}^d$ . Thus, in this non-coercive case, we do not interpret the graph of Figure 2 as a turnpike property, since the turnpike would depend on (and increase with)  $T$ . Rather,

FIGURE 2. For the experiment of Figure 1, we see that not only the error  $\mathcal{E}(\mathbf{x}_T(t))$  decays (at least polynomially), but the trajectories  $\mathbf{x}_T(t)$  too reach some stationary point which ought to be near argmin  $\mathcal{E}$ . (See the discussion just below.)



the trajectories  $\mathbf{x}(t)$  become stationary beyond time  $t \geq T^*$  to some point  $\bar{\mathbf{x}} \in \mathbb{R}^{d_x}$ , which is polynomially "sliding" to  $+\infty$  (the "argmin" of  $\mathcal{E}$ ) as  $T \rightarrow +\infty$ .

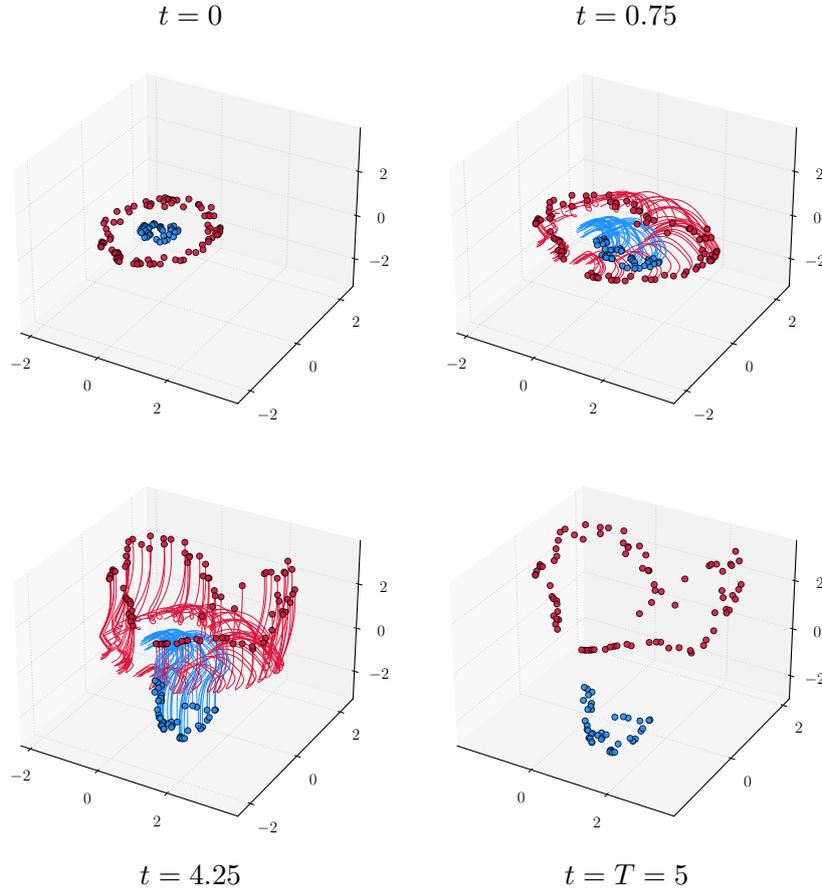


FIGURE 3. The evolution of the states  $\{\mathbf{x}_i(t)\}_{i \in [n]}$  solving (1.1), for the experiment of Figure 1. The states are stationary in a separation regime beyond  $t \geq T^*$ , as indicated by Figure 1.

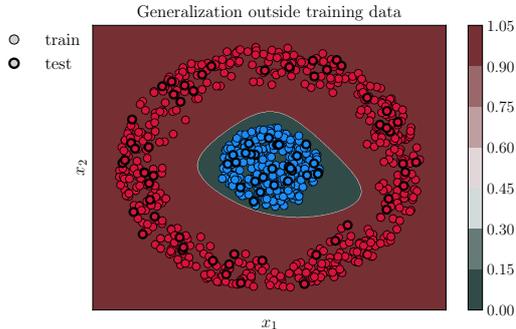


FIGURE 4. The predictor  $f_{\text{approx}}$  learned through the neural ODE flow. It captures the shape of the dataset given by  $f$ , accurately classifies the test data, thus ensuring satisfactory generalization.

**2.4. Discussion.** Let us provide a structured commentary regarding the different assumptions surrounding the above result, possible extensions, and novelty with respect to past literature on both neural ODEs and optimal control.

**Remark 1** ( $L^\infty$  constraint). *Penalizing the  $L^1$  norm in (2.5) enforces the use of sparse controls, which without an  $L^\infty$  constraint, would a priori concentrate near  $t = 0$  as a Dirac mass. We include the  $L^\infty$  constraint in the definition of  $\mathfrak{U}_{\text{ad},T}$  in order to prevent such degeneracy. One can then recover a Dirac mass centered at  $t = 0$  when  $M \rightarrow +\infty$ .*

**Remark 2** (Interpolation). *In the case where  $\mathcal{E}$  attains its infimum (here 0), (finite-time) interpolation as per Definition 2.1, which can be seen as simultaneous or ensemble controllability, has been shown to hold for the dynamics  $\mathbf{f}$  as considered here in several recent works [34, 15, 1, 39, 38, 3, 41]. We have stated it as an assumption in Theorem 2.1 to make transparent the ingredients used in the proof.*

*On another hand, as our setting includes losses which do not attain their infimum, one cannot expect exact interpolation to always hold. This is exemplified by the cross-entropy defined in (1.3), which motivates the asymptotic interpolation hypothesis. Under the assumption that there exists a control  $u \in L^\infty(0, T_0; \mathbb{R}^{d_u})$  for which the margin  $\gamma = \gamma(\mathbf{x}(T_0))$  defined as*

$$\gamma(\mathbf{x}(T_0)) := \min_{i \in [n]} \left\{ \left( P\mathbf{x}_i(T_0) \right)_{y^{(i)}} - \max_{\substack{j \in [m] \\ j \neq y^{(i)}}} \left( P\mathbf{x}_i(T_0) \right)_j \right\} \quad (2.12)$$

*is positive in some  $T_0 > 0$ , in [21, Proposition 7.4.2] asymptotic interpolation is shown to hold for the cross-entropy (1.3) with*

$$h(t) = \log \left( 1 + (m - 1)e^{-\gamma e^t} \right).$$

**Remark 3** (The dynamics). 

- *While there are several works in the literature which prove sparsity in time for controls found by minimizing some functional, even for systems with drifts (unlike ours), the theory is either done for linear systems ([48, 2, 22]), or nonlinear ones for specific regression functionals and/or differentiable dynamics and/or infinite time horizons ([30, 31, 44]). Similar considerations can be found in the literature on optimal control of multi-agent/mean-field systems ([6, 20, 7]). The setting we presented herein makes no such assumptions, and our results can then be seen as complementary to these works. Our consideration of divergences instead of distances*

in the optimization problem can be seen as a novelty in the optimal control context.

- More complicated neural ODEs of the form

$$\begin{cases} \dot{\mathbf{x}}_i(t) = w^2(t)\sigma(w^1(t)\mathbf{x}_i(t)) & \text{in } (0, T) \\ \mathbf{x}_i(0) = \mathbf{x}^0 \end{cases} \quad (2.13)$$

for  $i \in [n]$ , where  $w^2(t) \in \mathbb{R}^{d \times d_{hid}}$  and  $w^1(t) \in \mathbb{R}^{d_{hid} \times d}$  (we omit the translation control for simplicity), tend to perform well in experiments due to the higher number of controls. When  $\sigma$  is 1-homogeneous, and  $w^2(t) = \pm 1$  or is an orthogonal matrix for all  $t$ , Theorem 2.1 still holds due to the fact that Lemma 3.1 applies for such dynamics. When we remove such assumptions on  $w^2(t)$ , the technical impediment we encounter is the lack of invariance of the  $L^1(0, T; \mathbb{R}^{d_u})$  norm with respect to the natural scaling induced by the equation (Lemma 3.1). Indeed, if one sets  $w_1^1(t) := T^\alpha w^1(tT)$  and  $w_1^2(t) := T^{1-\alpha} w^2(tT)$  for  $t \in [0, 1]$  and some  $\alpha \in (0, 1)$ , then it can be seen that  $\mathbf{x}_i^1(t) := \mathbf{x}_i(tT)$  solves (2.13) on  $[0, 1]$ . Yet,

$$\int_0^T \|w^1(t)\|_1 dt + \int_0^T \|w^2(t)\|_1 dt = T^{\alpha-1} \int_0^1 \|w_1^1(s)\|_1 ds + T^{-\alpha} \int_0^1 \|w_1^2(s)\|_1 ds.$$

This is incompatible with our proof strategy. However, noting the above identity, one could investigate the applicability of our techniques to (2.13) and parameter regularizations of the form

$$\int_0^T \|w^1(t)\|_1^{1/\alpha} dt + \int_0^T \|w^2(t)\|_1^{1/1-\alpha} dt,$$

which would be invariant by the above scaling. In such a case, the sparsity pattern should be defined with respect to the regularization one considers. Due to the likely nontrivial nature of the proof, we leave it open.

**Remark 4** (Dependence on the data). Clearly from (2.6), we see that the amplitude of the optimal controls is not the appropriate measure for how these controls depend on the data (unlike the case of  $\ell^2$ -penalties studied in [15]). Similar conclusions apply to the corresponding optimal state, which is stationary (and in the interpolation regime, at least numerically) when the control vanishes. The parameter which does however strongly depend on the data is the stopping time  $T^*$ , through the constant  $\mathfrak{C}$ . Looking at the proof of Theorem 2.1, we see that this constant is explicit:

$$\mathfrak{C} := \|u_{T_0}\|_{L^\infty(0, T_0)} \max \left\{ 1, \int_0^{T_0} \mathcal{E}(\mathbf{x}_{T_0}(t)) dt \right\},$$

where  $T_0 > 0$  is arbitrary, and  $u_{T_0}$  is any control ensuring controllability in the sense of Definition 2.1. Such controls typically increase with the euclidean norm of the data in a continuous way, and without smallness assumptions, this dependence may be highly nonlinear. Note that this constant also depends on the ambient dimension  $d$  (the width), and clarifying the role of  $d$  in this context is an open problem.

**Remark 5** (What about (1.2)?). While we do not demonstrate any long-time pattern for global minima of (1.2), we may provide a numerical comparison with (1.5), in the setup of Figure 1. (See Figure 5.) The learned predictor is almost identical to that shown in Figure 3 (albeit learned after 20 layers), so we omit the plot.

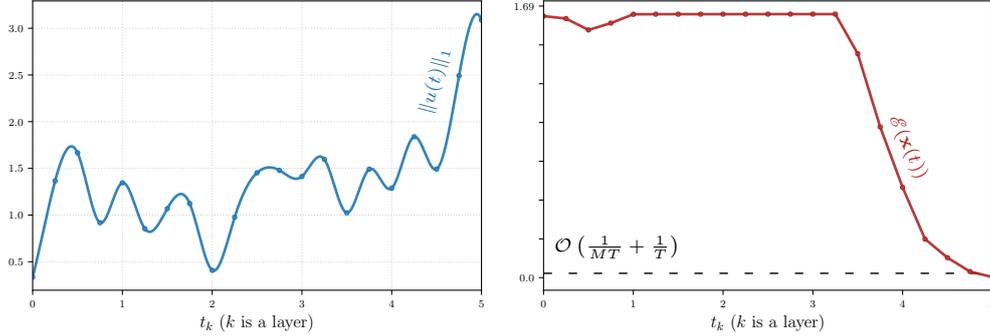


FIGURE 5. (Left) Optimal controls  $u(t)$  solving (1.2). (Right) The empirical risk  $\mathcal{E}(\mathbf{x}(t))$  of the optimal states  $\{\mathbf{x}_i(t)\}_{i \in [n]}$ . We do not see any stability, and the empirical risk is small only near the final time/layer.

**2.5. Related work.** The neural ODE lens has been used to great effect in practice. Examples of such use include adaptive ODE solvers [9, 11], symplectic schemes [8], or indirect training algorithms based on the Pontryagin Maximum Principle [33, 4]. Further applications include irregular time series modeling [37, 46], and generative modeling through normalizing flows [25, 36]. We refer the reader to the thesis [32] for an excellent review of various applications and state of the art numerical methods.

Typically in deep learning through neural networks, sparsity is explicitly enforced through the structure of the weights, in the mould of using convolutions with filters instead of matrix multiplications ([35]). We rather take the approach of considering a somewhat universal architecture (in the spirit of universal approximation setups, [5]), in view of obtaining a clearer picture on how different penalties affect the long-time properties of global minima (in this regard, see [15] for an  $\ell^2$ -penalty study).

### 3. PROOFS

In this section we provide the proof of Theorem 2.1. We shall split the proof into two parts. We first state and prove Proposition 3.1, which contains the first part of Theorem 2.1, concerning the sparsity of optimal controls. The proof of the latter is done throughout Section 3.1. We then provide the remainder of the proof in Section 3.2.

**3.1. Preliminary results.** The main goal of this subsection is to state and prove Proposition 3.1. A cornerstone of our forthcoming arguments is the possibility of rescaling any trajectory of (2.2) set in  $[0, T_0]$  to obtain the same trajectory set on  $[0, T]$ .

**Lemma 3.1.** *Let  $\mathbf{x}^0 \in \mathbb{R}^{d_x}$ ,  $T_0 > 0$ ,  $u_{T_0} \in L^1(0, T_0; \mathbb{R}^{d_u})$ , and let  $\mathbf{x}_{T_0}$  be the unique solution to (2.2) set on  $[0, T_0]$ , with control  $u_{T_0}$ . Let  $T > 0$ , and define*

$$u_T(t) := \frac{T_0}{T} u_{T_0} \left( t \frac{T_0}{T} \right) \quad \text{for } t \in [0, T],$$

and

$$\mathbf{x}_T(t) := \mathbf{x}_{T_0} \left( t \frac{T_0}{T} \right) \quad \text{for } t \in [0, T].$$

Then  $\mathbf{x}_T$  is the unique solution to (2.2) with control  $u_T$ .

We omit the proof, which is straightforward. We also summarize the notion of sparsity through the following definition.

**Definition 3.1** (Sparse controls). *Let  $M > 0$  and  $0 < T^* \leq T$  be fixed. We say that  $u \in \mathfrak{U}_{\text{ad},T}$  is sparse in  $(T^*, T)$  if*

$$\|u(t)\|_1 = M \quad \text{a.e. } t \in (0, T^*), \quad (3.1)$$

$$\|u(t)\|_1 = 0 \quad \text{a.e. } t \in (T^*, T). \quad (3.2)$$

For any  $T^* > 0$ , we shall denote by  $\mathfrak{U}_{\text{sp},T^*}$  the set consisting of all  $u \in \mathfrak{U}_{\text{ad},T}$  which are sparse in  $(T^*, T)$ , namely which satisfy (3.1) – (3.2).

**Proposition 3.1.** *Let  $T > 0$  and  $M > 0$  be fixed. Let  $u_T \in \mathfrak{U}_{\text{ad},T}$  be a global minimizer of  $\mathcal{J}_T$  defined in (2.5), and let  $\mathbf{x}_T$  be the corresponding unique solution to (2.2). Then  $u_T \in \mathfrak{U}_{\text{sp},T^*}$ , where  $T^*$  is defined as*

$$T^* := \min \left\{ t \in [0, T] : \mathcal{E}(\mathbf{x}_T(t)) = \min_{s \in [0, T]} \mathcal{E}(\mathbf{x}_T(s)) \right\}. \quad (3.3)$$

Note that the  $T^*$  is clearly well defined, as the set over which the min is taken is clearly bounded, and is also closed as the preimage of the singleton

$$\left\{ \min_{s \in [0, T]} \mathcal{E}(\mathbf{x}_T(s)) \right\}$$

under the continuous map  $t \mapsto \mathcal{E}(\mathbf{x}(t))$ . The core of the proof of Proposition 3.1 lies in the following lemma, which ensures that if a control  $u_T \in \mathfrak{U}_{\text{ad},T}$  does not saturate the  $L^\infty$ -constraint before some time  $T^*$ , then  $u_T$  is not optimal for  $\mathcal{J}_T$  and can always be "improved" through the scaling of Lemma 3.1.

**Lemma 3.2.** *Let  $T > 0$  and  $M > 0$  be fixed. Let  $u_T \in \mathfrak{U}_{\text{ad},T}$  be any admissible (but not necessarily optimal) control, and let  $T^* > 0$  be defined as in (3.3). Assume that, for some  $\theta \in (0, 1)$ , there exists a finite collection of disjoint non-empty intervals  $\{(a_j, b_j)\}_{j \in \mathfrak{J}}$  with  $(a_j, b_j) \subset (0, T^*)$  for which*

$$\|u_T(t)\|_1 \leq (1 - \theta)M \quad \text{for a.e. } t \in \mathbf{O}_{\mathfrak{J}}, \quad (3.4)$$

and

$$\mathcal{E}(\mathbf{x}_T(t)) - \mathcal{E}(\mathbf{x}_T(T^*)) \geq \theta \quad \text{for all } t \in \mathbf{O}_{\mathfrak{J}} \quad (3.5)$$

hold, where

$$\mathbf{O}_{\mathfrak{J}} := \bigcup_{j=1}^{\mathfrak{J}} (a_j, b_j).$$

Then there exists some  $\bar{u} \in \mathfrak{U}_{\text{ad},T}$  satisfying

$$\bar{u}(t) = 0 \quad \text{for a.e. } t \in (T^* - \tau, T), \quad (3.6)$$

and

$$\mathcal{J}_T(\bar{u}) \leq \mathcal{J}_T(u_T) - \theta\tau,$$

where

$$\tau := \theta \text{meas}(\mathbf{O}_{\mathfrak{J}}) = \theta \sum_{j=1}^{\mathfrak{J}} (b_j - a_j).$$

We may now provide the proof to Proposition 3.1.

*Proof of Proposition 3.1.* We argue by contradiction. Suppose that  $u_T \in \mathfrak{U}_{\text{ad},T}$  is a global minimizer of  $\mathcal{J}_T$  such that  $u_T \notin \mathfrak{U}_{\text{sp},T^*}$ , where  $T^* > 0$  is defined as in the statement. Hence, either condition (3.1) or condition (3.2) does not hold.

**Case 1: (3.2) does not hold.** Let us thus suppose that

$$\|u_T(t)\|_1 > 0 \quad \text{a.e. } t \in \Omega \quad (3.7)$$

holds for some  $\Omega \subsetneq (T^*, T)$  of positive Lebesgue measure. Consider

$$\bar{u}(t) = \begin{cases} u_T(t) & \text{for } t \in [0, T^*] \\ 0 & \text{for } t \in (T^*, T]. \end{cases}$$

Clearly  $\bar{u} \in \mathfrak{U}_{\text{ad},T}$ . Furthermore, we have

$$\bar{\mathbf{x}}(t) = \mathbf{x}_T(t) \quad \text{for } t \in [0, T^*],$$

and since  $\mathbf{f}(\cdot, 0) \equiv 0$ , also

$$\bar{\mathbf{x}}(t) = \bar{\mathbf{x}}(T^*) = \mathbf{x}_T(T^*), \quad \text{for } t \in [T^*, T].$$

Combining these facts with the definition (3.3) of  $T^*$ , we are lead to

$$\int_0^T \mathcal{E}(\bar{\mathbf{x}}(t)) dt = \int_0^{T^*} \mathcal{E}(\mathbf{x}_T(t)) dt + \int_{T^*}^T \mathcal{E}(\mathbf{x}_T(T^*)) dt \leq \int_0^T \mathcal{E}(\mathbf{x}_T(t)) dt.$$

By virtue of (3.7) we also find

$$\begin{aligned} \int_0^T \|\bar{u}(t)\|_1 dt &= \int_0^{T^*} \|u_T(t)\|_1 dt \\ &< \int_0^{T^*} \|u_T(t)\|_1 dt + \int_{T^*}^T \|u_T(t)\|_1 dt = \int_0^T \|u_T(t)\|_1 dt. \end{aligned}$$

Combining the two previous inequalities, we deduce that  $\mathcal{J}_T(\bar{u}) < \mathcal{J}_T(u_T)$ , which contradicts the optimality of  $u_T$ .

**Case 2: (3.1) does not hold.** The idea is to again construct an auxiliary control which improves  $u_T$  to deduce a contradiction. We now split the proof in three steps.

*Step 1.* If (3.1) is not fulfilled, then there must exist some  $\theta \in (0, 1)$  such that the set

$$\mathbf{A}_\theta := \left\{ t \in (0, T^*) : \|u_T(t)\|_1 \leq (1 - \theta)M \right\}$$

has positive Lebesgue measure, namely  $\text{meas}(\mathbf{A}_\theta) > 0$ . Now set  $\omega := \frac{\text{meas}(\mathbf{A}_\theta)}{2}$ , and using elementary set theory we find

$$\mathbf{A}_\theta \cap (0, T^* - \omega) = \mathbf{A}_\theta \setminus \left( (0, T^*) \setminus (0, T^* - \omega) \right) = \mathbf{A}_\theta \setminus [T^* - \omega, T^*),$$

whence the set

$$\mathbf{B}_\theta := \mathbf{A}_\theta \cap (0, T^* - \omega)$$

also has positive Lebesgue measure:  $\text{meas}(\mathbf{B}_\theta) > 0$ . By classical results in Lebesgue measure theory (see [45, Thm. 3.25]), for all  $\varepsilon > 0$  there exists a finite collection of disjoint nonempty intervals  $\{(a_j, b_j)\}_{j \in [n(\varepsilon)]}$ , with  $(a_j, b_j) \subset (0, T^* - \omega)$ , such that the set

$$\mathbf{O}_\varepsilon := \bigcup_{j=1}^{n(\varepsilon)} (a_j, b_j)$$

satisfies

$$\text{meas}(\mathbf{O}_\varepsilon \setminus \mathbf{B}_\theta) < \varepsilon \quad \text{and} \quad \text{meas}(\mathbf{B}_\theta \setminus \mathbf{O}_\varepsilon) < \varepsilon. \quad (3.8)$$

In particular,

$$\text{meas}(\mathbf{O}_\varepsilon) > \text{meas}(\mathbf{B}_\theta) - \varepsilon. \quad (3.9)$$

*Step 2.* Let  $\varepsilon \in (0, \text{meas}(\mathbf{B}_\theta))$  be arbitrary and to be chosen later, and let  $\{(a_j, b_j)\}_{j \in [n(\varepsilon)]}$  be the corresponding collection of disjoint intervals satisfying (3.8), with  $\mathbf{O}_\varepsilon$  denoting the union of these intervals as defined above. We now look to construct a control  $u^\varepsilon \in \mathfrak{U}_{\text{ad}, T}$  such that

$$\|u^\varepsilon(t)\|_1 \leq (1 - \theta^*)M$$

and

$$\mathcal{E}(\mathbf{x}^\varepsilon(t)) - \mathcal{E}(\mathbf{x}^\varepsilon(T_\bullet)) \geq \theta^*$$

for some  $\theta^* > 0$  and for all  $t \in \mathbf{O}_\varepsilon$ , where

$$T_\bullet := \min \left\{ t \in [0, T] : \mathcal{E}(\mathbf{x}^\varepsilon(t)) = \min_{s \in [0, T]} \mathcal{E}(\mathbf{x}^\varepsilon(s)) \right\}$$

should also satisfy  $T_\bullet \geq T^* - \omega$ . To this end, set

$$u^\varepsilon(t) := \begin{cases} u_T(t) & \text{for } t \in (0, T) \setminus (\mathbf{O}_\varepsilon \setminus \mathbf{B}_\theta) \\ 0 & \text{for } t \in \mathbf{O}_\varepsilon \setminus \mathbf{B}_\theta. \end{cases}$$

Since  $u_T \in \mathfrak{U}_{\text{ad}, T}$ , it may readily be seen that

$$\|u^\varepsilon(t)\|_1 \leq M \quad \text{for a.e. } t \in (0, T).$$

Hence  $u^\varepsilon \in \mathfrak{U}_{\text{ad}, T}$ . Now let  $\mathbf{x}^\varepsilon$  denote the solution to (2.2) associated to  $u^\varepsilon$ . By virtue of the specific form of  $\mathbf{f}$ , the Lipschitz continuity of  $\sigma$ , and the Grönwall inequality, we may readily deduce that there exists a constant  $C_1 = C_1(T, M, \sigma) > 0$  independent of  $\varepsilon$  such that

$$\|\mathbf{x}^\varepsilon(t) - \mathbf{x}_T(t)\|_1 \leq C_1 \int_0^T \|u^\varepsilon(s) - u_T(s)\|_1 ds \quad (3.10)$$

for all  $t \in [0, T]$ . On the other hand, by using (3.8), we also deduce that

$$\int_0^T \|u^\varepsilon(s) - u_T(s)\|_1 ds \leq M \text{meas}(\mathbf{O}_\varepsilon \setminus \mathbf{B}_\theta) < M\varepsilon. \quad (3.11)$$

Combining (3.10) and (3.11) leads us to

$$\|\mathbf{x}^\varepsilon(t) - \mathbf{x}_T(t)\|_1 < C_1 M \varepsilon$$

for  $t \in [0, T]$ . Now since  $\mathbf{x}_T \in C^0([0, T]; \mathbb{R}^{d_x})$ , the stacked trajectory  $\mathbf{x}_T(t)$  remains in a compact subset of  $\mathbb{R}^{d_x}$  for all  $t \in [0, T]$ . Due to (3.1), and since  $\varepsilon \leq \text{meas}(\mathbf{B}_\theta)$ , we also find that  $\mathbf{x}_\varepsilon$  remains in a slightly larger compact subset, independent of  $\varepsilon$ . Hence, by the locally Lipschitz character of  $\text{loss}(\cdot, y)$ , implying that of  $\mathcal{E}$ , the estimate

$$\left| \mathcal{E}(\mathbf{x}^\varepsilon(t)) - \mathcal{E}(\mathbf{x}_T(t)) \right| \leq C_2 M \varepsilon, \quad (3.12)$$

holds for some  $C_2 = C_2(T, M, \sigma, \mathcal{E}) > 0$  independent of  $\varepsilon$ , and for all  $t \in [0, T]$ . On the other hand, using only the definition (3.3) of  $T^*$ , we find that there exists some  $\lambda > 0$  such that

$$\mathcal{E}(\mathbf{x}_T(t)) \geq \mathcal{E}(\mathbf{x}_T(T^*)) + \lambda \quad (3.13)$$

for all  $t \in [0, T^* - \omega]$ . Estimate (3.12) combined with (3.13) yields

$$\begin{aligned} \mathcal{E}(\mathbf{x}^\varepsilon(T^*)) &\leq \mathcal{E}(\mathbf{x}_T(T^*)) + C_2 M \varepsilon \leq \mathcal{E}(\mathbf{x}_T(t)) - \lambda + C_2 M \varepsilon \\ &\leq \mathcal{E}(\mathbf{x}^\varepsilon(t)) - \lambda + 2C_2 M \varepsilon, \end{aligned} \quad (3.14)$$

for all  $t \in [0, T^* - \omega]$ , which, by choosing  $\varepsilon < \lambda/2C_2M$ , implies that  $T_\bullet \geq T^* - \omega$ , as desired. The computations done in (3.14) also yield

$$\begin{aligned} \mathcal{E}(\mathbf{x}^\varepsilon(t)) &\geq \mathcal{E}(\mathbf{x}^\varepsilon(T^*)) + \lambda - C_2M\varepsilon \\ &\geq \mathcal{E}(\mathbf{x}^\varepsilon(T_\bullet)) + \lambda - 2C_2M\varepsilon \end{aligned} \quad (3.15)$$

for all  $t \in [0, T^* - \omega]$ . As we chose  $\varepsilon < \lambda/2C_2M$ , we have that  $\lambda - 2C_2M\varepsilon > 0$ , and may then set

$$\theta^* := \min \{ \theta, \lambda - 2C_2M\varepsilon \},$$

so that  $\theta^* > 0$ . By virtue of (3.15),

$$\mathcal{E}(\mathbf{x}^\varepsilon(t)) - \mathcal{E}(\mathbf{x}^\varepsilon(T_\bullet)) \geq \theta^*$$

holds for all  $t \in \mathbf{O}_\varepsilon$ . Now, observe that  $u^\varepsilon$  also satisfies

$$\|u^\varepsilon(t)\|_1 \leq (1 - \theta^*)M$$

for a.e.  $t \in \mathbf{O}_\varepsilon$ . Indeed, if  $t \in \mathbf{O}_\varepsilon \setminus \mathbf{B}_\theta$ , then  $u^\varepsilon(t) = 0$  by definition, so the inequality clearly holds. On the other hand, if  $t \in \mathbf{O}_\varepsilon \cap \mathbf{B}_\theta$ , then  $t \in \mathbf{A}_\theta$ , and since  $\theta^* \geq \theta$ , the conclusion follows.

*Step 3.* We may now apply Lemma 3.2, which ensures the existence of some  $\bar{u}^\varepsilon \in \mathfrak{U}_{\text{ad},T}$  for which

$$\mathcal{J}_T(\bar{u}^\varepsilon) \leq \mathcal{J}_T(u^\varepsilon) - (\theta^*)^2 \text{meas}(\mathbf{O}_\varepsilon) \quad (3.16)$$

holds. As a consequence of (3.11) and (3.12), we have

$$\mathcal{J}_T(u^\varepsilon) \leq \mathcal{J}_T(u_T) + (1 + C_2T)M\varepsilon,$$

which, when combined with (3.16) and (3.9), yields

$$\mathcal{J}_T(\bar{u}^\varepsilon) < \mathcal{J}_T(u_T) + (1 + C_2T)M\varepsilon - (\theta^*)^2(\text{meas}(\mathbf{B}_\theta) - \varepsilon).$$

Looking at the above inequality, we may note that, by choosing  $\varepsilon > 0$  even smaller (namely taking

$$\varepsilon \leq \frac{(\theta^*)^2 \text{meas}(\mathbf{B}_\theta)}{(1 + C_2T)M}$$

would do), we may ensure that

$$\mathcal{J}_T(\bar{u}^\varepsilon) < \mathcal{J}_T(u_T),$$

which contradicts the optimality of  $u_T$ . This concludes the proof.  $\square$

We conclude this section with a proof of Lemma 3.2.

*Proof of Lemma 3.2.* We will argue by induction over the number of intervals  $\mathfrak{J} \geq 1$ , constructing appropriately the control  $\bar{u}$  explicitly in each step via affine transformations of  $u_T$  – the desired estimates will follow by using the time-scaling invariance of the  $L^1$ -norm of the controls.

**Step 1). Initialization.** Let us first assume that  $\mathfrak{J} = 1$ . Consider

$$\bar{u}(t) := \begin{cases} u_T(t) & \text{for } t \in (0, a_1) \\ \frac{b_1 - a_1}{c_1 - a_1} u_T \left( (t - a_1) \frac{b_1 - a_1}{c_1 - a_1} + a_1 \right) & \text{for } t \in [a_1, c_1) \\ u_T(t + b_1 - c_1) & \text{for } t \in [c_1, T^* - (b_1 - c_1)), \\ 0 & \text{for } t \in [T^* - (b_1 - c_1), T), \end{cases}$$

where  $c_1 \in (a_1, b_1)$  is chosen so that

$$\frac{b_1 - a_1}{c_1 - a_1}(1 - \theta) = 1,$$

which is equivalent to

$$b_1 - c_1 = \theta(b_1 - a_1) =: \tau.$$

Observe that as a consequence of (3.4), we clearly have  $\bar{u} \in \mathfrak{U}_{\text{ad},T}$ . In addition, by virtue of the choice of  $c_1$ , and the definition of  $\tau$ ,  $\bar{u}(t)$  also satisfies (3.6). Now, making use of the scaling provided by Lemma 3.1, and the fact that  $\mathbf{f}(\cdot, 0) \equiv 0$ , one can check that the state trajectory  $\bar{\mathbf{x}}(t)$  associated to  $\bar{u}(t)$  is exactly given by

$$\bar{\mathbf{x}}(t) = \begin{cases} \mathbf{x}_T(t) & \text{for } t \in [0, a_1) \\ \mathbf{x}_T\left((t - a_1)\frac{b_1 - a_1}{c_1 - a_1} + a_1\right) & \text{for } t \in [a_1, c_1) \\ \mathbf{x}_T(t + b_1 - c_1) & \text{for } t \in [c_1, T^* - (b_1 - c_1)), \\ \mathbf{x}_T(T^*) & \text{for } t \in [T^* - (b_1 - c_1), T]. \end{cases}$$

Moreover, observe that since  $\tau := b_1 - c_1$ ,

$$\mathcal{E}(\bar{\mathbf{x}}(t)) = \mathcal{E}(\mathbf{x}_T(T^*)) \quad \text{for } t \in [T^* - \tau, T]. \quad (3.17)$$

Let us now evaluate the functional  $\mathcal{J}_T$  along  $\bar{u}$ . We start by computing the  $L^1$  norm of  $\bar{u}$ :

$$\begin{aligned} \|\bar{u}\|_{L^1(0,T;\mathbb{R}^{d_u})} &= \int_0^{a_1} \|u_T(t)\|_1 dt + \int_{c_1}^{T^* - (b_1 - c_1)} \|u_T(t + b_1 - c_1)\|_1 dt \\ &\quad + \frac{b_1 - a_1}{c_1 - a_1} \int_{a_1}^{c_1} \left\| u_T\left((t - a_1)\frac{b_1 - a_1}{c_1 - a_1} + a_1\right) \right\|_1 dt \\ &= \int_0^{b_1} \|u_T(s)\|_1 ds + \int_{T^* - b_1}^{T^*} \|u_T(s)\|_1 ds \\ &\leq \|u_T\|_{L^1(0,T;\mathbb{R}^{d_u})}. \end{aligned} \quad (3.18)$$

On the other hand, by virtue of (3.17), (3.5), the definition (3.3) of  $T^*$ , and the same changes of variable used to deduce (3.18), we find

$$\begin{aligned} \int_0^T \left( \mathcal{E}(\bar{\mathbf{x}}(t)) - \mathcal{E}(\mathbf{x}_T(T^*)) \right) dt &= \int_0^{a_1} \left( \mathcal{E}(\mathbf{x}_T(t)) - \mathcal{E}(\mathbf{x}_T(T^*)) \right) dt \\ &\quad + \underbrace{\frac{c_1 - a_1}{b_1 - a_1}}_{1-\theta} \int_{a_1}^{b_1} \left( \mathcal{E}(\mathbf{x}_T(t)) - \mathcal{E}(\mathbf{x}_T(T^*)) \right) dt \\ &\quad + \int_{b_1}^{T^*} \left( \mathcal{E}(\mathbf{x}_T(t)) - \mathcal{E}(\mathbf{x}_T(T^*)) \right) dt \\ &\leq \int_0^T \left( \mathcal{E}(\mathbf{x}_T(t)) - \mathcal{E}(\mathbf{x}_T(T^*)) \right) dt - \theta^2(b_1 - a_1). \end{aligned}$$

By combining the above inequality with (3.18), it follows that

$$\mathcal{J}_T(\bar{u}) \leq \mathcal{J}_T(u_T) - \theta^2(b_1 - a_1).$$

The statement of the Lemma thus holds for  $\mathfrak{J} = 1$ .

**Step 2). Heredity.** Let us suppose that, for some  $n \geq 1$ , the statement of the lemma holds whenever  $\mathfrak{J} = n$ , and let  $u_T$  satisfy (3.4) and (3.5) with  $\mathfrak{J} = n + 1$ . Assume without loss of generality that  $a_1 > a_j$  for all  $j \in \{2, \dots, \mathfrak{J}\}$ . Using precisely the same argument as in Step 1, we can construct a control  $\bar{u}_1$  satisfying

$$\bar{u}_1(t) = 0 \quad \text{for a.e. } t \in (T^* - \tau_1, T)$$

with  $\tau_1 = \theta(b_1 - a_1)$ , and

$$\mathcal{J}_T(\bar{u}_1) \leq \mathcal{J}_T(u_T) - \theta^2(b_1 - a_1),$$

and which is such that  $\bar{u}_1(t) = u_T(t)$  for all  $t \in (0, t_1)$ . Now observe that, since  $a_1 > a_j$  for all  $j \geq 2$ , and in view of (3.17), it follows that  $\bar{u}_1$  satisfies (3.4) and (3.5) with  $\mathfrak{J} - 1 = n$  number of intervals and with  $T_1^* = T^* - \tau_1$  instead of  $T^*$ . By the induction hypothesis, we conclude that there exists some control  $\bar{u} \in \mathfrak{U}_{\text{ad}, T}$  such that

$$\bar{u}(t) = 0 \quad \text{for a.e. } t \in (T_1^* - \tau, T)$$

with  $\tau = \theta \sum_{j=2}^{\mathfrak{J}} (b_j - a_j)$ , and

$$\mathcal{J}_T(\bar{u}) \leq \mathcal{J}_T(\bar{u}_1) - \theta^2 \sum_{j=2}^{\mathfrak{J}} (b_j - a_j) \leq \mathcal{J}_T(u_T) - \theta^2 \sum_{j=1}^{\mathfrak{J}} (b_j - a_j).$$

The statement of the Lemma thus also holds for  $\mathfrak{J} = n + 1$ . This concludes the proof.  $\square$

### 3.2. Proof of Theorem 2.1.

*Proof of Theorem 2.1.* Properties (2.6) and (2.7) for the minimizers of  $\mathcal{J}_T$  follow directly from Proposition 3.1. Let us give the proof of the statements (i) and (ii) in Theorem 2.1.

**Proof of (i).** If the interpolation property holds, then there exist  $T_0 > 0$  and some control  $u_{T_0} \in L^\infty(0, T_0; \mathbb{R}^{d_u})$  such that the associated solution  $\mathbf{x}_{T_0} \in C^0([0, T_0]; \mathbb{R}^{d_x})$  to (2.2) satisfies  $\mathcal{E}(\mathbf{x}_{T_0}(T_0)) = 0$ . Set

$$T_1 := \frac{T_0 \|u_{T_0}\|_{L^\infty(0, T_0; \mathbb{R}^{d_u})}}{M}, \quad (3.19)$$

and consider

$$u_{T_1}(t) := \frac{M}{\|u_{T_0}\|_{L^\infty(0, T_0; \mathbb{R}^{d_u})}} u_{T_0} \left( t \frac{T_0}{T_1} \right) \quad \text{for } t \in (0, T_1).$$

Observe that  $u_{T_1} \in \mathfrak{U}_{\text{ad}, T_1}$ . Furthermore, in view of Lemma 3.1, the associated solution  $\mathbf{x}_{T_1}$  to (2.2), is given by

$$\mathbf{x}_{T_1}(t) = \mathbf{x}_{T_0} \left( t \frac{T_0}{T_1} \right) \quad \text{for } t \in (0, T_1),$$

and hence,

$$\mathcal{E}(\mathbf{x}_{T_1}(T_1)) = \mathcal{E}(\mathbf{x}_{T_0}(T_0)) = 0.$$

Now for any  $T > 0$ , we define

$$\bar{u}(t) = \begin{cases} u_{T_1}(t) & \text{for } t \in (0, T) \cap (0, T_1) \\ 0 & \text{for } t \in (0, T) \setminus (0, T_1). \end{cases}$$

Clearly  $\bar{u} \in \mathfrak{U}_{\text{ad},T}$ . By a simple change of variable, and using (3.19), one sees that

$$\begin{aligned} \mathcal{J}_T(\bar{u}) &\leq \int_0^{T_1} \mathcal{E}(\mathbf{x}_{T_1}(t)) dt + M T_1 \\ &= \frac{\|u_{T_0}\|_{L^\infty(0,T_0;\mathbb{R}^{d_u})}}{M} \int_0^{T_0} \mathcal{E}(\mathbf{x}_{T_0}(t)) dt + \|u_{T_0}\|_{L^\infty(0,T_0;\mathbb{R}^{d_u})} T_0 \\ &= \frac{C_1}{M} + C_2, \end{aligned} \tag{3.20}$$

holds, where  $C_1 > 0$  and  $C_2 > 0$  are independent of both  $T$  and  $M$ . In view of (2.6), any minimizer  $u_T$  of  $\mathcal{J}_T$  satisfies  $u_T \in \mathfrak{U}_{\text{sp},T^*}$  for some  $T^* \in (0, T]$ . Since  $\bar{u} \in \mathfrak{U}_{\text{ad},T}$ , using (3.20), we obtain

$$\mathcal{J}_T(u_T) = \int_0^T \mathcal{E}(\mathbf{x}_T(t)) dt + M T^* \leq \mathcal{J}_T(\bar{u}) \leq \frac{C_1}{M} + C_2. \tag{3.21}$$

Since  $\mathcal{E} \geq 0$ , using (3.21) we deduce that

$$T^* \leq \frac{C_1}{M^2} + \frac{C_2}{M}.$$

Moreover, using (2.7) in (3.21), we also deduce that

$$T \mathcal{E}(\mathbf{x}_T(T^*)) \leq \mathcal{J}_T(u_T) \leq \frac{C_1}{M} + C_2.$$

The last two estimates imply (i) in the statement of Theorem 2.1, as desired.

**Proof of (ii).** If the asymptotic interpolation property holds, then there exist  $T_0 > 0$ , a function  $h$  as in Definition 2.1, and some control  $u^\infty \in L^\infty(\mathbb{R}_+; \mathbb{R}^{d_u})$  such that the corresponding solution  $\mathbf{x}^\infty$  to (2.2) set on  $\mathbb{R}_+$  satisfies

$$\mathcal{E}(\mathbf{x}^\infty(t)) \leq h(t) \tag{3.22}$$

for all  $t \geq T_0$ . Combining this with the continuity of the map  $t \mapsto \mathcal{E}(\mathbf{x}^\infty(t))$ , we can readily deduce that there exists a constant  $C_0 > 0$  depending only on  $T_0 > 0$  such that

$$\mathcal{E}(\mathbf{x}^\infty(t)) \leq C_0 \tag{3.23}$$

for all  $t \geq 0$ . Let us henceforth set

$$\lambda := \frac{M}{\|u^\infty\|_{L^\infty(\mathbb{R}_+; \mathbb{R}^{d_u})}}.$$

For any  $T_1 > 0$ , we also define

$$u_{T_1}(t) = \begin{cases} \lambda u^\infty(\lambda t) & \text{for } t \in (0, T_1] \\ 0 & \text{for } t > T_1. \end{cases}$$

Observe that, by definition of  $\lambda$ , one has  $u_{T_1} \in \mathfrak{U}_{\text{ad},T}$  for any  $T > 0$ . By virtue of Lemma 3.1, the state associated to  $u_{T_1}$  is precisely

$$\mathbf{x}_{T_1}(t) = \begin{cases} \mathbf{x}^\infty(\lambda t) & \text{for } t \in (0, T_1) \\ \mathbf{x}^\infty(\lambda T_1) & \text{for } t \geq T_1. \end{cases}$$

Now, by virtue of the definition of  $u_{T_1}$ , for any  $T > 0$ , we have

$$\begin{aligned} \mathcal{J}_T(u_{T_1}) &\leq \int_0^{T_1} \mathcal{E}(\mathbf{x}^\infty(\lambda t)) dt + \max\{0, T - T_1\} \mathcal{E}(\mathbf{x}^\infty(\lambda T_1)) + M T_1 \\ &\leq (C_0 + M) T_1 + T \mathcal{E}(\mathbf{x}^\infty(\lambda T_1)). \end{aligned} \quad (3.24)$$

We now distinguish two cases. If  $T \leq 1/h(T_0)$ , then using (3.23), the optimality of  $u_T$  as well as the fact that  $u_{T_1} \in \mathfrak{U}_{\text{ad},T}$ , along with  $u_T \in \mathfrak{U}_{\text{sp},T^*}$ , and the definition (3.3) of  $T^*$ , through (3.24) we find

$$T \mathcal{E}(\mathbf{x}_T(T^*)) + M T^* \leq (C_0 + M) T_1 + \frac{C_0}{h(T_0)},$$

and choosing  $T_1 = 1$  leads us to the conclusion. Now suppose that  $T > 1/h(T_0)$ . By Definition 2.1, the decreasing function  $h$  is a bijection from  $(T_0, +\infty)$  onto its range  $(0, h(T_0))$ , and so  $h^{-1}(1/T)$  is well defined precisely for  $T > 1/h(T_0)$ . We set

$$T_1 := \frac{1}{\lambda} h^{-1}\left(\frac{1}{T}\right).$$

Combining the optimality of  $u_T$  with (3.24), and using the fact that  $u_T \in \mathfrak{U}_{\text{sp},T^*}$ , we find

$$\begin{aligned} \mathcal{J}_T(u_T) &= M T^* + \int_0^T \mathcal{E}(\mathbf{x}_T(t)) dt \leq \mathcal{J}_T(u_{T_1}) \\ &\leq \mathfrak{C}(M) h^{-1}\left(\frac{1}{T}\right) + T \mathcal{E}\left(\mathbf{x}^\infty\left(h^{-1}\left(\frac{1}{T}\right)\right)\right), \end{aligned} \quad (3.25)$$

where the constant

$$\mathfrak{C}(M) := \frac{(C_0 + M)}{\lambda}$$

is independent of  $T$ . Now since  $h^{-1} : (0, h(T_0)) \rightarrow (0, +\infty)$  is non-decreasing, and  $T > 1/h(T_0)$ , we have that  $h^{-1}(1/T) \geq T_0$ . Using this fact, along with (3.22) in (3.25), combined with the definition (3.3) of  $T^*$ , allows us to deduce that

$$T \mathcal{E}(\mathbf{x}_T(T^*)) + M T^* \leq \mathfrak{C}(M) h^{-1}\left(\frac{1}{T}\right) + 1.$$

The desired statement (ii) then follows also for  $T > 1/h(T_0)$ . This concludes the proof.  $\square$

#### 4. CONCLUDING REMARKS

**4.1. Epilogue.** We have presented a manifestation of an ordered sparsity pattern and approximation/stability properties for supervised learning problems for neural ODEs with  $L^1(0, T; \mathbb{R}^{d_u})$  penalties. Our main result ensures that any solution  $u_T$  to (2.5) is sparse in time, in the sense that  $u_T \equiv 0$  on  $(T^*, T)$  for some  $T^* \in (0, T]$ . Under appropriate controllability assumptions, we also provide estimates on the stopping time  $T^*$ , and on the empirical risk  $\mathcal{E}(\mathbf{x}_T(t))$  for  $t \geq T^*$ .

4.2. **Outlook.** We comment some questions that remain regarding our study.

1. The existence of minimizers for (2.5)–(2.4) remains unclear. It can be ensured if one replaces the  $L^1$  penalty by a BV one, for which compactness of minimizing sequences holds. BV controls fit in the setting of ordered sparsity, unlike  $W^{1,1}$  ones, which are continuous. The BV norm is also invariant with respect to the scaling of Lemma 3.1. But a complete extension of our arguments to this case would require further work.
2. It is gripping that, when seen in the classical  $L^2$  tracking context (i.e. the loss is the squared  $\ell^2$  distance) with an  $L^1$  penalty for the controls, Theorem 2.1 only provides a polynomial turnpike estimate for the state. This is different to the  $L^2$  penalty context, presented in [15, 16], in which an exponential turnpike/stabilization estimate for the state is shown. There is reason to believe that for more specific loss functions, our stability results can be sharpened.
3. As a matter of fact, since  $u_T(t) = 0$  for  $t \geq T^*$ , and our numerical experiments show that the state is stable in a regime in which the error  $\mathcal{E}$  is 0, one could also stipulate that a result of the mould  $\mathcal{E}(\mathbf{x}_T(t)) = 0$  for  $t \geq T^*$  holds. Such an exact turnpike property for the state has been obtained in the linear setting in [27]. However, the transfer of the techniques of the latter paper to our setting does not appear straightforward.

**Acknowledgments.** We thank Dario Pighin and Enrique Zuazua for insightful discussions.

**Funding:** This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No.765579-ConFlex and from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement NO. 694126-DyCon).

#### REFERENCES

- [1] AGRACHEV, A., AND SARYCHEV, A. Control on the manifolds of mappings with a view to the deep learning. *Journal of Dynamical and Control Systems* (2021), 1–20.
- [2] ALT, W., AND SCHNEIDER, C. Linear-quadratic control problems with  $L^1$ -control cost. *Optimal Control Appl. Methods* 36, 4 (2015), 512–534.
- [3] BÁRCENA-PETISCO, J. A. Optimal control for neural ODE in a long time horizon and applications to the classification and simultaneous controllability problems.
- [4] BENNING, M., CELLEDONI, E., EHRHARDT, M. J., OWREN, B., AND SCHÖNLIEB, C.-B. Deep learning as optimal control problems: Models and numerical methods. *J. Comput. Dyn.* 6, 2 (2019), 171.
- [5] BÖLCSKEI, H., GROHS, P., KUTYNIOK, G., AND PETERSEN, P. Optimal approximation with sparsely connected deep neural networks. *SIAM J. Math. Data Sci.*, 1 (2020), 8–45.
- [6] CAPONIGRO, M., FORNASIER, M., PICCOLI, B., AND TRÉLAT, E. Sparse stabilization and optimal control of the Cucker-Smale model. *Mathematical Control and Related Fields* 3, 4 (2013), 447–466.
- [7] CAPONIGRO, M., FORNASIER, M., PICCOLI, B., AND TRÉLAT, E. Sparse stabilization and control of alignment models. *Mathematical Models and Methods in Applied Sciences* 25, 03 (2015), 521–564.

- [8] CELLEDONI, E., EHRHARDT, M. J., ETMANN, C., MCLACHLAN, R. I., OWREN, B., SCHÖNLIEB, C.-B., AND SHERRY, F. Structure-preserving deep learning. *European Journal of Applied Mathematics* 32, 5 (2021), 888–936.
- [9] CHEN, T. Q., RUBANOVA, Y., BETTENCOURT, J., AND DUVENAUD, D. K. Neural ordinary differential equations. In *Advances in Neural Information Processing Systems* (2018), pp. 6571–6583.
- [10] CHIZAT, L., AND BACH, F. On the global convergence of gradient descent for over-parameterized models using optimal transport. *Advances in neural information processing systems* 31 (2018).
- [11] DUPONT, E., DOUCET, A., AND TEH, Y. W. Augmented Neural ODEs. In *Advances in Neural Information Processing Systems* (2019), pp. 3134–3144.
- [12] E, W. A proposal on machine learning via dynamical systems. *Commun. Math. Stat.* 5, 1 (2017), 1–11.
- [13] EFFLAND, A., KOBLER, E., KUNISCH, K., AND POCK, T. Variational networks: An optimal control approach to early stopping variational methods for image restoration. *J. Math. Imaging Vision* (2020), 1–21.
- [14] ELSAYED, G., KRISHNAN, D., MOBAHI, H., REGAN, K., AND BENGIO, S. Large margin deep networks for classification. *Advances in neural information processing systems* 31 (2018).
- [15] ESTEVE-YAGÜE, C., GESHKOVSKI, B., PIGHIN, D., AND ZUAZUA, E. Large-time asymptotics in deep learning. *arXiv preprint arXiv:2008.02491* (2021).
- [16] ESTEVE-YAGÜE, C., GESHKOVSKI, B., PIGHIN, D., AND ZUAZUA, E. Turnpike in Lipschitz—nonlinear optimal control. *Nonlinearity* 35, 4 (2022), 1652.
- [17] FAULWASSER, T., AND BONVIN, D. Exact turnpike properties and economic nmpc. *European Journal of Control* 35 (2017), 34–41.
- [18] FAULWASSER, T., AND GRÜNE, L. Turnpike properties in optimal control. *Numerical Control: Part A* 23 (2022), 367.
- [19] FAULWASSER, T., HEMPEL, A.-J., AND STREIF, S. On the turnpike to design of deep neural nets: Explicit depth bounds. *arXiv preprint arXiv:2101.03000* (2021).
- [20] FORNASIER, M., PICCOLI, B., AND ROSSI, F. Mean-field sparse optimal control. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 372, 2028 (2014), 20130400.
- [21] GESHKOVSKI, B. *Control in moving interfaces and deep learning*. PhD thesis, Universidad Autónoma de Madrid, 2021.
- [22] GESHKOVSKI, B., AND ZUAZUA, E. Optimal actuator design via Brunovsky’s normal form. *IEEE Transactions on Automatic Control* (2022).
- [23] GESHKOVSKI, B., AND ZUAZUA, E. Turnpike in optimal control of PDEs, ResNets, and beyond. *Acta Numerica* 31 (2022), 135–263.
- [24] GOODFELLOW, I., BENGIO, Y., AND COURVILLE, A. *Deep learning*. 2016.
- [25] GRATHWOHL, W., CHEN, R. T., BETTENCOURT, J., SUTSKEVER, I., AND DUVENAUD, D. Ffjord: Free-form continuous dynamics for scalable reversible generative models. *arXiv preprint arXiv:1810.01367* (2018).
- [26] GRÜNE, L., SCHALLER, M., AND SCHIELA, A. Sensitivity analysis of optimal control for a class of parabolic PDEs motivated by model predictive control. *SIAM Journal on Control and Optimization* 57, 4 (2019), 2753–2774.
- [27] GUGAT, M., SCHUSTER, M., AND ZUAZUA, E. The finite-time turnpike phenomenon for optimal control problems: Stabilization by non-smooth tracking terms.

- In *Stabilization of Distributed Parameter Systems: Design Methods and Applications*. Springer, 2021, pp. 17–41.
- [28] HABER, E., AND RUTHOTTO, L. Stable architectures for deep neural networks. *Inverse Problems* *34*, 1 (2017), 014004.
- [29] HE, K., ZHANG, X., REN, S., AND SUN, J. Deep residual learning for image recognition. In *Proceedings of the IEEE conference on computer vision and pattern recognition* (2016), pp. 770–778.
- [30] KALISE, D., KUNISCH, K., AND RAO, Z. Infinite horizon sparse optimal control. *J. Optim. Theory Appl.* *172*, 2 (2017), 481–517.
- [31] KALISE, D., KUNISCH, K., AND RAO, Z. Sparse and switching infinite horizon optimal controls with mixed-norm penalizations. *ESAIM Control Optim. Calc. Var.* *26* (2020), 61.
- [32] KIDGER, P. On neural differential equations. *arXiv preprint arXiv:2202.02435* (2022).
- [33] LI, Q., CHEN, L., TAI, C., AND E, W. Maximum principle based algorithms for deep learning. *J. Mach. Learn. Res.* *18*, 1 (2017), 5998–6026.
- [34] LI, Q., LIN, T., AND SHEN, Z. Deep learning via dynamical systems: An approximation perspective. *Journal of the European Mathematical Society* (2022).
- [35] MALLAT, S. Understanding deep convolutional networks. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* *374*, 2065 (2016), 20150203.
- [36] PAPAMAKARIOS, G., NALISNICK, E. T., REZENDE, D. J., MOHAMED, S., AND LAKSHMINARAYANAN, B. Normalizing flows for probabilistic modeling and inference. *J. Mach. Learn. Res.* *22*, 57 (2021), 1–64.
- [37] RUBANOVA, Y., CHEN, R. T., AND DUVENAUD, D. K. Latent ordinary differential equations for irregularly-sampled time series. In *Advances in Neural Information Processing Systems* (2019), pp. 5320–5330.
- [38] RUIZ-BALET, D., AFFILI, E., AND ZUAZUA, E. Interpolation and approximation via momentum ResNets and neural ODEs. *Systems & Control Letters* *162* (2022), 105182.
- [39] RUIZ-BALET, D., AND ZUAZUA, E. Neural ODE control for classification, approximation and transport. *arXiv preprint arXiv:2104.05278* (2021).
- [40] SANTOSA, F., AND SYMES, W. Linear inversion of band-limited reflection seismograms. *SIAM J. Sci. Statist. Comput.* *7* (1986), 1307–1330.
- [41] TABUADA, P., AND GHARESIFARD, B. Universal approximation power of deep residual neural networks through the lens of control. *IEEE Transactions on Automatic Control* (2022).
- [42] TIBSHIRANI, R. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society: Series B (Methodological)* *58*, 1 (1996), 267–288.
- [43] TRÉLAT, E., AND ZUAZUA, E. The turnpike property in finite-dimensional nonlinear optimal control. *Journal of Differential Equations* *258*, 1 (2015), 81–114.
- [44] VOSSEN, G., AND MAURER, H. On  $L^1$ -minimization in optimal control and applications to robotics. *Optimal Control Applications and Methods* *27*, 6 (2006), 301–321.
- [45] YEH, J. *Real analysis: theory of measure and integration second edition*. World Scientific Publishing Company, 2006.
- [46] YOON, T., SHIN, S., AND YANG, E. Learning polymorphic Neural ODEs with

- time-evolving mixture. *IEEE Transactions on Pattern Analysis and Machine Intelligence* (2022).
- [47] ZHANG, C., BENGIO, S., HARDT, M., RECHT, B., AND VINYALS, O. Understanding deep learning (still) requires rethinking generalization. *Communications of the ACM* 64, 3 (2021), 107–115.
- [48] ZUAZUA, E. Switching control. *J. Eur. Math. Soc.* 13, 1 (2010), 85–117.

**Carlos Esteve-Yagüe**

Department of Applied Mathematics  
and Theoretical Physics  
University of Cambridge  
Cambridge  
CB3 0WA, UK  
e-mail: [ce423@cam.ac.uk](mailto:ce423@cam.ac.uk)

**Borjan Geshkovski**

Department of Mathematics  
Massachusetts Institute of Technology  
Simons Building, Room 246C  
77 Massachusetts Avenue  
Cambridge  
MA  
02139-4307 USA  
e-mail: [borjan@mit.edu](mailto:borjan@mit.edu)