# Direct Product Primality Testing of Graphs is GI-hard 

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#### Abstract

We investigate the computational complexity of the graph primality testing problem with respect to the direct product (also known as Kronecker, cardinal or tensor product). In [1] Imrich proves that both primality testing and a unique prime factorization can be determined in polynomial time for (finite) connected and nonbipartite graphs. The author states as an open problem how results on the direct product of nonbipartite, connected graphs extend to bipartite connected graphs and to disconnected ones. In this paper we partially answer this question by proving that the graph isomorphism problem is polynomial-time many-one reducible to the graph compositeness testing problem (the complement of the graph primality testing problem). As a consequence of this result, we prove that the graph isomorphism problem is polynomial-time Turing reducible to the primality testing problem. Our results show that connectedness plays a crucial role in determining the computational complexity of the graph primality testing problem.


Keywords: Kronecker product, graphs factorization, graphs isomorphism, GI complexity

## 1. Introduction

Factorization is a fundamental task in mathematics and in many other disciplines including computer science, physics and engineering. The notion of product among mathematical objects not only enables the creation of new objects from smaller ones, but also naturally addresses the more complex task of decomposing an object as the product of simpler components. Factoring a mathematical object is therefore one of the the main methods for deeply understanding its structure.

Integer factorization is by far the most widely known and studied factorization problem; however, many other types of mathematical objects have been extensively studied in order to understand if and how they can be factored. Specifically, graph factorization with respect to several notions of product has been thoroughly investigated both from the theoretical and from the practical point of view.

In this paper we investigate the computational complexity of graph factorization with respect to the direct product (see Definition 2.2) which is one of the most widely studied graph product. Some authors refer to the direct product as the Kronecker, tensor or cardinal product. We will

[^0]name it direct product and we will denote it by the operator $\times$ according to the notation used in the recent book by Hammack, Imrich and Klavžar [2].

Direct product is one of the three products (the other two being the Cartesian and Strong products) that satisfies the following fundamental algebraic properties ( $\simeq$ stands for "isomorphic to"):

1. Commutativity: $G_{1} \times G_{2} \simeq G_{2} \times G_{1}$
2. Associativity: $G_{1} \times\left(G_{2} \times G_{3}\right) \simeq\left(G_{1} \times G_{2}\right) \times G_{3}$
3. Projections from a product to its factors are weak homomorphisms

A fourth product have been considered in the literature, namely the lexicographic product. Lexicographic product does not satisfy properties 1 and 3.

We both consider the primality testing problem and the factorization problem. Informally, primality testing is a decision problem that, given a graph $G$, answers the question: "is $G$ the product of smaller, nontrivial graphs?". Graph factorization aims at decomposing $G$ into the product of smaller nontrivial graphs (more formal definitions will be given in the next section). Although factorization of general with respect to the direct product is not unique, Imrich [1] proved that if a graph is connected and nonbipartite, then its factorization with respect to the direct product is unique and can be computed in polynomial time. In this paper we address the following question posed by Imrich at the end of his paper.

## How do results on the cardinal product of nonbipartite, connected graphs extend to bipartite connected graphs and to disconnected ones?

We prove (Theorem 4.11) that the graph isomorphism problem reduces to the problem of testing the compositeness of possibly unconnected, nonbipartite graphs. Since the reduction we use is a polynomial time many-one reduction, we show (Corollary 4.12) that testing the primality of a graph is $G I$-hard. In other words, we prove that testing the primality of a graph in polynomial time would provide a polynomial time algorithm for testing graph isomorphism, which is widely considered to be not feasible, although no formal proof exists. It remains an open question whether testing primality of bipartite, connected graphs can be done in polynomial time

This paper is organized as follows. In section 2 we introduce the notation and definition of terms used in the rest of this work. In section 3 we review the relevant literature related to the graph factorization problem. Section 4 presents the main result of this paper. Finally, conclusions and future research directions are discussed in section 5 .

## 2. Notation and Basic Definitions

In this section we give basic notation and definitions that will be used throughout the paper. An undirected graph $G=(V, E)$ is described as a finite set $V$ of nodes $V=\left\{v_{1}, \ldots, v_{n}\right\}$ and a finite set of edges $E \subseteq V \times V$, where an edge $e \in E$ is an unordered pair of nodes $e=\{u, v\}, u, v \in V$. Given a graph $G, V(G)$ and $E(G)$ denote the set of nodes and edges of $G$, respectively. We denote by $G_{1} \cup G_{2}$ the disjoint union of graphs $G_{1}$ and $G_{2}$, i.e., the graph with node set $V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and edge set $E\left(G_{1}\right) \cup E\left(G_{2}\right)$. Disjoint means that $V\left(G_{1}\right)$ and $V\left(G_{2}\right)$ satisfy $V\left(G_{1}\right) \cap V\left(G_{2}\right)=\emptyset$.

The set of edges of a graph $G$ can be represented also as an adjacency matrix M. If $G$ has $n$ nodes, $\mathbf{M}$ is a $n \times n$ binary matrix, such that $\mathbf{M}_{i j}=1$ if and only if $\left\{v_{i}, v_{j}\right\} \in E$. The adjacency
matrix for undirected graphs is symmetric, since every edge $\left\{v_{i}, v_{j}\right\}$ can also be written as $\left\{v_{j}, v_{i}\right\}$. As a shorthand notation, we denote with $\operatorname{Adj}(G)$ the adjacency matrix of graph $G$.

We use the symbol $\Gamma$ to denote the set of finite, undirected graphs where no self-loops are allowed. The symbol $\Gamma_{0}$ denotes the set of finite, undirected graphs where self-loops are allowed; a self-loop is an edge of the form $\{v, v\}$, for some $v \in V(G)$.

Four types of graph products have been investigated in the literature: Cartesian product, Direct product, Strong product and Lexicographic product. In all cases, the product of two graphs $G_{1}, G_{2}$ is a new graph $G$ whose set of nodes is the Cartesian product of $V\left(G_{1}\right)$ and $V\left(G_{2}\right)$ :

$$
V(G)=V\left(G_{1}\right) \times V\left(G_{2}\right)=\left\{\{u, v\} \mid u \in V\left(G_{1}\right) \wedge v \in V\left(G_{2}\right)\right\}
$$

The edge set $E(G)$ is defined according to the notion of graph product as follows.
Definition 2.1 (Cartesian product). The Cartesian product of two graphs $G_{1}, G_{2}$ is denoted as $G=G_{1} \square G_{2}$, where $V(G)=V\left(G_{1}\right) \times V\left(G_{2}\right)$ and

$$
E(G)=\left\{\left\{\{x, y\},\left\{x^{\prime}, y^{\prime}\right\}\right\} \mid\left(x=x^{\prime} \wedge\left\{y, y^{\prime}\right\} \in E\left(G_{2}\right)\right) \vee\left(\left\{x, x^{\prime}\right\} \in E\left(G_{1}\right) \wedge y=y^{\prime}\right)\right\}
$$

Definition 2.2 (Direct product). The direct product of two graphs $G_{1}, G_{2}$ is denoted as $G=$ $G_{1} \times G_{2}$, where $V(G)=V\left(G_{1}\right) \times V\left(G_{2}\right)$ and

$$
E(G)=\left\{\left\{\{x, y\},\left\{x^{\prime}, y^{\prime}\right\}\right\} \mid\left\{x, x^{\prime}\right\} \in E\left(G_{1}\right) \wedge\left\{y, y^{\prime}\right\} \in E\left(G_{2}\right)\right\}
$$

The direct product is also known as Kronecker or cardinal product.
Definition 2.3 (Strong product). The strong product of two graphs $G_{1}, G_{2}$ is denoted as $G=$ $G_{1} \boxtimes G_{2}$, where $V(G)=V\left(G_{1}\right) \times V\left(G_{2}\right)$ and

$$
E(G)=E\left(G_{1} \square G_{2}\right) \cup E\left(G_{1} \times G_{2}\right)
$$

Definition 2.4 (Lexicographic product). The lexicographic product of two graphs $G_{1}, G_{2}$ is denoted as $G=G_{1} \circ G_{2}$, where $V(G)=V\left(G_{1}\right) \times V\left(G_{2}\right)$ and

$$
E(G)=\left\{\left\{\{x, y\},\left\{x^{\prime}, y^{\prime}\right\}\right\} \mid\left\{x, x^{\prime}\right\} \in E\left(G_{1}\right) \vee\left(x=x^{\prime} \wedge\left\{y, y^{\prime}\right\} \in E\left(G_{2}\right)\right)\right\}
$$

Figure 1 shows the Cartesian, direct, strong and lexicographic product of two graphs $G_{1}, G_{2}$. A nontrivial graph $G \in \Gamma_{0}$ is a graph with more than one node $(|V(G)|>1)$. We say that a graph $G$ is prime according to a given graph product $\odot$ if $G$ is nontrivial and $G=G_{1} \odot G_{2}$ implies that either $G_{1}$ or $G_{2}$ are trivial, i.e., one of them has exactly one node.

The direct product of $G_{1}, G_{2}$ can be specified in terms of the Kronecker product of their adjacency matrices. Given a $n \times m$ matrix $\mathbf{A}$ and a $p \times q$ matrix $\mathbf{B}$, the Kronecker product $\mathbf{C}=\mathbf{A} \otimes \mathbf{B}$ is a $n p \times m q$ matrix obtained from the scalar multiplication between each element of $\mathbf{A}$ and the whole matrix $\mathbf{B}$ :

$$
\mathbf{C}=\mathbf{A} \otimes \mathbf{B}=\left(\begin{array}{cccc}
a_{11} \mathbf{B} & a_{12} \mathbf{B} & \ldots & a_{1 m} \mathbf{B} \\
a_{21} \mathbf{B} & a_{22} \mathbf{B} & \ldots & a_{2 m} \mathbf{B} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n 1} \mathbf{B} & a_{n 2} \mathbf{B} & \ldots & a_{n m} \mathbf{B}
\end{array}\right)
$$



Figure 1: Example of the different types of graph products.

It can be easily shown [2] that the adjacency matrix $\operatorname{Adj}(G)$ of the graph $G=G_{1} \times G_{2}$ is strongly related (see Observation 4.5) to the adjacency matrices $\operatorname{Adj}\left(G_{1}\right)$ and $\operatorname{Adj}\left(G_{2}\right)$.

We finally recall the definition of many-one reducibility and Turing reducibility. Given two sets $S_{1}, S_{2} \subseteq \mathbb{N}$, we say that $S_{1}$ is many-one reducible to $S_{2}$, if there exists a total computable function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that [3]

$$
n \in S_{1} \Longleftrightarrow f(n) \in S_{2}
$$

A polynomial time many-one reduction (denoted by $\leq_{M}$ ) is a many-one reduction with the additional constraint that $f$ is computable in polynomial time.

Turing reducibility is a weaker form of many-one reducibility. Informally, $S_{1}$ is Turing reducible to $S_{2}$ if there exists an oracle for testing membership in $S_{1}$ relying on another oracle for testing membership in $S_{2}$ [4]. In other words, $S_{1}$ is Turing reducible to $S_{2}$ if it is possible to answer the question "is $n \in S_{1}$ " given the existence of an effective procedure for answering the question "is $m \in S_{2}$ " for any $m \in \mathbb{N}$ [3].

A polynomial time Turing reduction (denoted by $\leq_{T}$ ) is a Turing reduction satisfying the following two additional constraints:

1. the oracle for testing membership in $S_{1}$ makes at most a polynomial number of calls to the oracle for testing membership in $S_{2}$ and
2. the overall computational cost of the oracle for testing membership in $S_{1}$ (excluding the calls to the oracle for testing membership in $S_{2}$ ) is polynomially bounded.
As a final consideration, throughout the paper we intend graphs to be finite and undirected, unless otherwise specified. Table 1 summarizes the notation used in this paper.

| Symbol | Description |
| :--- | :--- |
| $\Gamma$ | The set of finite, undirected graphs, without self-loops |
| $\Gamma_{0}$ | The set of finite, undirected graphs, self-loops allowed |
| $\mathbf{A d j}(G)$ | The adjacency matrix of a graph $G$ |
| $\mathbf{I}_{n}$ | The $n \times n$ identity matrix |
| $\mathbf{0}_{n}$ | The $n \times n$ zero matrix |
| $\times$ | The direct graph product operator |
| $\square$ | The Cartesian graph product operator |
| $\boxtimes$ | The strong graph product operator |
| $\circ$ | The lexicographic graph product operator |
| $\otimes$ | The Kronecker matrix product operator |
| $\simeq$ | The graphs isomorphism operator |
| $\leq_{\mathrm{M}}$ | Polynomial many-one reducibility |
| $\leq_{\mathrm{T}}$ | Polynomial Turing reducibility |
| $\cup$ | Disjoint union of graphs |

Table 1: Basic notation.

## 3. Related works

In this section we list the main results on graph factorization that are strictly related to the work presented in this paper. The interested reader may find a comprehensive review of the theory of graph factorizations in the recent book by Hammack, Imrich and Klavžar [2].

Direct product. Prime factorization of connected, nonbipartite graphs in $\Gamma_{0}$ is unique up to isomorphism and the order of the factors, and can be computed in polynomial time [1].

Cartesian product. Prime factorization of connected graphs is unique up to isomorphism and the order of the factors [5, 6]. Prime factorization is not unique in the class of possibly disconnected simple graphs. Following Sabidussi's approach, Feigenbaum et al. [7] derived a polynomial-time algorithm that computes the prime factors of a connected graph. A different polyonimial-time algorithm for connected graphs has been independently discovered by Winkler [8].

Strong product. Prime factorization of connected graphs is unique up to reorderings and isomorphisms of factors and it can be computed in polynomial time [1].

Lexicographic product. Determining whether a connected graph is prime is at least as difficult as the graph isomorphism problem [9].

An interesting observation relating graph factorization and graph isomorphism problem can be found at the end of page of [2, p. 229]. The authors claim that, if $X$ is the disjoint union of graphs $G$ and $H$, then $G \simeq H$ if and only if $X=D_{2} \square G=D_{2} \boxtimes G=D_{2} \circ G$ where $D_{2}$ denotes the graph with two nodes and two self-loops. They conclude that "testing whether a disconnected graph is decomposable with respect to any of these three products is at least as hard as the graph isomorphism problem". They do not give a formal proof of their claim and in particular they do not explain how they get rid of the case in which $X$ is the disjoint union of two non isomorphic graphs $G_{1}$ and $G_{2}$ and, at the same time, $X$ admits as a factor a graph with two nodes

| Graph type | Product type |  |  |  |
| ---: | :---: | :---: | :---: | :---: |
|  | Direct | Cartesian | Strong | Lexicographic |
| Connected, nonbipartite | $\mathbf{P}[1]$ | $\mathrm{P}[7,8]$ | $\mathrm{P}[10]$ | $\bullet$ |
| Connected | $\bullet$ | $\mathbf{P}[7,8]$ | $\mathbf{P}[10]$ | GI-Hard [9] |
| Unconnected, nonbipartite | GI-Hard (our results) | $\bullet$ | $\bullet$ | $\bullet$ |
| Nonbipartite | GI-Hard (our results) | $\bullet$ | $\bullet$ | $\bullet$ |

Table 2: Complexity of the graph factorization problem for different types of graphs considered in the literature (connected, unconnected, nonbipartite) and different notions of graph product (direct, cartesian, strong and lexicographic product); $\mathbf{P}$ stands for polynomially time solvable. Table cells reported in light gray can be easily inferred from another cells in the same column depending on the relation between the corresponding classes of graphs. For instance, a polynomial-time solvable problem for connected graphs is polynomial-time solvable within a restricted class of graphs (e.g., connected and nonbipartite). Dots denote the cases that, to our knowledge, have not yet been explored.
that is not isomorphic to $D_{2}$. In that case the decomposability test would lead to erroneously declare $G_{1} \simeq G_{2}$. Moreover, if $X$ admits more than one factorization, also computing a single factorization could not be enough for testing isomorphism.

In Section 4 we show (see Figures 2 and 3) that when the direct product is used, both these cases may occur.

## 4. Main Results

In this section we prove that testing whether two graphs $G_{1}, G_{2}$ are isomorphic is not harder than testing whether an undirected graph $G \in \Gamma_{0}$ is $\times$-composite, i.e., $G$ admits nontrivial factors with respect to the direct product decomposition. More formally, we show that graph isomorphism problem is polynomially many-one reducible to the problem of testing $\times$-compositeness of graphs.

Before starting, we formally define the following three decision problems in terms of their admissible inputs and related outputs.

Definition $4.1(\mathbf{G I}[S])$. Let $S$ be any set of graphs. $\mathbf{G I}[S]$ is defined as follows.
Input: $G_{1}, G_{2} \in S$
Output: YEs if $G_{1}$ is isomorphic to $G_{2}$, no otherwise.
Definition 4.2 (Primality[S]). Let $S$ be any set of graphs. Primality $[S]$ is defined as follows.
Input: $G \in S$
Output: yes if $G$ is prime with respect to the direct product, no otherwise.
Definition 4.3 (Compositeness[S]). Let $S$ be any set of graphs. Compositeness[S] is defined as follows.
Input: $G \in S$
Output: YEs if $G$ is composite with respect to the direct product, no otherwise.
Definition 4.4 (GI-hard problem). A decisional problem P is GI-hard if and only if

$$
\text { GI[general graphs] } \leq_{T} P
$$

It is easy to observe that GI[general graphs] $\leq_{T} \mathbf{G I}$ [connected graphs] so that, by transitivity, we can conclude that a decisional problem $P$ is $G I$-hard if and only if

$$
\mathbf{G I}[\text { connected graphs }] \leq_{T} P
$$

The following observation highlights the strong relation between the direct product of graphs and the Kronecker product of their adjacency matrices.

Observation 4.5. Let $G_{1}$ and $G_{2}$ be graphs. Then

$$
\operatorname{Adj}\left(G_{1} \times G_{2}\right)=\mathbf{P}^{\boldsymbol{\top}}\left(\mathbf{A d j}\left(G_{1}\right) \otimes \operatorname{Adj}\left(G_{2}\right)\right) \mathbf{P}
$$

where $\mathbf{P}$ is a suitable permutation matrix.
In the following lemma we prove that two graphs $G_{1}$ and $G_{2}$ are isomorphic if and only if there exists a permutation matrix $\mathbf{P}$ that transforms the adjacency matrix of the disjoint union of $G_{1}$ and $G_{2}$ into the Kronecker product of the identity matrix $\mathbf{I}_{2}$ and a suitable binary matrix $\mathbf{B}$.

Lemma 4.6. Let $G_{1}, G_{2}$ be undirected, connected graphs with n nodes. Let $\mathbf{M}_{1}=\boldsymbol{\operatorname { A d j }}\left(G_{1}\right)$ and $\mathbf{M}_{2}=\operatorname{Adj}\left(G_{2}\right)$. Let $\mathbf{M}$ denote the adjacency matrix of the disjoint union $G=G_{1} \cup G_{2}$. Without loss of generality, we may write $\mathbf{M}$ as

$$
\mathbf{M}=\left(\begin{array}{cc}
\mathbf{M}_{1} & \mathbf{0}_{n}  \tag{1}\\
\mathbf{0}_{n} & \mathbf{M}_{2}
\end{array}\right)
$$

Then, $G_{1}$ and $G_{2}$ are isomorphic $\left(G_{1} \simeq G_{2}\right)$ if and only if there exists a $2 n \times 2 n$ permutation matrix $\mathbf{P}$ and a $n \times n$ binary matrix $\mathbf{B}$ such that

$$
\begin{equation*}
\mathbf{P}^{\top} \mathbf{M P}=\mathbf{I}_{2} \otimes \mathbf{B} \tag{2}
\end{equation*}
$$

where $I_{2}$ denotes the $2 \times 2$ identity matrix.
Proof. ( $\Longrightarrow$ ) We first prove that if $G_{1}$ and $G_{2}$ are isomorphic, then Eq. (2) holds. If $G_{1} \simeq G_{2}$ then there exists a $n \times n$ permutation matrix $\mathbf{Q}$ that transforms the adjacency matrix $\mathbf{M}_{2}$ of $G_{2}$ in the adjacency matrix $\mathbf{M}_{1}$ of $G_{1}$ :

$$
\begin{equation*}
\mathbf{M}_{1}=\mathbf{Q}^{\top} \mathbf{M}_{2} \mathbf{Q} \tag{3}
\end{equation*}
$$

Let us define

$$
\mathbf{P}=\left(\begin{array}{ll}
\mathbf{I}_{n} & \mathbf{0}_{n}  \tag{4}\\
\mathbf{0}_{n} & \mathbf{Q}
\end{array}\right)
$$

It follows that

$$
\begin{array}{rlr}
\mathbf{P}^{\top} \mathbf{M P} & =\left(\begin{array}{cc}
\mathbf{I}_{n} & \mathbf{0}_{n} \\
\mathbf{0}_{n} & \mathbf{Q}^{\top}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{M}_{1} & \mathbf{0}_{n} \\
\mathbf{0}_{n} & \mathbf{M}_{2}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{I}_{n} & \mathbf{0}_{n} \\
\mathbf{0}_{n} & \mathbf{Q}
\end{array}\right) \quad \text { by (4) and (1) } \\
& =\left(\begin{array}{cc}
\mathbf{M}_{1} & \mathbf{0}_{n} \\
\mathbf{0}_{n} & \mathbf{Q}^{\top} \mathbf{M}_{2} \mathbf{Q}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\mathbf{M}_{1} & \mathbf{0}_{n} \\
\mathbf{0}_{n} & \mathbf{M}_{1}
\end{array}\right) \\
& =\mathbf{I}_{2} \otimes \mathbf{M}_{1} & \text { by (3) } \\
\end{array}
$$

$(\Longleftarrow)$ We prove that if Eq. (2) holds, then $G_{1}, G_{2}$ are isomorphic.
Observe that the transformation $\mathbf{P}^{\top} \mathbf{M P}$ consists of relabeling the nodes of $G$ according to the permutation matrix $\mathbf{P}$. Let us define this relabeling as the bijective function $\pi:\{1,2, \ldots, 2 n\} \rightarrow$ $\{1,2, \ldots, 2 n\}$. Therefore, $\pi(i)=j$ if and only if the node $i$ of $G$ is relabeled as $j$. Note also that $\mathbf{P}^{\boldsymbol{T}} \mathbf{M P}$ is symmetric, since it represents the adjacency matrix of an undirected graph.
Since we are assuming that $G_{1}$ is connected, then there always exists a path from node $i$ to node $j$, $1 \leq i, j \leq n$. Thus, should the permutation contain a mapping such that $\pi(i) \leq n$ and $\pi(j)>n$, the relabeled adjacency matrix $\mathbf{P}^{\boldsymbol{\top}} \mathbf{M P}$ would contain at least one 1 in the upper-right quadrant and (by symmetry) in the lower-left one. However, this contradicts the hypothesis (2), since the upper right and lower left quadrants of $\mathbf{I}_{2} \otimes \mathbf{B}$ are the zero matrix $\mathbf{0}_{n}$. The same considerations apply to $G_{2}$.
Thus, from Eq. (2) we observe that $\pi$ maps the sets $\{1,2, \ldots, n\}$ and $\{n+1, n+2, \ldots, 2 n\}$ into themselves, and therefore $\mathbf{P}$ must have a block structure:

$$
\mathbf{P}=\left(\begin{array}{ll}
\mathbf{P}_{1} & \mathbf{0}_{n}  \tag{5}\\
\mathbf{0}_{n} & \mathbf{P}_{2}
\end{array}\right)
$$

Let $G_{3}$ be the undirected graph such that $\operatorname{Adj}\left(G_{3}\right)=\mathbf{B}$. Combining (5) and (2) we can conclude that $G_{1} \simeq G_{3}$ and $G_{2} \simeq G_{3}$, because the adjacency matrices $\mathbf{M}_{1}, \mathbf{M}_{2}$ can be transformed into $\mathbf{B}$ via the permutation matrices $\mathbf{P}_{1}, \mathbf{P}_{2}$, respectively. By transitivity we conclude $G_{1} \simeq G_{2}$.

Lemma 4.6 ensures that the adjacency matrix of the disjoint union of two isomorphic graphs may always be written as $\mathbf{I}_{2} \otimes \mathbf{B}$; note that $\mathbf{I}_{2}$ is the adjacency matrix of $D_{2}$, the graph with two nodes and two self-loops. Unfortunately, simply testing primality of the disjoint union $X=$ $G_{1} \cup G_{2}$ of two graphs $G_{1}$ and $G_{2}$ is not enough for deciding whether $G_{1}$ and $G_{2}$ are isomorphic or not. In fact, as mentioned at the end of Section 3, the graph $X$ could admit as a factor a graph with two nodes different from $D_{2}$. For example, Figure 3 shows a graph $X$ such that

- $X$ is the disjoint union of two non isomorphic graphs (connected and having the same number of nodes and edges) and
- $X$ admits as a factor a graph with two nodes different from $D_{2}$.

Moreover, the idea of factorizing $X=G_{1} \cup G_{2}$ to check whether $D_{2}$ is a factor (or, equivalently, $G_{1} \simeq G_{2}$ ) might fail due to the fact that $X$ could admit two different factorizations $F_{1}$ and $F_{2}$ where $F_{1}$ contains $D_{2}$ while $F_{2}$ does not. Figure 2 shows an example of a graph $X$ such that

- $X$ is the disjoint union of two isomorphic graphs (both connected and with a prime number of nodes)
- $X$ admits two distinct factorizations $F_{1}$ and $F_{2}$ where $F_{1}$ contains $D_{2}$ while $F_{2}$ does not.

In order to prove the main result of this paper, we define a class of graphs $\Theta$ as follows.
Definition 4.7 (Class $\Theta$ ). A graph $G=(V, E)$ with $n$ nodes and $m$ edges belongs to the class $\Theta \subset \Gamma_{0}$ if and only if:

## P1 G is undirected, connected and not bipartite;

$\boldsymbol{P 2}$ The number of nodes $n$ is prime;


Figure 2: $G_{2}$ is a connected graph with a prime number of nodes. $G_{3}$ is the disjoint union of two copies of $G_{2} . G_{3}$ admits two different factorizations, namely, $G_{1} \times G_{2}$ and $D_{2} \times G_{2}$.


Figure 3: $G_{3}$ is the disjoint union of two connected graphs with the same number of nodes and edges. $G_{3}$ is the is the direct product of $G_{1}$ and $G_{2} . G_{3}$ admits as a factor $G_{1}$ which is a graph with two nodes different from $D_{2} . G_{3}$ does not admit $D_{2}$ as a factor

P3 The number s of self-loops is strictly less than the number of edges, i.e., $s<m$;
$\boldsymbol{P 4}(2 m-s)$ is not divisible by 2 ;
P5 $(2 m-s)$ is not divisible by 3;
In the following theorem we give a polynomial time many-one reduction from $\mathbf{G I}[\boldsymbol{\Theta}]$ to Compositeness $[\boldsymbol{\Theta}]$. A consequence of this result is that the existence of a polynomial-time algorithm to determine whether a given graph in $\Theta$ is composite with respect to the direct product would imply the existence of a polynomial-time algorithm for the graph isomorphism problem between any two graphs in $\Theta$.

## Theorem 4.8.

$$
\mathbf{G}[\mathbf{[}] \leq_{M} \operatorname{Compositeness}[\boldsymbol{\Theta}]
$$

Proof. Let $\mathcal{A}$ be an algorithm for testing compositeness for graphs in $\Theta$. The following algorithm solves the isomorphism problem for graphs in $\Theta$ relying on a single call of $\mathcal{A}$, therefore providing a polynomial time many-one reduction from $\mathbf{G I}[\boldsymbol{\Theta}]$ to Compositeness $[\boldsymbol{\Theta}]$.

```
\(\Theta\)-Graph-Isomorphism \(\left(G_{1}, G_{2}\right)\)
if \(\left|V\left(G_{1}\right)\right| \neq\left|V\left(G_{2}\right)\right|\) or \(\left|E\left(G_{1}\right)\right| \neq\left|E\left(G_{2}\right)\right|\)
    return No
\(G=G_{1} \cup G_{2} / /\) graphs disjoint union
return \(\mathcal{A}(G)\)
```

Let us prove that $\Theta$-Graph-Isomorphism is correct. To this end, we consider two cases:
$G$ is prime. Let $G_{1}, G_{2} \in \Theta$ and $\mathbf{M}=\mathbf{A d j}\left(G_{1} \cup G_{2}\right)$. According to Lemma 4.6 and to Ob servation 4.5 , if $G$ is prime we can conclude that $G_{1}$ and $G_{2}$ are not isomorphic, since if they were, there should exist a permutation matrix $\mathbf{P}$ and a suitable adjacency matrix $\mathbf{B}$ such that $\mathbf{P}^{\top} \mathbf{M P}=\mathbf{I}_{2} \otimes \mathbf{B}$ and then $G_{1} \cup G_{2}$ would be composite.
$G$ is composite. Let $G_{1}, G_{2} \in \Theta$ and $\mathbf{M}=\mathbf{A d j}\left(G_{1} \cup G_{2}\right)$. Let $n=\left|V\left(G_{1}\right)\right|=\left|V\left(G_{2}\right)\right|$ be the number of nodes of either $G_{1}$ or $G_{2}$. Since $G_{1} \in \Theta, n$ is prime ( $\mathbf{P 2}$ ). Consequently, the number of nodes of $G=G_{1} \cup G_{2}$ is $2 n$ and its adjacency matrix $\mathbf{M}$ has size $2 n \times 2 n$. Therefore, the only possible factorization of $\mathbf{M}$ is $\mathbf{M}=\mathbf{A} \otimes \mathbf{B}$, where $\mathbf{A}$ has size $2 \times 2$ and $\mathbf{B}$ has size $n \times n$. Additionally, since $G_{1}, G_{2} \in \Theta$, their adjacency matrices have exactly $2 m-s$ nonzero elements each, where $m=\left|E\left(G_{1}\right)\right|=\left|E\left(G_{2}\right)\right|$ is the number of edges of either $G_{1}$ or $G_{2}$ and $s$ is the number of self-loops. Let us consider each possible configuration of the matrix $\mathbf{A}$ :

$$
\begin{array}{ll}
\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right) \\
\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) & \left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) \\
\left(\begin{array}{ll}
0 & 1 \\
1 & 0 \\
1 & 0
\end{array}\right) & \left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)
\end{array}\left(\begin{array}{ll}
1 & 1 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right) .
$$

Since $G_{1}$ and $G_{2}$ are undirected, $G$ is undirected as well, so its adjacency matrix $\mathbf{M}$ must be symmetric. From 4.6 we deduce that $\mathbf{A}$ must be symmetric, since $\mathbf{B}$ is a nonzero matrix. Therefore, we exclude all configurations of matrix $\mathbf{A}$ that are not symmetrix.

$$
\begin{aligned}
& \left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
0 & 0
\end{array}\right) \\
& \left(\begin{array}{ll}
0 & 1 \\
1 & 0 \\
1 & 1
\end{array}\right)
\end{aligned}\left(\begin{array}{ll}
0 & 1 \\
1 & 0 \\
0 & 1
\end{array}\right) .\left(\begin{array}{ll}
0 & 0 \\
1 & 1
\end{array}\right)\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right) .
$$

Since $G_{1}$ and $G_{2}$ are connected, the degrees of all their nodes must be strictly grater than zero. Therefore, A must contain more than a single nonzero element as, conversely, the resulting matrix $\mathbf{M}$ would have at least $n$ unconnected nodes with zero degree. Therefore, we exclude all configurations of $\mathbf{A}$ that have less than two nonzero elements.

$$
\begin{aligned}
& \left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \\
& \left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) \\
& \left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right)
\end{aligned}\left(\begin{array}{ll}
0 & 1 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)
$$

Let us now consider the number $b$ of nonzero elements in B. Should A have three nonzero elements, the resulting number of nonzero elements in $\mathbf{M}$ would be $3 b$, which is divisible by 3 . Moreover, the number of nonzero elements of $\mathbf{M}$ is equal to the number of nonzero elements of the adjacency matrices of $G_{1}$ and $G_{2}$ :

$$
\begin{equation*}
3 b=2(2 m-s) \tag{6}
\end{equation*}
$$

Since $G_{1}, G_{2} \in \Theta$, we know that the number of nonzero elements ( $2 m-s$ ) in their adjacency matrices must not be divisible by $3(\mathbf{P 5})$. Thus, $2(2 m-s)$ must not be divisible by 3 either, contradicting (6). We conclude that A can not contain three nonzero elements.

$$
\left.\begin{array}{l}
\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right) \\
\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right)
\end{array}\left(\begin{array}{ll}
0 & 1 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)\right)\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right) .
$$

Similarly, should the matrix $\mathbf{A}$ have four nonzero elements, the number of nonzero elements in $\mathbf{M}$ would be $4 b$; from the same reasoning above, we get:

$$
\begin{equation*}
4 b=2(2 m-s) \tag{7}
\end{equation*}
$$

However, from P4 we have that $(2 m-s)$ is not divisible by 2 , and therefore $2(2 m-s)$ is not divisible by 4 . We conclude that $\mathbf{A}$ can not have four nonzero elements.


Finally, we point out that since $G_{1}$ and $G_{2}$ are not bipartite $(\mathbf{P} 1), G$ is not bipartite as well. Should $\mathbf{A}=\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$, matrix $\mathbf{M}=\mathbf{A} \otimes \mathbf{B}$ would represent a bipartite graph, where the first $n$ nodes are only connected to the other $n$ nodes and viceversa. Therefore, A can not be in that form.

$$
\left.\begin{array}{l}
\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
0 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right) \\
\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
0 & 1 \\
0 & 1
\end{array}\right) \\
0
\end{array} 1\right)\left(\begin{array}{ll}
0 & 0 \\
1 & 1
\end{array}\right)\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right)\left(\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right) .
$$

We conclude that $\mathbf{A}=\mathbf{I}_{2}$. Thus, according to Lemma 4.6, $G_{1}$ and $G_{2}$ are isomorphic.
Theorem 4.8 shows that, within the class $\Theta$, there exists an intimate relation between graph primality with respect to the direct product and graph isomorphism. In what follows we will extend Theorem 4.8 to the class of graph $\Gamma_{0}$ by describing a polynomial-time, isomorphismpreserving transformation that maps any connected graph $G$ into a graph in $\Theta$. Before doing so, we need to prove a small technical lemma.

Lemma 4.9. For each $n \in \mathbb{Z}$ there exists $d \in\{0,1,2,3\}$ such that $(n+d)$ is not divisible by two nor by three; formally, $(n+d) \bmod 2 \neq 0$ and $(n+d) \bmod 3 \neq 0$.

Proof. Let us denote with $\llbracket n \rrbracket_{k}$ the equivalence class of all integers that are congruent to $n$ modulo $k$ (also called residual class): $\llbracket n \rrbracket_{k}=\{\ldots, n-2 k, n-k, n, n+k, n+2 k, \ldots\}$. The statement of the lemma can then be rephrased as: for each $n \in \mathbb{Z}$ there exists $d \in\{0,1,2,3\}$ such that $(n+d) \notin\left(\llbracket 0 \rrbracket_{2} \cup \llbracket 0 \rrbracket_{3}\right)$.
Using well-known properties of residual classes we can derive the following table, that shows the value of $d$ for any possible combination of residual classes modulo 2 and modulo 3 that $n$ may belong to.

| $n$ | $\llbracket 0 \rrbracket_{3}$ | $\llbracket 1 \rrbracket_{3}$ | $\llbracket 2 \rrbracket_{3}$ |
| :---: | :---: | :---: | :---: |
| $\llbracket 0 \rrbracket_{2}$ | $d=1$ | $d=1$ | $d=3$ |
| $\llbracket 1 \rrbracket_{2}$ | $d=2$ | $d=0$ | $d=0$ |

For example, if $n \in \llbracket 1 \rrbracket_{2} \cap \llbracket 0 \rrbracket_{3}$, then $(n+2) \in \llbracket 1 \rrbracket_{2}$ and $(n+2) \in \llbracket 2 \rrbracket_{3}$.

Theorem 4.10. There exists a mapping $f: \Gamma_{0} \rightarrow \Theta$ such that for every two connected graphs $G_{1}, G_{2} \in \Gamma_{0}$ with the same number of nodes and edges, $G_{1} \simeq G_{2}$ if and only if $f\left(G_{1}\right) \simeq f\left(G_{2}\right)$. Furthermore, $f(G)$ can be computed in polynomial time with respect to the size of $G$.

Proof. Given a connected graph $G \in \Gamma_{0}$, let us define $G^{\prime}=f(G)$. We show how $G^{\prime}$ is computed. Let $m=|E(G)|$ and $n=|V(G)|$. According to the Bertrand-Chebyshev theorem [11], for any integer $q>1$ there exists a prime in the set $\{q+1, \ldots, 2 q-1\}$, and such prime can be found in polynomial time [12]. Therefore, there is a prime $p$ such that $2 n<p<4 n$.
The vertex set of $G^{\prime}$ is defined as

$$
V\left(G^{\prime}\right)=V(G) \cup\left\{v_{n+1}, v_{n+2}, \ldots, v_{p}\right\}
$$

where $v_{n+1}, v_{n+2}, \ldots v_{p}$ are new nodes.


The edge set $E\left(G^{\prime}\right)$ is constructed incrementally from $E(G)$, as follows. Let

$$
\begin{aligned}
C_{f} & =\left\{\left\{x, v_{n+1}\right\} \mid x \in V(G)\right\} \\
C_{c} & =\left\{\left\{v_{n+1}, v_{n+2}\right\},\left\{v_{n+2}, v_{n+3}\right\}, \ldots,\left\{v_{p}, v_{n+1}\right\}\right.
\end{aligned}
$$

that is, $C_{f}$ is a set of new edges that connect each node in $V(G)$ to the first newly created node $v_{n+1}$, and $C_{c}$ is a set of new edges that form a cycle within the new nodes. Since we have chosen $p$ such that $2 n<p<4 n$, the length of the cycle in $C_{f}$ is greater than $n$.


We finally add a number $s$ of self-loops within the new nodes in order to meet conditions $\mathbf{P 4}$ and P5. Lemma 4.9 guarantees that $s$ is at most 3 . Thus:

$$
\begin{aligned}
& s=0 \Longrightarrow C_{s}=\emptyset \\
& s=1 \Longrightarrow C_{s}=\left\{\left\{v_{n+2}, v_{n+2}\right\}\right\} \\
& s=2 \Longrightarrow C_{s}=\left\{\left\{v_{n+2}, v_{n+2}\right\},\left\{v_{n+3}, v_{n+3}\right\}\right\} \\
& s=3 \Longrightarrow C_{s}=\left\{\left\{v_{n+2}, v_{n+2}\right\},\left\{v_{n+3}, v_{n+3}\right\},\left\{v_{n+4}, v_{n+4}\right\}\right\}
\end{aligned}
$$

The edge set $E\left(G^{\prime}\right)$ is therefore defined as $E\left(G^{\prime}\right)=E(G) \cup C_{f} \cup C_{c} \cup C_{s}$.
Observe that, since $G$ is connected, the edge subset $C_{f}$ induces at least one odd cycle (specifically, at least one cycle of length three), and therefore $G^{\prime}$ is not bipartite ${ }^{1}$. Therefore we conclude that $G^{\prime} \in \Theta$.


We now prove that $G_{1} \simeq G_{2} \Longleftrightarrow f\left(G_{1}\right) \simeq f\left(G_{2}\right)$.
$(\Longrightarrow)$ Assume $G_{1} \simeq G_{2}$. Then, the isomorphism can be trivially extended to $f\left(G_{1}\right)$ and $f\left(G_{2}\right)$ since these graphs are obtained from $G_{1}, G_{2}$ by adding an identical structure.
$(\Longleftarrow)$ Assume $f\left(G_{1}\right) \simeq f\left(G_{2}\right)$. The only possible isomorphisms are those that map one of the cycles $C_{c}$ to the corresponding one on the other graph. Since in the transformation we have chosen $p>2 n$, the cycle length of $C_{c}$ is larger than $n$, and therefore is larger than any simple cycle in $G_{1}$ (or $G_{2}$ ). Consequently, the isomorphism between $f\left(G_{1}\right)$ and $f\left(G_{2}\right)$ can be restricted to an isomorphism between $G_{1}$ and $G_{2}$.

Theorem 4.10 allows us to assert the main result of this paper, that is the relation between primality test and graphs isomorphism.

In what follows we denote by $C$ and $U$ be the sets of connected and unconnected graphs, respectively and by $N B$ be the set of nonbiparite graphs.

[^1]
## Theorem 4.11.

## $\mathbf{G I}[C] \leq_{\mathrm{M}}$ Compositeness $[U \cap N B]$

Proof. Assume that there exists an algorithm $\mathcal{A}$ that solves the Compositeness $[U \cap N B]$ decision problem. Then, the following algorithm solves the GI[C] decision problem and, at the same time, provides a polynomial time many-one reduction from GI[C] to Compositeness $[U \cap N B]$.
$\operatorname{Graph}-\operatorname{Isomorphism}\left(G_{1}, G_{2}\right)$

```
if \(\left|V\left(G_{1}\right)\right| \neq\left|V\left(G_{2}\right)\right|\) or \(\left|E\left(G_{1}\right)\right| \neq\left|E\left(G_{2}\right)\right|\)
    return no
\(G_{3}=f\left(G_{1}\right) / /\) Theorem 4.10
\(G_{4}=f\left(G_{2}\right) / /\) Theorem 4.10
\(G=G_{3} \cup G_{4} / /\) graphs disjoint union
return \(\mathcal{A}(G)\)
```

In fact, by Theorem 4.10, $G_{1}$ is isomorphic to $G_{2}$ if and only if $f\left(G_{1}\right)$ is isomorphic to $f\left(G_{2}\right)$. Since both $f\left(G_{1}\right)$ and $f\left(G_{2}\right)$ belong to $\Theta$, then by Theorem 4.8, $G=G_{3} \cup G_{4}$ is decomposable if and only if $G_{3}$ is isomorphic to $G_{4}$.
It is easy to verify that Graph-Isomorphism is a polynomial time many-one reduction.

Note that Compositeness[ $U \cap N B$ ] remains $G I$-hard even if we relax the undirected constraint or the nonbipartite one, as the resulting class of graphs would be larger than the one which was considered throughout our discussion.

Corollary 4.12. Primality $[U \cap N B]$ is GI-hard or, equvalently,

$$
\mathbf{G I}[C] \leq_{\mathrm{T}} \operatorname{Primality}[U \cap N B]
$$

Proof. The proof follows directly from the proof of Theorem 4.11 by inverting the result provided by the oracle $\mathcal{A}$.

## 5. Conclusions

In this paper we proved that primality testing of unconnected, nonbipartite grahps with respect to direct product is at least as hard as deciding graph isomorphism. The same result also applies to the computation of a prime factorization of a graph. This result answer a long standing open question posed in [1] and shows the crucial role played by connectedness in decomposing a graph.

It would be of some interest to investigate the reversed question, i.e., whether deciding graph isomorphism is at least as hard as primality testing or not. Another interesting research direction is the study and the implementation of efficient heuristics for computing a prime factorization or its approximation of large, possibly unconnected and/or weighted graphs knowing that a polynomial time algorithm for computing such a prime factorization is unlikely to exist.

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[^1]:    ${ }^{1}$ A well known result in graph theory states that a graph is bipartite if and only if it has no odd cycles

