Population Monotonic Allocation Schemes for Vertex Cover Games^{*}

Han Xiao^{†1}, Qizhi Fang¹, and Ding-Zhu Du²

¹School of Mathematical Sciences, Ocean University of China, Qingdao, China ²Department of Computer Science, University of Texas at Dallas, Richardson, TX 75080, USA

Abstract

For vertex cover games (introduced by Deng et al., Math. Oper. Res., 24:751-766, 1999 [2]), we investigate population monotonic allocation schemes (introduced by Sprumont, Games Econ. Behav., 2: 378-394, 1990 [11]). We show that the existence of a population monotonic allocation scheme (PMAS for short) for vertex cover games can be determined efficiently and that a PMAS, if exists, can be constructed accordingly. We also show that integral PMAS-es for vertex cover games can be characterized with stable matchings and be enumerated by employing Gale-Shapley algorithm (introduced by Gale and Shapley, Amer. Math. Monthly, 69:9-15, 1962 [4]).

Keywords: population monotonic allocation scheme, cross-monotonic cost-sharing scheme, vertex cover, stable matching.

Mathematics Subject Classification: 05C57, 91A12, 91A43, 91A46.

1 Introduction

Cooperative game theory lays out a theoretical framework for analyzing cooperation among independent participants. An essential issue in a cooperative game is to find an adequate allocation to distribute the expected cost of the coalition to individual participants. There are many criteria for evaluating how "good" an allocation is, such as fairness, stability, and so on. Emphases on different criteria lead to different solution concepts, e.g., the core, the Shapley value, the nucleolus, the bargaining set, and the von Neumann-Morgenstern solution. Among those solution concepts, the core which addresses the issue of stability is one of the most attractive solution concepts.

^{*}This work was supported in part by National Natural Science Foundation of China (11871442, 11971447) and Fundamental Research Funds for the Central Universities (201713051, 201964006).

[†]Corresponding author. Email: hxiao@ouc.edu.cn.

The core in a cooperative game is the set of allocations for the grand coalition (i.e., the coalition of all participants), under which no participant can derive a better payoff by leaving the grand coalition, either individually or as a subgroup. However, an allocation that lies in the core does not necessarily guarantee the unhindered formation of a coalition, as the cost allocated to participants in the current coalition may increase when a new participant joins in.

To study allocations in an expanding coalition, population monotonic allocation schemes (also known as cross-monotonic cost-sharing schemes) were introduced, under which no participant of any coalition derives a worse payoff after a new participant joins in. A PMAS gives no incentive to any participant to block the expansion of coalition and hence the grand coalition is always achieved. Besides, PMAS-es shift the attention from allocations only for the grand coalition to allocation schemes, which deal with partial cooperation and provide allocations for any coalition. Moreover, the set of allocations for the grand coalition that can be reached through a PMAS can be seen as a refinement of the core: the core provides allocations in a sense of static stability, while PMAS-es provide allocations in a sense of dynamic stability.

Populations monotonic allocation schemes were first studied by Sprumont [11], where some characterizations on PMAS-es were provided. In particular, Sprumont proved that submodularity is sufficient for existence of a PMAS. Grahn and Voorneveld [5] showed that every bankruptcy game admits a PMAS by indicating bankruptcy rules that give rise to a PMAS. Norde et al. [8] presented a combinatorial algorithm for computing PMAS-es in minimum cost spanning tree games. Hamers et al. [7] characterized the class of coloring games admitting a PMAS and provided an algorithm enumerating all integral PMAS-es. Motivated by the work of Hamers et al. [7], we investigate PMAS-es for vertex cover games by generalizing the characterization of submodular vertex cover games [9]. Our results are in the same spirit as the work of Hamers et al. [7], and are also inspired by the work of Chen et al. [1]. We provide an efficient characterization for the class of vertex cover games admitting a PMAS and show that integral PMAS-es can be characterized with stable matchings and be enumerated by employing Gale-Shapley algorithm.

Vertex cover games studied in this paper fall into the scope of combinatorial optimization games [2, 3], which arise from cost allocations in the minimum vertex cover problem. Vertex cover games were first studied by Deng et al. [2], where the algorithmic aspect of the core was investigated and a complete characterization for the balancedness of vertex cover games was presented. In a following work [3], Deng et al. gave a necessary and sufficient condition for the total balancedness of vertex cover games. As opposed to the model of Deng et al. [2, 3] where players are edges, Gusev [6] introduced a different class of vertex cover games where players are vertices, and investigated the application to transport networks. In this paper, we stick to the vertex cover game introduced by Deng et al. [2] and investigate PMAS-es.

The rest of this paper is organized as follows. Section 2 is a preliminary section introducing the relevant concepts of game theory and graph theory. In Section 3, an efficient characterization for the class of vertex cover games admitting a PMAS is presented. Section 4 offers a dual-based description of PMAS-es for vertex cover games. In Section 5, we characterize and enumerate integral PMAS-es for vertex cover games with stable matchings. Section 6 concludes the results in this paper and addresses some complexity issues in computing PMAS-es.

2 Preliminaries

This section first reviews some concepts from game theory and graph theory, and then introduces the definition and some known results for vertex cover games.

2.1 Cooperative game theory

A cooperative game is a tuple $\Gamma = (N, \gamma)$, where N is the set of players and $\gamma : 2^N \to \mathbb{R}$ is the characteristic function with the convention $\gamma(\emptyset) = 0$. Any subset S of N is called a *coalition*, where N is called the grand coalition. For coalition S, $\gamma(S)$ represents the total cost charged to S. A cooperative game $\Gamma = (N, \gamma)$ is said monotonic if $\gamma(S) \leq \gamma(T)$ for any $S, T \in 2^N$ with $S \subseteq T$. A cooperative game $\Gamma = (N, \gamma)$ is said submodular if the characteristic function γ is submodular, i.e., $\gamma(S) + \gamma(T) \geq \gamma(S \cup T) + \gamma(S \cap T)$ for any $S, T \in 2^N$. The subgame of Γ corresponding to coalition T, denoted by Γ_T , is a game (T, γ_T) with $\gamma_T(S) = \gamma(S)$ for any $S \subseteq T$.

A cost allocation of $\Gamma = (N, \gamma)$ is a vector $\boldsymbol{a} = (a_i)_{i \in N}$, which consists of proposed costs to be paid by players in the grand coalition. A cost allocation \boldsymbol{a} is said *efficient* if $\sum_{i \in N} a_i = \gamma(N)$, and said group rational if $\sum_{i \in S} a_i \leq \gamma(S)$ for any $S \subseteq N$. In particular, \boldsymbol{a} is said *individual rational* if $a_i \leq \gamma(\{i\})$ for any $i \in N$. An *imputation* of Γ is a cost allocation that is efficient and individual rational. The core of Γ , denoted by $\mathcal{C}(\Gamma)$, is the set of imputations that are group rational. A core allocation is a cost allocation in the core, which satisfies all players in the grand coalition and no player has an incentive to split off from the grand coalition. A game Γ is said balanced if $\mathcal{C}(\Gamma) \neq \emptyset$ and total balanced if $\mathcal{C}(\Gamma_T) \neq \emptyset$ for any nonempty $T \subseteq N$.

A population monotonic allocation scheme (PMAS for short) of $\Gamma = (N, \gamma)$ is a vector $\boldsymbol{a} = (a_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ satisfying the following two conditions:

- efficiency: $\sum_{i \in S} a_{S,i} = \gamma(S)$ for any $S \in 2^N \setminus \{\emptyset\}$;

- monotonicity: $a_{S,i} \ge a_{T,i}$ for any $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$ and any $i \in S$.

Let $\mathcal{P}(\Gamma)$ be the set of PMAS-es for Γ . When $\mathcal{P}(\Gamma) \neq \emptyset$, Γ is said *population monotonic* (also known as *cross-monotonic*). Let $\boldsymbol{a} = (a_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ be a PMAS for Γ . For any $S \in 2^N \setminus \{\emptyset\}$, denote by $a_S = (a_{S,i})_{i \in S}$ the restriction of a to S, which is a core allocation of subgame Γ_S . Notice that even total balancedness is not sufficient for the population monotonicity, as a PMAS provides for every coalition a core allocation in a cross-monotonic way. Hamers et al. [7] proved that PMAS-es of monotonic cooperative games are always nonnegative. We refer to [11] for more about PMAS-es.

Lemma 1 (Hamers et al. [7]). Let \boldsymbol{a} be a PMAS for a monotonic cooperative game $\Gamma = (N, \gamma)$. Then $a_{S,i} \geq 0$ for any $S \in 2^N \setminus \{\emptyset\}$ and any $i \in S$.

2.2 Graph theory

Throughout, a graph is always finite, undirected and simple. Let $n \in \mathbb{N}$. We use K_n to denote the complete graph with n vertices, use C_n to denote the graph which is a cycle with n vertices, and use P_n to denote the graph which is a path with n vertices. Let H be a graph. We use V(H) to denote the vertex set of H and use E(H) to denote the edge set of H. A graph is said H-free if it contains no subgraph isomorphic to H. A graph is *bipartite* if it is odd-cycle-free. A graph is a forest if it is cycle-free. A forest is a tree if it is connected. A vertex is pendant if it has degree one. An edge is pendant if it is incident to a pendant vertex.

Let G = (V, E) be a graph. The *distance* of two vertices u and v in G is the minimum number of edges in a path connecting them. The *diameter* of G is the largest distance between any two vertices in G. For any $S \subseteq E$, G[S] denotes the edge-induced subgraph of G, V_S denotes the vertex set of G[S], and $\delta_S(v)$ denotes the set of edges incident to v in G[S]. A vertex cover of G is a vertex set $C \subseteq V$ such that each edge of G intersects C. The vertex cover number of G, denoted by $\tau(G)$, is the minimum size of vertex covers in G. A matching of G is an edge set $M \subseteq E$ without common vertices. The matching number of G, denoted by $\nu(G)$, is the maximum size of matchings in G. Clearly, $\nu(G) \leq \tau(G)$, since every vertex in a vertex cover only covers at most one edge in a matching. It is well known that equality $\nu(G) = \tau(G)$ holds when G is bipartite [10].

2.3 Vertex cover games

A vertex cover game has players on edges and the game value is defined by the vertex cover number. Formally, the vertex cover game on a graph G = (V, E) is a cooperative game $\Gamma_G = (N, \gamma)$, where N = E and $\gamma(S) = \tau(G[S])$ for any $S \subseteq N$.

Lemma 2. Every vertex cover game is monotonic.

Proof. Let $\Gamma_G = (N, \gamma)$ be the vertex cover game on a graph G. Let $S \subseteq T \subseteq N$. Notice that every vertex cover of G[T] is also a vertex cover of G[S]. It follows that $\gamma(S) \leq \gamma(T)$. \Box

Lemma 3 (Deng et al. [2]). The vertex cover game Γ_G on a graph G is balanced if and only if $\nu(G) = \tau(G)$.

Lemma 4 (Deng et al. [3]). The vertex cover game Γ_G on a graph G is totally balanced if and only if G is bipartite.

Lemma 5 (Okamoto [9]). The vertex cover game Γ_G on a graph G is submodular if and only if G is (K_3, P_4) -free.

3 An efficient characterization for population monotonicity

In this section, we show that a graph induces a vertex cover game admitting PMAS-es if and only if the graph is (K_3, C_4, P_5) -free. We decompose our proof into several lemmas.



Figure 1: Forbidden subgraphs

Lemma 6. Let $\Gamma_G = (N, \gamma)$ be the vertex cover game on a graph G. If $\mathcal{P}(\Gamma_G) \neq \emptyset$, then G is (K_3, C_4, P_5) -free.

Proof. Let $a \in \mathcal{P}(\Gamma_G)$. We show that any of K_3 , C_4 and P_5 yields a contradiction.

We first consider K_3 . Suppose $E(K_3) = \{1, 2, 3\}$. Note that $\gamma(\{1, 2, 3\}) = 2$ and $\gamma(\{1, 2\}) = \gamma(\{2, 3\}) = \gamma(\{1, 3\} = 1)$. By efficiency and monotonicity, we have

$$\begin{split} &4 = \gamma(\{1,2,3\}) + \gamma(\{1,2,3\}) \\ &= a_{\{1,2,3\},1} + a_{\{1,2,3\},2} + a_{\{1,2,3\},3} + a_{\{1,2,3\},1} + a_{\{1,2,3\},2} + a_{\{1,2,3\},3} \\ &\leq a_{\{1,2\},1} + a_{\{1,2\},2} + a_{\{2,3\},3} + a_{\{1,3\},1} + a_{\{2,3\},2} + a_{\{1,3\},3} \\ &= \gamma(\{1,2\}) + \gamma(\{2,3\}) + \gamma(\{1,3\}) \\ &= 3. \end{split}$$

which yields a contradiction.

Now we check C_4 and P_5 . Let H be either C_4 or P_5 . Suppose $E(H) = \{1, 2, 3, 4\}$, where i and i + 1 are incident in H. Note that $\gamma(\{1, 2, 3\}) = \gamma(\{2, 3, 4\}) = 2$ and $\gamma(\{1, 2\}) = \gamma(\{2, 3\}) = \gamma(\{3, 4\}) = 1$. By efficiency and monotonicity, we have

$$\begin{split} &4 = \gamma(\{1,2,3\}) + \gamma(\{2,3,4\}) \\ &= a_{\{1,2,3\},1} + a_{\{1,2,3\},2} + a_{\{1,2,3\},3} + a_{\{2,3,4\},2} + a_{\{2,3,4\},3} + a_{\{2,3,4\},4} \\ &\leq a_{\{1,2\},1} + a_{\{1,2\},2} + a_{\{2,3\},3} + a_{\{2,3\},2} + a_{\{3,4\},3} + a_{\{3,4\},4} \\ &= \gamma(\{1,2\}) + \gamma(\{2,3\}) + \gamma(\{3,4\}) \\ &= 3, \end{split}$$

which yields a contradiction.

The following lemma gives an alternative characterization for (K_3, C_4, P_5) -free graphs.

Lemma 7. A graph G is (K_3, C_4, P_5) -free if and only if each component of G is a tree of diameter at most 3.

Proof. We first prove the "only if" part. Notice that a (K_3, P_5) -free graph does not contain any odd cycle and that a (C_4, P_5) -free graph does not contain any even cycle. It follows that every (K_3, C_4, P_5) -free graph is a forest. Since any tree of diameter larger than 3 contains a P_5 , each component of a (K_3, C_4, P_5) -free graph is a tree of diameter at most 3.

Now we prove the "if" part. Let G be a graph whose components are trees of diameter at most 3. Thus G is (K_3, C_4) -free. Since any graph of diameter at most 3 is P_5 -free, G is (K_3, C_4, P_5) -free.



Figure 2: An example of stars. The dark Figure 3: An example of pisceses. The dark vertices are the bases and the dashed edge is a free rider.

Before proceeding, we introduce some notions for simplicity. A tree of diameter 2 is called a *star*. The unique non-pendant vertex of a star is called the *center* (see Figure 2). Clearly, the center of a star is a minimum vertex cover for the star. A K_2 can also be viewed as a star but only one endpoint can be viewed as the center. A tree of diameter 3 is called a *pisces*. A pisces can be obtained from two stars by joining their centers with an edge. The two non-pendant vertices in a pisces are called the *bases* which form a minimum vertex cover for the pisces. The unique non-pendant edge in a pisces is called a *free rider* which has special significance for vertex cover games (see Figure 3). To see this, consider the vertex cover game on a pisces. For any coalition without the non-pendant edge in a pisces can alway take a free ride and get covered by a minimum vertex cover of other edges.

Let G = (V, E) be (K_3, C_4, P_5) -free. Lemma 7 implies that every component of G is either a star or a pisces. Let $C^* \subseteq V$ be the set of centers of stars and bases of pisceses in G. Then C^* is a minimum vertex cover of G. For any nonempty $S \subseteq E$, there is a minimum vertex cover $C_S^* \subseteq C^*$ of G[S], as every component of G[S] is either a star or a pisces. We will use these notations repeatedly in the rest of this paper. Now we are ready to present one of our main results.

Theorem 1. Let $\Gamma_G = (N, \gamma)$ be the vertex cover game on a graph G. Then Γ_G is population monotonic if and only if G is (K_3, C_4, P_5) -free.

Proof. The "only if" part follows from Lemma 6. Now we prove the "if" part. Assume that G is (K_3, C_4, P_5) -free and Γ_G is the vertex cover game on G. By Lemma 7, every component of G is either a star or a pisces. Let C^* be the set of centers of stars and bases of pisceses in G. For any $S \in 2^N \setminus \{\emptyset\}$, let $C_S^* \subseteq C^*$ be a minimum vertex cover of G[S], and define the cost allocation by

$$a_{S,i} = \begin{cases} 0 & \text{if } i \text{ is a free rider incident to other edges in } S, \\ 1 & \text{if } i \text{ is a free rider not incident to other edges in } S, \\ \frac{1}{\lambda_S(i)} & \text{otherwise,} \end{cases}$$

where $\lambda_S(i)$ is the number of non-free rider edges in S incident to the vertex in C_S^* that covers i. The idea behind this allocation scheme is simple. For any $S \in 2^N \setminus \{\emptyset\}$, split every vertex in C_S^* equally among non-free rider edges in S covered by the vertex, unless a free rider is the unique edge in S covered by it. It remains to show that $(a_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ is a PMAS for Γ_G .

We first prove efficiency. Let $S \in 2^N \setminus \{\emptyset\}$. By the construction, we have $\sum_{i \in \delta_S(v)} a_{S,i} = 1$ for any $v \in C_S^*$. Further notice that free riders are the only possible edges incident to more than one vertex in C_S^* . Since $a_{S,i} = 0$ for any free rider *i* that is incident to other edges in *S*, we have

$$\sum_{i \in S} a_{S,i} = \sum_{v \in C_S^*} \sum_{i \in \delta_S(v)} a_{S,i} = |C_S^*| = \gamma(S).$$

We now check monotonicity. Let $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$ and let $i \in S$. We distinguish two cases of *i*. First assume that *i* is a free rider. We have $1 = a_{S,i} \ge a_{T,i} \ge 0$ if *i* is not incident to other edges in *S* and $a_{S,i} = a_{T,i} = 0$ otherwise. It follows that $a_{S,i} \ge a_{T,i}$ when *i* is a free rider. Now assume that *i* is not a free rider. Since $S \subseteq T$, we have $\lambda_S(i) \le \lambda_T(i)$, implying that

$$a_{S,i} = \frac{1}{\lambda_S(i)} \ge \frac{1}{\lambda_T(i)} = a_{T,i}$$

Therefore, $a_{S,i} \ge a_{T,i}$ follows in either case.

Our proof for Theorem 1 is constructive, which provides a PMAS for every population monotonic vertex cover game and motivates our subsequent work. The PMAS in our proof is based on a simple principle: for every vertex in a special minimum vertex cover, share the cost equally among non-free rider players covered by the vertex. By Lemma 7, every component of a (K_3, C_4, P_5) -free graph has at most two non-pendant vertices. Hence a (K_3, C_4, P_5) -free graph can be recognized efficiently, which implies that the population monotonicity of vertex cover games can be determined efficiently.

Corollary 2. The population monotonicity of vertex cover games can be determined in polynomial time and a PMAS, if exists, can be constructed accordingly.

4 A dual-based description of PMAS-es with free riders

LP(S):

Let G = (V, E) be a graph and $S \subseteq E$ be a nonempty edge set. Denote by LP(S) the following linear program defined on G[S].

$$\min \sum_{v \in V} y_v$$
s.t. $y_u + y_v \ge 1, \quad \forall \ i_{uv} \in S,$
(1)

$$y_v \ge 0, \quad \forall \ v \in V_S.$$
 (2)

The incidence vector of any minimum vertex cover in G[S] is a feasible solution of LP(S). It follows that $\tau(G[S])$ is lower bounded by the optimum of LP(S). König Theorem states that the gap between $\tau(G[S])$ and the optimum of LP(S) is closed when G[S] is bipartite. Denote by DP(S)the dual of LP(S). Hence $\tau(G[S])$ equals the optimum of DP(S) when G[S] is bipartite.

$$\max \sum_{i \in S} x_{S,i}$$

$$DP(S): \qquad \text{s.t.} \quad \sum_{i \in \delta_S(v)} x_{S,i} \le 1, \quad \forall \ v \in V_S,$$

$$x_{S,i} \ge 0, \quad \forall \ i \in S.$$
(4)

Further assume that G is (K_3, C_4, P_5) -free. Let $\Gamma_G = (N, \gamma)$ be the vertex cover game on G. For any $S \in 2^N \setminus \{\emptyset\}, \gamma(S)$ equals the optimum of DP(S). Moreover, we have the following observation for PMAS-es of vertex cover games.

Lemma 8. Let $\Gamma_G = (N, \gamma)$ be the vertex cover game on a graph G and **a** be a PMAS for Γ_G . Then \mathbf{a}_S is an optimal solution of DP(S) for any $S \in 2^N \setminus \{\emptyset\}$.

Proof. We first prove that a_S is feasible to DP(S). Since vertex cover games are monotonic, the nonnegativity of a_S follows from Lemma 1. It remains to show that $\sum_{i \in \delta_S(v)} a_{S,i} \leq 1$ for any $v \in V_S$. Let $v \in V_S$. Clearly, $\delta_S(v) \subseteq S$. By efficiency and monotonicity, we have

$$\sum_{i \in \delta_S(v)} a_{S,i} \le \sum_{i \in \delta_S(v)} a_{\delta_S(v),i} = \gamma(\delta_S(v)) = 1.$$

Hence \boldsymbol{a}_S is feasible to DP(S).

Now we prove that a_S is optimal to DP(S). By efficiency, we have

$$\sum_{i \in S} a_{S,i} = \gamma(S).$$

Theorem 1 implies that G[S] is (K_3, C_4, P_5) -free, which is a special bipartite graph. Hence $\gamma(S)$ equals the optimum of DP(S), implying that a_S is an optimal solution of DP(S).

Lemma 8 implies that for vertex cover games, allocations for each coalition in a PMAS are dual optimal solutions of corresponding minimum vertex cover problem. In the following, we present a more precise dual-based description for PMAS-es with free riders. Let C^* be the set of centers of stars and bases of pisceses in G and $C_S^* \subseteq C^*$ be a minimum vertex cover of G[S]. Let $\pi(S)$ be the linear system (3)-(4) and $\pi^*(S)$ be the linear system obtained from $\pi(S)$ by

- setting (3) to $\sum_{i \in \delta_S(v)} x_{S,i} = 1$ for any $v \in C_S^*$, and
- setting (4) to $x_{S,i} = 0$ for any free rider *i* incident to other edges in *S*.

Clearly, $\pi^*(S) \subseteq \pi(S)$. Moreover, $\pi^*(S)$ is a subset of optimal solutions of DP(S).

Lemma 9. Let G = (V, E) be a (K_3, C_4, P_5) -free graph. Then every vector in $\pi^*(S)$ is an optimal solutions of DP(S) for any nonempty $S \subseteq E$.

Proof. Let C^* be the set of centers of stars and bases of pisceses in G and $S \subseteq E$ be a nonempty edge set. Let $C_S^* \subseteq C^*$ be a minimum vertex cover of G[S] and $\mathbf{x}_S \in \pi^*(S)$. Since $\pi^*(S) \subseteq \pi(S)$, we have $\mathbf{x}_S \in \pi(S)$. It remains to prove the optimality of \mathbf{x}_S . Lemma 7 implies that every component of G[S] is either a star or a pisces. Hence free riders in S are the only possible edges incident to more than one vertex in C_S^* . Since $\mathbf{x}_{S,i} = 0$ for any free rider i incident to other edges in S, it follows that

$$\sum_{i \in S} x_{S,i} = \sum_{v \in C_S^*} \sum_{i \in \delta_S(v)} x_{S,i} = |C_S^*| = \gamma(S).$$

Therefore, \boldsymbol{x}_S is an optimal solution of DP(S).

Now we strengthen Lemma 8 and present a dual-based description of PMAS-es with free riders.

Theorem 3. Let $\Gamma_G = (N, \gamma)$ be the vertex cover game on a graph G and \mathbf{a} be a PMAS for Γ_G . Then $\mathbf{a}_S \in \pi^*(S)$ for any $S \in 2^N \setminus \{\emptyset\}$.

Proof. Let a be a PMAS for Γ_G . Theorem 1 and Lemma 7 imply that every component of G is either a star or a pisces. Let C^* be the set of centers of stars and bases of pisceses in G. Let $S \in 2^N \setminus \{\emptyset\}$ and $C_S^* \subseteq C^*$ be a minimum vertex cover of G[S]. Lemma 8 implies that a_S is an optimal solution of DP(S). Notice that any minimum vertex cover for a graph is a union of minimum vertex covers for every component. Hence we assume that G[S] is connected. We show that a_S satisfies all equality constraints in $\pi^*(S)$ by distinguishing three cases. Case 1: G[S] is a star without free riders. Let v^* be the center of G[S]. Notice that $\delta_S(v^*) = S$ and $C_S^* = \{v^*\}$. By efficiency, we have

$$\sum_{i \in \delta_S(v^*)} a_{S,i} = \sum_{i \in S} a_{S,i} = \gamma(S) = 1.$$

Case 2: G[S] is a pisces. Let i^* be the free rider in S and v_1^*, v_2^* be the endpoints of i^* . Hence $C_S^* = \{v_1^*, v_2^*\}, \, \delta_S(v_1^*) \cup \delta_S(v_2^*) = S$ and $\delta_S(v_1^*) \cap \delta_S(v_2^*) = \{i^*\}$. By efficiency and monotonicity, we have

$$\begin{aligned} 2 &= \gamma(S) = \sum_{i \in S} a_{S,i} \\ &\leq \sum_{i \in \delta_S(v_1^*)} a_{\delta_S(v_1^*),i} + \sum_{i \in \delta_S(v_2^*)} a_{\delta_S(v_2^*),i} - a_{\delta_S(v_k^*),i^*} \\ &= \gamma(\delta_S(v_1^*)) + \gamma(\delta_S(v_2^*)) - a_{\delta_S(v_k^*),i^*} \\ &= 2 - a_{\delta_S(v_k^*),i^*} \end{aligned}$$

for k = 1, 2. By monotonicity, we have

$$a_{S,i^*} = a_{\delta_S(v_k^*), i^*} = 0$$

and hence

$$\sum_{i \in \delta_S(v_k^*)} a_{S,i} = \sum_{i \in \delta_S(v_k^*)} a_{\delta_S(v_k^*),i} = 1$$

for k = 1, 2.

Case 3: G[S] is a star with a free rider. Let v_1^* be the center of G[S]. Notice that $\delta_S(v_1^*) = S$ and $C_S^* = \{v_1^*\}$. Since G[S] contains a free rider, there exists $T \subseteq N$ such that G[T] is a pisces and $\delta_S(v_1^*) = \delta_T(v_1^*)$. Let i^* be the free rider in $S \subsetneq T$ and v_1^*, v_2^* be the endpoints of i^* . Let $C_T^* = \{v_1^*, v_2^*\}$. Clearly, $C_T^* \subseteq C^*$ is a minimum vertex cover of G[T]. By monotonicity, we have

$$\sum_{i \in \delta_S(v_1^*)} a_{S,i} = \sum_{i \in \delta_T(v_1^*)} a_{T,i} = 1$$

and hence

$$a_{S,i^*} = a_{T,i^*} = 0.$$

5 Integral PMAS-es and stable matchings

In this section, we first show that every integral PMAS for a vertex cover game Γ_G is an extreme point of $\mathcal{P}(\Gamma_G)$, then use stable matchings to characterize integral PMAS-es for Γ_G , and finally conclude that integral PMAS-es for Γ_G can be enumerated by employing Gale-Shapley algorithm.

Theorem 4. Let $\Gamma_G = (N, \gamma)$ be the vertex cover game on a graph G with $\mathcal{P}(\Gamma_G) \neq \emptyset$. Then every integral PMAS for Γ_G is an extreme point of $\mathcal{P}(\Gamma_G)$.

Proof. Let \boldsymbol{a} be an integral PMAS for Γ_G . Suppose $\boldsymbol{a} = \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$, where $\boldsymbol{b}, \boldsymbol{c} \in \mathcal{P}(\Gamma_G)$. Let $S \in 2^N \setminus \{\emptyset\}$. It follows that $\boldsymbol{a}_S = \frac{1}{2}\boldsymbol{b}_S + \frac{1}{2}\boldsymbol{c}_S$. By Lemma 8, $\boldsymbol{a}_S, \boldsymbol{b}_S$ and \boldsymbol{c}_S are all optimal solutions of DP(S), implying that $0 \leq a_{S,i} \leq 1$, $0 \leq b_{S,i} \leq 1$, and $0 \leq c_{S,i} \leq 1$ for any $i \in S$. Since \boldsymbol{a}_S is integral, we have $a_{S,i} = b_{S,i} = c_{S,i}$ for any $i \in S$. Therefore, \boldsymbol{a} is an extreme point of $\mathcal{P}(\Gamma_G)$. \Box

Theorem 4 states that every integral PMAS for a vertex cover game Γ_G is an extreme point of $\mathcal{P}(\Gamma_G)$. Unfortunately, not all extreme points of $\mathcal{P}(\Gamma_G)$ are integral even when G is a star. Consider the vertex cover game $\Gamma_{K_{1,n}} = (N, \gamma)$ where $n \geq 4$. Notice that $\gamma(S) = 1$ for any $S \in 2^N \setminus \{\emptyset\}$. Thus $\Gamma_{K_{1,n}}$ falls into the scope of *unit games* investigated by Hamers et al. [7], where they showed that $\mathcal{P}(\Gamma_{K_{1,n}})$ has more than $(n-2) \cdot n!$ non-integral extreme points. Hence for $\mathcal{P}(\Gamma_G)$, instead of all extreme points, we focus on integral extreme points.

Before proceeding, we introduce the notion of preference systems and stable matchings. Let G = (V, E) be a graph. For any $v \in V$, let \prec_v be a strict linear order on edges in $\delta(v)$. We call \prec_v the preference of v. For any $i, j \in \delta(v)$, we say that i dominates j (at v) if $i \prec_v j$. We use \prec to denote the set of preferences \prec_v for any $v \in V$, and call the ordered pair (G, \prec) a preference system. In particular, (G, \prec) is bipartite if G is bipartite. A stable matching in (G, \prec) is a matching M of G such that every edge in $E \setminus M$ is dominated by an edge in M. For any $S \subseteq E$, we use \prec_S to denote the restriction of \prec to S, and hence $(G[S], \prec_S)$ is also a preference system. Gale and Shapley [4] proved that every bipartite preference system admits a stable matching by providing an efficient algorithm, namely Gale-Shapley algorithm, for computing stable matchings. Now we are ready to characterize integral PMAS-es for vertex cover games with stable matchings.

Theorem 5. Let $\Gamma_G = (N, \gamma)$ be the vertex cover game on a graph G with $\mathcal{P}(\Gamma_G) \neq \emptyset$. Then the following statements are equivalent.

- (i) **a** is an integral PMAS for Γ_G .
- (ii) There is a preference system (G, \prec) , where every free rider has the lowest rank, such that \mathbf{a}_S is the incidence vector of a stable matching in $(G[S], \prec_S)$ for any $S \in 2^N \setminus \{\emptyset\}$.

Proof. Theorem 1 and Lemma 7 imply that every component of G is either a star or a pisces.

 $(i) \Rightarrow (ii)$. Let \boldsymbol{a} be an integral PMAS for Γ_G . We first define a preference system (G, \prec) from \boldsymbol{a} . Let C^* be the set of centers of stars and bases of pisceses in G. To define a preference system (G, \prec) , it suffices to define a preference \prec_v for any $v \in C^*$. Let v^* be a vertex in C^* . For any nonempty set $S \subseteq \delta(v^*)$, we have $\sum_{i \in S} a_{S,i} = \gamma(S) = 1$. We define a preference \prec_{v^*} for v^* from \boldsymbol{a} as follows. Start with $S = \delta(v^*)$. Let $i^* \in S$ be the edge with $a_{S,i^*} = 1$. Define partial orders of \prec_{v^*} by

$$i^* \prec_{v^*} j, \ \forall \ j \in S \setminus \{i^*\}.$$

Update S with $S \setminus \{i^*\}$ and repeat the process above until $S = \emptyset$. We claim that when S becomes empty, \prec_{v^*} is a well-defined strict linear order on $\delta(v^*)$. Indeed, consider any two edges i, j in $\delta(v^*)$. Clearly, $G[\{i, j\}]$ is a star. Since $a_{\{i, j\}, i} + a_{\{i, j\}, j} = \gamma(\{i, j\}) = 1$, we may assume that $a_{\{i, j\}, i} = 1$ and $a_{\{i, j\}, j} = 0$. For any $S \subseteq N$ with $\{i, j\} \subseteq S$, the monotonicity implies that $a_{S, j} = a_{\{i, j\}, j} = 0$. Hence $i \prec_{v^*} j$ always holds. Thus **a** determines a unique preference system (G, \prec) .

Now we show that every free rider in (G, \prec) has the lowest rank in any preference. Let i^* be a free rider in G and v_1^*, v_2^* be the endpoints of i^* . Clearly, $v_1^*, v_2^* \in C^*$. By Theorem 3, $a_{S,i^*} = 0$ for any S containing other edges incident to i^* . Thus i^* has the lowest rank in both $\prec_{v_1^*}$ and $\prec_{v_2^*}$.

Therefore, \boldsymbol{a} defines a unique preference system (G, \prec) where every free rider has the lowest rank. Since every component of G is either a star or a pisces, it is easy to see that $(G[S], \prec_S)$ has a unique stable matching M_S with incidence vector \boldsymbol{a}_S for any $S \in 2^N \setminus \{\emptyset\}$.

 $(ii) \Rightarrow (i)$. Let (G, \prec) be a preference system where every free rider has the lowest rank. Since every component of G is either a star or a pisces, $(G[S], \prec_S)$ has a unique stable matching M_S for any $S \in 2^N \setminus \{\emptyset\}$. We show that $\boldsymbol{a} = (a_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ is an integral PMAS for Γ_G , where $\boldsymbol{a}_S = (a_{S,i})_{i \in S}$ is the incidence vector of M_S .

We first check efficiency. Let $S \in 2^N \setminus \{\emptyset\}$ and M_S be the unique stable matching of $(G[S], \prec_S)$. Since every free rider has the lowest rank in any preference, M_S is also a maximum matching of G[S]. Then we have

$$\sum_{i \in S} a_{S,i} = |M_S| = \nu(G[S]) = \tau(G[S]) = \gamma(S).$$

Hence the efficiency follows.

We now prove monotonicity. Let $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$. Let M_S and M_T be the unique stable matching of $(G[S], \prec_S)$ and $(G[T], \prec_T)$, respectively. Let $i \in S$. If $i \in M_T$, then we have $i \in M_S$, implying that

$$a_{S,i} = a_{T,i} = 1.$$

Otherwise, we have

$$a_{S,i} \ge a_{T,i} = 0.$$

In either case we have $a_{S,i} \ge a_{T,i}$. Hence the monotonicity follows.

Theorem 5 reveals the principle behind integral PMAS-es for vertex cover games: the new player always takes the full cost of the vertex covering it in a minimum vertex cover. More specifically, consider the following coalition growing process. The coalition starts with an empty set. A new player joins the coalition by deciding whether or not to connect to a vertex in the current coalition. But every new player is only allowed to connect to a vertex of an existing minimum vertex cover or a pendant vertex of an existing star in the current coalition. However, the new player always has to pay for the full cost of the vertex covering it, no matter whether or not the vertex belongs to a minimum vertex cover for the coalition before the new player joins. Theorem 5 also suggests that for vertex cover games, there is a one-to-one correspondence between integral PMAS-es and preference systems such that every free rider has the lowest rank. Since such preference system defined on a (K_3, C_4, P_5) -free graph has a unique stable matching, we have the following corollary.

Corollary 6. Integral PMAS-es for vertex cover games can be enumerated by employing Gale-Shapely algorithm.

6 Concluding remarks

This paper investigates PMAS-es for vertex cover games. We show that the population monotonicity can be determined in polynomial time and a PMAS, if exists, can be constructed accordingly. We also show that integral PMAS-es can be characterized with stable matchings and be enumerated by employing Gale-Shapley algorithm. However, neither computing a PMAS nor determining whether a given vector is a PMAS can be done efficiently, as both problems have exponential size. Nevertheless, integral allocations for each coalition in a PMAS can be computed efficiently by Gale-Shapley algorithm.

Acknowledgments

We would like to thank Xin Chen for the valuable help at the early stage of this paper. We would also like to thank Bo Li for bringing up the notion of free riders and thank Dachuan Xu and Donglei Du for their helpful discussion, which greatly improved the presentation of this paper.

References

- X. Chen, X. Gao, Z. Hu, and Q. Wang. Population monotonicity in newsvendor games. Management Science, 65(5):2142–2160, 2019.
- [2] X. Deng, T. Ibaraki, and H. Nagamochi. Algorithmic aspects of the core of combinatorial optimization games. *Mathematics of Operations Research*, 24(3):751–766, 1999.
- [3] X. Deng, T. Ibaraki, H. Nagamochi, and W. Zang. Totally balanced combinatorial optimization games. *Mathematical Programming*, 87(3):441–452, 2000.
- [4] D. Gale and L. S. Shapley. College admissions and the stability of marriage. American Mathematical Monthly, 69(1):9–15, 1962.
- [5] S. Grahn and M. Voorneveld. Population monotonic allocation schemes in bankruptcy games. Annals of Operations Research, 109(1-4):317–329, 2002.
- [6] V. Gusev. The vertex cover game: application to transport networks. Omega, article ID 102102, 10 pages, 2019.
- [7] H. Hamers, S. Miquel, and H. Norde. Monotonic stable solutions for minimum coloring games. Mathematical Programming, 145(1-2):509-529, 2014.
- [8] H. Norde, S. Moretti, and S. Tijs. Minimum cost spanning tree games and population monotonic allocation schemes. *European Journal of Operational Research*, 154(1):84–97, 2004.
- [9] Y. Okamoto. Submodularity of some classes of the combinatorial optimization games. *Mathematical Methods of Operations Research*, 58(1):131–139, 2003.
- [10] A. Schrijver. Combinatorial Optimization: Polyhedra and Efficiency. Springer, 2003.
- [11] Y. Sprumont. Population monotonic allocation schemes for cooperative games with transferable utility. *Games and Economic Behavior*, 2(4):378–394, 1990.