# ON STURMIAN SUBSTITUTIONS CLOSED UNDER DERIVATION

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ABSTRACT. Occurrences of a factor $w$ in an infinite uniformly recurrent sequence $\mathbf{u}$ can be encoded by an infinite sequence over a finite alphabet. This sequence is usually denoted $\mathbf{d_u}(w)$ and called the derived sequence to $w$ in $\mathbf{u}$. If $w$ is a prefix of a fixed point $\mathbf{u}$ of a primitive substitution $\varphi$, then by Durand's result from 1998, the derived sequence $\mathbf{d_u}(w)$ is fixed by a primitive substitution $\psi$ as well. For a non-prefix factor $w$, the derived sequence $\mathbf{d_u}(w)$ is fixed by a substitution only exceptionally. To study this phenomenon we introduce a new notion: A finite set $M$ of substitutions is said to be closed under derivation if the derived sequence $\mathbf{d_u}(w)$ to any factor $w$ of any fixed point $\mathbf{u}$ of $\varphi \in M$ is fixed by a morphism $\psi \in M$. In our article we characterize the Sturmian substitutions which belong to a set $M$ closed under derivation. The characterization uses either the slope and the intercept of its fixed point or its S-adic representation.

*Keywords:* return word, derived sequence, Sturmian word, S-adic representation, fixed point, primitive morphism
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## 1. Introduction

In combinatorics of words, the notion of return words to a factor of an infinite word is an analogue to first return map in dynamical systems. Given an infinite word $\mathbf{u} = u_0u_1u_2\ldots$ with $u_i$ being an element of a finite alphabet, we say that $u_iu_{i+1}\cdots u_{j-1}$ is a return word to a factor $w$ if for each $k$ satisfying $i \leq k \leq j$, the factor $w$ is a prefix of the infinite word $u_ku_{k+1}u_{k+2}\ldots$ only for $k = i$ and $k = j$. We study infinite words $\mathbf{u}$ such that every factor of $\mathbf{u}$ occurs infinitely many times and has a finite number of return words. These words are called uniformly recurrent. This class of words includes purely periodic words. Obviously, any factor $w$ of a purely periodic word $\mathbf{u}$ which is longer than the period has just one return word. On the other hand, if a uniformly recurrent word $\mathbf{u}$ has a factor having only one return word, then $\mathbf{u}$ is purely periodic.

In this article, we focus on uniformly recurrent words which have exactly 2 return words to each factor. As shown by Vuillon in [16], such words are exactly the infinite Sturmian words, i.e., aperiodic words having the least factor complexity possible.

If a factor $w$ of a uniformly recurrent word $\mathbf{u}$ has $k$ return words, then the order of their occurrences in $\mathbf{u}$ can be coded by an infinite word over a $k$-letter alphabet. This word is denoted by $\mathbf{d_u}(w)$ and called the derived word of $w$ in $\mathbf{u}$. Derived words to prefixes $w$ of $\mathbf{u}$ were introduced by Durand in [5] in order to characterize primitively substitutive infinite words.

Among other, Durand showed that if $w$ is a prefix of an infinite word $\mathbf{u}$ fixed by a primitive substitution, then $\mathbf{d_u}(w)$ is fixed by a primitive substitution as well. Taking all such prefixes $w$, the set of derived words $\mathbf{d_u}(w)$ is finite, and thus the set of primitive substitutions fixing these derived words to prefixes is finite. Moreover, if we consider derived words to a prefix of a derived word $\mathbf{d_u}(w)$, we obtain again a derived word to some prefix $w'$ of the original word $\mathbf{u}$. Thus, we observe that the finite set of primitive substitutions fixing the derived words to prefixes is invariant under taking derived word to a prefix and considering its fixing primitive substitution.

If $w$ is not a prefix, then $\mathbf{d_u}(w)$ need not be fixed by a substitution at all. To study this phenomenon, we introduce the following definition.

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**Definition 1.** *Let $M$ be a finite set of primitive substitutions. The set $M$ is said to be* closed under derivation *if the derived word $\mathbf{d_u}(w)$ with respect to any factor $w$ of any fixed point $\mathbf{u}$ of $\varphi \in M$ is fixed by a primitive substitution $\psi \in M$.*

*A primitive substitution $\xi$ is called* closeable under derivation *if $\xi$ belongs to a set $M$ closed under derivation.*

Substitutions having Sturmian words as fixed points are very well described (see [13]) and there exists a handy tool to characterize which Sturmian words are fixed by a substitution (see [17]). It is therefore convenient to start the study of sets closed under derivation by considering Sturmian words and Sturmian substitutions. In this article, we fully solve this question: in Theorems 20 and 24 we characterize Sturmian substitutions that are closeable under derivation; the characterization is presented in the terms of the representation in the special Sturmian monoid (generated by the morphisms given in (2) below) and also alternatively in terms of the slope and the intercept of its fixed point (Theorem 33).

The article is organized as follows. Section 2 contains necessary definitions and notions. In Sections 3 and 4 we introduce needed results on Sturmian words and morphisms. Section 5 contains auxiliary lemmas required in the last Sections 6 and 7, where we deal with the case of Sturmian morphism that are not closed under derivation and Sturmian morphisms that are closed under derivation, respectively.

## 2. Preliminaries

Let $\mathcal{A}$ denote an *alphabet*, a finite set of symbols called *letters*. A *finite word* of length $n$ over $\mathcal{A}$ is a concatenation of $n$ letters, i.e., $u = u_0u_1 \cdots u_{n-1}$ with $u_i \in \mathcal{A}$. The *length* of $u$ equals $n$ and is denoted by $|u|$. The set of all finite words over the alphabet $\mathcal{A}$ and the operation of word concatenation form a monoid $\mathcal{A}^*$. The unique word of length 0, the *empty word* $\varepsilon$, is its neutral element. The *cyclic shift* of the word $u$ is the word

$$\operatorname{cyc}(u) = u_1u_2 \cdots u_{n-1}u_0. \tag{1}$$

An *infinite word* over $\mathcal{A}$ is a sequence $\mathbf{u} = u_0u_1u_2 \cdots = (u_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ with $u_i \in \mathcal{A}$ for all $i \in \mathbb{N} = \{0, 1, 2, \ldots\}$. A finite word $w$ is a *factor* of $\mathbf{u}$ if there exists an integer $i$ such that $w = u_iu_{i+1}u_{i+2} \cdots u_{i+|w|-1}$. The index $i$ is an *occurrence* of $w$ in $\mathbf{u}$. The *language* $\mathcal{L}(\mathbf{u})$ *of* $\mathbf{u}$ is the set of all its factors. A factor $w$ is a *right special* factor if there exist at least two distinct letters $a, b \in \mathcal{A}$ such that $wa, wb \in \mathcal{L}(\mathbf{u})$. A *left special* factor is defined analogously. A factor is *bispecial* if it is left and right special.

Given a word $u$, finite or infinite, and finite words $p, v$ and a word $s$ such that $u = pvs$, then we say that $p$ is a *prefix* of $u$ and $s$ is its *suffix*. The prefix $p$ is *proper* if $p \neq \varepsilon$ and $p \neq u$.

If each factor of $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$ has infinitely many occurrences in $\mathbf{u}$, the word $\mathbf{u}$ is *recurrent*. Given a recurrent infinite word $\mathbf{u}$ and its factor $w$, a *return word of* $w$ *in* $\mathbf{u}$ is a factor $v \in \mathcal{L}(\mathbf{u})$ such that $vw \in \mathcal{L}(\mathbf{u})$ and the factor $w$ occurs in $vw$ exactly twice — once as a prefix and once as a suffix. Assume there is an integer $k$ such that $r_0, r_1, \ldots r_k$ are all return words of $w$ in $\mathbf{u}$. We can write $\mathbf{u} = pr_{s_0}r_{s_1}r_{s_2}\ldots$ with $|p|$ equal the least occurrence of $w$ in $\mathbf{u}$ and $s_i \in \{0, 1, \ldots, k\}$. We say that the word $(s_i)_{i=0}^{+\infty}$ is the *derived word of* $\mathbf{u}$ *with respect to* $w$, denoted $\mathbf{d_u}(w)$. For $w$ being a prefix, these words were introduced in [5]. For a general factor $w$, they are investigated in [7].

In this article, we consider derived words up to a permutation of letters, i.e., we do take into account the indexing of the return words when comparing derived words.

A mapping $\psi : \mathcal{A}^* \to \mathcal{A}^*$ is a *morphism* over $\mathcal{A}^*$ if $\psi(uv) = \psi(u)\psi(v)$ for each $u, v \in \mathcal{A}^*$. The domain of $\psi$ is extended to $\mathcal{A}^{\mathbb{N}}$ naturally by $\psi(\mathbf{u}) = \psi(u_0u_1u_2 \ldots) = \psi(u_0)\psi(u_1)\psi(u_2)\ldots$ for $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$. If $\psi(\mathbf{u}) = \mathbf{u}$, we say that $\mathbf{u}$ is a *fixed point* of $\psi$. A morphism $\psi$ is *primitive* if there exists an integer $k$ such that for each pair of letters $a, b \in \mathcal{A}$ the word $\psi^k(a)$ contains the letter $b$.

In [5], a morphism $\psi$ over $\mathcal{A}$ is called *substitution* if there exists a letter $a \in \mathcal{A}$ such that $\psi(a) = aw$ for some non-empty word $w$ and the length of the $n^{th}$ iteration of $\psi$ applied to $a$ tends to infinity, i.e., $|\psi^n(a)| \to +\infty$. Clearly, any substitution has at least one fixed point, namely $\mathbf{u} = aw\psi(w)\psi^2(w)\psi^3(w)\cdots$. This fixed point is usually denoted as $\lim_{n\to\infty} \psi^n(a)$. A primitive morphism $\psi$ has some power $\psi^k$ which is a substitution. For example, a morphism given by $\varphi(0) = 100$ and $\varphi(1) = 0$ is not a substitution, but it is primitive as $\varphi^2(0) = 0100100$ and $\varphi^2(1) = 100$. The morphism $\varphi^2$ is a substitution and has two fixed points, namely $\lim_{n\to\infty} \varphi^{2n}(0)$ and $\lim_{n\to\infty} \varphi^{2n}(1)$.

If $\mathbf{u}$ is a fixed point by $\varphi$, then it is also fixed by $\varphi$ for all $k \in \mathbb{N}$. An infinite word $\mathbf{u}$ is *rigid* if the set of all morphisms which fix $\mathbf{u}$ is of the form $\left\{\varphi^k : k \in \mathbb{N}\right\}$ for some morphism $\varphi$.

## 3. Sturmian words

Sturmian words are infinite words over a two letter alphabet having the least unbounded factor complexity possible. In other words, an infinite word is *Sturmian* if for each $n \in \mathbb{N}$ the number of its factors of length $n$ equals $n+1$. There are many other characterizations of Sturmian words. For the phenomenon that we investigate, the characterization based on the notion of interval exchange transformation is the most suitable.

For a given parameters $\ell_0, \ell_1 > 0$, we consider the partition of the interval $I = [0, \ell_0 + \ell_1)$ into $I_0 = [0, \ell_0)$ and $I_1 = [\ell_0, \ell_0 + \ell_1)$ or the partition of $I = (0, \ell_0 + \ell_1]$ into $I_0 = (0, \ell_0]$ and $I_1 = (\ell_0, \ell_0 + \ell_1]$. The transformation $T : I \to I$ defined by

$$T(x) = \begin{cases} x + \ell_1 & \text{if } x \in I_0, \\ x - \ell_0 & \text{if } x \in I_1 \end{cases}$$

is a *two interval exchange transformation*, or shortly *2iet*. If we take an initial point $\rho \in I$, the sequence $\mathbf{u} = u_0 u_1 u_2 \cdots \in \{0,1\}^{\mathbb{N}}$ defined by

$$u_n = \begin{cases} 0 & \text{if } T^n(\rho) \in I_0, \\ 1 & \text{if } T^n(\rho) \in I_1 \end{cases}$$

is a *2iet sequence* with the *parametrs* $\ell_0, \ell_1, \rho$. In other words, a 2iet sequence is a coding of itineraries $(T^n(\rho))_{n=0}^{+\infty}$ with respect to the partition $I_0 \cup I_1$. The value $\gamma = \frac{\ell_1}{\ell_0+\ell_1}$ is called the *slope* of $\mathbf{u}$. It is well known that the set of all 2iet sequences having an irrational slope coincides with the set of all Sturmian words (see for instance [13]). If we need to distinguish whether a Sturmian word comes from a transformation with the domain $I = [0, \ell_0 + \ell_1)$ or with the domain $I = (0, \ell_0 + \ell_1]$ we use the names *lower and upper Sturmian word*, respectively. For most of the parameters $\rho \in (0, \ell_0 + \ell_1)$ the lower Sturmian word with the parameters $\ell_0, \ell_1, \rho$ equals to the upper Sturmian word with the same parameters.

Clearly, lower (upper) Sturmian words corresponding to the triplets $(\ell_0, \ell_1, \rho)$ and $(c\ell_0, c\ell_1, c\rho)$ coincide for any positive constant $c$. It is the reason for the triplet of parameters $\ell_0, \ell_1, \rho$ to be often normalized into the form

$$\Big(\frac{\ell_0}{\ell_0+\ell_1}, \frac{\ell_1}{\ell_0+\ell_1}, \frac{\rho}{\ell_0+\ell_1}\Big) = (1-\gamma, \gamma, \delta),$$

where $\gamma$ is the slope. The lower Sturmian word with parameters $(1-\gamma, \gamma, \delta)$ where $\delta \in [0,1)$ is in [13] denoted by $\mathbf{s}_{\gamma,\delta}$ and the upper Sturmian word with parameters $(1-\gamma, \gamma, \delta)$ where $\delta \in (0,1]$ is denoted by $\mathbf{s'}_{\gamma,\delta}$.

The language $\mathcal{L}(\mathbf{u})$ of a Sturmian word $\mathbf{u}$ is independent of the parameter $\rho$, it depends only on the slope $\gamma = \frac{\ell_1}{\ell_0+\ell_1}$. Any Sturmian word is uniformly recurrent. Frequencies of the letters 0 and 1 are $1-\gamma$ and $\gamma$, respectively. Among all Sturmian words with a fixed irrational slope $\gamma = \frac{\ell_1}{\ell_0+\ell_1}$, the sequence with the triplet of parameters $(\ell_0, \ell_1, \ell_1)$ plays a special role. Such a sequence is called a *standard Sturmian word* and it is usually denoted by $\mathbf{c}_\gamma$. Any prefix of $\mathbf{c}_\gamma$ is a left special factor. In other words, a Sturmian word $\mathbf{u} \in \{0,1\}^{\mathbb{N}}$ is standard if both sequences $0\mathbf{u}$, $1\mathbf{u}$ are Sturmian.

The *shift operator* $\sigma$ maps an infinite word $\mathbf{u} = u_0 u_1 u_2 \ldots$ to the word $\sigma(\mathbf{u}) = u_1 u_2 u_3 \ldots$, i.e., $\sigma$ erases the starting letter of the word $\mathbf{u}$. If $\mathbf{u}$ is a Sturmian word coding an initial point $\rho$ under a two interval exchange transformation $T$, then $\sigma(\mathbf{u})$ is coding of the initial point $T(\rho)$.

**Observation 2.** *If $\mathbf{u}$ is a lower Sturmian word with parameters $\ell_0, \ell_1$, and $\rho$, then $\sigma(\mathbf{u})$ is a lower Sturmian word with parameters $\ell_0, \ell_1$, and $\rho'$, where*

$$\rho' = \begin{cases} \rho + \ell_1 & \text{if } \rho \in [0, \ell_0), \\ \rho - \ell_0 & \text{if } \rho \in [\ell_0, \ell_0 + \ell_1). \end{cases}$$

*If* $\mathbf{u}$ *is an upper Sturmian word with parameters* $\ell_0, \ell_1$*, and* $\rho$*, then* $\sigma(\mathbf{u})$ *is an upper Sturmian word with parameters* $\ell_0, \ell_1$*, and* $\rho'$*, where*

$$\rho' = \begin{cases} \rho + \ell_1 & \text{if } \rho \in (0, \ell_0], \\ \rho - \ell_0 & \text{if } \rho \in (\ell_0, \ell_0 + \ell_1]. \end{cases}$$

## 4. Sturmian morphisms

In this article, we work with these four elementary morphisms:

$$\varphi_a : \begin{cases} 0 \to 0 \\ 1 \to 10 \end{cases}, \quad \varphi_b : \begin{cases} 0 \to 0 \\ 1 \to 01 \end{cases}, \quad \varphi_\alpha : \begin{cases} 0 \to 01 \\ 1 \to 1 \end{cases}, \quad \varphi_\beta : \begin{cases} 0 \to 10 \\ 1 \to 1 \end{cases}. \tag{2}$$

Each of these 4 morphisms is a so-called Sturmian morphism, that is a morphism such that any its image of a Sturmian word is a again a Sturmian word. Moreover, the morphisms $\varphi_b$ and $\varphi_\beta$ map a standard Sturmian word to a standard Sturmian word and are thus called *standard Sturmian morphisms*. Let $\mathcal{M}$ be the monoid generated by the four morphisms, i.e. $\mathcal{M} = \langle \varphi_a, \varphi_b, \varphi_\alpha, \varphi_\beta \rangle$. The monoid $\mathcal{M}$ is usually called the *special Sturmian monoid*. For a non-empty word $w = w_0 \cdots w_{n-1}$ over the alphabet $\{a, b, \alpha, \beta\}$ we set

$$\varphi_w = \varphi_{w_0} \varphi_{w_1} \cdots \varphi_{w_{n-1}}.$$

Each morphism $\varphi_w$ maps a lower (upper) Sturmian word to a lower (upper) Sturmian word. A morphism $\varphi_w$ is primitive if and only if $w$ contains at least one Latin letter and at least one Greek letter. If $\varphi_w$ is primitive, then $\varphi_w$ is a substitution. To obtain the monoid of all Sturmian morphisms we have to extend the set of generators by the morphism $E : 0 \mapsto 1, 1 \mapsto 0$. This morphism maps a lower (upper) Sturmian word to an upper (lower) Sturmian word. If $\psi$ is a Sturmian morphism from the monoid $\langle E, \varphi_a, \varphi_b, \varphi_\alpha, \varphi_\beta \rangle$, then $\psi^2 \in \mathcal{M}$.

The four elementary morphisms $\varphi_a, \varphi_b, \varphi_\alpha, \varphi_\beta$ serve as a basis for a representation of any Sturmian word.

**Theorem 3** ([9])**.** *An infinite binary word* $\mathbf{u}$ *is Sturmian if and only if there exists an infinite word* $\mathbf{w} = w_0 w_1 w_2 \cdots$ *over the alphabet* $\{a, b, \alpha, \beta\}$ *and an infinite sequence* $(\mathbf{u}_i)_{i \geq 0}$ *of Sturmian words such that* $\mathbf{u} = \mathbf{u}_0$ *and* $\mathbf{u}_i = \varphi_{w_i}(\mathbf{u}_{i+1})$ *for all* $i \in \mathbb{N}$.

Usually, the sequence $(\varphi_{w_i})$ is called an *S-adic representation* of $\mathbf{u}$. In our context, the sequence $(w_i)$ shall be simply called an S-adic representation of $\mathbf{u}$.

The monoid $\mathcal{M}$ is a proper submonoid of the monoid of all Sturmian morphisms and it is not free. It is easy to show that for any $k \in \mathbb{N}$ we have

$$\varphi_{\alpha a^k \beta} = \varphi_{\beta b^k \alpha} \quad \text{and} \quad \varphi_{a \alpha^k b} = \varphi_{b \beta^k a}.$$

In fact, these rules give the presentation of the monoid:

**Theorem 4** ([15, 10])**.** *Let* $w, v \in \{a, b, \alpha, \beta\}^*$*. The morphism* $\varphi_w$ *equals* $\varphi_v$ *if and only if the word* $v$ *can be obtained from* $w$ *by possibly repeated application of the rewriting rules*

$$\alpha a^k \beta = \beta b^k \alpha \quad \textit{and} \quad a \alpha^k b = b \beta^k a \qquad \textit{for any } k \in \mathbb{N}\,. \tag{3}$$

Note that the rules (3) preserve positions in $w \in \{a, b, \alpha, \beta\}^*$ of Latin and Greek letters. Thus, by setting $a < b$ and $\alpha < \beta$ we may define a lexicographic order on all equivalent words in $\{a, b, \alpha, \beta\}^*$. In [11], this lexicographic order allowed to describe the derived words with respect to the prefixes of Sturmian words. It is the reason to use a non-traditional notation of the elementary morphisms, which are in [13] denoted as follows:

$$\varphi_b = G, \quad \varphi_a = \widetilde{G}, \quad \varphi_\beta = D, \quad \varphi_\alpha = \widetilde{D}. \tag{4}$$

**Definition 5.** *Let* $w \in \{a, b, \alpha, \beta\}^*$*. The lexicographically largest word in* $\{a, b, \alpha, \beta\}^*$ *which can be obtained from* $w$ *by application of rewriting rules* (3) *is denoted* $N(w)$*. If* $\psi = \varphi_w$*, then the word* $N(w)$ *is the* normalized name *of the morphism* $\psi$ *and it is also denoted by* $N(\psi) = N(w)$*.*

The next lemma is a direct consequence of Theorem 4.

**Lemma 6.** *Let $w \in \{a, b, \alpha, \beta\}^*$. We have $w = N(w)$ if and only if $w$ does not contain $\alpha a^k \beta$ or $a\alpha^k b$ as a factor for any $k \in \mathbb{N}$. In particular, if $w \in \{a, b, \alpha, \beta\}^* \setminus \{a, \alpha\}^*$, the normalized name $N(w)$ has prefix either $a^i\beta$ or $\alpha^i b$ for some $i \in \mathbb{N}$.*

The following definition is helpful in the description of derived words with respect to prefixes.

**Definition 7.** *Let $w \in \{a, b, \alpha, \beta\}^* \setminus \{a, \alpha\}^*$ be the normalized name of a morphism $\psi$. Put*

$$\Delta(w) = \begin{cases} N(w'a^k\beta) & \text{if } \ w = a^k\beta w', \\ N(w'\alpha^k b) & \text{if } \ w = \alpha^k b w' \end{cases}$$

*with $k \in \mathbb{N}$. For $\psi = \varphi_w$ we set*

$$\Delta(\psi) = \varphi_{\Delta(w)}.$$

**Example 8.** *Let $v = aa\beta\alpha\beta\beta a$. We have $N(v) = aa\beta\beta\beta\alpha a$ and $\Delta(N(v)) = N(\beta\beta\alpha aa\beta\beta) = \beta\beta\beta bb\beta\alpha$.*

**Remark 9.** *Let us point out several properties of the operation $\Delta$. Assume $w \in \{a, b, \alpha, \beta\}^* \setminus \{a, \alpha\}^*$ such that $\varphi_w$ is primitive.*

- *(I) If only two letters from $\{a, b, \alpha, \beta\}$ occur in $w$, i.e., $w \in \{b, \beta\}^* \cup \{b, \alpha\}^* \cup \{a, \beta\}^*$, then $\Delta(w)$ is an iteration of the cyclic shift.*
- *(II) The function $\Delta$ preserves the length of a word. The number of the letters $b$ and $\beta$ in the normalized name $N(w)$ of a word $w$ is never smaller than the number of these letters in $w$. Thus the sequence $\Delta^i(w)$ of iterations of $\Delta$ is eventually periodic and for each sufficiently large index $i$, $\Delta^{i+1}(w)$ of can be computed from $\Delta^i(w)$ by cyclic shift without using normalization.*
- *(III) If $w$ contains at least one letter from $\{a, \alpha\}$, then the form of rewriting rules (3) implies that $N(w)$ and $\Delta(w)$ contains at least one letter from $\{a, \alpha\}$.*
- *(IV) If $w$ contains at least 3 letters from $\{a, b, \alpha, \beta\}$, then by Example 31 of [11], $\Delta^i(w)$ contains both letters $\beta$ and $b$ for each sufficiently large $i$.*

**Theorem 10** ([11])**.** *Let $\psi \in \langle\varphi_a, \varphi_b, \varphi_\alpha, \varphi_\beta\rangle$ be a primitive morphism and $N(\psi) = w \in \{a, b, \alpha, \beta\}^* \setminus \{a, \alpha\}^*$ be its normalized name. If $\mathbf{u}$ is the fixed point of $\psi$, then $\mathbf{x}$ is (up to a permutation of letters) a derived word of $\mathbf{u}$ with respect to one of its prefixes if and only if $\mathbf{x}$ is the fixed point of the morphism $\Delta^j(\psi)$ for some $j \in \mathbb{N}$.*

**Example 11** (Example 8 continued)**.** *Taking $w = aa\beta\beta\beta\alpha a$, we have*

$$\begin{aligned}
\Delta(w) &= \beta\beta\beta bb\beta\alpha, & \Delta^7(w) &= \beta\beta\beta\alpha bb\beta, \\
\Delta^2(w) &= \beta\beta bb\beta\beta\alpha, & \Delta^8(w) &= \beta\beta\alpha bb\beta\beta, \\
\Delta^3(w) &= \beta bb\beta\beta\beta\alpha, & \Delta^9(w) &= \beta\alpha bb\beta\beta\beta, \\
\Delta^4(w) &= bb\beta\beta\beta\beta\alpha, & \Delta^{10}(w) &= \alpha bb\beta\beta\beta\beta, \\
\Delta^5(w) &= b\beta\beta\beta\beta\alpha b, & \Delta^{11}(w) &= b\beta\beta\beta\beta\alpha b = \Delta^5(w). \\
\Delta^6(w) &= \beta\beta\beta\beta\alpha bb, & &
\end{aligned}$$

*Note that for $i > 4$, the element $\Delta^i(w)$ is always a cyclic shift of the previous element $\Delta^{i-1}(w)$, illustrating Items (II) and (IV) of Remark 9.*

In [11], we considered only derived words to non-empty prefixes. If we include in our considerations also the empty prefix $\varepsilon$, then the derived word to $\varepsilon$ in $\mathbf{u}$ is $\mathbf{u}$ itself and it is fixed by $\psi = \Delta^0(\psi)$.

The objective of this article is to detect the sets of primitive Sturmian morphisms that are closed under derivation. In order to do that, a tool deciding whether a Sturmian word is fixed by such a morphism is needed. It is easy to see that a Sturmian word $\mathbf{u}$ is fixed by a primitive morphism if and only if $\mathbf{u}$ has a purely periodic $S$-adic representation. Yasutomi [17] found a characterization of such Sturmian words using algebraic properties of their parameters. To quote his result we recall that a number $\lambda$ is quadratic if it is an irrational root of a quadratic equation $Ax^2 + Bx + C = 0$ with rational coefficients $A \neq 0, B, C$.

Let $\mathbb{Q}(\lambda)$ denote the minimal number field containing $\mathbb{Q}$ and $\lambda$. If $\lambda$ is quadratic, then $\mathbb{Q}(\lambda) = \{c + d\lambda : c, d \in \mathbb{Q}\}$. Let $\overline{\lambda}$ be the other root of $Ax^2 + Bx + C = 0$, i.e., the algebraic conjugate of $\lambda$. Since the mapping $z = c + d\lambda \mapsto \overline{z} = c + d\overline{\lambda}$ is an automorphism of the field $\mathbb{Q}(\lambda)$, we have $\overline{z + y} = \overline{z} + \overline{y}$ and $\overline{z \cdot y} = \overline{z} \cdot \overline{y}$ for each $z, y \in \mathbb{Q}(\lambda)$.

**Theorem 12** ([17])**.** *Let $\gamma, \delta \in [0,1]$ and $\gamma$ be irrational. A Sturmian word coding the two interval exchange transformation with parameters $\ell_0 = 1-\gamma, \ell_1 = \gamma, \rho = \delta$ is fixed by a primitive morphism if and only if*

*(1) $\gamma$ and $\delta$ belong to the same quadratic field $\mathbb{Q}(\gamma)$; and*
*(2) $\overline{\gamma} \notin (0,1)$; and*
*(3) If $\overline{\gamma} > 1$, then $\overline{\delta} \in [1-\overline{\gamma}, \overline{\gamma}]$;  if $\overline{\gamma} < 0$, then $\overline{\delta} \in [\overline{\gamma}, 1-\overline{\gamma}]$.*

A quadratic number $\gamma \in (0,1)$ with conjugate $\overline{\gamma} \notin (0,1)$ is called a *Sturm number*. The notion Sturm number was originally defined via properties of coefficients in the continued fraction expansion of $\gamma$, later Allauzen [1] found an algebraic characterization of Sturm numbers.

The parameters $(\ell_0, \ell_1, \rho)$ of a Sturmian word in the previous theorem satisfy $\ell_0 + \ell_1 = 1$, i.e., the parameter $\ell_1$ equals the slope. We rewrite this theorem to a form which is more convenient for our considerations. We normalize the parameters $\ell_0, \ell_1$ of a two interval exchange $T$ to satisfy the condition that the longer interval is of length 1 and the shorter one is of length $\theta$. Clearly, $\theta \in (0,1)$ and the slope $\gamma$ equals $\frac{\theta}{1+\theta}$ or $\frac{1}{1+\theta}$. This kind of normalization is also used in [6, Chapter 6] in order to reveal the relation of Sturmian words to the Ostrowski numeration system.

**Theorem 13.** *Let $\theta \in (0,1)$ be irrational and $\rho \in [0, 1+\theta]$. A Sturmian word with parameters $1, \theta, \rho$ or $\theta, 1, \rho$ is fixed by a primitive morphism if and only if*

*(1) $\theta$ and $\rho$ belong to the same quadratic field; and*
*(2) $\overline{\theta} < 0$; and*
*(3) $\overline{\theta} \leq \overline{\rho} \leq 1$.*

*Proof.* The relation between the parameters $\gamma$ and $\delta$ in Theorem 12 and the parameters in our modification is $\gamma = \frac{\theta}{1+\theta}$ or $\gamma = \frac{1}{1+\theta}$ and $\delta = \frac{\rho}{1+\theta}$. Equivalently,

$$\Big(\theta = \tfrac{1-\gamma}{\gamma} \ \text{ and } \ \rho = \tfrac{\delta}{\gamma}\Big) \quad \text{or} \quad \Big(\theta = \tfrac{\gamma}{1-\gamma} \ \text{ and } \ \rho = \tfrac{\delta}{1-\gamma}\Big).$$

Clearly, $\gamma$ and $\delta$ belong to the same quadratic field if and only if $\theta$ and $\rho$ belong to the same quadratic field. The fact that the quadratic slope $\gamma$ is a Sturm number implies

$$\overline{\gamma} \notin (0,1) \iff \overline{\gamma}(1-\overline{\gamma}) < 0 \iff \tfrac{\overline{\gamma}}{1-\overline{\gamma}} < 0 \quad \text{and} \quad \tfrac{1-\overline{\gamma}}{\overline{\gamma}} < 0 \iff \overline{\theta} < 0\,.$$

If $\overline{\gamma} > 1$, Item (3) of Theorem 13 can be equivalently rewritten

$$1 - \overline{\gamma} \leq \overline{\delta} \leq \overline{\gamma} \iff 1 \geq \tfrac{\overline{\delta}}{1-\overline{\gamma}} \geq \tfrac{\overline{\gamma}}{1-\overline{\gamma}} \quad \text{and} \quad \tfrac{1-\overline{\gamma}}{\overline{\gamma}} \leq \tfrac{\overline{\delta}}{\overline{\gamma}} \leq 1 \iff 1 \geq \overline{\rho} \geq \overline{\theta}\,.$$

If $\overline{\gamma} < 0$, then

$$\overline{\gamma} \leq \overline{\delta} \leq 1 - \overline{\gamma} \iff \tfrac{\overline{\gamma}}{1-\overline{\gamma}} \leq \tfrac{\overline{\delta}}{1-\overline{\gamma}} \leq 1 \quad \text{and} \quad 1 \geq \tfrac{\overline{\delta}}{\overline{\gamma}} \geq \tfrac{1-\overline{\gamma}}{\overline{\gamma}} \iff \overline{\theta} \leq \overline{\rho} \leq 1.$$

∎

## 5. Auxiliary lemmas

**Lemma 14.** *Let $w$ be a factor of an aperiodic word $\mathbf{u}$.*

*(1) There exists $s$ such that $ws$ is right special in $\mathbf{u}$, and $ws'$ is not right special for any proper prefix $s'$ of $s$. Moreover, $\mathbf{d_u}(w) = \mathbf{d_u}(ws)$.*
*(2) There exists $p$ such that $pw$ is left special in $\mathbf{u}$, and $p'w$ is not left special for any proper suffix $p'$ of $u$. Moreover, if $p'w$ is a prefix of $\mathbf{u}$ for some proper suffix $p'$ of $p$, then $\mathbf{d_u}(w) = \mathbf{d_u}(p'w)$. Otherwise, $\mathbf{d_u}(w) = \mathbf{d_u}(pw)$.*

*Proof.* Item (1): As $\mathbf{u}$ is aperiodic, the factor $w$ is a factor of some right special factor. Let $ws$ be the shortest such right special factor. Thus, the word $s$ is unique. Let $r_1, r_2, \ldots, r_k$ be all the return words of $w$. As the factor $s$ always occurs after $w$, the word $r_i ws \in \mathcal{L}(\mathbf{u})$. Since $w$ is a prefix of $r_i w$, the word $ws$ is a prefix of $r_i ws$, and so $r_i$ is a return word of $ws$. To conclude, the return words of $w$ and $ws$ are identical, and thus so are the derived words with respect to $w$ and $ws$.

Item (2): The factor $w$ is a factor of some left special factor. Let $pw$ be the shortest such left special factor. Let $r_1, r_2, \ldots, r_k$ be all the return words of $w$. Similarly to the previous case, the word $p' r_i w$ contains exactly two occurrences of $p'w$ for each prefix $p'$ of $p$. Thus, $p' r_i$ is a return word of $p'w$. The occurrences of $r_i$ are occurrences of $p' r_i$ shifted by $p'$ for all $i$ maybe except for the first occurrence. Thus, if $p'w$ is a prefix of $\mathbf{u}$ for some suffix of $p'$ of $p$, then $\mathbf{d_u}(w) = \mathbf{d_u}(p'w)$. In the other case, $\mathbf{d_u}(w) = \mathbf{d_u}(pw)$. □

As a consequence of the last lemma, to describe the derived words to all factors of $\mathbf{u}$ we can restrict our study to factors $w$ such that $w$ is either a right special prefix of $\mathbf{u}$ or $w$ is a bispecial factor of $\mathbf{u}$ but not a prefix of $\mathbf{u}$.

The next proposition is borrowed from our previous article, where we investigated derived words to prefixes of Sturmian words.

**Proposition 15** ([11])**.** *Let $\mathbf{u}$ and $\mathbf{u}'$ be Sturmian words such that $\mathbf{u} = \varphi_b(\mathbf{u}')$ and let $w'$ be a non-empty right special prefix of $\mathbf{u}'$. We have $d_{\mathbf{u}'}(w') = d_{\mathbf{u}}(w)$ with $w = \varphi_b(w')0$.*

**Corollary 16.** *Let $\mathbf{w}$ be a Sturmian word and $k \in \mathbb{N}$.*

*(1) Denote $\mathbf{u} = \varphi_b^k(\mathbf{w})$. The word $0^k$ is a bispecial prefix of $\mathbf{u}$ and $\mathbf{d_u}(0^k) = \mathbf{w}$.*
*(2) If $1\mathbf{w}$ is a Sturmian word, then $\varphi_a^k(1\mathbf{w}) = 1\varphi_b^k(\mathbf{w})$.*

*Proof.* Item (1): The proof is done by induction on $k$. If $k = 0$, then the empty word is right special factor of $\mathbf{u} = \mathbf{w}$ and the derived word with respect to the empty word is the word $\mathbf{w}$ itself.

Now $k > 1$. First, write the explicit form of the morphism $\varphi_b^k : 0 \mapsto 0$ and $1 \mapsto 0^k 1$. Therefore, in the word $\mathbf{u}$ two neighbouring occurrences of the letter 1 are separated either by the block $0^k$ or $0^{k+1}$. It means that $0^k$ is a bispecial factor of $\mathbf{u}$. Let us assume that $\mathbf{u}' = \varphi_b^{k-1}(\mathbf{w})$ has a right special prefix $0^{k-1}$ and $\mathbf{d}_{\mathbf{u}'}(0^{k-1}) = \mathbf{w}$. Clearly, $0^k = \varphi_b(0^{k-1})0$ is a right special prefix of $\mathbf{u} = \varphi_b(\mathbf{u}') = \varphi_b^k(\mathbf{w})$. By Proposition 15, $\mathbf{d_u}(0^k) = \mathbf{d}_{\mathbf{u}'}(0^{k-1}) = \mathbf{w}$.

Item (2): We proceed by induction on $k$. The case $k = 0$ is trivial. If $k > 0$, we use the fact that the morphisms $\varphi_a$ and $\varphi_b$ are conjugate, in particular $0\varphi_a(x) = \varphi_b(x)0$ for every $x \in \{0,1\}^*$. It implies $0\varphi_a(\mathbf{x}) = \varphi_b(\mathbf{x})$ for every $\mathbf{x} \in \{0,1\}^{\mathbb{N}}$.

Applying this property to the word $1\mathbf{w}$ and using the induction hypothesis, we obtain $\varphi_a^k(1\mathbf{w}) = \varphi_a\big(\varphi_a^{k-1}(1\mathbf{w})\big) = \varphi_a\big(1\varphi_b^{k-1}(\mathbf{w})\big) = \varphi_a(1)\big(\varphi_b^{k-1}(\mathbf{w})\big) = 10\varphi_a\big(\varphi_b^{k-1}(\mathbf{w})\big) = 1\varphi_b\big(\varphi_b^{k-1}(\mathbf{w})\big) = 1\varphi_b^k(\mathbf{w})$. □

**Lemma 17.** *Let $\mathbf{u}$ be a Sturmian word with parameters $\ell_0$, $\ell_1$ and $\rho$. The Sturmian word*

- *$\varphi_b(\mathbf{u})$ has parameters $\ell_0 + \ell_1$, $\ell_1$ and $\rho$;*
- *$\varphi_a(\mathbf{u})$ has parameters $\ell_0 + \ell_1$, $\ell_1$ and $\rho + \ell_1$;*
- *$\varphi_\beta(\mathbf{u})$ has parameters $\ell_0$, $\ell_0 + \ell_1$ and $\rho + \ell_0$;*
- *$\varphi_\alpha(\mathbf{u})$ has parameters $\ell_0$, $\ell_0 + \ell_1$ and $\rho$.*

*Proof.* [3, Lemma 2.2.18] claims that $G = \varphi_b$ maps the lower Sturmian word $\mathbf{s}_{\gamma,\delta}$ to the word $\mathbf{s}_{\frac{\gamma}{1+\gamma},\frac{\delta}{1+\gamma}}$. And analogously, the upper Sturmian word $\mathbf{s}'_{\gamma,\delta}$ is mapped by $G$ to to the upper Sturmian word $\mathbf{s}'_{\frac{\gamma}{1+\gamma},\frac{\delta}{1+\gamma}}$. Thus, using our notation, a Sturmian word with the triplet of parameters $\frac{1}{\ell_0+\ell_1}(\ell_0, \ell_1, \rho) = (1-\gamma, \gamma, \delta)$ is mapped by $\varphi_b$ to a Sturmian word with the triplet of parameters

$$(\ell_0^{new}, \ell_1^{new}, \rho^{new}) = c(1 - \tfrac{\gamma}{1+\gamma}, \tfrac{\gamma}{1+\gamma}, \tfrac{\delta}{1+\gamma}),$$

where $c$ is an arbitrary positive constant. If we choose $c = 1 + \gamma$, we obtain the triplet

$$\ell_0^{new} = 1 = \ell_0 + \ell_1, \quad \ell_1^{new} = \gamma = \ell_1 \quad \text{and} \quad \rho^{new} = \delta = \rho.$$

Proof of the remaining part of the lemma is analogous. □

**Lemma 18.** *Let the two word $u = u_1u_2\cdots u_n \in \{a,b,\alpha,\beta\}^*$ and $v = v_1v_2\cdots v_n \in \{a,b,\alpha,\beta\}^*$ satisfy for each $k = 1,2,\ldots,n$ :*

$$\text{if } v_k \in \{a,b\}, \text{ then } u_k \in \{a,b\} \qquad \text{and} \qquad \text{if } v_k \in \{\alpha,\beta\}, \text{ then } u_k \in \{\alpha,\beta\}. \tag{5}$$

*If $\mathbf{u}$ and $\mathbf{v}$ are Sturmian words with the same slope, then the slopes of $\varphi_u(\mathbf{u})$ and $\varphi_v(\mathbf{v})$ are equal.*

*Proof.* By Lemma 17, both morphisms $\varphi_b$ and $\varphi_a$ change the original slope $\frac{\ell_1}{\ell_0+\ell_1}$ to the same new slope $\frac{\ell_1}{\ell_0+2\ell_1}$. And analogously, both morphisms $\varphi_\beta$ and $\varphi_\alpha$ change the original slope $\frac{\ell_1}{\ell_0+\ell_1}$ to the same new slope $\frac{\ell_1}{2\ell_0+\ell_1}$. □

## 6. Sturmian morphisms non-closeable under derivation

The following example shows that not every Sturmian morphism is closeable under derivation.

**Example 19.** *Consider a Sturmian morphism $\Psi = \varphi_{ab\beta}$. We have*

$$\Psi : \begin{cases} 0 \mapsto 100, \\ 1 \mapsto 10010. \end{cases}$$

*Let $\mathbf{u}$ be its fixed point:*

$$\mathbf{u} = 1001010010010010100\ldots.$$

*We claim that $\mathbf{u}$ is a Sturmian word with parameters $\vec{x} = \left(\sqrt{3}-1, 2-\sqrt{3}, \frac{3-\sqrt{3}}{2}\right)$.*

*Let $\mathbf{v}$ be the lower Sturmian word with parameters $\vec{x}$. By Lemma 17,*

- *$\varphi_\beta(\mathbf{v})$ has parameters $\left(\sqrt{3}-1, 1, \frac{1+\sqrt{3}}{2}\right)$,*
- *$\varphi_{b\beta}(\mathbf{v})$ has parameters $\left(\sqrt{3}, 1, \frac{1+\sqrt{3}}{2}\right)$,*
- *$\varphi_{ab\beta}(\mathbf{v})$ has parameters $\left(\sqrt{3}+1, 1, \frac{3+\sqrt{3}}{2}\right) = \frac{1}{2-\sqrt{3}}\vec{x}$.*

*Thus, $\mathbf{v}$ is fixed by $\Psi$, and so $\mathbf{u} = \mathbf{v}$.*

*Next, we show that the derived word with respect to $0 \in \mathcal{L}(\mathbf{u})$ is not fixed by any primitive substitution, which implies that $\Psi$ is not closeable under derivation.*

*The factor $0$ is not a prefix of $\mathbf{u}$. It is a prefix of $\sigma(\mathbf{u})$. By Observation 2, the word $\sigma(\mathbf{u})$ has parameters $\left(\sqrt{3}-1, 2-\sqrt{3}, \frac{7-3\sqrt{3}}{2}\right)$. The return words to $0$ in $\mathbf{u}$ (and $\sigma(\mathbf{u})$) are $r_0 = 0$ and $r_1 = 01$. Thus, we may write*

$$\sigma(\mathbf{u}) = r_0r_1r_1r_0r_1r_0r_1r_0r_1r_1r_0\ldots.$$

*Since $r_0 = \varphi_b(0)$ and $r_1 = \varphi(1)$, we obtain*

$$\sigma(\mathbf{u}) = \varphi_b(\mathbf{d_u}(0)).$$

*By Lemma 17, the derived word $\mathbf{d_u}(0)$ has parameters $\left(2\sqrt{3}-3, 2-\sqrt{3}, \frac{7-3\sqrt{3}}{2}\right)$. In order to use Theorem 12, we normalize the parameters to $\frac{1}{\sqrt{3}-1}\left(2\sqrt{3}-3, 2-\sqrt{3}, \frac{7-3\sqrt{3}}{2}\right)$. Using the notation of Theorem 12, we have $\gamma = \frac{2-\sqrt{3}}{\sqrt{3}-1} = \frac{\sqrt{3}-1}{2}$ and $\rho = \frac{7-3\sqrt{3}}{2(\sqrt{3}-1)} = \frac{2\sqrt{3}-1}{2}$. Considering the algebraic conjugates $\overline{\gamma} = \frac{-\sqrt{3}-1}{2}$ and $\overline{\rho} = \frac{-2\sqrt{3}-1}{2}$, we notice*

$$\overline{\gamma} < 0 \qquad \text{and} \qquad \overline{\rho} < \overline{\gamma}.$$

*Therefore, the third condition of Theorem 12 is not satisfied, and thus $\mathbf{d_v}(0) = \mathbf{d_u}(0)$ is not fixed by a primitive substitution.*

In the general case, we prove later the following theorem on Sturmian substitutions that are not closeable under derivation.

**Theorem 20.** *Let $\psi = \varphi_w$ be a Sturmian morphism such that $w \in \{a,b,\alpha,\beta\}^*$. If at least three distinct letters from $\{a,b,\alpha,\beta\}$ occur in $w$, then $\psi$ is not closeable under derivation.*

First we prepare several auxiliary statements exploited in the proof of the above theorem.

**Lemma 21.** *Let a Sturmian word* $\mathbf{u}$ *with parameters* $(\ell_0, \ell_1, \rho)$ *be fixed by a primitive morphism* $\varphi_w$*.*

*(1) If* $w \in \{b, \beta\}^*$*, then* $\rho = \ell_1$*;*
*(2) If* $w \in \{b, \alpha\}^*$*, then* $\rho = 0$*;*
*(3) If* $w \in \{a, \beta\}^*$*, then* $\rho = \ell_0 + \ell_1$*;*
*(4) If* $w \in \{a, \alpha\}^*$*, then* $\rho = \ell_0$*.*

*Proof.* We define four planes

$$P_1 = \{(x,y,z) \in \mathbb{R}^3 \colon z = y\}, \qquad P_2 = \{(x,y,z) \in \mathbb{R}^3 \colon z = 0\},$$
$$P_3 = \{(x,y,z) \in \mathbb{R}^3 \colon z = x + y\}, \quad \text{and} \quad P_4 = \{(x,y,z) \in \mathbb{R}^3 \colon z = x\}.$$

Applying Lemma 17, it is straightforward to verify that if the triplet of parameters $(\ell_0, \ell_1, \rho)$ of a Sturmian word $\mathbf{v}$ belongs

(i) to $P_1$, then the parameters of $\varphi_b(\mathbf{v})$ and the parameters of $\varphi_\beta(\mathbf{v})$ belong to $P_1$;
(ii) to $P_2$, then the parameters of $\varphi_b(\mathbf{v})$ and the parameters of $\varphi_\alpha(\mathbf{v})$ belong to $P_2$;
(iii) to $P_3$, then the parameters of $\varphi_a(\mathbf{v})$ and the parameters of $\varphi_\beta(\mathbf{v})$ belong to $P_3$;
(iv) to $P_4$, then the parameters of $\varphi_a(\mathbf{v})$ and the parameters of $\varphi_\alpha(\mathbf{v})$ belong to $P_4$.

Let us start with Item (4) and assume that $\mathbf{u}$ is fixed by a morphism $\varphi_w$ with $w \in \{a, \alpha\}^*$. Since the word $\varphi_w(0)$ has a prefix 0 and $\varphi_w(1)$ has a prefix 1, the morphism $\varphi_w$ has two fixed points. Denote $\mathbf{u}^{(1)}$ the lower Sturmian word coding the two interval exchange with the domain $[0, \ell_0 + \ell_1)$ and the initial point $\rho = \ell_0 \in I_1 = [\ell_0, \ell_0 + \ell_1)$. Let $\mathbf{u}^{(0)}$ denote the upper Sturmian word coding two interval exchange with the domain $(0, \ell_0 + \ell_1]$ and the initial point $\rho = \ell_0 \in I_0$. It means that $\mathbf{u}$, $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(1)}$ have the same slope $\gamma = \frac{\ell_1}{\ell_0 + \ell_1}$. The word $\mathbf{u}$ is fixed by $\varphi_w$ and thus the slopes $\varphi(\mathbf{u})$ and $\mathbf{u}$ are the same, namely $\gamma$. By Lemma 18, the slope of $\varphi_w(\mathbf{u}^{(0)})$ and $\varphi_w(\mathbf{u}^{(1)})$ is $\gamma$ as well. Moreover, parameters of $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(1)}$ belong to the plane $P_4$, which is preserved under the action of $\varphi_w$. It follows that $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(1)}$ are the two fixed points of $\varphi_w$ and thus $\mathbf{u}$ equals $\mathbf{u}^{(0)}$ or $\mathbf{u}^{(1)}$. Both these words have the initial point $\rho = \ell_0$.

The proof of Items (1)–(3) is analogous. The only difference is that the morphism $\varphi_w$ has only one fixed point.

□

**Lemma 22.** *Let* $\theta$ *be a quadratic irrational such that* $0 < \theta < 1$*,* $\overline{\theta} < 0$ *and let* $\rho \in \mathbb{R}$*,* $0 \le \rho \le 1 + \theta$*. Let* $\mathbf{u}$ *be a Sturmian word with parameters* $\ell_0 = 1, \ell_1 = \theta$ *and* $\rho$*, or with parameters* $\ell_0 = \theta, \ell_1 = 1$ *and* $\rho$*.*

*(1) If* $\rho = \ell_1$*, then* $\mathbf{u}$ *is fixed by a primitive morphism* $\varphi_w$ *with* $w \in \{b, \beta\}^*$*;*
*(2) If* $\rho = 0$*, then* $\mathbf{u}$ *is fixed by a primitive morphism* $\varphi_w$ *with* $w \in \{b, \alpha\}^*$*;*
*(3) If* $\rho = \ell_0 + \ell_1$*, then* $\mathbf{u}$ *is fixed by a primitive morphism* $\varphi_w$ *with* $w \in \{a, \beta\}^*$*;*
*(4) If* $\rho = \ell_0$*, then* $\mathbf{u}$ *is fixed by a primitive morphism* $\varphi_w$ *with* $w \in \{a, \alpha\}^*$*.*

*Proof.* We assume without loss of generality that $\mathbf{u}$ has parameters $\ell_0 = 1, \ell_1 = \theta$ and $\rho$. In particular, the slope of $\mathbf{u}$ is $\gamma = \frac{\theta}{1+\theta}$. The parameter $\theta$ and all four possible choices for the parameter $\rho$ from the set $\{\ell_1, 0, \ell_0 + \ell_1, \ell_0\} = \{1, \theta, 0, \theta + 1\}$ satisfy the Yasutomi condition in Theorem 13 and thus in all four cases $\mathbf{u}$ is fixed by a primitive morphism.

Item (1): If $\rho = \ell_1 = \theta$, then $\mathbf{u}$ is a standard Sturmian word. By [4], it is fixed by a standard substitution, that is, by a substitution $\varphi_u$ with $u = u_1 u_2 \cdots u_n \in \{b, \beta\}^*$.

Item (2): We use the word $u = u_1 u_2 \cdots u_n \in \{b, \beta\}^*$ from the proof of Item (1) to define the word $v = v_1 v_2 \cdots v_n \in \{b, \alpha\}^*$. We set $v_k = b$ if $u_k = b$, and $v_k = \alpha$ if $u_k = \beta$. Let $\mathbf{v}$ denote the fixed point of $\varphi_v$ By Lemma 18, the fixed point of $\varphi_v$ has the same slope as the fixed point of $\varphi_u$, namely $\gamma = \frac{\theta}{1+\theta}$. By Lemma 21, the third parameter of $\mathbf{v}$ is 0. It means that $\mathbf{v}$ has parameters $(\ell_0, \ell_1, 0)$ and coincides with $\mathbf{u}$. Consequently, the word $\mathbf{u}$ is fixed by $\varphi_v$ with $v \in \{b, \alpha\}^*$.

The proof of the remaining two parts is analogous. □

**Lemma 23.** *Let* $\mathbf{v}$ *be a Sturmian word fixed by a primitive substitution* $\psi = \varphi_w$*, where* $w = \beta e b a^k$ *for some* $e \in \{a, b, \alpha, \beta\}^*$ *and* $k \ge 0$*. No primitive substitution fixes the Sturmian word* $\sigma(\mathbf{v})$*.*

*Proof.* As $w$ starts with $\beta$, the letter 1 is the more frequent letter in $\mathbf{v}$ and 1 is the starting letter of $\mathbf{v}$. Thus, $\mathbf{v}$ is coding of the interval exchange with parameters $\ell_0 = \theta \in (0,1)$, $\ell_1 = 1$ and $\rho$ satisfying $\theta \leq \rho \leq 1+\theta$. The word $\mathbf{v}$ can be written in the form $1\mathbf{v}'$, where $\mathbf{v}' = \sigma(\mathbf{v})$. By Observation 2, the word $\mathbf{v}'$ has parameters $\ell_0' = \theta$, $\ell_1' = 1$ and $\rho' = \rho - \theta$.

We prove by contradiction that no substitution fixes $\mathbf{v}'$. Let us assume that $\mathbf{v}'$ is fixed by a primitive substitution. Theorem 13 gives

$$\overline{\theta} \leq \overline{\rho} - \overline{\theta} \leq 1 \quad \text{and} \quad \overline{\theta} < 0. \tag{6}$$

Put $w^{(1)} = \beta e$ and $w^{(2)} = ba^k$ and denote $\psi_1 = \varphi_{w^{(1)}}$ and $\psi_2 = \varphi_{w^{(2)}}$. Clearly, $\psi = \psi_1 \circ \psi_2$ and

$$\mathbf{v} = \psi_1\left(\psi_2(\mathbf{v})\right) \implies \psi_2(\mathbf{v}) = \psi_2\psi_1\left(\psi_2(\mathbf{v})\right).$$

Thus the word $\mathbf{v}'' = \psi_2(\mathbf{v})$ is fixed by the primitive substitution $\psi_2 \circ \psi_1$. By Lemma 17, the word $\mathbf{v}''$ has parameters $1+k+\theta, 1, k+\rho$, which we normalize to $\ell_0'' = 1, \ell_1'' = \frac{1}{1+k+\theta}$ and $\rho'' = \frac{k+\rho}{1+k+\theta}$. These parameters satisfy the condition given by Theorem 13, i.e.,

$$\frac{1}{1+k+\overline{\theta}} < 0 \quad \text{and} \quad \frac{1}{1+k+\overline{\theta}} \leq \frac{k+\overline{\rho}}{1+k+\overline{\theta}} \leq 1. \tag{7}$$

By (7) and (6) we obtain $\overline{\rho} \leq \overline{\theta}+1$ and $\overline{\rho} \geq \overline{\theta}+1$, respectively. It means that $\rho = 1+\theta = \ell_0+\ell_1$. By Item (3) of Lemma 22, the word $\mathbf{v}$ is fixed by a substitution $\varphi_u$ with $u \in \{\beta, a\}^*$. Since such word $u$ cannot be rewritten using (3), by Theorem 4 the substitution $\varphi_u$ and every its power are elements of $\langle \varphi_\beta, \varphi_a \rangle$. Similarly by Theorem 4 we obtain $\varphi_w^k \notin \langle \varphi_\beta, \varphi_a \rangle$ for all $k \in \mathbb{N}$. Since every Sturmian word is rigid (see [14]), we obtain a contradiction as the Sturmian word $\mathbf{v}$ cannot be fixed by both $\varphi_u$ and $\varphi_w$. □

*Proof of Theorem 20.* Let $\mathbf{w}$ be a fixed point of $\psi = \varphi_w$. By Theorem 10, the derived word to any prefix of $\mathbf{w}$ is fixed by a substitution $\Delta^i(w)$ for some $i$. By Remark 9 there exists $i_0 \in \mathbb{N}$ such that for each $i > i_0$, both letters $b$ and $\beta$ occur in $\Delta^i(w)$ and $\Delta(\Delta^i(w)) = \mathrm{cyc}_j(\Delta^i(w))$ for some $j$. By the same remark, at least one of the letters from $\{a, \alpha\}$ occurs in $\Delta^i(w)$. Therefore, there exists $i > i_0$ such that

$$\text{for some } \ e \in \{a,b,\alpha,\beta\}^* \text{ and } k \in \mathbb{N}, k \geq 1, \quad \text{either } \Delta^i(w) = a^k\beta eb \ \text{ or } \ \Delta^i(w) = \alpha^k be\beta.$$

We first treat the case $\Delta^i(w) = a^k\beta eb$. Let $\mathbf{u}$ denote the fixed point of $\varphi_{\Delta^i(w)}$. It follows that $\mathbf{u}$ has an S-adic representation $(a^k\beta eb)^\omega$.

Let $\mathbf{v}$ denote the Sturmian word with the S-adic representation $(\beta eba^k)^\omega$. The word $\mathbf{v}$ begins with the letter 1 and thus $\mathbf{v} = 1\mathbf{v}'$ for some Sturmian word $\mathbf{v}'$. Moreover,

$$\mathbf{u} = \varphi_a^k(\mathbf{v}) = \varphi_a^k(1\mathbf{v}').$$

By Corollary 16 Item (2), $\mathbf{u} = 1\varphi_b^k(\mathbf{v}')$. By Corollary 16 Item (1), $0^k$ is a factor of $\mathbf{u}$ and $\mathbf{d_u}(0^k) = \mathbf{v}' = \sigma(\mathbf{v})$. Lemma 23 implies that $\mathbf{v}'$ is not fixed by any primitive substitution.

To sum up:

- $\mathbf{u}$ is a derived word to a prefix of the fixed point $\mathbf{w}$ of the substitution $\psi$;
- $0^k$ is a factor of $\mathbf{u}$ and the derived word of $\mathbf{d_u}(0^k)$ is not fixed by any primitive substitution.

It implies that the fixed point $\mathbf{w}$ of the primitive substitution $\psi$ is not closeable under derivation.

The second case $\Delta^i(w) = \alpha^k be\beta$ follows easily from the first case by exchanging the letters 0 and 1. □

## 7. Closeable under derivation Sturmian substitutions

The aim of this section is to prove the following theorem.

**Theorem 24.** *If $\psi = \varphi_w$ is a primitive Sturmian substitution such that $w \in \{b,\beta\}^* \cup \{b,\alpha\}^* \cup \{a,\beta\}^* \cup \{a,\alpha\}^*$, then $\psi$ is closeable under derivation.*

The proof will be split into four cases according to the couple of letters which appear in the name $w$ determining the substitution $\varphi_w$. To abbreviate the notation we set

$$C(w) = \{\varphi_v \colon v = \mathrm{cyc}^k(w), k \in \mathbb{N}\} \qquad \text{for } w \in \{a,b,\alpha,\beta\}^*.$$

7.1. **Case** $w \in \{b, \beta\}^*$**.** The substitution $\varphi_w$ is composed from the standard Sturmian substitutions $\varphi_\beta$ and $\varphi_b$. Therefore, the fixed point $\mathbf{u}$ of $\varphi_w$ is a standard Sturmian word. Any bispecial factor of a standard Sturmian word is one of its prefixes. Thus by Lemma 14 the derived word to an arbitrary factor of $\mathbf{u}$ coincides with the derived word to a prefix of $\mathbf{u}$. Theorem 10 implies that the derived word to a prefix of $\mathbf{u}$ is fixed by a substitution $\varphi_v$ with $v = \mathrm{cyc}^k(w)$ for some $k$, i.e., $\varphi_v \in C(w)$. Clearly, $v$ belongs to $\{b, \beta\}^*$ and thus $\varphi_v$ is again a standard Sturmian substitution. Thus, we may repeat this argument, and we can conclude the following.

**Claim 25.** *For any $w \in \{b, \beta\}^*$, the set $C(w)$ is closed under derivation.*

Discussion of the other cases uses a consequence of Lemmas 21 and 22. Recall that the shift operator $\sigma$ erases the starting letter of an infinite word, i.e., maps the word $\mathbf{u} = u_0u_1u_2\ldots$ to the word $\sigma(\mathbf{u}) = u_1u_2u_3\ldots$. To abbreviate the notation we define two projections $H$ and $F$: $\{a, b, \alpha, \beta\}^* \to \{a, b, \alpha, \beta\}^*$ by

$$H(a) = H(b) = b, H(\alpha) = \alpha, H(\beta) = \beta \quad \text{and} \quad F(a) = a, F(b) = b, F(\alpha) = F(\beta) = \beta.$$

**Lemma 26.** *Let $\mathbf{u}$ be fixed by a substitution $\varphi_w$.*

*(1) If $w \in \{a, \beta\}^*$, then $\sigma(\mathbf{u})$ is a standard Sturmian word which is fixed by the substitution $\varphi_{H(w)}$.*
*(2) If $w \in \{b, \alpha\}^*$, then $\sigma(\mathbf{u})$ is a standard Sturmian word which is fixed by the substitution $\varphi_{F(w)}$.*
*(3) If $w \in \{a, \alpha\}^*$ and $\mathbf{u}$ has a prefix 1, then $\sigma(\mathbf{u})$ is fixed by the substitution $\varphi_{H(w)}$.*
*(4) If $w \in \{a, \alpha\}^*$ and $\mathbf{u}$ has a prefix 0, then $\sigma(\mathbf{u})$ is fixed by the substitution $\varphi_{F(w)}$.*

*Proof.* Let $(\ell_0, \ell_1, \rho)$ be the parameters of $\mathbf{u}$.

Item (1): By Lemma 21, $\mathbf{u}$ has parameters $\ell_0, \ell_1, \rho = \ell_0 + \ell_1$. In particular, $\mathbf{u}$ is an upper Sturmian word. By Observation 2, $\sigma(\mathbf{u})$ has parameters $\ell_0, \ell_1, \rho = \ell_1$, i.e., $\sigma(\mathbf{u})$ is a standard Sturmian word. Clearly, the slopes of $\sigma(\mathbf{u})$ and $\mathbf{u}$ coincide. Using Lemmas 18 and 22, the word $\sigma(\mathbf{u})$ is fixed by the substitution $\varphi_{H(w)}$.

Item (2): Analogous to the proof of Item (1).

Item (3): The substitution $\varphi_w$ has two fixed points, one starting with the letter 0 and one starting with the letter 1. By Lemma 21, both fixed points represent a coding of a two interval exchange transformation $T$ with parameter $\rho$ satisfying $\rho = \ell_0$. If $\mathbf{u}$ starts with 1, then $\mathbf{u}$ is a coding of the transformation $T$ with the domain $[0, \ell_0 + \ell_1)$. In this case, $T(\rho) = 0$ and by Lemma 22 the word $\sigma(\mathbf{u})$ is fixed by the substitution $\varphi_{H(w)}$.

Item (4): Analogous to the proof of Item (3). ☐

7.2. **Case** $w \in \{b, \alpha\}^*$**.** Theorem 10 implies that a derived word to a prefix of $\mathbf{u}$ is fixed by one of the substitutions from $C(w)$. Using Lemma 14, we can focus on derived words to bispecial non-prefixes of $\mathbf{u}$. By Lemma 26, the word $\sigma(\mathbf{u})$ is a standard Sturmian word and thus any bispecial factor $v$ of $\mathbf{u}$ occurs as a prefix of $\sigma(\mathbf{u})$. Therefore, $\mathbf{d}_{\mathbf{u}}(v) = \mathbf{d}_{\sigma(\mathbf{u})}(v)$. By Lemma 26, the word $\sigma(\mathbf{u})$ is fixed by the standard substitution $\varphi_{F(w)}$, i.e., $F(w) \in \{b, \beta\}^*$. Using Claim 25 we conclude the following.

**Claim 27.** *For any $w \in \{b, \alpha\}^*$, the set $C(w) \cup C\big(F(w)\big)$ is closed under derivation.*

7.3. **Case** $w \in \{a, \beta\}^*$. This case is analogous to the previous one. Indeed, the words $\mathbf{u}$ and $E(\mathbf{u})$ have the same (up to the permutation of letters) set of derived words. Since $\mathbf{u}$ is fixed by $\varphi_w$, the word $E(\mathbf{u})$ is fixed by $E\varphi_w E$. As $\varphi_a = E\varphi_\alpha E$ and $\varphi_b = E\varphi_\beta E$, the substitutions $E\varphi_w E = \varphi_v$, where $v \in \{b, \alpha\}^*$.

**Claim 28.** *For any $w \in \{a, \beta\}^*$, the set $C(w) \cup C\big(H(w)\big)$ is closed under derivation.*

7.4. **Case** $w \in \{a, \alpha\}^*$. The substitution $\varphi_w$ has two fixed points. The derived words to prefixes of the fixed points of $\varphi_w$ are described in [11]:

**Proposition 29** ([11])**.** *Let $a$ be the first letter of the word $w \in \{a, \alpha\}^*$.*

*(i) Let $\mathbf{u}$ be the fixed point of $\varphi_w$ starting with 0 and $p$ be a non-empty prefix of $\mathbf{u}$. Denote $v = b^{-1}N(wb) \in \{a, \beta\}^*$. The derived word $\mathbf{d}_{\mathbf{u}}(p)$ equals a derived word $\mathbf{d}_{\mathbf{v}}(q)$, where $\mathbf{v}$ is the unique fixed point of the substitution $\varphi_v$ and $q$ is a prefix of $\mathbf{v}$.*

*(ii) Let $\mathbf{u}$ be the fixed point of $\varphi_w$ starting with $1$ and $p$ be a non-empty prefix of $\mathbf{u}$. Put $v = \mathrm{cyc}(w)$. The word $\mathbf{d_u}(p)$ equals a derived word $\mathbf{d_v}(q)$, where $\mathbf{v}$ is the fixed point of the substitution $\varphi_v$ starting with $1$ and $q$ is a prefix of $\mathbf{v}$.*

An analogous proposition can be stated if $\alpha$ is the first letter of $w \in \{a, \alpha\}^*$. In this case, the roles of the letters 0 and 1 in Items (i) and (ii) are then interchanged. In particular, the word $v$ in Item (i) is defined by $v = \beta^{-1}N(w\beta) \in \{b, \alpha\}^*$. Nevertheless, in the following example we show that in both cases the word $v$ belongs either to $\{\mathrm{cyc}^k(F(w)) : k \in \mathbb{N}\}$ or to $\{\mathrm{cyc}^k(H(w)) : k \in \mathbb{N}\}$.

**Example 30.** *Consider $w = a^4\alpha^2a^2\alpha a^3 \in \{a, \alpha\}^*$. The starting letter of this word is $a$. The word $v = b^{-1}N(wb)$ from Proposition 29 satisfies $v = a^3\beta^2a^2\beta a^4$. Therefore, we have $v = \mathrm{cyc}(F(w))$.*

*Consider $w = \alpha a^3\alpha^4 a \in \{a, \alpha\}^*$. It follows that $v = \beta^{-1}N(w\beta) = b^3\alpha^4 b$, and thus $v = \mathrm{cyc}(H(w))$.*

Proposition 29 has the following direct corollary.

**Corollary 31.** *Let $\mathbf{u}$ be a fixed point of $\varphi_w$ with $w \in \{a, \alpha\}^*$ and $p \neq \varepsilon$ be a prefix of $\mathbf{u}$. The word $\mathbf{d_u}(p)$ is fixed by a substitution from $C(F(w)) \cup C(H(w))$.*

**Claim 32.** *For any $w \in \{a, \alpha\}^*$, the set $C(w) \cup C\big(H(w)\big) \cup C\big(F(w)\big) \cup C\big(HF(w)\big)$ is closed under derivation.*

*Proof.* We deduce a stronger statement, namely that the set $M := \{\varphi_w\} \cup C\big(H(w)\big) \cup C\big(F(w)\big) \cup C\big(HF(w)\big)$ is closed under derivation.

First, we realize that $H(w) \in \{b, \alpha\}^*$, $F(w) \in \{a, \beta\}^*$ and $HF(w) \in \{b, \beta\}^*$. By virtue of Claims 25, 27 and 28, the set $N := M\big(H(w)\big) \cup C\big(F(w)\big) \cup C\big(HF(w)\big)$ is closed under derivation. To demonstrate that $M = \{\varphi_w\} \cup N$ is closed under derivation, we only need to show that the derived word to any factor $v$ of any fixed point $\mathbf{u}$ of $\varphi_w$ is fixed by a morphism from $M$. Indeed:

a) if $v = \varepsilon$, then $\mathbf{d_u}(\varepsilon) = \mathbf{u}$ and thus $\mathbf{d_u}(\varepsilon)$ is fixed by $\varphi_w \in M$;
b) if $v$ is a non-empty prefix of $\mathbf{u}$, then by Corollary 31 $\mathbf{d_u}(v)$ is fixed by a substitution from $N \subset M$;
c) if $v$ is a non-prefix factor of $\mathbf{u}$, then $v$ is a factor of $\sigma(\mathbf{u})$ and $\mathbf{d_u}(v) = \mathbf{d}_{\sigma(\mathbf{u})}(v)$. By Items (3) and (4) of Lemma 26, the word $\sigma(\mathbf{u})$ is fixed by a substitution from $N$. Since $N$ is closed under derivation, $\mathbf{d_u}(v)$ is fixed by a substitution from $N \subset M$. □

The following theorem is a direct consequence of Lemmas 21 and 23 and Theorem 20.

**Theorem 33.** *Let $\mathbf{u}$ code the two interval exchange transformation with parameters $\ell_0 = 1 - \gamma, \ell_1 = \gamma, \rho = \delta$, where $\gamma, \delta \in [0, 1]$ and $\gamma$ irrational. If $\mathbf{u}$ is fixed by a primitive substitution $\varphi$, then $\varphi$ is closeable under derivation if and only if $\delta \in \{0, \gamma, 1 - \gamma, 1\}$.*

## 8. Comments

We presented examples of finite sets $M$ of Sturmian substitutions that are closed under derivation. We used two tools: $S$-adic representation of Sturmian words and algebraic characterization of Sturmian words that are fixed by a primitive substitution. Probably the most explored class of words generalizing Sturmian words is the class of ternary Arnoux–Rauzy words. Therefore, it is natural to generalize our results to this class, more specifically, to the substitutions fixing ternary Arnoux–Rauzy words. However, there is no analogue of the Yasutomi's characterization of fixed points in this class. Another well studied class that generalizes Sturmian words are words coding $k$-interval exchange transformations. In the case $k = 3$ and the permutation of interval exchange being (321), an analogue to Yasutomi's conditions is provided in [2].

Our example of sets $M$ which are closed under derivation are composed of Sturmian substitutions and thus all elements of $M$ act on the same alphabet. The same property would hold for substitutions fixing ($k$-ary) Arnoux–Rauzy words and for substitutions fixing three interval exchange with the permutation (321). However, there exists an example of a set $M$ closed under derivation which contains substitutions acting on a binary alphabet and substitutions acting on a ternary alphabet such that no proper subset of $M$ is closed under derivation. This example is given in [12] and the set $M$ contains substitutions fixing derived words to non-empty factors of the period doubling sequence. Recall that the period doubling sequence is fixed by the substitution $a \mapsto ab, b \mapsto aa$. The derived words of the period doubling sequence were described in [8].

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