On square factors and critical factors of k-bonacci words on infinite alphabet

N. Ghareghani[a, 1], P. Sharifani [b]

[a] *Department of Industrial design, College of Fine Arts, University of Tehran, Tehran, Iran*

[b] *School of Mathematics, Institute for Research in Fundamental Sciences* (IPM), *Tehran, Iran*

Emails: ghareghani@ut.ac.ir, ghareghani@ipm.ir, pouyeh.sharifani@gmail.com.

Abstract

For any integer $k > 2$, the infinite k-bonacci word $W^{(k)}$, on the infinite alphabet is defined as the fixed point of the morphism $\varphi_k : \mathbb{N} \to \mathbb{N}^2 \cup \mathbb{N}$, where

$$\varphi_k(ki + j) = \begin{cases} (ki)(ki + j + 1) & \text{if } j = 0, \cdots, k-2, \\ (ki + j + 1) & \text{if } j = k-1. \end{cases}$$

The finite k-bonacci word $W_n^{(k)}$ is then defined as the prefix of $W^{(k)}$ whose length is the $(n+k)$-th k-bonacci number. We obtain the structure of all square factors occurring in $W^{(k)}$. Moreover, we prove that the critical exponent of $W^{(k)}$ is $3 - \frac{3}{2^k - 1}$. Finally, we provide all critical factors of $W^{(k)}$.

Keywords: k-bonacci words, words on infinite alphabet, square, critical exponent, critical factor.

1 Introduction

The infinite Fibonacci word and finite Fibonacci words are well-studied in the literature and satisfy several extremal properties, see [6, 8, 18, 9, 19]. The infinite Fibonacci word $F^{(2)}$ is the unique fixed point of the binary morphism $0 \to 01$ and $1 \to 0$. The n-th finite Fibonacci word

[1] `Corresponding author`

$F_n^{(2)}$ is the prefix of of length f_{n+2} of $F^{(2)}$, where f_n is the n-th Fibonacci number. A natural generalization of Fibonacci words are k-bonacci words which are defined on the k-letter alphabet $\{0, 1, \ldots, k-1\}$. The infinite k-bonacci word $F^{(k)}$ is the unique fixed point of the morphism $\phi_k(0) = 01, \phi_k(1) = 02, \ldots, \phi_k(k-2) = 0(k-1), \phi_k(k-1) = 0$ (see [20]). The n-th finite k-bonacci word $F_n^{(k)}$ is defined to be $\phi_k^n(0)$ or equivalently, the prefix of length $f_{n+k}^{(k)}$ of $F^{(k)}$, where $f_{n+k}^{(k)}$ denotes the $(n+k)$-th k-bonacci number. While the Fibonacci words are good examples of binary words, k-bonacci words are good examples of words over k-letter alphabet and they have many interesting properties (see [20, 1, 4, 12]).

In [22], authors defined the infinite Fibonacci word on infinite alphabet $\mathbb{N}$ as the fixed point of the morphism $\varphi_2 : (2i) \to (2i)(2i+1)$ and $\varphi_2 : (2i+1) \to (2i+2)$. We denote the infinite Fibonacci word on infinite alphabet by $W^{(2)}$. The n-th finite Fibonacci word $W_n^{(2)}$ is then defined similar as $F_n^{(2)}$. It is trivial that if digits (letters) of $W^{(2)}$ are computed mod 2, then the resulting word is the ordinary infinite Fibonacci word $F^{(2)}$. Zhang et al. studied some properties of word $W^{(2)}$. They studied the growth order and digit sum of $W^{(2)}$ and gave several decompositions of $W^{(2)}$ using singular words. Glen et al. considered more properties of $W^{(2)}$ [14]. Among other results, they investigated the structure of palindrome factors and square factors of $W^{(2)}$.In [11], authors introduced the finite (infinite) k-bonacci word over infinite alphabet, for $k > 2$. The n-th finite (res. infinite) k-bonacci word over infinite alphabet is denoted by $W^{(k)}$ (resp. $W_n^{(k)}$). They studied some properties of these words and classified all palindrome factors of $W^{(k)}$, for $k \geq 3$.

For a finite word W and a positive integer n, W^n is simply obtained by concatenating the word W, n times with itself and W^ω is defined as the concatenation of W with itself, infinitely many times; That is $W^\omega = W.W.W\ldots$. For a rational number r with $r.|W| \in \mathbb{N}$, the fractional power W^r is defined to be the prefix of length $r.|W|$ of the infinite word W^ω. For example if $W = 0102$ then $W^{\frac{5}{2}} = 0102010201$. The index of a factor U of word W is defined as

$$\mathrm{INDEX(U, W) = max\{r \in \mathbb{Q} : U^r \prec W\}}.$$

Then the *critical exponent* $E(W)$ of an infinite word W is given by

$$E(W) = \sup\{\mathrm{INDEX(U, W) : U \in F(W) \setminus \{\epsilon\}}\}.$$

A word U is a critical factor of W if $E(W) = \mathrm{INDEX(U, W)}$. The study of the existence of a factor of the form U^r in a long word and specially computing the critical exponent of a long word is the subject of many papers for example see [17, 21, 2, 5, 7, 16, 3]. Specially, in the case of infinite k-bonacci word $F^{(k)}$, it is proved that $E(F^{(k)}) = 2 + \frac{1}{\alpha_k - 1}$ (see [13]), where α_k, the k-th generalized golden ratio, is the (unique) positive real root of the k-th degree polynomial $x^k - x^{k-1} - \ldots - x - 1$. It is proved that $2 - \frac{1}{k} < \alpha_k < 1$ [10, 15]. Hence, $3 < E(F^{(k)}) < 3 + \frac{1}{k-1}$, and $E(F^{(2)}) = 2 + \frac{\sqrt{5}+1}{2}$.

In this work we first investigate some properties of $W_n^{(k)}$. Then, using them, we explore the structure of all square factors of $W_n^{(k)}$. More precisely, we prove that all square factors of $W^{(k)}$ are of the form $ki \oplus C^j(W_n^{(k)})$, for some integers $i > 0$ and $j \geq 0$, where $C^j(U)$ denoted the j-th conjugate of word U. Finally, using the structure of square factors of $W^{(k)}$, we prove that the critical exponent of $W^{(k)}$ is $3 - \frac{3}{2^k - 1}$.

2 Preliminaries

In this section we give more definitions and notations that are used in the paper. We denote the alphabet, which is a finite or countable infinite set, by $\mathcal{A}$. When $\mathcal{A}$ is a countable infinite set, we simply take $\mathcal{A} = \mathbb{N}$; Then each element of $\mathcal{A}$ is called a digit (instead of a letter). We denote by $\mathcal{A}^*$ the set of finite words over $\mathcal{A}$ and we let $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\epsilon\}$, where ϵ the empty word. We denote by $\mathcal{A}^\omega$ the set of all infinite words over $\mathcal{A}$ and we let $\mathcal{A}^\infty = \mathcal{A}^* \cup \mathcal{A}^\omega$. If $a \in \mathcal{A}$ and $W \in \mathcal{A}^\infty$, then the symbols $|W|$ and $|W|_a$ denote the length of W, and the number of occurrences of letter a in W, respectively.

For a finite word $W = w_1 w_2 \dots w_n$, with $w_i \in \mathcal{A}$ and for $1 \le j \le j' \le n$, we denote $W[j, j'] = w_j \dots w_{j'}$, and for simplicity we denote $W[j, j]$ by $W[j]$. Let $U_i \in \mathcal{A}^*$, for $1 \le i \le n$, then $\prod_{i=n}^{1} U_i$ is defined to be $U_n U_{n-1} \dots U_1$. For a finite word W and an integer n, $n \oplus W$ denotes the word obtained by adding n to each digit of W. For example, let $W = 01020103$ and $n = 5$, then $n \oplus W = 56575658$. Similarly, if every digit of W is grater than $n - 1$, then $W \ominus n$ denotes the word obtained by subtracting n from each digit of W.

A word $V \in \mathcal{A}^+$ is a factor of a word $W \in \mathcal{A}^\infty$, if there exist $U \in \mathcal{A}^*$ and $U' \in \mathcal{A}^\infty$, such that $W = UVU'$. Similarly, a word $V \in \mathcal{A}^\infty$ is a factor of $W \in \mathcal{A}^\infty$ if there exists $U \in \mathcal{A}^*$ such that $W = UV$. When V is a factor of W then we denote it as $V \prec W$. A word $V \in \mathcal{A}^+$ (resp. $V \in \mathcal{A}^\infty$) is said to be a *p*refix (resp. *s*uffix) of a word $W \in \mathcal{A}^\infty$, denoted as $V \lhd W$ (resp. $V \rhd W$), if there exists $U \in \mathcal{A}^\infty$ (resp. $U \in \mathcal{A}^*$) such that $W = VU$ (resp. $W = UV$). If $W \in \mathcal{A}^*$ and $W = VU$ (resp. $W = UV$,) we write $V = WU^{-1}$ (resp. $V = U^{-1}W$). The set of all factors of a word w is denoted by $F(w)$. If $W = w_1 \dots w_n$ be a finite word and $0 \le j \le n - 1$, then the *j-th conjugatae* of W is defined as $C^j(W) = w_{j+1} \dots w_n w_1 \dots w_j$. For example the word 0130102 is the 4-th conjugate of 0102013. A word V is a *conjugate* of W if there exists $0 \le j \le n - 1$ such that $V = C^j(W)$. A factor of the form UU in W is called a square factor or simply a square. For a square factor $UU = W[t, t + 2|u|]$ of W, the *center* of the square UU in W is defined to be $c_s(U^2, W) = t + |U| + \frac{1}{2}$.

The n-th k-bonacci number defined as

$$f_n^{(k)} = \begin{cases} 0 & \text{if } \ n = 0, \cdots, k-2, \\ 1 & \text{if } \ n = k-1, \\ \sum_{i=n-1}^{n-k} f_i^{(k)} & \text{if } n \ge k. \end{cases} \tag{1}$$

The finite (resp. infinite) k-bonacci words $W_n^{(k)}$ (resp. $W^{(k)}$) on infinite alphabet $\mathbb{N}$ is defined in [11], using the morphism φ_k given below

$$\varphi_k(ki+j) = \begin{cases} (ki)(ki+j+1) & \text{if } j = 0, \cdots, k-2 \\ (ki+j+1) & \text{otherwise .} \end{cases}$$

More precisely, $W_n^{(k)} = \varphi_k^n(0)$ and $W^{(k)} = \varphi_k^\omega(0)$ (Note that $W_0^{(k)} = F_0^{(k)} = 0$). For a fixed value of k, the k-bonacci words over infinite alphabet are reduced to k-bonacci words over finite alphabet when the digits are calculated $\mod k$. It is easy to show that for $n \geq 0$,

$$|F_n^{(k)}| = |W_n^{(k)}| = f_{n+k}^{(k)}. \tag{2}$$

3 Some properties of $W_n^{(k)}$

In this section we provide some basic properties $W_n^{(k)}$, some of which are proved in [11]. All of these properties are useful for the rest of the work.

Lemma 1. [Lemma 4 of [11]] *Let $n \geq 0$ and $k > 2$. The finite word $W_n^{(k)}$ contains no factor* 00*.*

Following two lemmas give recursive formulas for computing $W_n^{(k)}$.

Lemma 2. [Lemma 5 of [11]] *For $1 \leq n \leq k-1$,*

$$W_n^{(k)} = \prod_{i=n-1}^{0} W_i^{(k)}\ n. \tag{3}$$

Lemma 3. [Lemma 7 of [11]] *For $n \geq k$,*

$$W_n^{(k)} = \prod_{i=n-1}^{n-k+1} W_i^{(k)}\ (k \oplus W_{n-k}^{(k)}). \tag{4}$$

The following corollary is a direct consequence of Lemmas 2 and 3 and can be proved using induction on i.

Corollary 4. *Let i and n be two non-negative integers, then $W_n^{(k)} \oplus ki \prec W_{n+ki}^{(k)}$.*

Considering the recurrence relations (3) and (4) we have the following definitions which are very useful in the next sections.

Definition 1. Let j be a nonnegative integer, then a factor A of $W_n^{(k)}$ is called a *bordering factor of type* j, for some $n-k+1 \leq j \leq n-1$ if $j0 \prec A \prec W_j^{(k)} W_{j-1}^{(k)} \ldots W_m^{(k)}$, where $m = \max\{0, n-k+1\}$. Moreover, a *bordering square factor* of $W_n^{(k)}$ is a bordering factor of $W_n^{(k)}$ which is also a square.

Definition 2. Let $n \geq k$, then a factor A of $W_n^{(k)}$ is called a *straddling factor* of $W_n^{(k)}$ if $A = A_1 A_2$, for some nonempty words A_1 and A_2, with $A_1 \rhd W_{n-1}^{(k)} \dots W_{n-k+1}^{(k)}$ and $A_2 \lhd k \oplus W_{n-k}^{(k)}$. Moreover, if a straddling factor of $W_n^{(k)}$ is also a square, it is called an *straddling square factor.*

Lemma 5. [Lemma 10 of [11]] *For any $n \geq 1$, the digit n is the largest digit of $W_n^{(k)}$ and appears once at the end of this word.*

Lemma 6. *For every integer $i < n$ we have $i0 \prec W_n^{(k)}$.*

Proof. Since $i + 1 \leq n$, we have $W_{i+1}^{(k)} \prec W_n^{(k)}$. By Lemmas 2 and 3, $W_i^{(k)} W_{i-1}^{(k)} \lhd W_{i+1}^{(k)}$. Hence, $i0 \prec W_{i+1}^{(k)} \prec W_n^{(k)}$ and the result follows. □

Lemma 7. *Let $0 < i < n$ and $i0 = W_n^{(k)}[t, t + 1]$, for some $t \in \mathbb{N}$. Then we have*

$$W_n^{(k)}[t - |W_i^{(k)}| + 1, t + 1] = W_i^{(k)}0.$$

In other words, if $i0$ appears in $W_n^{(k)}$, then this i appeared as the last digit of a factor $W_i^{(k)}$ of $W_n^{(k)}$.

Proof. We prove this by induction on n. If $n = 2$, then $W_2^{(k)} = 0102$ and in this case the only possibility for t is $t = 2$ and $W_2^{(k)}[1, 3] = W_1^{(k)}0 = 010$, as desired. We suppose that the claim is true for all $m \leq n$ we want to prove this for the case $n+1$. If $n+1 \geq k$, then by Lemma 3 we have

$$W_{n+1}^{(k)} = \prod_{t=n}^{n-k+2} W_t^{(k)} \, (k \oplus W_{n+1-k}^{(k)}). \tag{5}$$

Let $i0$ occurs in $W_{n+1}^{(k)}$, then either $i0 \prec W_t^{(k)}$, for some $n - k + 2 \leq t \leq n$, or $i0$ is a bordering factor of $W_{n+1}^{(k)}$. If $i0 \prec W_t^{(k)}$, then by induction hypothesis this i should be the last digit of some factor $W_i^{(k)}$ of $W_{n+1}^{(k)}$. If $i0$ is a bordering factor of $W_{n+1}^{(k)}$, then it is clear that i is the end digit of a factor $W_i^{(k)}$ of $W_{n+1}^{(k)}$.

In the case $n + 1 < k$, using similar argument as the previous case and Lemma 2 we obtain the result. □

Lemma 8. *Let $2 < k < n$ and $B \rhd W_n^{(k)}$ with $|B| = |W_{n-k}^{(k)}| + |W_{n-k-1}^{(k)}|$. Then*

(i) *If $n = k + 1$, then $|B|_2 = 1$.*

(ii) *If $n > k + 1$, then $|B|_0 > 0$.*

Proof. (i) If $n = k + 1$, then by (3), $B = 2.k.(k + 1)$, so the result follows.

(ii) If $n > k+1$, then by (4),

$$W_{n-k+1}^{(k)}(k \oplus W_{n-k}^{(k)}) \rhd W_n^{(k)}. \tag{6}$$

Let $D \rhd W_{n-k+1}^{(k)}$ and $|D| = |W_{n-k-1}^{(k)}|$. Then by (6), to prove the lemma it is suffices to show that $|D|_0 > 0$. If $k = 3$, when $n = 5, 6$ it is clear that $|D|_0 > 0$. So, If $n \geq 7$, then by Equation (4), we have

$$\begin{aligned} W_{n-2}^{(n-k+1)} = W_{n-2}^{(k)} &= W_{n-3}^{(k)} W_{n-4}^{(k)}(k \oplus W_{n-5}^{(k)}) \\ &= W_{n-3}^{(k)} W_{n-5}^{(k)} \underbrace{W_{n-6}^{(k)}(k \oplus W_{n-7}^{(k)})(k \oplus W_{n-5}^{(k)})}_{|W_{n-4}^{(k)}|=|D|} \end{aligned} \tag{7}$$

By (7), it is clear that $|D|_0 > 0$.

If $k > 3$ and $n < 2k-1$, then by Lemma 2, we have $0(n-k+1) \rhd W_{n-k+1}^{(k)}$. Since $n > k+1$, we $|W_{n-k-1}^{(k)}| \geq 2$ and hence, $|D|_0 > 0$.

If $k > 3$ and $n \geq 2k-1$, then by Lemma 3, $W_{n-2k+2}^{(k)}(k \oplus W_{n-2k+1}^{(k)}) \rhd W_{n-k+1}^{(k)}$. Since $k > 3$, $|W_{n-k+1}^{(k)}| > |W_{n-2k+2}^{(k)}| + |W_{n-2k+1}^{(k)}|$ and $|W_{n-2k+2}^{(k)}|_0 > 0$, we conclude that $|D|_0 > 0$.

$\square$

Lemma 9. *Let n, k and j be nonnegative integers with $3 \leq k \leq n$ and $n-k+3 \leq j \leq n$. Then $B = W_{n-k+2}^{(k)} k$ is not a factor of $W_j^{(k)}$.*

Proof. If $j < k$, then $|W_j^{(k)}|_k = 0$ and so B is not a factor of $W_j^{(k)}$.

Hence, we shall prove the result for $k \leq j \leq n$. We prove this part by bounded induction on j. Let $p = \max\{k, n-k+3\}$. Since, $j \geq k$ and $n-k+3 \leq j$, the first step of induction is $j = p$. If $p = k$, then the only occurrence of k in $W_j^{(k)}$, is in its last digit, we conclude that if $B \prec W_j^{(k)}$, then $B \rhd W_j^{(k)} = W_k^{(k)}$. Using Lemma 3, we have $W_j^{(k)} = W_{j-1}^{(k)} \dots W_1^{(k)} k$. Since $(n-k+2)k \rhd B \rhd W_k^{(k)}$, we provide $n-k+2 = 1$, so $n = k-1 < k$, which is a contradiction.

If $p = n-k+3$, then by (4), we have

$$W_{n-k+3}^{(k)} = W_{n-k+2}^{(k)} W_{n-k+1}^{(k)} \dots W_{n-2k+4}^{(k)}(k \oplus W_{n-2k+3}^{(k)}) \tag{8}$$

If $B \prec W_j^{(k)} = W_{n-k+3}^{(k)}$, then there exist integers s and t such that $B = W_{n-k+3}^{(k)}[s, t+1]$, $W_{n-k+3}^{(k)}[t] = n-k+2$ and $W_{n-k+3}^{(k)}[t+1] = k$. Using Lemma 5 and Equation (8), either $t = |W_{n-k+2}^{(k)}|$ or $t \geq |W_{n-k+3}^{(k)}| - |W_{n-2k+3}^{(k)}| + 1$. If $t = |W_{n-k+2}^{(k)}|$, then $W_{n-k+3}^{(k)}[t+1] = 0$, which is a contradiction. If $t \geq |W_{n-k+3}^{(k)}| - |W_{n-2k+3}^{(k)}| + 1$, then using (8) and the fact that $|W_{n-k+3}^{(k)}| \leq 2|W_{n-k+2}^{(k)}|$ and $B = |W_{n-k+2}^{(k)}| + 1$, we conclude that $s < |W_{n-k+2}^{(k)}|$. Which implies that $|W_{n-k+3}^{(k)}[s, t+1]|_{n-k+2} \geq 2$. But by definition of B and using Lemma 5, we have $|B|_{n-k+2} = 1$, which is a contradict. Therefore, the first step of induction is true.

We are going to prove that B is not a factor of $W_{j+1}^{(k)}$. For contrary let $B \prec W_{j+1}^{(k)}$. By (4), we have

$$W_{j+1}^{(k)} = W_j^{(k)} \dots W_{j-k+2}^{(k)}(k \oplus W_{j-k+1}^{(k)}). \tag{9}$$

By induction hypothesis B is not a factor of $W_i^{(k)}$ for $j-k+1 \le i \le j$. Since B contains the digit 0 and $(k \oplus W_{j-k+1}^{(k)})$ does not contain it, there are two following possible cases for B:

- **Case 1.** B is a bordering factor of $W_{j+1}^{(k)}$; Let ℓ be largest integer such that $B \prec W_j^{(k)} \dots W_\ell^{(k)}$. Since for every integer i, $W_i^{(k)}$ start with 0, we have $(n-k+2)k \prec W_\ell^{(k)}$ which means that $\ell > n-k+2$. Therefore, $W_{n-k+3}^{(k)} \lhd W_\ell^{(k)}$. Hence, $W_{n-k+2}^{(k)} 0 \lhd W_\ell^{(k)}$. Since $(n-k+2)k \prec W_\ell^{(k)}$, there exists integer α such that $W_\ell^{(k)}[\alpha, \alpha+1] = (n-k+2)k$. By Lemma 5, $\alpha > |W_{n-k+2}^{(k)}|$. Therefore, $B \prec W_\ell^{(k)}$, which contradicts to the definition of bordering factor. Hence, B is not a bordering factor of $W_{j+1}^{(k)}$.

- **Case 2.** B is a straddling factor of $W_{j+1}^{(k)}$; By definition of straddling factor, there exists nonempty word S which is the suffix of B and a prefix of $k \oplus W_{j-k+1}^{(k)}$. Since $j < n$, we have $j-k+2 < n-k+2$ and hence using (9), we provide that the last two digits of B occur in $k \oplus W_{j-k+1}^{(k)}$. Let $|S| = t+1$, for some $t > 0$, this means that $(k \oplus W_{j-k+1}^{(k)})[t, t+1] = (n-k+2)k$. By definition of S we obtain

$$BS^{-1} \rhd W_{j+1}^{(k)}[(k \oplus W_{j-k+1}^{(k)})]^{-1} \tag{10}$$

On the other hand, since $(n-k+2) \prec k \oplus W_{j-k+1}^{(k)}$, we have $n-k+2 \ge k$, or $n \ge 2k-2$. If $n = 2k-2$, then $W_{j-k+1}^{(k)}[t, t+1] = 00$, which contradicts to Lemma 1. Therefore, $n \ge 2k-3$, now, using Equations (3) and (4), and the fact that $n-k+3 \le j$, we have

$$W_{n-2k+2}^{(k)} W_{n-2k+1}^{(k)} \lhd W_{n-2k+3}^{(k)} \lhd W_{j-k+1}^{(k)} \tag{11}$$

By Lemma 5 and Equation (11), we conclude that either $t = |W_{n-2k+2}^{(k)}|$ or $t > |W_{n-2k+2}^{(k)}| + |W_{n-2k+1}^{(k)}|$. First suppose that $t = |W_{n-2k+2}^{(k)}|$. Then by (11), we have $S = k \oplus (W_{n-2k+2}^{(k)} 0)$. Using (10), we provide

$$B[(k \oplus W_{n-2k+2}^{(k)})k]^{-1} \rhd W_{j+1}^{(k)}[(k \oplus W_{j-k+1}^{(k)})]^{-1}$$
$$W_{n-k+1}^{(k)} \dots W_{n-2k+3}^{(k)} \rhd W_j^{(k)} \dots W_{j-k+2}^{(k)}$$

Hence, $n-2k+3 = j-k+2$, or $j = n-k+1$, which contradicts to our assumption $n-k+3 \le j$. Now, suppose that $t > |W_{n-2k+2}^{(k)}| + |W_{n-2k+1}^{(k)}|$. Let $D \rhd B$, and $|D| = |W_{n-2k+2}^{(k)}| + |W_{n-2k+1}^{(k)}|$, then $D \prec (k \oplus W_{j-k+1}^{(k)})$. On the other hand using Lemma 8 either $|D|_0 > 0$ or $|D|_2 > 0$, which is a contradiction.

□

4 Squares in $W_n^{(k)}$

In this section we give the structure of all square factors of $W_n^{(k)}$. We first prove that when $n < 2k-1$, $W_n^{(k)}$ has no square factor. Then we characterize all square factors of $W^{(k)}$.

Lemma 10. *For two positive integers n and k, there is no bordering square in $W_n^{(k)}$.*

Proof. For contrary suppose that there exists $n-k+2 \leq j \leq n-1$, for which $W_n^{(k)}$ contains a bordering square of type j; we denote this word by A. By Definition 1, $j0 \prec A \prec W_j^{(k)} W_{j-1}^{(k)} \dots W_{n-k+1}^{(k)}$. Since A is a square word, so $|A|_j \geq 2$. but by Lemma 5,

$$|W_j^{(k)} W_{j-1}^{(k)} \dots W_{n-k+1}^{(k)}|_j = 1.$$

This is a contradiction. □

Lemma 11. *If $n < k+1$, then $W_n^{(k)}$ contains no square factor.*

Proof. We prove this by bounded induction on n. By definition $W_0^{(k)} = 0$ does not contain any square. Suppose that for any integer i, $0 \leq i \leq n < k$, $W_n^{(k)}$ does not contain any square. For contrary suppose that B is a square factor of $W_{n+1}^{(k)}$. By (3) and (4) we have

$$W_{n+1}^{(k)} = \begin{cases} W_{n+1}^{(k)} = W_n^{(k)} W_{n-1}^{(k)} \dots W_0^{(k)}(n+1) & \text{if } n+1 < k, \\ W_{n+1}^{(k)} = W_n^{(k)} W_{n-1}^{(k)} \dots W_1^{(k)}(n+1) & \text{if } n+1 = k. \end{cases} \tag{12}$$

Using induction hypothesis and Lemma 10, we provide that B is a straddling square. By Definition 2 and Equation (12), $|B|_{n+1} \geq 2$, which contradicts with Lemma 5. □

Lemma 12. *let $n > k$ and A^2 be a straddling square of $W_n^{(k)}$. Then $c_s(A^2, W_n^{(k)}) > |W_n^{(k)}| - |W_{n-k}^{(k)}|$.*

Proof. By (4), we have

$$W_n^{(k)} = W_{n-1}^{(k)} \dots W_{n-k+2}^{(k)} \boxed{W_{n-k+1}^{(k)}} (k \oplus W_{n-k}^{(k)}) \tag{13}$$

For contrary suppose that $c_s(A^2, W_n^{(k)}) < |W_n^{(k)}| - |W_{n-k}^{(k)}|$. Let $A^2 = W_n^{(k)}[s_1, s_2]$. Since A^2 is a straddling square of $W_n^{(k)}$, $s_2 \geq |W_n^{(k)}| - |W_{n-k}^{(k)}| + 1$ and $s_1 < |W_n^{(k)}| - |W_{n-k}^{(k)}|$.

If $c_s(A^2, W_n^{(k)}) < |W_n^{(k)}| - |W_{n-k}^{(k)}| - |W_{n-k+1}^{(k)}|$, then using (13), we conclude that $B = W_{n-k+1}^{(k)} k \prec W_{n-1}^{(k)} \dots W_{n-k+2}^{(k)}$. By Lemma 9 for any $n-k+2 \leq j \leq n-1$, B is not a factor $W_j^{(k)}$. Let $s < n$ be largest integer such that $B \prec W_{n-1}^{(k)} \dots W_s^{(k)}$. Since $0 \lhd W_s^{(k)}$, we have $(n-k+1)k \prec W_s^{(k)}$. Let

$W_s^{(k)}[\alpha, \alpha+1] = (n-k+1)k$. By Lemma 5, $s > n-k+1$. Therefore, $W_{n-k+1}^{(k)}0 \lhd W_{n-k+2}^{(k)} \lhd W_s^{(k)}$ and by Lemma 5, $\alpha > |W_{n-k+1}^{(k)}|$. Therefore, $B \prec W_s^{(k)}$, which contradicts to Lemma 9. Hence,

$$|W_n^{(k)}| - |W_{n-k}^{(k)}| - |W_{n-k+1}^{(k)}| + 1 < c_s(A^2, W_n^{(k)}) < |W_n^{(k)}| - |W_{n-k}^{(k)}|. \tag{14}$$

This means that the center of A^2 happens in $W_{n-k+1}^{(k)}$ which is distinguished by a box in (13). We denote the first occurrences of A in A^2 by A_1 and the last occurrence of A in A^2 by A_2. By (14), we conclude that there exist non-empty words U_1 and U_2, such that $W_{n-k+1}^{(k)} = U_1U_2$ and $U_2 \lhd A_2$. By Lemma 5, $|U_1|_{n-k+1} = 0$ and the only occurrence of $n-k+1$ in U_2. which is its last digit. We conclude that $|A_2|_{n-k+1} > 0$ and all digits of A_2 which appear after $n-k+1$ are greater than $k-1$. Hence $|A_1|_{n-k+1} > 0$ and all digits of A_1 which appear after $n-k+1$ should be also greater than $k-1$. Since $|U_1|_{n-k+1} = 0$, we conclude that $U_1 \rhd A_1$ and all occurrence of $n-k+1$ are before the first digit of U_1. But this is a contradiction, because $|U_1|_0 > 0$ and hence there is a digit 0 which appears after all digit $n-k+1$ in A_1. $\square$

Corollary 13. *let A^2 be a straddling square of $W_n^{(k)}$. Then for each $i \leq k-1$, $|A|_i = 0$.*

Proof. By Lemma 12, $c_s(A^2, W_n^{(k)}) > |W_n^{(k)}| - |W_{n-k}^{(k)}|$. Therefore, using Equation 4, we conclude that $A \prec (k \oplus W_{n-k}^{(k)})$. This means that all digits of A are greater than $k-1$, as desired. $\square$

Lemma 14. *Let $i < n$ and $n \geq k+1$, then $W_n^{(k)}$ contains no factor of the form $W_i^{(k)}W_i^{(k)}$.*

Proof. We prove this by induction on n. If $n = 1$, then $W_1^{(k)} = 01$ contains no factor $W_0^{(k)}W_0^{(k)} = 00$. By (4), we have $W_n^{(k)} = \prod_{i=n-1}^{n-k+1} W_i^{(k)} \, (k \oplus W_{n-k}^{(k)})$. For the contrary suppose that $W_i^{(k)}W_i^{(k)} \prec W_n^{(k)}$ for some $i < n$. By induction hypothesis for any $j < n$, $W_i^{(k)}W_i^{(k)}$ is not a factor of $W_j^{(k)}$. Now, by Definitions 1 and 2, $W_i^{(k)}W_i^{(k)}$ should be either a bordering square or a straddling square. By Lemma 10, $W_n^{(k)}$ contains no bordering square. Therefore, $W_i^{(k)}W_i^{(k)}$ is a straddling square of $W_n^{(k)}$ which can not be occurred by Corollary 13. Hence, there is no factor of the form $W_i^{(k)}W_i^{(k)}$ in $W_n^{(k)}$. $\square$

Lemma 15. *If $n < 2k-1$, then $W_n^{(k)}$ contains no square factor.*

Proof. For contrary suppose that there exists a straddling square A^2 in $W_n^{(k)}$. By definition of straddling factor A contains the digit $n-k+1$. Hence by Corollary 13, $n-k+1 \geq k$ and $n \geq 2k-1$, which is a contradiction. $\square$

Lemma 16. *kk is the only square of $W_{2k-1}^{(k)}$.*

Proof. Let $A^2 = W_{2k-1}^{(k)}[t, t+|A^2|]$ be a square factor of $W_{2k-1}^{(k)}$. By Lemmas 15 and 11, we conclude that for every $j < 2k-1$, $W_j^{(k)}$ contains no square factor. By Lemma 10, $W_{2k-1}^{(k)}$ has no bordering square. Hence, A^2 is a straddling square.

$$W^{(k)}_{2k-1} = W^{(k)}_{2k-2} \dots W^{(k)}_{k} (k \oplus W^{(k)}_{k-1}) \tag{15}$$

By Corollary 13 and definition of straddling factor of $W^{(k)}_{2k-1}$, $t = |W^{(k)}_{2k-1}| - |W^{(k)}_{k-1}|$. Hence, $kk \lhd A^2$. Hence, either $A^2 = kk$ or the number of occurrences of kk in A^2 is at least two. If the number of occurrences of kk in A^2 is at least two, then $kk \prec (k \oplus W^{(k)}_{k-1})$ which means that $00 \prec W^{(k)}_{k-1}$, which contradicts with Lemma 11. Hence, the only possibility for A^2 is kk. □

Lemma 17. *Let A^2 be a square of $W^{(k)}_m$. Then there exists integers i and $n \leq m$ such that $A^2 \ominus ki$ is a straddling square of $W^{(k)}_n$.*

Proof. We prove this using induction on m. If $m < 2k-1$, then by Lemma 15, $W^{(k)}_m$ contains no square factor and the result follows. If $m = 2k-1$, then by Lemma 16, kk is the only square of $W^{(k)}_m$ which is a straddling square. If $m > 2k-1$, then by (4) and using the induction hypothesis and Lemma 10 the result follows. □

The following corollary is a direct consequence of Lemma 17.

Corollary 18. *Let A^2 be a square of $W^{(k)}$. Then there exists integers i and n, such that $A^2 \ominus ki$ is a straddling square of $W^{(k)}_n$.*

Lemma 19. *let A^2 be a straddling square of $W^{(k)}_n$. Then*

$$c_s(A^2, W^{(k)}_n) \leq |W^{(k)}_n| - |W^{(k)}_{n-k}| + |W^{(k)}_{n-2k+1}| + \frac{1}{2}.$$

Proof. By (4), we have

$$W^{(k)}_n = \prod_{i=n-1}^{n-k+1} W^{(k)}_i \, (k \oplus W^{(k)}_{n-k}) \tag{16}$$

$$= \prod_{i=n-1}^{n-k+2} W^{(k)}_i (W^{(k)}_{n-k} \cdot \dots . W^{(k)}_{n-2k+2}) \, (k \oplus W^{(k)}_{n-2k+1}) \, (k \oplus W^{(k)}_{n-k}). \tag{17}$$

We remined that $k \oplus W^{(k)}_{n-2k+1} \prec k \oplus W^{(k)}_{n-k}$. Hence, if $c_s(A^2, W^{(k)}_n) > |W^{(k)}_n| - |W^{(k)}_{n-k}| + |W^{(k)}_{n-2k+1}|$, then $k \oplus ((n-2k+1) W^{(k)}_{n-2k+1}) \prec k \oplus W^{(k)}_{n-k}$ it means that $(n-2k+1) W^{(k)}_{n-2k+1} \prec W^{(k)}_{n-k}$. Using Lemma 6 we conclude that $W^{(k)}_{n-2k+1} W^{(k)}_{n-2k+1} \prec W^{(k)}_{n-k}$ which contradicts to Lemma 14. □

Lemma 20. *let $A^2 = W^{(k)}_n[t, t+j]$ be a straddling square of $W^{(k)}_n$. Then $t > |W^{(k)}_n| - |W^{(k)}_{n-k}| - |W^{(k)}_{n-2k+1}|$.*

Proof. For contrary suppose that $A^2 = W_n^{(k)}[t, t+j] \prec W_n^{(k)}$ for some $t < |W_n^{(k)}| - |W_{n-k}^{(k)}| - |W_{n-2k+1}^{(k)}|$. By Lemma 12, $c_s(A^2, W_n^{(k)}) > |W_n^{(k)}| - |W_{n-k}^{(k)}|$. Hence,

$$W_n^{(k)}[t_1, t_2] \prec A \prec k \oplus W_{n-k}^{(k)}. \tag{18}$$

Where $t_1 = |W_n^{(k)}| - |W_{n-k}^{(k)}| - |W_{n-2k+1}^{(k)}| - 1$ and $t_2 = |W_n^{(k)}| - |W_{n-k}^{(k)}|$. By equation (17) an definition of t_1 and t_2, we have $W_n^{(k)}[t_1, t_2] = (n-2k+2)(k \oplus W_{n-2k+1}^{(k)})$. Therefore, by Equation (18) $(n-2k+2)(k \oplus W_{n-2k+1}^{(k)}) \prec k \oplus W_{n-k}^{(k)}$, this means that $n - 2k + 2 \geq k$. So, $(n-3k+2)W_{n-2k+1}^{(k)} \prec W_{n-k}^{(k)}$. Now, by Lemma 6, we conclude that $W_{n-3k+2}^{(k)} W_{n-2k+1}^{(k)} \prec W_{n-k}^{(k)}$. Since, $W_{n-3k+2}^{(k)} \lhd W_{n-2k+1}^{(k)}$ we conclude that $W_{n-3k+2}^{(k)} W_{n-3k+2}^{(k)} \prec W_{n-k}^{(k)}$, which is impossible by Lemma 14. $\square$

By Lemma 19, $A^2 \prec k \oplus (W_{n-2k+1}^{(k)} W_{n-k}^{(k)})$. Now using Equation(4), we have

$$A^2 \prec k \oplus \left(W_{n-2k+1}^{(k)} W_{n-k}^{(k)}\right) = k \oplus \left(W_{n-2k+1}^{(k)} W_{n-2k+3}^{(k)}[(W_{n-2k+3}^{(k)})^{-1} W_{n-k}^{(k)}]\right)$$

Definition 3. Let n, k be two nonnegative integers with $k \geq 3$ and $n > 2k - 1$. We define the word $V_n^{(k)}$ as follows:

- If $2k - 1 < n < 3k - 2$, then $V_n^{(k)} = W_{n-2k}^{(k)} W_{n-2k-1}^{(k)} \cdots W_0^{(k)}$;
- If $n \geq 3k - 2$, then $V_n^{(k)} = W_{n-2k}^{(k)} W_{n-2k-1}^{(k)} \cdots W_{n-3k+3}^{(k)}$.

Lemma 21. *Let n, k be two positive integers with $k \geq 3$ and $n > 2k - 1$.*

(i) *If $n < 3k - 2$, then*

$$W_{n-2k+2}^{(k)} = W_{n-2k+1}^{(k)} V_n^{(k)} (n - 2k + 2), \tag{19}$$
$$W_{n-2k+1}^{(k)} = V_n^{(k)} (n - 2k + 1). \tag{20}$$

(ii) *If $n = 3k - 2$, then*

$$W_{n-2k+2}^{(k)} = W_{n-2k+1}^{(k)} V_n^{(k)} k, \tag{21}$$
$$W_{n-2k+1}^{(k)} = V_n^{(k)} 0k. \tag{22}$$

(iii) *If $n > 3k - 2$, then*

$$W_{n-2k+2}^{(k)} = W_{n-2k+1}^{(k)} V_n^{(k)} (k \oplus W_{n-3k+2}^{(k)}), \tag{23}$$
$$W_{n-2k+1}^{(k)} = V_n^{(k)} W_{n-3k+2}^{(k)} (k \oplus W_{n-3k+1}^{(k)}). \tag{24}$$

Lemma 22. *Let n, k be two positive integers with $k \geq 3$ and $n > 2k - 1$. Then $V_n^{(k)}$ occurs exactly once in $W_{n-2k+1}^{(k)}$.*

Proof. If $2k-1<n<3k-2$, then using Definition 3 and the fact that $|V_n^{(k)}|_{n-2k+1}=0$ we conclude that $V_n^{(k)}$ occurs exactly once in $W_{n-2k+1}^{(k)}$. If $n\geq 3k-2$, then by (4) we have

$$W_{n-2k+1}^{(k)}=W_{n-2k}^{(k)}\dots W_{n-3k+2}^{(k)}(k\oplus W_{n-2k}^{(k)})=V_n^{(k)}W_{n-3k+2}^{(k)}(k\oplus W_{n-2k}^{(k)}).$$

By definition of $V_n^{(k)}$, $|V_n^{(k)}|_{n-2k}=1$ and $(n-2k)0\prec V_n^{(k)}$. Using the facts that $|W_{n-3k+2}^{(k)}|_{n-2k}=0$ and $|k\oplus W_{n-2k}^{(k)}|_0=0$. Hence, $(n-2k)0$ occurs one time in $W_{n-2k+1}^{(k)}$. So, $V_n^{(k)}$ also occurs once in $W_{n-2k+1}^{(k)}$. □

Corollary 23. *Let n,k be two positive integers with $k\geq 3$. If $2k-1<n<3k-2$, then $V_n^{(k)}$ in $W_{n-2k+1}^{(k)}$ always is followed by digit $n-2k+1$. If $n\geq 3k-2$, then $V_n^{(k)}$ in $W_{n-2k+1}^{(k)}$ always is followed by digit 0.*

Proof. If $2k-1<n<3k-2$, then the result follows using Lemma 22 and Definition 3. If $n\geq 3k-2$, then by (4), $V_n^{(k)}0\prec W_{n-2k+1}^{(k)}$. On the other hand, by Lemma 22, $V_n^{(k)}$ occurs exactly once in $W_{n-2k+1}^{(k)}$. Hence, the result follows. □

Lemma 24. *Let n,k be two nonnegative integers with $k\geq 3$ and $n>2k-1$. Then*

$$|W_{n-k}^{(k)}|\geq 3|W_{n-2k+1}^{(k)}|. \tag{25}$$

Proof. We can check easily that (25), holds in the cases $k=3$ and $5\leq n\leq 8$. If $k>3$ or $k=3$ and $n\geq 7$, then using (4), we get

$$|W_{n-k}^{(k)}|\geq |W_{n-k-1}^{(k)}|+|W_{n-k-2}^{(k)}|+|W_{n-k-3}^{(k)}| \tag{26}$$

If $k>3$, then $n-k-3\geq n-2k+1$. Hence, Equation (26) yields the inequality $|W_{n-k}^{(k)}|\geq 3|W_{n-2k+1}^{(k)}|$, as desired. If $k=3$, then

$$\begin{aligned}|W_{n-3}^{(k)}| &=|W_{n-4}^{(k)}|+|W_{n-5}^{(k)}|+|W_{n-6}^{(k)}|\\ &=2|W_{n-5}^{(k)}|+2|W_{n-6}^{(k)}|+|W_{n-7}^{(k)}|\\ &=3|W_{n-5}^{(k)}|+|W_{n-6}^{(k)}|-|W_{n-8}^{(k)}|\\ &>3|W_{n-5}^{(k)}|.\end{aligned}$$

□

To find all straddling squares of $W_n^{(k)}$ we need to give the following definition.

Definition 4. Let n,k be two nonnegative integers with $k\geq 3$ and $n\geq 2k-1$. We define the word $U_n^{(k)}$ to be the prefix of $W_{n-2k+1}^{(k)}W_{n-k}^{(k)}$ of size $4|W_{n-2k+1}^{(k)}|$.

We note that Definition 4 is well-defined using Lemma 24.

Lemma 25. *Let n, k be two nonnegative integers with $k \geq 3$ and $2k-1 < n < 3k-2$. Then*

$$U_n^{(k)}(k) = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)}(n-2k+2) W_{n-2k+1}^{(k)}.$$

Proof. Since $W_{n-2k+3}^{(k)} \lhd W_{n-k}^{(k)}$ we have

$$W_{n-2k+1}^{(k)} W_{n-k}^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+3}^{(k)} [(W_{n-2k+3}^{(k)})^{-1} W_{n-k}^{(k)}] \tag{27}$$

If $k = 3$, then applying Equation (4) for $W_{n-2k+3}^{(k)}$ and using (19) and (20) we get

$$\begin{aligned} W_{n-2k+3}^{(k)} =& W_{n-2k+2}^{(k)} W_{n-2k+1}^{(k)} (k \oplus W_{n-2k}^{(k)}) \\ =& \underbrace{W_{n-2k+1}^{(k)}}_{|W_{n-2k+1}^{(k)}|} \underbrace{V_n^{(k)}(n-2k+2)}_{|W_{n-2k+1}^{(k)}|} \underbrace{W_{n-2k+1}^{(k)}}_{|W_{n-2k+1}^{(k)}|} (k \oplus W_{n-2k}^{(k)}) \end{aligned} \tag{28}$$

Using Definition 4 and Equations (27) and (28) we conclude that

$$U_n^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)}(n-2k+2) W_{n-2k+1}^{(k)}.$$

If $k > 3$, then applying Equation (4) for $W_{n-2k+3}^{(k)}$ we get

$$W_{n-2k+2}^{(k)} W_{n-2k+1}^{(k)} W_{n-2k}^{(k)} \lhd W_{n-2k+3}^{(k)}$$

Hence, using (19) and (20) we provide

$$\underbrace{W_{n-2k+1}^{(k)}}_{|W_{n-2k+1}^{(k)}|} \underbrace{V_n^{(k)}(n-2k+2)}_{|W_{n-2k+1}^{(k)}|} \underbrace{W_{n-2k+1}^{(k)}}_{|W_{n-2k+1}^{(k)}|} W_{n-2k}^{(k)} \lhd W_{n-2k+3}^{(k)}$$

Therefore, by Definition 4 and Equation (27) we get

$$U_n^{(k)}(k) = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)}(n-2k+2) W_{n-2k+1}^{(k)}.$$

□

Lemma 26. *Let n, k be two nonnegative integers with $k \geq 3$ and $n > 3k-2$. Then*

(i) *If $k = 3$, then*

$$U_n^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)}(k \oplus W_{n-3k+2}^{(k)}) W_{n-2k+1}^{(k)} (k \oplus W_{n-3k+1}^{(k)}),$$

(ii) *If $k > 3$, then*

$$U_n^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)}(k \oplus W_{n-3k+2}^{(k)}) W_{n-2k+1}^{(k)} W_{n-3k+1}^{(k)}.$$

Proof. By Equation (4) we have

$$W_{n-2k+1}^{(k)} W_{n-k}^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+3}^{(k)} [(W_{n-2k+3}^{(k)})^{-1} W_{n-k}^{(k)}] \tag{29}$$

If $k = 3$, then applying Equation (4) for $W_{n-2k+3}^{(k)}$ and using (23) and (24) we get

$$\begin{aligned} W_{n-2k+3}^{(k)} =& W_{n-2k+2}^{(k)} W_{n-2k+1}^{(k)} (k \oplus W_{n-2k}^{(k)}) \\ =& \underbrace{W_{n-2k+1}^{(k)}}_{|W_{n-2k+1}^{(k)}|} \underbrace{V_n^{(k)} (k \oplus W_{n-3k+2}^{(k)}) W_{n-2k+1}^{(k)} (k \oplus W_{n-3k+1}^{(k)})}_{2|W_{n-2k+1}^{(k)}|} [(k \oplus W_{n-3k+1}^{(k)})^{-1} (k \oplus W_{n-2k}^{(k)})] \end{aligned} \tag{30}$$

Using Definition 4 and Equations (29) and (30) we conclude that

$$U_n^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)} (k \oplus W_{n-3k+2}^{(k)}) W_{n-2k+1}^{(k)} (k \oplus W_{n-3k+1}^{(k)}).$$

If $k > 3$, then applying Equation (4) for $W_{n-2k+3}^{(k)}$ we get

$$W_{n-2k+2}^{(k)} W_{n-2k+1}^{(k)} W_{n-2k}^{(k)} \lhd W_{n-2k+3}^{(k)}$$

Where,

$$W_{n-2k+2}^{(k)} W_{n-2k+1}^{(k)} W_{n-2k}^{(k)} = \underbrace{W_{n-2k+1}^{(k)}}_{|W_{n-2k+1}^{(k)}|} \underbrace{V_n^{(k)} (k \oplus W_{n-3k+2}^{(k)}) W_{n-2k+1}^{(k)} W_{n-3k+1}^{(k)}}_{2|W_{n-2k+1}^{(k)}|} [(W_{n-3k+1}^{(k)})^{-1} W_{n-2k}^{(k)}] \tag{31}$$

Using Definition 4 and Equations (29) and (31) we conclude that

$$U_n^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)} (k \oplus W_{n-3k+2}^{(k)}) W_{n-2k+1}^{(k)} W_{n-3k+1}^{(k)}.$$

□

In the next lemma we give a formula for $U_{3k-2}^{(k)}$, the proof is similar to the proof of Lemma 26 so it is omitted.

Lemma 27. *Let n, k be two nonnegative integers with $k \geq 3$ and $n = 3k - 2$. Then*

(i) *If $k = 3$, then*
$$U_n^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)} k W_{n-2k+1}^{(k)} k,$$

(ii) *If $k > 3$, then*
$$U_n^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)} k W_{n-2k+1}^{(k)} 0.$$

Corollary 28. *If A^2 is a straddling square of $W_n^{(k)}$, then*

(i) $A^2 \prec k \oplus U_n^{(k)}$,

(ii) $c_s(A^2 \ominus k, U_n^{(k)}) \leq 2|W_{n-2k+1}^{(k)}| + \frac{1}{2}$.

Proof. According to Definition 4 we have

(i) This is the direct consequence of Lemma 20 and equation (17).

(ii) This can be deducted easily from Lemma 19.

□

Lemma 29. *Let $n < 3k-1$, then the word $(n-2k+1)V_n^{(k)}(n-2k+1)$ occurs exactly once in $U_n^{(k)}$.*

Proof. Using Lemma 25 and Equation (20) we have

$$U_n^{(k)}(k) = V_n^{(k)}(n-2k+1)V_n^{(k)}(n-2k+1)V_n^{(k)}(n-2k+2)V_n^{(k)}(n-2k+1). \tag{32}$$

By Deifinition3, it is clear that $0 \lhd V_n^{(k)}$, $|V_n^{(k)}|_{n-2k+1} = 0$ and $|V_n^{(k)}|_{n-2k+2} = 0$. Hence, using (32) we conclude that $V_n^{(k)}$ occurs exactly four times in $U_n^{(k)}$. By Equation (32), the word $(n-2k+1)V_n^{(k)}(n-2k+1)$ occurs exactly once in $U_n^{(k)}$. □

Lemma 30. *Let $n \leq 3k-1$, then the word $(n-2k+1)V_n^{(k)}0$ occurs exactly once in $U_n^{(k)}$.*

Proof. By Deifinition3, it is clear that $0 \lhd V_n^{(k)}$, $|V_n^{(k)}|_{n-2k+1} = 0$ and $|V_n^{(k)}|_{n-2k+2} = 0$. Hence, using Lemmas 27 and 26 and using Lemma 22, we conclude that $V_n^{(k)}$ occurs exactly four times in $U_n^{(k)}$. Therefore, by Equations (22) and (24) it is easy to see that $(n-2k+1)V_n^{(k)}0$ occurs exactly once in $U_n^{(k)}$.

□

Lemma 31. *Let $n \geq 2k-1$ and A^2 be a straddling square of $W_n^{(k)}$ and let $A'^2 = A^2 \ominus k$. Then*

$$c_s(A'^2, U_n^{(k)}) \leq |W_{n-2k+1}^{(k)}| + |V_n^{(k)}| + \frac{1}{2}. \tag{33}$$

Proof. For contrary suppose that $c_s(A'^2, U_n^{(k)}) > |W_{n-2k+1}^{(k)}| + |V_n^{(k)}| + \frac{1}{2}$. We divide the proof in the following cases:

- If $n < 3k-2$, then by Lemma 25 and Equation (20) we conclude that

$$W_{n-2k+1}^{(k)}V_n^{(k)}(n-2k+1) \lhd U_n^{(k)}. \tag{34}$$

Using (34) and the fact that $A'^2 \oplus k$ is a straddling square of $W_n^{(k)}$, we conclude that $(n-2k+1)V_n^{(k)}(n-2k+1)$ should occurs at least twice in $A'^2 \prec U_n^{(k)}$. This is a contradiction with Lemma 29.

- If $n < 3k-2$, then by Lemma 25 and Equation (20) we conclude that

$$W_{n-2k+1}^{(k)} V_n^{(k)} 0 \lhd U_n^{(k)}. \tag{35}$$

Using (35) and the fact that $A'^2 \oplus k$ is a straddling square of $W_n^{(k)}$, we conclude that $(n-2k+1)V_n^{(k)}0$ should occurs at least twice in $A'^2 \prec U_n^{(k)}$. This is a contradiction with Lemma 30.

□

The following corollary is the direct consequence of Lemma 31.

Corollary 32. *let A^2 be a straddling square of $W_n^{(k)}$. Then $c_s(A^2, W_n^{(k)}) < |W_n^{(k)}| - |W_{n-k}^{(k)}| + |V_n^{(k)}|$.*

Lemma 33. *Let $n \geq 2k-1$, $P_n^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)}$. Then A^2 is a straddling square of $W_n^{(k)}$ if and only if $A'^2 = A^2 \ominus k$ is a square of $P_n^{(k)}$ satisfying following properties:*

(i) $|W_{n-2k+1}^{(k)}| \leq c_s(A'^2, P_n^{(k)}) \leq |W_{n-2k+1}^{(k)}| + |V_n^{(k)}|$,

(ii) *Let $A'^2 = P_n^{(k)}[t, t+|A'^2|]$. Then, $t < |W_{n-2k+1}^{(k)}|$.*

Proof. Let A^2 is a straddling square of $W_n^{(k)}$. Then by Corollary 28 we have $A^2 \ominus k \prec U_n^{(k)}$. Using Lemmas 12 and 31 we conclude that $|W_{n-2k+1}^{(k)}| \leq c_s(A'^2, P_n^{(k)}) \leq |W_{n-2k+1}^{(k)}| + |V_n^{(k)}|$. Moreover, since A^2 is a straddling factor of $W_n^{(k)}$ A'^2 satisfying (ii).

Now, let A'^2 is a square of $P_n^{(k)}$ which satisfies (i) and (ii), then we prove that $A^2 = A'^2 \oplus k$ is a straddling square of $W_n^{(k)}$. By Lemma 15, we conclude that $n \geq 2k-1$. Using Equation (4) for $W_n^{(k)}$ and $W_{n-k+1}^{(k)}$ we have

$$\begin{aligned} W_n^{(k)} &= W_{n-1}^{(k)} \dots W_{n-k+1}^{(k)} (k \oplus W_{n-k}^{(k)}) \\ W_n^{(k)} &= W_{n-1}^{(k)} \dots W_{n-k+2}^{(k)} (W_{n-k}^{(k)} \dots W_{n-2k+2}^{(k)} (k \oplus W_{n-2k+1}^{(k)}))(k \oplus W_{n-k}^{(k)}) \end{aligned}$$

Since $W_{n-2k+1}^{(k)} V_n^{(k)} \lhd W_{n-k}^{(k)}$, we conclude that

$$W_n^{(k)} = W_{n-1}^{(k)} \dots W_{n-k+2}^{(k)} W_{n-k}^{(k)} \dots W_{n-2k+2}^{(k)} \underbrace{(k \oplus W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)})}_{P_n^{(k)}} (k \oplus (W_{n-2k+1}^{(k)} V_n^{(k)})^{-1} W_{n-k}^{(k)}) \tag{36}$$

Using the fact that A'^2 satisfies (i) and (ii) and as shown in Equation (36), we conclude that $A^2 = A'^2 \oplus k$ is a straddling square of $W_n^{(k)}$. □

Theorem 34. *Let $0 \le j \le |V_n^{(k)}|$. Then $(C^{(j)}(k \oplus W_{n-2k+1}^{(k)}))^2$ is a straddling square of $W_n^{(k)}$. Moreover, every straddling square A^2 of $W_n^{(k)}$ is of the form $A^2 = (C^{(j)}(k \oplus W_{n-2k+1}^{(k)}))^2$, for some $0 \le j \le |V_n^{(k)}|$.*

Proof. By Lemma 21, $V_n^{(k)} \lhd W_{n-2k+1}^{(k)}$. Hence, there exists suffix V' of $W_{n-2k+1}^{(k)}$ such that $W_{n-2k+1}^{(k)} = V_n^{(k)} V'$. Let $0 \le j \le |V_n^{(k)}|$ and $V_1 = V_n^{(k)}[1, j]$ and $V_2 = V_n^{(k)}[j+1, |V_n^{(k)}|]$. Then

$$P_n^{(k)} = V_1 \quad \overbrace{V_2 V' V_1}^{C^{(j)}(W_{n-2k+1}^{(k)})} \quad \underbrace{V_2 V' V_1}_{C^{(j)}(W_{n-2k+1}^{(k)})} \quad V_2 \tag{37}$$

Now, using Lemma 33 and Equation (37), $(C^{(j)}(k \oplus W_{n-2k+1}^{(k)}))^2$ is a straddling square of $W_n^{(k)}$.

Moreover, let A^2 be a straddling square of $W_n^{(k)}$. Hence the first A in A^2 should contains $n-k+1$. By Lemma 33 $A'^2 = A^2 \ominus k$ is a square factor of $P_n^{(k)}$ satisfying the conditions of the lemma. Therefore, $|A'|_{n-2k+1} \ge 1$. By Lemma 33, we can assume that $c_s(A'^2, P_n^{(k)}) = |W_{n-2k+1}^{(k)}| + j + \frac{1}{2}$ for some $0 \le j \le |V_n^{(k)}|$.

Again using Lemma 21, $V_n^{(k)} \lhd W_{n-2k+1}^{(k)}$. Let $V' \rhd W_{n-2k+1}^{(k)}$ such that $W_{n-2k+1}^{(k)} = V_n^{(k)} V'$, $V_1 = V_n^{(k)}[1, j]$ and $V_2 = V_n^{(k)}[j+1, |V_n^{(k)}|]$. Therefore, for the first A' in A'^2 we have $A' \rhd V_1 V_2 V' V_1$ and for the last A' in A'^2, $A' \lhd V_2 V' V_1 V_2$. On the other hand $V_1 V_2 V' V_1 \prec W_{n-2k+1}^{(k)} V_n^{(k)}$ and by Definition 3 and Lemma 5, $|W_{n-2k+1}^{(k)} V_n^{(k)}|_{n-2k+1} = 1$, hence $|A'|_{n-2k+1} = 1$. Since the first place that $n-2k+1$ occurs in $V_2 V' V_1 V_2$ is $|V_2 V'|$, hence $V_2 V' \lhd A'$. Since the number of occurrences of $V_2 V'$ in $V_1 V_2 V' V_1$ is once. We conclude that $A' = V_2 V' V_1$. $\square$

Theorem 35. *Let $k \ge 3$. Then A^2 is a square of $W^{(k)}$ if and only if $A \in \{ki \oplus C^j(W_{n-2k+1}^{(k)}) : 0 \le j \le |V_n^{(k)}|, i > 0, n \ge 0\}$.*

Proof. If $A = ki \oplus C^j(W_{n-2k+1}^{(k)})$, for some $0 \le j \le |V_n^{(k)}|, i > 0, n \ge 0$, then by Theorem 34 $k \oplus C^j(W_{n-2k+1}^{(k)}) = (A^2 \ominus k(i-1)) \prec W_n^{(k)}$ or equivalently $A^2 \prec W_n^{(k)} \oplus k(i-1)$. By Corollary 4, we conclude that $A^2 \prec W_{n+k(i-1)}^{(k)} \prec W^{(k)}$.

On the other hand, if A^2 is a square of $W^{(k)}$, then by Corollary 18, there exist $n > 2k-1, i > 0$, such that $A^2 \ominus k(i-1)$ is a straddling square of $W_n^{(k)}$. By Corollary 4, we conclude that $A^2 \prec W_{n+k(i-1)}^{(k)}$. $\square$

We finish this section with the following example.

Example 1. In this example we provide all square factors of $W_{11}^{(3)}$, which is given bellow. All of these squares are listed in Table 1 according to Theorem 35. We note that letters a and b stand for the digits 10 and 11.

$$
\begin{aligned}
W_{11}^{(3)} = {}&0102013010234010201334350102013010234343534601020130102340102013343534353463435 67\\
&0102013010234010201334350102013010234343534634353463435673435346676801020130102 34\\
&0102013343501020130102343435346010201301023401020133435343534634356734353463435 67\\
&3435346676834353463435676768679010201301023401020133435010201301023434353460102 013\\
&0102340102013343534353463435670102013010234010201334350102013010234343534634353 463\\
&4356734353466768343534634356734353466768343534634356767686793435346343567343534 667\\
&68676867967689a010201301023401020133435010201301023434353460102013010234010201334 3\\
&5343534634356701020130102340102013343501020130102343435346343534634356734353466 768\\
&0102013010234010201334350102013010234343534601020130102340102013343534353463435 673\\
&4353463435673435346676834353463435676768679343534634356734353466768343534634356 767\\
&686793435346343567343534667686768679676 89a3435346343567343534667683435346343567 676\\
&8679676867967689a67686799a9b
\end{aligned}
$$

Table 1: Square factors of $W_{11}^{(3)}$

	j \ i	1	2	3
$C^{(j)}(ki \oplus W_0^{(3)})$	0	3	6	9
$C^{(j)}(ki \oplus W_1^{(3)})$	0	34	67	-
	1	43	76	-
$C^{(j)}(ki \oplus W_2^{(3)})$	0	3435	6768	-
	1	4353	7686	-
	2	3534	6867	-
$C^{(j)}(ki \oplus W_3^{(3)})$	0	3435346	6768679	-
	1	4353463	7686796	-
	2	3534634	6867967	-
	3	5346343	8679676	-
	4	3463435	6796768	-
$C^{(j)}(ki \oplus W_4^{(3)})$	0	3435346343567	-	-
	1	4353463435673	-	-
	2	3534634356734	-	-
	3	5346343567343	-	-
	4	3463435673435	-	-
	5	4634356734353	-	-
	6	6343567343534	-	-
	7	3435673435346	-	-
$C^{(j)}(ki \oplus W_4^{(3)})$	0	3435346343567343534667 68	-	-
	1	4353463435673435346676 83	-	-
	2	3534634356734353466768 34	-	-
	3	5346343567343534667683 43	-	-
	4	3463435673435346676834 35	-	-
	5	4634356734353466768343 53	-	-
	6	6343567343534667683435 34	-	-
	7	3435673435346676834353 46	-	-
	8	4356734353466768343534 63	-	-
	9	3567343534667683435346 34	-	-
	10	5673435346676834353463 43	-	-
	11	6734353466768343534634 35	-	-
	12	7343534667683435346343 56	-	-
	13	3435346676834353463435 67	-	-

	j \ i	1	2	3
$C^{(j)}(ki \oplus W_4^{(3)})$	0	34353463435673435346676834353463435676768679	-	-
	1	43534634356734353466768343534634356767686793	-	-
	2	35346343567343534667683435346343567676867934	-	-
	3	53463435673435346676834353463435676768679343	-	-
	4	34634356734353466768343534634356767686793435	-	-
	5	46343567343534667683435346343567676867934353	-	-
	6	63435673435346676834353463435676768679343534	-	-
	7	34356734353466768343534634356767686793435346	-	-
	8	43567343534667683435346343567676867934353463	-	-
	9	35673435346676834353463435676768679343534634	-	-
	10	56734353466768343534634356767686793435346343	-	-
	11	67343534667683435346343567676867934353463435	-	-
	12	73435346676834353463435676768679343534634356	-	-
	13	34353466768343534634356767686793435346343567	-	-
	14	43534667683435346343567676867934353463435673	-	-
	15	35346676834353463435676768679343534634356734	-	-
	16	53466768343534634356767686793435346343567343	-	-
	17	34667683435346343567676867934353463435673435	-	-
	18	46676834353463435676768679343534634356734353	-	-
	19	66768343534634356767686793435346343567343534	-	-
	20	67683435346343567676867934353463435673435346	-	-
	21	76834353463435676768679343534634356734353466	-	-
	22	68343534634356767686793435346343567343534667	-	-
	23	83435346343567676867934353463435673435346676	-	-

5 Critical Exponent and Critical Factors of $W^{(k)}$

Lemma 36. *let A^2 be a straddling square of $W_n^{(k)}$. Then*

$$\mathrm{INDEX(A, W_n^{(k)})} = \begin{cases} 3 - \frac{1}{2^{n-2k+1}} & \text{if } 2k-1 \leq n \leq 3k-3, \\ 3 - \frac{1}{2^{k-2}} & \text{if } n = 3k-2, \\ 3 - \frac{|W_{n-3k+2}^{(k)}| + |W_{n-3k+1}^{(k)}|}{|W_{n-2k+1}^{(k)}|} & \text{if } n > 3k-2. \end{cases}$$

Proof. By Lemma 33, $A^2 \ominus k \prec P_n^{(k)} = W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)}$. If $2k-1 \leq n \leq 3k-3$, then by Definition 3.

$$\begin{aligned} P_n^{(k)} &= W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} V_n^{(k)} \\ &= W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} W_{n-2k}^{(k)} \ldots W_0^{(k)} \\ &= W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} W_{n-2k+1}^{(k)} (n-2k+1)^{-1} \\ &= (W_{n-2k+1}^{(k)})^{3 - \frac{1}{2^{n-2k+1}}}. \end{aligned}$$

Where the last equality holds since $|W_{n-2k+1}^{(k)}| = 2^{n-2k+1}$. If $n = 3k-2$, then by Definition 3.

$$\begin{aligned} P_{3k-2}^{(k)} &= W_{k-1}^{(k)} W_{k-1}^{(k)} V_{3k-2}^{(k)} \\ &= W_{k-1}^{(k)} W_{k-1}^{(k)} W_{k-2}^{(k)} \ldots W_1^{(k)} \\ &= W_{k-1}^{(k)} W_{k-1}^{(k)} W_{k-1}^{(k)} (0(k-1))^{-1} \\ &= (W_{k-1}^{(k)})^{3 - \frac{1}{2^{k-2}}}. \end{aligned}$$

In the case $n > 3k-2$, again by using Definition 3, we conclude that

$$P_n^{(k)} = (W_{n-2k+1}^{(k)})^{3 - \frac{|W_{n-3k+2}^{(k)}| + |W_{n-3k+1}^{(k)}|}{|W_{n-2k+1}^{(k)}|}}.$$

□

In the following example for $k = 5$ and $9 \leq n \leq 17$, we show that how Lemma 36 works.

Example 2. In this example we listed all $P_n^{(5)}$, when $9 \leq n \leq 17$ and for all values of n we present the corresponding power r. We note that letters a and b stand for the digits 10 and 11.

As we can see in Table 2 in Example 2, the largest power of $W_{n-9}^{(5)} \oplus k$ in $P_n^{(5)} \oplus k$ is $3 - \frac{3}{31}$. This power happens when $n = 14$, which is the critical exponent of $P_n^{(5)} \oplus k$. Moreover, in the following Theorem we show that this r is also the critical exponent of $W^{(5)}$.

Table 2: Powers of $W_{n-9}^{(5)}$ in $W_n^{(5)}$

n	$P_n^{(5)} \oplus k = (W_{n-9}^{(5)} \oplus k)^r$	r
9	55	2
10	56565	$3-\frac{1}{2}$
11	56575657565	$3-\frac{1}{4}$
12	5657565856575658565757565	$3-\frac{1}{8}$
13	565756585657565956575658565756595657565856575 6	$3-\frac{1}{8}$
14	565756585657565956575658565756a565756585657565956575658565756a 5657565856575659565756585657	$3-\frac{3}{31}$
15	565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a565756585657565956575658	$3-\frac{6}{61}$
16	565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a565756585657565956575658abac 565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a565756585657565956575658abac 565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a5657565856575659	$3-\frac{12}{120}$
17	565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a565756585657565956575658abac 565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a5657565856575659abacabad 565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a565756585657565956575658abac 565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a5657565856575659abacabad 565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a565756585657565956575658abac 565756585657565956575658565756a5657565856575659565756585657ab 565756585657565956575658565756a	$3-\frac{24}{236}$

Theorem 37. *Let $k \geq 3$, then the critical exponent of $W^{(k)}$ equals to $3 - \frac{3}{2^k-1}$. Moreover, the set of all critical factors of $W^{(k)}$ is $\{P^{(k)}_{3k-1} \oplus ki\}$.*

Proof. By Theorem 34, for all $n \geq 2k-1$, $W^{(k)}_n$ always contains a square factor. Hence $E(W^{(k)}) \geq 2$. Let $A \in F(W^{(k)})$ and $r = \mathrm{INDEX}(A) \geq 2$. We will prove that $r \leq 3 - \frac{3}{2^{k-1}}$. Since $r \geq 2$, A^2 is a square factor of $W^{(k)}$. By Corollary 18 there exisit integers i and n such that $A^2 \ominus ki$ is a straddling square of $W^{(k)}_n$. Let $m_1 = \max\{3 - \frac{1}{2^{n-2k+1}} : 2k-1 \leq n \leq 3k-3\}$, $m_2 = 3 - \frac{1}{2^{k-2}}$ and $m_3 = \max\{3 - \frac{|W^{(k)}_{n-3k+2}| + |W^{(k)}_{n-3k+1}|}{|W^{(k)}_{n-2k+1}|} : n \geq 3k-1\}$. Now, using Lemma 36 we conclude that $r \leq \max\{m_1, m_2, m_3\}$. It is easy to check that $m_1 = m_2 = 3 - \frac{1}{2^{k-2}}$. Since $g(n) = 3 - \frac{|W^{(k)}_{n-3k+2}| + |W^{(k)}_{n-3k+1}|}{|W^{(k)}_{n-2k+1}|}$ is a decreasing function of n, we conclude that $m_3 = g(3k-1) = 3 - \frac{3}{2^k-1}$. Hence $r \leq 3 - \frac{3}{2^k-1}$. On the other hand, by Lemma 33, $P^{(k)}_{3k-1} \oplus k = (W^{(k)}_k)^{3-\frac{3}{2^k-1}}$. This implies that the set of all critical factors of $W^{(k)}$ equals to $\{P^{(k)}_{3k-1} \oplus ki : i \geq 1\}$. □

Example 3. In Table 3, we compute the critical exponent and one of the critical factors of $W^{(k)}$, for $3 \leq k \leq 8$, according to Theorem 37. We note that the digits $10, 11, \ldots, 16$ are denoted by the letters $a, b, \ldots g$, respectively.

Table 3: the critical exponent and one of the critical factors of $W^{(k)}$

k	$P^{(k)}_{3k-1} \oplus k = (W^{(k)}_k \oplus k)^r$	$r = 3 - \frac{3}{2^k-1}$
3	343534634353463435	$3 - \frac{3}{7}$
4	4546454745464584546454745464584546454745 46	$3 - \frac{3}{15}$
5	5657565856575659565756585657566a5657565856575659565756585657566a 565756585657565956575658565 7	$3 - \frac{3}{31}$
6	676867696768676a676867696768676b676867696768676a67686769676867c 676867696768676a676867696768676b676867696768676a67686769676867c 676867696768676a676867696768676b676867696768676a676867696768	$3 - \frac{3}{63}$
7	7879787a7879787b7879787a7879787c7879787a7879787b7879787a7879787d 7879787a7879787b7879787a7879787c7879787a7879787b7879787a787978e 7879787a7879787b7879787a7879787c7879787a7879787b7879787a7879787d 7879787a7879787b7879787a7879787c7879787a7879787b7879787a787978e 7879787a7879787b7879787a7879787c7879787a7879787b7879787a7879787d 7879787a7879787b7879787a7879787c7879787a7879787b7879787a7879	$3 - \frac{3}{127}$
8	898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a898e 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a898f 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a898e 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a89g 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a898e 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a898f 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a898e 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a89g 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a898e 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a898f 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a898e 898a898b898a898c898a898b898a898d898a898b898a898c898a898b898a	$3 - \frac{3}{255}$

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