# A Metalanguage for Guarded Iteration<sup>\*,\*\*</sup>

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## Abstract

Notions of guardedness serve to delineate admissible recursive definitions in various settings in a compositional manner. In recent work, we have introduced an axiomatic notion of guardedness in symmetric monoidal categories, which serves as a unifying framework for various examples from program semantics, process algebra, and beyond. In the present paper, we propose a generic *metalanguage for guarded iteration* based on combining this notion with the fine-grain callby-value paradigm, which we intend as a unifying programming language for guarded and unguarded iteration in the presence of computational effects. We give a generic (categorical) semantics of this language over a suitable class of strong monads supporting guarded iteration, and show it to be in touch with the standard operational behaviour of iteration by giving a concrete big-step operational semantics for a certain specific instance of the metalanguage and establishing soundness and (computational) adequacy for this case.

*Keywords:* Computational monads, metalanguage, guarded iteration, computational adequacy

### 1. Introduction

Guardedness is a recurring theme in programming and semantics, fundamentally distinguishing the view of computations as processes unfolding in time from the view that identifies computations with a final result they may eventually produce. Historically, the first perspective is inherent to process algebra (e.g. [33]), where the main attribute of a process is its *behaviour*, while the second is inherent to classical denotational semantics via domain theory [44], where the only information properly infinite computations may communicate to the outer world is the mere fact of their divergence. This gives rise to a distinction between *intensional* and *extensional* paradigms in semantics [1].

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<sup>\*\*</sup>This article is a revised version of [19].

```
handle r in

(handleit e = \star in  // start a loop

print ("think of a number") &  // execute the loop guard

(do y \leftarrow rand();

    z \leftarrow read();

    if (y = 42) then raise<sub>r</sub> \star else  // 42 is the ultimate answer

    if (z = y) then ret \star else raise<sub>e</sub> \star ))  // continue, unless

    // number guessed correctly
```

with print ("the answer!")

Figure 1: Example of a guarded loop.

For example, in CCS [33] a process is guarded in a variable x if every occurrence of x in this process is preceded by an action. One effect of this constraint is that guarded recursive specifications can be solved uniquely, e.g. the equation  $x = \bar{a}.x$ , whose right-hand side is guarded in x, has the infinite stream  $\bar{a}.\bar{a}...$  as its unique solution. If we view  $\bar{a}$  as an action of producing an output, we can also view the process specified by  $x = \bar{a}.x$  as *productive* and the respective solution  $\bar{a}.\bar{a}...$  as a *trace* obtained by collecting its outputs. The view of guardedness as productivity is pervasive in programming and reasoning with coinductive types [12, 15, 16, 23] as implemented in dependent type environments such as Coq and Agda. Semantic models accommodate this idea in various ways, e.g. from a modal [35, 2, 32], (ultra-)metric [13, 26], and a unifying topos-theoretic perspective [5, 10].

In recent work, we have proposed a new *axiomatic* approach to unifying notions of guardedness [22, 20], where the main idea is to provide an *abstract* notion of guardedness applicable to a wide range of (mutually irreducible) models, including, e.g., complete partial orders, complete metric spaces, and infinitedimensional Hilbert spaces, instead of designing a concrete model carrying a specific notion of guardedness. A salient feature of axiomatic guardedness is that it varies in a large spectrum starting from *total guardedness* (everything is guarded) and ending at *vacuous guardedness* (very roughly, guardedness in a variable means essentially non-occurrence of this variable in the defining expression) with proper examples as discussed above lying between these two extremes. The fact that axiomatic guardedness can be varied so broadly indicates that it can be used for bridging the gap between the intensional and extensional paradigms, which is indeed the perspective we are pursuing here by introducing a *metalanguage for guarded iteration*.

The developments in [20] are couched in terms of a special class of monoidal categories called *guarded traced symmetric monoidal categories*, equipped with a monoidal notion of guardedness and a monoidal notion of feedback allowing only such cyclic computations that are guarded in the corresponding sense. In

the present work we explore a refinement of this notion by instantiating guarded traces to *Kleisli categories* of computational monads in the sense of Moggi [34], with coproduct (inherited from the base category) as the monoidal structure. The feedback operation is then equivalently given by *guarded effectful iteration*, i.e. a (partial) operator

$$\frac{f: X \to T(Y+X)}{f^{\dagger}: X \to TY} \tag{1}$$

to be thought of as iterating f over X until a result in Y is reached [22]. As originally argued by Moggi, strong monads can be regarded as representing computational effects, such as nondeterminism, exceptions, or process algebra actions, and thus the corresponding internal language of strong monads, the *computational metalanguage* [34], can be regarded as a generic programming language over these effects. We extend this perspective by parametrizing such a language with a notion of guardedness and equipping it with guarded iteration. In doing so, we follow the approach of Geron and Levy [14] who already explored the case of unguarded iteration by suitably extending a fine-grain call-by-value language [29], a refined variant of Moggi's original computational  $\lambda$ -calculus.

A key insight we borrow from [14] is that effectful iteration can be efficiently organized via throwing and handling exceptions (also called *labels* in this context) in a loop, leading to a more convenient programming style in comparison to the one directly inspired by the typing of the iteration operator (1). We show that the exception handling metaphor seamlessly extends to the guarded case and is compatible with the axioms of guardedness. A quick illustration is presented in Fig. 1 where the handleit command implements a loop in which the raise command indexed with the corresponding exception e identifies the tail call. The *print* operation acts as a guard and makes the resulting program well-typed. We also involve two operations rand and read for random number generation and for reading a user input from the console correspondingly. Apart from the non-standard use of exceptions via the handleit construct, they can be processed in a standard way with the handle command, and therefore in the example, we can break from the loop by throwing exception r when the random number appears to be 42 (the answer to the ultimate question of life, the universe, and everything).

To interpret our metalanguage we derive and explore a notion of *strong* guarded iteration and give a generic (categorical) denotational semantics, for which the main subtlety are functional abstractions of guarded morphisms. We then define a big-step operational semantics for a concrete (simplistic) instance of our metalanguage and show an adequacy result w.r.t. a concrete choice of the underlying category and the strong monad.

*Related work.* We have already mentioned work by Geron and Levy [14]. The instance of operational semantics we explore here is chosen so as to give the simplest proper example of guarded iteration, i.e. the one giving rise to infinite traces, making the resulting semantics close to one explored in a line of work by Nakata and Uustalu [37, 38, 36, 39]. We regard our operational semantics

as a showcase for the denotational semantics, and do not mean to address the notorious issue of undecidability of program termination, which is the main theme of Nakata and Uustalu's work. We do however see our work as a stepping stone both for deriving more sophisticated styles of operational semantics and for developing concrete denotational models for addressing the operational behaviour as discussed in op.cit. The guarded  $\lambda$ -calculus [10] is a recently introduced language for guarded recursion (as apposed to guarded iteration), on the one hand much more expressive than ours, but on the other hand capturing a very concrete model, the topos of trees [5].

This paper extends a previous conference publication [19] by giving full proofs and additional explanations and example material. Also, we consolidate the treatment of iteration-in-context by showing the necessity of conditions relating the strength to guardedness and iteration (Theorem 6). The version of the metalanguage we present here (Fig. 4) improves slightly on the original conference version by modifying the formation rules for gcase and handle; this, in particular, allows us to type more terms, and handle "unguarded exceptions".

*Plan of the paper.* In Section 2 we give the necessary technical preliminaries, and discuss and complement the semantic foundations for guarded iteration [22, 20]. In Sections 3 and 4 we present our metalanguage for guarded iteration (without functional types) and its generic denotational semantics. In Section 5 we identify conditions for interpreting functional types and extend the denotational semantics to this case. In Section 6 we consider an instance of our metalanguage (for a specific choice of signature), give a big-step operational semantics and prove a corresponding adequacy result. Conclusions are drawn in Section 7.

# 2. Monads for Effectful Guarded Iteration

We use the standard language of category theory [30]. Some conventions regarding notation are in order. By  $|\mathbf{C}|$  we denote the class of objects of a category  $\mathbf{C}$ , and by  $\mathsf{Hom}_{\mathbf{C}}(A, B)$  (or  $\mathsf{Hom}(A, B)$ , if no confusion arises) the set of morphisms  $f: A \to B$  from  $A \in |\mathbf{C}|$  to  $B \in |\mathbf{C}|$ . We tend to omit object indices on natural transformations.

Coproduct summands and distributive categories. We call a pair  $\sigma = \langle \sigma_1 : Y_1 \to X, \sigma_2 : Y_2 \to X \rangle$  of morphisms a summand of X, denoted  $\sigma : Y_1 \to X$ , if it forms a coproduct cospan, i.e. X is a coproduct of  $Y_1$  and  $Y_2$  with  $\sigma_1$  and  $\sigma_2$  as coproduct injections. Each summand  $\sigma = \langle \sigma_1, \sigma_2 \rangle$  thus determines a complement summand  $\bar{\sigma} = \langle \sigma_2, \sigma_1 \rangle \colon Y_2 \to X$ . We often identify a summand  $\langle \sigma_1, \sigma_2 \rangle$  with its first component when  $\sigma_2$  is predetermined canonically, clear from the context, or irrelevant. Summands of a given object X are naturally preordered by taking  $\langle \sigma_1, \sigma_2 \rangle$  to be smaller than  $\langle \theta_1, \theta_2 \rangle$  iff  $\sigma_1$  factors through  $\theta_1$ . In the presence of an initial object  $\emptyset$ , with unique morphisms  $!: \emptyset \to X$ , this preorder has a greatest element  $\langle \operatorname{id}_X, ! \rangle \colon X \to X$  and a least element  $\langle !, \operatorname{id}_X \rangle \colon \emptyset \hookrightarrow X$ . By writing  $X_1 + \ldots + X_n$  we designate the latter as a coproduct of the  $X_i$  and assign the canonical names  $\operatorname{in}_i \colon X_i \hookrightarrow X_1 + \ldots + X_n$ 

to the corresponding summands; if  $\sigma: Y_1 \hookrightarrow X_1, \vartheta: Y_2 \hookrightarrow X_2$  are summands, then so is  $\sigma + \vartheta: Y_1 + Y_2 \hookrightarrow X_1 + X_2$ . Dually to summands, we write  $\operatorname{pr}_i: X_1 \times \ldots \times X_n \to X_i$  for canonical projections (without introducing a special arrow notation); by  $\Delta$  we abbreviate the diagonal natural transformation  $\langle \operatorname{id}_A, \operatorname{id}_A \rangle: A \to A \times A$ . Note that in an extensive category [8], the second component of any coproduct summand  $\langle \sigma_1, \sigma_2 \rangle$  is determined by the first up to isomorphism. However, we do not generally assume extensiveness, working instead with the weaker assumption of distributivity [11]: a category with finite products and coproducts (including a final and an initial object) is distributive if the natural transformation

$$X \times Y + X \times Z \xrightarrow{[\mathsf{id} \times \mathsf{in}_1, \mathsf{id} \times \mathsf{in}_2]} X \times (Y + Z)$$

is an isomorphism, whose inverse we denote by  $dist_{X,Y,Z}$ , or usually just dist. Then dist is natural in X, Y, Z, and moreover compatible with the coproduct structure in the expected sense; in particular,

$$\begin{aligned} \mathsf{dist} \left( \mathsf{id} \times \mathsf{in}_1 \right) &= \mathsf{in}_1 \\ \mathsf{dist} \left( \mathsf{id} \times \mathsf{in}_2 \right) &= \mathsf{in}_2 \\ [h \times f, h \times g] \; \mathsf{dist} &= h \times [f, g] \end{aligned}$$

for  $h: X \to U$ ,  $f: Y \to W$ ,  $g: Z \to W$ . In proofs, we summarily refer to such properties by the keyword *distributivity*.

Strong monads. Following Moggi [34], we identify a monad **T** on a category **C** with the corresponding Kleisli triple  $(T, \eta, (-)^*)$  on **C** consisting of an endomap T on  $|\mathbf{C}|$ , a  $|\mathbf{C}|$ -indexed class of morphisms  $\eta_X \colon X \to TX$ , called the unit of **T**, and the Kleisli lifting maps  $(-)^* \colon \operatorname{Hom}(X, TY) \to \operatorname{Hom}(TX, TY)$  such that

$$\eta^{\star} = \mathrm{id} \qquad \qquad f^{\star}\eta = f \qquad \qquad (f^{\star}g)^{\star} = f^{\star}g^{\star}.$$

These definitions imply that T is an endofunctor (with  $Tf = (\eta f)^*$ ) and  $\eta$  is a natural transformation. Provided that **C** has finite products, a monad **T** on **C** is *strong* if it is equipped with *strength*, i.e. a natural transformation  $\tau_{X,Y}: X \times TY \to T(X \times Y)$  satisfying the following standard coherence conditions (e.g. [34]):

$$(\mathbf{trv}) \quad \frac{f \colon X \to TY}{(T \text{ in}_1)f \colon X \to_{\text{in}_2} T(Y+Z)} \qquad (\mathbf{sum}) \quad \frac{f \colon X \to_{\sigma} TZ}{[f,g] \colon X+Y \to_{\sigma} TZ}$$
$$(\mathbf{cmp}) \quad \frac{f \colon X \to_{\text{in}_2} T(Y+Z)}{[g,h]^* f \colon X \to_{\sigma} TV} \qquad h \colon Z \to TV}{(\mathbf{str}) \quad \frac{f \colon X \to_{\sigma} TY}{\tau (\text{id}_Z \times f) \colon Z \times X \to_{\text{id} \times \sigma} T(Z \times Y)}$$

Figure 2: Axioms of abstract guardedness.

where  $f: Y \to TZ$ .

Morphisms of the form  $f: X \to TY$  constitute the *Kleisli category* of **T**, which has the same objects as **C**, units  $\eta_X: X \to TX$  as identities, and composition  $(f, g) \mapsto f^*g$ , also called *Kleisli composition*.

In programming language semantics, both the strength  $\tau$  and the distributivity transformation dist essentially serve to propagate context variables. We often need to combine them into

$$\delta = (T\operatorname{dist})\tau \colon X \times T(Y+Z) \to T(X \times Y + X \times Z).$$

In what follows we will make extensive use of the following simple property of  $\delta$ :

$$\delta \langle \mathsf{pr}_1, \delta \langle \mathsf{pr}_1, f \rangle \rangle = T(\langle \mathsf{pr}_1, \mathsf{id}_{X \times Z} \rangle + \langle \mathsf{pr}_1, \mathsf{id}_{X \times W} \rangle) \,\delta \,\langle \mathsf{pr}_1, f \rangle. \tag{2}$$

for  $f: X \times Y \to T(Z + W)$  (where the morphisms in the equation have type  $X \times Y \to T(X \times X \times Z + X \times X \times W)$ ).

Guarded Iteration. Let us fix a distributive category  $\mathbf{C}$  and a strong monad  $\mathbf{T}$ on  $\mathbf{C}$ . The monad  $\mathbf{T}$  is (abstractly) guarded if it is equipped with a notion of guardedness, i.e. with a relation between Kleisli morphisms  $f: X \to TY$  and summands  $\sigma: Y' \to Y$  closed under the rules in Fig. 2, where  $f: X \to_{\sigma} TY$ denotes the fact that f and  $\sigma$  are in the relation in question, in which case f is also called  $\sigma$ -guarded. We denote by  $\operatorname{Hom}_{\sigma}(X,TY)$  (or, more precisely,  $\operatorname{Hom}_{\mathbf{C},\sigma}(X,TY)$ ) the subset of  $\operatorname{Hom}(X,TY)$  consisting of the morphisms  $X \to_{\sigma} TY$ . We also write  $f: X \to_i TY$  for  $f: X \to_{\operatorname{in}_i} TY$ . More generally, we use the notation  $f: X \to_{p,q,\ldots} TY$  to indicate guardedness in the union of injections  $\operatorname{in}_p, \operatorname{in}_q, \ldots$  where  $p, q, \ldots$  are sequences over  $\{1, 2\}$  identifying the corresponding coproduct summand in Y. For example, we write  $f: X \to_{12,2} T((Y + Z) + Z)$  to mean that f is  $[\operatorname{in}_1 \operatorname{in}_2, \operatorname{in}_2]$ -guarded.

The above formulation of the notion of guardedness is necessitated by the standard categorical view of (binary) coproducts as a *property*: a binary coproduct is any object that satisfies the corresponding universal property; therefore, coproducts are defined up to isomorphism, and intrinsically refer to the specified coproduct injections. The alternative is to treat coproducts as a *structure*, i.e.

work with canonical coproducts. It is then also possible to adapt the formulation of guardedness and guarded iteration to comply with this view [28].

The axioms (trv), (sum) and (cmp) come from [22]. Intuitively, (trv) says that if a program does not output anything via a summand of the output type then it is guarded in that summand. Rule (cmp) asserts that guardedness is preserved by composition: if the unguarded part of the output of a program is postcomposed with a  $\sigma$ -guarded program then the result is  $\sigma$ -guarded, no matter how the guarded part is transformed. Finally, rule (sum) says that putting two guarded equation systems side by side again produces a guarded system. Here, we also add the rule (str) stating compatibility of guardedness and strength. Note that since **C** is distributive,  $id_Z \times \sigma \colon Z \times Y' \to Z \times Y$  is actually a summand whose canonical complement we take to be  $\mathsf{id}_Z \times \bar{\sigma}$ .

Let us record some simple consequences of the axioms in Fig. 2.

**Lemma 1.** The following rules are derivable:

(iso) 
$$\frac{f: X \to_{\sigma} TY \quad \vartheta: Y \simeq Y'}{(T\vartheta) f: X \to_{\vartheta\sigma} TY'} \qquad (\text{wkn}) \quad \frac{f: X \to_{\sigma} TY}{f: X \to_{\sigma\vartheta} TY}$$

$$(\mathbf{cmp}^{\star}) \ \frac{f \colon X \to_{\sigma + \mathsf{id}} T(Y + Z) \quad g \colon Y \to TV \quad h \colon Z \to TV \quad g\bar{\sigma} \colon Y' \to_{\vartheta} TV}{[g,h]^{\star} f \colon X \to_{\vartheta} TV}$$

(cdm) 
$$\frac{g \colon X \to TY \quad f \colon Y \to_{\sigma} TZ}{f^* g \colon X \to_{\sigma} TZ}$$

*Proof.* The rule (cdm) is obtained from (cmp<sup>\*</sup>) by instantiating Z with  $\emptyset$  and  $\sigma$ with !.

Let us show (iso). Let w.l.o.g.  $Y_1 + Y_2 = Y$  and  $f: X \rightarrow_{in_2} T(Y_1 + Y_2)$ , i.e.  $\sigma = in_2$ . Since  $\vartheta$  is an isomorphism, we have  $\vartheta = [\vartheta_1, \vartheta_2] \colon Y_1 + Y_2 \to Y'$  and hence we derive

**T** 7

$$\frac{f \colon X \to_{\mathsf{in}_2} T(Y_1 + Y_2) \qquad \eta \,\vartheta_2 \colon Y_2 \to TY' \qquad \frac{\eta \colon Y_1 \to TY_1}{(T\vartheta_1) \,\eta \colon Y_1 \to_{\vartheta_2} TY'} \quad (\mathbf{trv})}{[\eta \,\vartheta_1, \eta \,\vartheta_2]^{\star} \, f \colon X \to_{\vartheta_2} TY'} \quad (\mathbf{cmp})$$

Next, we check (**cmp**<sup>\*</sup>). Let w.l.o.g.  $\sigma = in_2$  and  $\vartheta = in_2$ . Note that by (iso),  $(T \operatorname{assoc}^{-1}) f \colon X \to_2 T(Y' + (Y'' + Z))$  where assoc is the associativity isomorphism  $Y' + (Y'' + Z) \cong (Y' + Y'') + Z$ . Then

$$[g,h]^{\star}f = [[g\bar{\sigma},g\sigma],h]^{\star}f = [g\bar{\sigma},[g\sigma,h]]^{\star}(T\operatorname{assoc}^{-1})f$$

is  $\vartheta$ -guarded by (**cmp**).

The rule (wkn) is obtained from (cmp<sup>\*</sup>) by instantiating Z with  $\emptyset$ , g with  $\eta$ and  $\vartheta$  with  $\sigma\vartheta$ . The induced non-trivial premise becomes  $\eta\bar{\sigma}: Y' \to_{\sigma\vartheta} TV$ , and it is verified as follows:  $\eta \bar{\sigma} = (T \bar{\sigma}) \eta = (T \bar{\sigma} \vartheta) (T \xi) \eta$  which is  $\sigma \vartheta$ -guarded by (trv) and (cdm). Here we used the fact that  $\bar{\sigma}$  factors as  $\sigma \vartheta \xi$  with some  $\xi$ , for, dually,  $\sigma\vartheta$  factors through  $\sigma$ . 



Figure 3: Axioms of guarded Elgot iteration.

**Definition 2** (Guarded (pre-)iterative/Elgot monads). A strong monad  $\mathbf{T}$  on a distributive category is *guarded pre-iterative* if it is equipped with a guarded iteration operator

$$\frac{f \colon X \to_2 T(Y+X)}{f^{\dagger} \colon X \to TY} \tag{3}$$

satisfying the

• fixpoint law:  $f^{\dagger} = [\eta, f^{\dagger}]^* f$ .

We call a pre-iterative monad **T** guarded Elgot [28] if it satisfies

- naturality:  $g^{\star}f^{\dagger} = ([(T \operatorname{in}_1) g, \eta \operatorname{in}_2]^{\star}f)^{\dagger}$  for  $f: X \to_2 T(Y + X), g: Y \to TZ;$
- codiagonal:  $(T[\mathsf{id},\mathsf{in}_2]f)^{\dagger} = f^{\dagger\dagger}$  for  $f: X \to_{12,2} T((Y+X)+X);$
- uniformity: f h = T(id + h) g implies  $f^{\dagger} h = g^{\dagger}$  for  $f: X \to_2 T(Y + X)$ ,  $g: Z \to_2 T(Y + Z)$  and  $h: Z \to X$ ;

• strength:  $\tau (\operatorname{id}_W \times f^{\dagger}) = (\delta (\operatorname{id}_W \times f))^{\dagger}$  for  $f: X \to_2 T(Y + X)$ .

and guarded iterative if  $f^{\dagger}$  is a unique solution of the fixpoint law (the remaining axioms then are granted [22]).

The above axioms of iteration are standard (cf. [6]), except *strength*, which we need here for the semantics of computations in multivariable contexts. To understand the axiom, observe that the right-hand side iterates over  $W \times X$ leaving the *W*-component unchanged, and eventually returns the *W*-component as part of the result, while the left-hand side iterates over *X* and subsequently pairs the result with the originally given element of *W*. These axioms, again except strength, can be presented in an intuitive graphical form as equations of flowchart diagrams – see Fig. 3. Here, the orange boxes identify Kleisli morphisms and blue boxes identify morphisms of the underlying category **C**. We indicate the scopes of feedback loops, representing applications of the iteration operator, by shaded green frames. Finally, we indicate by black bullets those outputs in which a corresponding Kleisli morphism is guarded.

The notion of (abstract) guardedness is a common generalization of various special cases occurring in practice. Every monad can be equipped with a least notion of guardedness, called vacuous guardedness and defined as follows:  $f: X \to_2 T(Y + Z)$  iff f factors through  $T \operatorname{in}_1: Y \to T(Y + Z)$ ; that is, intuitively speaking, the definitions of elements of X given by f do not mention variables in Z, or more precisely speaking can be rewritten to ensure this. On the other hand, the greatest notion of guardedness is total guardedness, defined by taking  $f: X \to_2 T(Y + Z)$  for every  $f: X \to T(Y + Z)$ . This addresses total iteration operators on  $\mathbf{T}$ , whose existence depends on special properties of  $\mathbf{T}$ , such as being enriched over complete partial orders. Our motivating examples are mainly those that lie properly between these two extreme situations, e.g. completely iterative monads for which guardedness is defined via monad modules and the iteration operator is partial, but uniquely satisfies the fixpoint law [31]. For illustration, we consider several instances of guarded iteration.

**Example 3.** We fix the category of sets and functions **Set** as an ambient distributive category in the following examples.

1. (Finitely branching processes) Let  $TX = \nu\gamma$ .  $\mathcal{P}_{\omega}(X + \operatorname{Act} \times \gamma)$ , the final  $\mathcal{P}_{\omega}(X + \operatorname{Act} \times -)$ -coalgebra with  $\mathcal{P}_{\omega}$  being the finite powerset functor. Thus, TX is equivalently described as the set of finitely branching nondeterministic trees with edges labelled by elements of  $\operatorname{Act}$  and with terminal nodes possibly labelled by elements of X (otherwise regarded as nullary nondeterminism, i.e. deadlock), taken modulo bisimilarity. Every  $f: X \to T(Y + X)$  can be viewed as a family  $(f(x) \in T(Y + X))_{x \in X}$  of trees whose terminal nodes are labelled in the disjoint union of X and Y. Each tree f(x) thus can be seen as a recursive process definition for the process name x relative to the names in X + Y. The notion of guardedness borrowed from process algebra requires that every  $x' \in X$  occurring in f(x) must be preceded by a transition, and if this condition is satisfied, we can calculate a unique solution  $f^{\dagger}: X \to TY$  of the system of definitions

 $(f(x): T(Y + X))_{x \in X}$ . In other words, **T** is guarded iterative with  $f: X \to_2 T(Y + Z)$  iff

out 
$$f: X \to \mathcal{P}_{\omega}((Y+Z) + \operatorname{Act} \times T(Y+Z))$$

factors through  $\mathcal{P}_{\omega}(in_1 + id)$  where out:  $TX \cong \mathcal{P}_{\omega}(X + Act \times TX)$  is the canonical final coalgebra isomorphism. As a result, **T** is a guarded iterative monad (more specifically *completely iterative* [31]).

2. (Countably branching processes) A variation of the previous example is obtained by replacing finite nondeterminism with countable nondeterminism, i.e. by replacing  $\mathcal{P}_{\omega}$  with the countable powerset functor  $\mathcal{P}_{\omega_1}$ . Note that in the previous example we could not extend the iteration operator to a total one, because unguarded systems of recursive process equations may define infinitely branching processes [4]. The monad  $TX = \nu\gamma$ .  $\mathcal{P}_{\omega_1}(X + \operatorname{Act} \times \gamma)$  does however support both partial guarded iteration in the sense of the previous example, and total iteration extending the former. This monad is therefore both guarded iterative in the former sense, but only guarded Elgot in the latter sense, for under total iteration, the fixpoints  $f^{\dagger}$  are no longer unique. This setup is analysed more generally in detail in previous work [21, 22].

3. A very simple example of total guarded iteration is obtained from the (full) powerset monad  $T = \mathcal{P}$ . The corresponding Kleisli category is enriched over complete partial orders and continuous functions and therefore admits total iteration calculated via least fixpoints. This yields an example of a guarded Elgot monad which is not guarded iterative.

4. (Complete finite traces) Let  $TX = \mathcal{P}(\mathsf{Act}^* \times X)$  be the monad obtained from  $\mathcal{P}$  by an obvious modification ensuring that the first elements of the pairs from  $\mathsf{Act}^* \times X$ , i.e. finite traces, are concatenated along Kleisli composition [9]. Like  $\mathcal{P}$ , this monad is order-enriched and thus supports a total iteration operator via least fixpoints (see e.g. [18]). From this, a guarded iteration operator is obtained by restricting to the guarded category with  $f: X \to_2 \mathcal{P}(\mathsf{Act}^* \times (Y+Z))$ iff f factors through the map

$$\mathcal{P}(\mathsf{Act}^{\star} \times Y + \mathsf{Act}^{+} \times Z) \xrightarrow{\mathcal{P}(\mathsf{id} + \iota \times \mathsf{id})} \mathcal{P}(\mathsf{Act}^{\star} \times Y + \mathsf{Act}^{\star} \times Z) \cong \mathcal{P}(\mathsf{Act}^{\star} \times (Y + Z))$$

induced by the inclusion  $\iota: \mathsf{Act}^+ \hookrightarrow \mathsf{Act}^*$ . Like in Clause 3, we obtain a guarded Elgot monad with a total iteration operator.

5. Finally, an example of partial guarded iteration can be obtained from Clause 3 above by replacing  $\mathcal{P}$  with the *non-empty powerset monad*  $\mathcal{P}^+$ . Total iteration as defined in Clause 3 does not restrict to total iteration on  $\mathcal{P}^+$ , because empty sets can arise from solving systems not involving empty sets, e.g.  $\eta \operatorname{in}_2: 1 \to \mathcal{P}^+(1+1)$  would not have a solution in this sense. However, it is easy to see that total iteration does restrict to guarded iteration for  $\mathcal{P}^+$  with the notion of guardedness defined as follows:  $f: X \to_2 \mathcal{P}^+(Y+Z)$  iff for every x, f(x) contains at least one element from Y. Therefore,  $\mathcal{P}^+$  is a guarded Elgot monad, which is not guarded iterative and with properly partial iteration. For a pre-iterative monad  $\mathbf{T}$ , we derive a *strong iteration operator*:

$$\frac{f: W \times X \to_2 T(Y+X)}{f^{\ddagger} = \left(T(\mathsf{pr}_2 + \mathsf{id})\,\delta\,\langle\mathsf{pr}_1, f\rangle\right)^{\dagger}: W \times X \to TY} \tag{4}$$

which essentially generalizes the original operator  $(-)^{\dagger}$  to morphisms extended with a context via  $W \times (-)$ . This will become essential in Section 3 for the semantics of our metalanguage.

**Lemma 4.** For every strong guarded Elgot monad  $\mathbf{T}$ , strong iteration (4) satisfies  $\tau \langle \mathsf{pr}_1, f^{\ddagger} \rangle = (\delta \langle \mathsf{pr}_1, f \rangle)^{\dagger}$  for every  $f \colon W \times X \to_2 T(Y + X)$ .

*Proof.* Let us rewrite the left hand side as follows:

$$\begin{aligned} \tau \langle \mathsf{pr}_1, f^{\ddagger} \rangle &= T(\mathsf{pr}_1 \times \mathsf{id}) \tau (\mathsf{id} \times f^{\ddagger}) \Delta \\ &= T(\mathsf{pr}_1 \times \mathsf{id}) \tau (\mathsf{id} \times (T(\mathsf{pr}_2 + \mathsf{id}) \,\delta\langle\mathsf{pr}_1, f\rangle)^{\dagger}) \Delta \qquad /\!\!/ \text{ defn. of } (-)^{\ddagger} \\ &= T(\mathsf{pr}_1 \times \mathsf{id}) \tau (\mathsf{id} \times (T\,\mathsf{pr}_2)(\delta\langle\mathsf{pr}_1, f\rangle)^{\dagger}) \Delta \qquad /\!\!/ \text{ naturality} \\ &= T(\mathsf{pr}_1 \times \mathsf{pr}_2) \tau (\mathsf{id} \times (\delta\langle\mathsf{pr}_1, f\rangle)^{\dagger}) \Delta \qquad /\!\!/ \text{ naturality} \\ &= T(\mathsf{pr}_1 \times \mathsf{pr}_2) (\delta(\mathsf{id} \times \delta\langle\mathsf{pr}_1, f\rangle))^{\dagger} \Delta \qquad /\!\!/ \text{ strength} \\ &= (T(\mathsf{pr}_1 \times \mathsf{pr}_2 + \mathsf{id}) \,\delta(\mathsf{id} \times \delta\langle\mathsf{pr}_1, f\rangle))^{\dagger} \Delta. \qquad /\!\!/ \text{ naturality} \end{aligned}$$

Note that

$$\begin{split} T(\mathsf{id} \times \mathsf{pr}_2 + \mathsf{id}) \, \delta \left( \mathsf{id} \times \delta \langle \mathsf{pr}_1, f \rangle \right) \left( \mathsf{pr}_1 \times \mathsf{id} \right) \\ &= T(\mathsf{id} \times \mathsf{pr}_2 + \mathsf{id}) \, \delta \left( \mathsf{pr}_1 \times \delta \langle \mathsf{pr}_1, f \rangle \right) \\ &= T(\mathsf{pr}_1 \times \mathsf{pr}_2 + \mathsf{pr}_1 \times \mathsf{id}) \, \delta \left( \mathsf{id} \times \delta \langle \mathsf{pr}_1, f \rangle \right) \\ &= T(\mathsf{id} + \mathsf{pr}_1 \times \mathsf{id}) \, T(\mathsf{pr}_1 \times \mathsf{pr}_2 + \mathsf{id}) \, \delta \left( \mathsf{id} \times \delta \langle \mathsf{pr}_1, f \rangle \right), \end{split}$$

and therefore, by *uniformity* (instantiating the equation from Definition 2 with  $f = T(id \times pr_2 + id) \delta(id \times \delta \langle pr_1, f \rangle)$ ,  $g = T(pr_1 \times pr_2 + id) \delta(id \times \delta \langle pr_1, f \rangle)$ , and  $h = pr_1 \times id)$ ,

$$\begin{split} \tau \left< \mathsf{pr}_1, f^{\ddagger} \right> &= \left( T(\mathsf{id} \times \mathsf{pr}_2 + \mathsf{id}) \,\delta \left( \mathsf{id} \times \delta \left< \mathsf{pr}_1, f \right> \right) \right)^{\dagger} (\mathsf{pr}_1 \times \mathsf{id}) \,\Delta \\ &= \left( T(\mathsf{id} \times \mathsf{pr}_2 + \mathsf{id}) \,\delta \left( \mathsf{id} \times \delta \left< \mathsf{pr}_1, f \right> \right) \right)^{\dagger} \left< \mathsf{pr}_1, \mathsf{id} \right>. \end{split}$$

Finally, observe that

$$\begin{split} T(\mathsf{id} \times \mathsf{pr}_2 + \mathsf{id}) \,\delta\left(\mathsf{id} \times \delta\langle \mathsf{pr}_1, f\rangle\right) \langle \mathsf{pr}_1, \mathsf{id}\rangle \\ &= T(\mathsf{id} \times \mathsf{pr}_2 + \mathsf{id}) \,\delta\left\langle \mathsf{pr}_1, \delta\langle \mathsf{pr}_1, f\rangle\right\rangle \\ &= T(\mathsf{id} \times \mathsf{pr}_2 + \mathsf{id}) \,T(\langle \mathsf{pr}_1, \mathsf{id}\rangle + \langle \mathsf{pr}_1, \mathsf{id}\rangle) \,\delta\langle \mathsf{pr}_1, f\rangle \qquad /\!\!/ (2) \\ &= T(\langle \mathsf{pr}_1, \mathsf{pr}_2\rangle + \langle \mathsf{pr}_1, \mathsf{id}\rangle) \,\delta\langle \mathsf{pr}_1, f\rangle \\ &= T(\mathsf{id} + \langle \mathsf{pr}_1, \mathsf{id}\rangle) \,\delta\langle \mathsf{pr}_1, f\rangle \end{split}$$

and therefore, by uniformity,

$$\begin{split} \tau \left< \mathsf{pr}_1, f^{\ddagger} \right> &= (T(\mathsf{id} \times \mathsf{pr}_2 + \mathsf{id}) \,\delta \,(\mathsf{id} \times \delta \left< \mathsf{pr}_1, f \right>))^{\dagger} \left< \mathsf{pr}_1, \mathsf{id} \right> \\ &= (\delta \left< \mathsf{pr}_1, f \right>)^{\dagger} \end{split}$$

as desired.

Strength and simple slices. To clarify the role of strong iteration (4), we characterize it as iteration in a simple slice category [24]  $\mathbf{C} / W$  arising for every fixed  $W \in |\mathbf{C}|$  as the co-Kleisli category of the product comonad [7]  $W \times -$ : that is,  $|\mathbf{C} / / W| = |\mathbf{C}|$ ,  $\operatorname{Hom}_{\mathbf{C} / / W}(X, Y) = \operatorname{Hom}_{\mathbf{C}}(W \times X, Y)$ , identities in  $\mathbf{C} / W$  are projections  $\operatorname{pr}_2 \colon W \times X \to X$ , and the composite of  $g \colon W \times X \to Y$ and  $f: W \times Y \to Z$  is  $f(\operatorname{pr}_1, g): W \times X \to Z$ . We often indicate composition in  $\mathbb{C} /\!\!/ W$  by  $\circ^W$  for clarity. We note that  $\mathbb{C} /\!\!/ 1$  is isomorphic to  $\mathbb{C}$ , and  $(\mathbf{C} / \! / W) / \! / V$  is isomorphic to  $\mathbf{C} / \! / W \times V$ , where the isomorphism just rebrackets products. The assignment  $W \mapsto \mathbb{C} / W$  in fact extends to a strict indexed category: A morphism  $k: W \to V$  induces a functor  $\mathbb{C} / / k: \mathbb{C} / / V \to \mathbb{C} / / W$  which acts as identity on objects and maps  $f \in \operatorname{Hom}_{\mathbf{C}/\!/ V}(X,Y) = \operatorname{Hom}_{\mathbf{C}}(X \times V,Y)$ to  $f(k \times id) \in \operatorname{Hom}_{\mathbf{C}/\!/W}(X, Y) = \operatorname{Hom}_{\mathbf{C}}(W \times X, Y)$ . Moreover, we have embeddings  $J^W : \mathbb{C} \to \mathbb{C}/\!\!/ W$ , given by  $J^W X = X$  and  $J^W f = f \operatorname{pr}_2$ , which commute with the functors  $\mathbb{C}/\!\!/ k$ , i.e.  $(\mathbb{C}/\!\!/ k)J^V = J^W$ . In particular, up to the isomorphism  $J^1: \mathbb{C} \cong \mathbb{C} /\!\!/ 1$  the functor  $J^W$  coincides with  $\mathbb{C} /\!\!/ !$  where ! is the unique **C**-morphism  $W \to 1$ . Of course,  $J^W$  is the right adjoint to the forgetful functor  $U^W : \mathbf{C} / / W \to \mathbf{C}$ , which acts on objects as  $U^W X = W \times X$  and on morphisms  $f: X \to Y$  as  $U^W f = \langle \mathsf{pr}_1, f \rangle$ . To avoid confusion with the unit of **T**, we write  $j_X \in \operatorname{Hom}_{\mathbf{C}/\!/W}(X, W \times X)$  for the unit of this adjunction, which is the **C**-morphism id:  $W \times X \to W \times X$ . Like in all co-Kleisli categories, the adjoint transpose map  $\operatorname{Hom}_{\mathbf{C}/\!\!/W}(X,Y) = \operatorname{Hom}_{\mathbf{C}/\!\!/W}(X,J^WY) \cong \operatorname{Hom}_{\mathbf{C}}(U^WX,Y) =$  $Hom_{\mathbf{C}}(W \times X, Y)$  is just identity; we thus have

$$f = (J^W f) \circ^W j_X \tag{5}$$

(as also easily verified directly) for each  $f: X \to Y$  in  $\mathbb{C}/\!\!/ W$ , i.e.  $f: W \times X \to Y$  in  $\mathbb{C}$ .

The monad **T** being strong means in particular that for every  $W \in |\mathbf{C}|, \tau$  yields a distributive law of the monad **T** over the comonad  $W \times -$ , which extends **T** from **C** to  $\mathbf{C} / / W$  [7]. We state this more precisely, and complement it with similar statements on propagation of guardedness and iteration:

**Theorem 5.** Let  $\mathbf{T}$  be a strong monad on a distributive category  $\mathbf{C}$ . Then the following hold.

- 1. For every  $W \in |\mathbf{C}|$ ,  $\mathbf{C} / / W$  is distributive, and  $\mathbf{T}$  coherently extends to a strong monad over  $\mathbf{C} / W$ : For every  $k \colon W \to V$ , the functor  $\mathbf{C} / k$  strictly preserves the monad structure, i.e. if  $\mathbf{T}^W$  and  $\mathbf{T}^V$  denote the extensions of  $\mathbf{T}$  to  $\mathbf{C} / W$  and  $\mathbf{C} / V$  respectively, then  $(\mathbf{C} / k) T^V = T^W \mathbf{C} / k$ , and the pair consisting of  $\mathbf{C} / k$  and the identity natural transformation on  $(\mathbf{C} / k) T^V$  is a monad morphism. The same holds for the functors  $J^W$ .
- 2. If **T** is guarded, then so is its extension  $\mathbf{T}^W$  to  $\mathbf{C}/\!\!/W$ , with the same notion of guardedness (i.e.  $\operatorname{Hom}_{\mathbf{C}/\!\!/W,\sigma}(X, T^WY) = \operatorname{Hom}_{\mathbf{C},\sigma}(W \times X, TY)$ ), and all functors  $\mathbf{C}/\!\!/k$ , as well as the functors  $J^W$ , preserve guardedness.
- 3. If **T** is guarded pre-iterative on **C** then so is the extension of **T** to  $\mathbf{C} /\!\!/ W$ , under the same definition of guardedness and with iteration defined as

strong iteration (4). If moreover **T** satisfies uniformity, then all functors  $\mathbf{C} \mathscr{/} k$ , as well as the functors  $J^W$ , preserve iteration.

- 4. If **T** is guarded Elgot on **C** then so is the extension of **T** to  $\mathbf{C} /\!\!/ W$ .
- 5. If **T** is guarded iterative then so is the extension of **T** to  $\mathbf{C} / / W$ .

Moreover, we have partial converses to the above claims, which further justify the axioms and definitions regarding strength, specifically the  $(\mathbf{str})$  rule for guardedness, the definition of strong iteration, and the strength law for iteration. Only for purposes of the statement and proof of the following theorem, we introduce notions of guardedness, guarded iterativity etc. for monads that are not assumed to be strong; these are axiomatized in the expected way, i.e. by just removing the axioms and rules referring to strength. We designate these notions as weak, and the standard versions as strong for clarity. E.g. a weakly guarded monad is a monad  $\mathbf{T}$  equipped with distinguished subsets  $\operatorname{Hom}_{\sigma}(X, TY)$ , indexed over summands  $\sigma: Y' \hookrightarrow Y$ , that satisfy axioms  $(\mathbf{trv})$ ,  $(\mathbf{sum})$  and  $(\mathbf{cmp})$ , and a strongly guarded monad is a weakly guarded strong monad satisfying axiom  $(\mathbf{str})$ .

**Theorem 6.** Let **T** be a monad on **C**, and assume that **T** extends coherently to monads  $\mathbf{T}^W$  on all  $\mathbf{C} /\!\!/ W$ , in the sense that  $(\mathbf{C} /\!\!/ k) T^V = T^W (\mathbf{C} /\!\!/ k)$  for every  $k: W \to V$ . Then the following hold

- The monad T is strong. In fact, the construction of the strength and the opposite construction from Theorem 5.1 (which induces the T<sup>W</sup> from a given strength) are mutually inverse.
- 2. If **T** is weakly guarded,  $\mathbf{T}^W$  is weakly guarded, and  $J^W$  preserves guardedness, then  $\operatorname{Hom}_{\mathbf{C} \not = W, \sigma}(X, T^W Y) \supseteq \operatorname{Hom}_{\mathbf{C}, \sigma}(W \times X, TY)$ .
- 3. If **T** is weakly guarded, and putting  $\operatorname{Hom}_{\mathbf{C} \rtimes W,\sigma}(X, T^WY) = \operatorname{Hom}_{\mathbf{C},\sigma}(W \times X, TY)$  makes each **T**<sup>W</sup> into a weakly guarded monad, then **T** is a strongly guarded monad, i.e. satisfies (str).
- 4. If **T** is strongly guarded and pre-iterative, each  $\mathbf{T}^W$  is pre-iterative and satisfies uniformity, and  $J^W : \mathbf{C} \to \mathbf{C} / \! / W$  preserves iteration, then iteration on  $\mathbf{T}^W$  is strong iteration (4).
- 5. If **T** is strongly guarded and pre-iterative, each  $\mathbf{T}^W$ , made into a preiterative monad by equipping it with iteration defined as strong iteration on **T**, satisfies naturality, and  $J^W$  preserves iteration, then **T** satisfies the strength law (Definition 2).

We prove Theorem 5 first but in fact occasionally make use of the converse statements recorded in Theorem 6 (whose proof will not depend on Theorem 5). Specifically, to establish a property regarding strength, we apply the current implication to conclude a weak (i.e. strength-free) property of  $\mathbf{C} /\!\!/ W \times V \cong (\mathbf{C} /\!\!/ W) /\!\!/ V$ , and then apply Theorem 6 to obtain a property of  $\mathbf{C} /\!\!/ W$  referring to strength.

*Proof (Theorem 5).* 1. Being a co-Kleisli category,  $\mathbf{C} / / W$  inherits finite products from  $\mathbf{C}$ . Finite coproducts are inherited thanks to  $\mathbf{C}$  being distributive; e.g.

$$\begin{split} \mathsf{Hom}_{\mathbf{C}/\!\!/W}(X+Y,Z) &= \mathsf{Hom}_{\mathbf{C}}(W \times (X+Y),Z) \\ &\cong \mathsf{Hom}_{\mathbf{C}}(W \times X + W \times Y,Z) \\ &\cong \mathsf{Hom}_{\mathbf{C}}(W \times X,Z) \times \mathsf{Hom}_{\mathbf{C}}(W \times Y,Z) \\ &= \mathsf{Hom}_{\mathbf{C}/\!\!/W}(X,Z) \times \mathsf{Hom}_{\mathbf{C}/\!\!/W}(Y,Z). \end{split}$$

Since both products and coproducts in  $\mathbb{C} / / W$  are inherited from  $\mathbb{C}$ , so is distributivity. We have already noted that  $\mathbb{T}$  lifts to  $\mathbb{C} / / W$  because the strength yields a distributive law of  $\mathbb{T}$  over the product comonad [7]. The lifted monad is explicitly described as follows. The unit is just  $\eta \operatorname{pr}_2 : W \times X \to TX$  where  $\eta$  is the unit of  $\mathbb{T}$  in  $\mathbb{C}$ , and the Kleisli lifting of  $f \in \operatorname{Hom}_{\mathbb{C} / / W}(X, TY)$  is  $f^*\tau$  where  $f^* : T(W \times X) \to TY$  is the Kleisli lifting of  $f : W \times X \to TY$  in  $\mathbb{C}$  and  $\tau$  is the strength of  $\mathbb{T}$  in  $\mathbb{C}$ . We note in particular that this implies  $T^W f = Tf\tau$  for  $f : X \to Y$  in  $\mathbb{C} / / W$  (hence  $f : W \times X \to Y$  in  $\mathbb{C}$ ). We defer consideration of the strength, and tackle coherence first.

We need to show that  $(\mathbb{C} /\!\!/ k) T^V = T^W (\mathbb{C} /\!\!/ k)$ . So let  $f \colon X \to Y$  in  $\mathbb{C} /\!\!/ W$ , i.e.  $f \colon W \times X \to Y$  in  $\mathbb{C}$ . Then

$$(\mathbf{C} /\!\!/ k) (T^V f) = (Tf) \tau_{V,X} (k \times \mathsf{id}_{TX}) /\!\!/ \text{ definitions}$$
$$= (Tf) T(k \times \mathsf{id}_X) \tau_{W,X} /\!\!/ \text{ naturality of } \tau$$
$$= T^W (\mathbf{C} /\!\!/ k) f. /\!\!/ \text{ definitions}$$

Preservation of the monad structure is then clear by the above description of this structure. The claim for  $J^W$  follows as a special case, since the isomorphism of **C** and **C** // 1 clearly extends to the corresponding monads.

We conclude by the initially mentioned strategy that  $\mathbf{T}^W$  is strong: By the above,  $\mathbf{T}$  extends to a monad  $\mathbf{T}^{W \times V}$  on  $\mathbf{C} /\!\!/ W \times V$ , which transfers along the isomorphism  $\mathbf{C} /\!\!/ W \times V \cong (\mathbf{C} /\!\!/ W) /\!\!/ V$  to a monad  $(\mathbf{T}^W)^V$  on  $(\mathbf{C} /\!\!/ W) /\!\!/ V$  acting on morphisms  $f \in \operatorname{Hom}_{(\mathbb{C} /\!\!/ W) /\!\!/ V}(X, Y) = \operatorname{Hom}_{\mathbf{C}}(W \times V \times X, Y)$  by  $(T^W)^V f = (Tf)\tau_{W \times V,X}$ . Since under the isomorphism  $\mathbf{C} /\!\!/ W \times V \cong (\mathbf{C} /\!\!/ W) /\!\!/ V$ , the embedding of  $\mathbf{C} /\!\!/ W$  into  $(\mathbf{C} /\!\!/ W) /\!\!/ V$  corresponds to  $\mathbf{C} /\!\!/ \operatorname{pr}_1 : \mathbf{C} /\!\!/ W \to \mathbf{C} /\!\!/ W \times V$ , the above preservation property for functors  $\mathbf{C} /\!\!/ k$  implies that  $(\mathbf{T}^W)^V$  extends  $\mathbf{T}^W$ ; again by the preservation properties already established, these extensions are coherent. By Theorem 6.1, it follows that  $\mathbf{T}^W$  is strong. The strength  $V \times T^W X \to T^W (V \times X)$  constructed in the proof of Theorem 6.1 is  $(T^W)^V k_X$  (understood as a  $\mathbf{C} /\!\!/ W$ -morphism), where  $k_X$  is the  $\mathbf{C} /\!\!/ W$ -identity on  $V \times X$  taken as a  $(\mathbf{C} /\!\!/ W) /\!\!/ V$ -morphism  $X \to V \times X$ , which as a  $\mathbf{C}$ -morphism  $W \times V \times X \to V \times X$  projects to the second and third component. By the above description of  $(T^W)^V$ , we have, eliding associativity isomorphisms,  $(T^W)^V k_X = (Tk_X) \tau_{W \times V,X} = \tau \operatorname{pr}_2$  where  $\operatorname{pr}_2 : W \times (V \times TX) \to V \times TX$ , using standard coherence properties of  $\tau$ . It follows that the  $\mathbf{C} /\!\!/ k$  preserve also the strength.

2. We need to verify that the extension of **T** to  $\mathbf{C} /\!\!/ W$  satisfies the axioms of guardedness from Fig. 2.

• (trv) Given  $f: W \times X \to TY$ , we need to check that  $T(\operatorname{in}_1 \operatorname{pr}_2) \tau \langle \operatorname{pr}_1, f \rangle : W \times X \to_2 T(Y + Z)$ . Indeed,  $T(\operatorname{in}_1 \operatorname{pr}_2) \tau \langle \operatorname{pr}_1, f \rangle$  reduces to  $(T \operatorname{in}_1) f$  and we are done by the original (trv) for C.

• (sum) Given  $f: W \times X \to_{\sigma} TZ$ ,  $g: W \times Y \to_{\sigma} TZ$ , by (sum) for **C**,  $[f,g]: W \times X + W \times Y \to_{\sigma} TZ$ . After precomposing the result with the isomorphism dist, we are done by Proposition 1.

• (cmp) Let  $f: W \times X \to_{\text{in}_2} T(Y+Z), g: W \times Y \to_{\sigma} TV, h: W \times Z \to TV$ and we need to show that  $[g,h]^* \delta(\text{pr}_1, f): W \times X \to_{\sigma} TV$ . The latter morphism equals the composite

$$\begin{split} W \times X & \xrightarrow{\tau \langle \mathsf{id}, f \rangle} T((W \times X) \times (Y + Z)) \\ & \xrightarrow{(\eta \operatorname{dist}(\mathsf{pr}_1 \times \mathsf{id}))^*} T(W \times Y + W \times Z) \xrightarrow{[g,h]^*} TV. \end{split}$$

By (cmp), we reduce to the problem of showing

 $(\eta \operatorname{dist} (\operatorname{pr}_1 \times \operatorname{id}))^{\star} \tau \langle \operatorname{id}, f \rangle \colon W \times X \to_2 T(W \times Y + W \times Z).$ 

Note that by (str),  $\tau \langle id, f \rangle \colon W \times X \to_{id \times in_2} T((W \times X) \times (Y + Z))$ . Now  $(W \times X) \times (Y + Z)$  is a coproduct of  $(W \times X) \times Y$  and  $(W \times X) \times Z$ , and  $\eta \operatorname{dist}(\operatorname{pr}_1 \times \operatorname{id})$ , regarded as a universal morphism induced by this coproduct structure, yields  $\eta \operatorname{in}_1(\operatorname{pr}_1 \times \operatorname{id}) = (T \operatorname{in}_1) \eta(\operatorname{pr}_1 \times \operatorname{id})$  by composition with the corresponding left coproduct injection; the latter morphism is in<sub>2</sub>-guarded by (trv). We are therefore done by (cmp).

• (str) As indicated above, we go via Theorem 6: The guardedness structure of the monad  $(\mathbf{T}^W)^V$  on  $(\mathbf{C}/\!\!/ W)/\!\!/ V$  is clearly the same as the one of the monad  $\mathbf{T}^{W \times V}$  on  $\mathbf{C}/\!\!/ W \times V$ , hence satisfies (trv), (sum), and (cmp) by the above. By Theorem 6.3, it follows that  $\mathbf{T}^W$  satisfies (str).

It remains to show that given  $k: W \to V$ ,  $\mathbb{C} / / k: \mathbb{C} / / V \to \mathbb{C} / / W$  preserves guardedness: If  $f: X \to_{\sigma} T^{V}Y$  in  $\mathbb{C} / / V$ , then by definition  $f: V \times X \to TY$ in  $\mathbb{C}$ . By (cdm), it follows that  $f(k \times id): W \times X \to_{\sigma} TY$ , so by definition  $(\mathbb{C} / / k) f: X \to_{\sigma} T^{W}Y$  in  $\mathbb{C} / / W$ . The claim for  $J^{W}$  follows as a special case, since  $\mathbb{C}$  and  $\mathbb{C} / / 1$  clearly remain isomorphic as guarded monads.

3. We have to verify the fixpoint law. Suppose that  $f: W \times X \to_2 T(Y+X)$  and check that  $f^{\ddagger} = [\eta \operatorname{pr}_2, f^{\ddagger}]^{\star} \delta \langle \operatorname{pr}_1, f \rangle$ . Indeed,

$$\begin{aligned} f^{\ddagger} &= (T(\mathsf{pr}_2 + \mathsf{id}) \,\delta\langle\mathsf{pr}_1, f\rangle)^{\dagger} & \text{// definition} \\ &= [\eta, (T(\mathsf{pr}_2 + \mathsf{id}) \,\delta\langle\mathsf{pr}_1, f\rangle)^{\dagger}]^{\star} \,T(\mathsf{pr}_2 + \mathsf{id}) \,\delta\langle\mathsf{pr}_1, f\rangle & \text{// fixpoint} \\ &= [\eta\,\mathsf{pr}_2, f^{\ddagger}]^{\star}\delta\langle\mathsf{pr}_1, f\rangle. \end{aligned}$$

It remains to show that for  $k: W \to V$ ,  $\mathbb{C} / k: \mathbb{C} / V \to \mathbb{C} / W$  preserves iteration, so let  $f: V \times X \to T(Y + X)$  in  $\mathbb{C}$ ; expanding the definition of  $\mathbb{C} / k$  and strong iteration, we have to show that

$$(T(\mathsf{pr}_2 + \mathsf{id}_{V \times X})\delta\langle\mathsf{pr}_1, f\rangle)^{\dagger}(k \times \mathsf{id}_X) = (T(\mathsf{pr}_2 + \mathsf{id}_{W \times X})\delta\langle\mathsf{pr}_1, f(k \times \mathsf{id})\rangle)^{\dagger}.$$

By uniformity, this equation follows from commutativity of (the outer frame in) the following diagram

$$\begin{array}{c|c} W \times X & \xrightarrow{k \times \mathrm{id}} & V \times X \\ \langle \mathsf{pr}_1, f(k \times \mathrm{id}) \rangle & & & \downarrow \langle \mathsf{pr}_1, f \rangle \\ W \times T(Y + X) & \xrightarrow{k \times \mathrm{id}} & V \times T(Y + X) \\ & \delta & & \downarrow \delta \\ T(W \times Y + W \times X) & \xrightarrow{T(k \times \mathrm{id} + k \times \mathrm{id})} & T(V \times Y + V \times X) \\ & & T(\mathsf{pr}_2 + \mathrm{id}) & & \downarrow T(\mathsf{pr}_2 + \mathrm{id}) \\ T(Y + W \times X) & \xrightarrow{T(\mathrm{id} + k \times \mathrm{id})} & T(Y + V \times X) \end{array}$$

in which the middle square commutes by naturality of  $\delta$  and commutativity of the other two squares is obvious.

The claim for  $J^W$  follows as a special case as soon as we show that **T** and **T**<sup>1</sup> are isomorphic as guarded pre-iterative monads. This is by uniformity w.r.t. the isomorphisms  $\langle !, \mathsf{id}_X \rangle \colon X \to 1 \times X$  (composition with which defines the isomorphism  $\mathbf{C} // 1 \to \mathbf{C}$ ), with the application condition checked in a very similar calculation as above.

- 4. We check the laws one by one.
- (naturality) We have to show that

$$g^{\star}\tau\langle \mathsf{pr}_1, f^{\ddagger} \rangle = ([(T \operatorname{in}_1) g, \eta \operatorname{in}_2 \mathsf{pr}_2]^{\star}\delta\langle \mathsf{pr}_1, f \rangle)^{\ddagger}$$

with  $f: W \times X \to_2 T(Y + X)$ ,  $g: W \times Y \to TZ$ . Let us rewrite the left-hand side as follows, using the definition of  $(-)^{\ddagger}$  and *naturality* of  $(-)^{\ddagger}$ , with steps marked by capital letters explained in detail afterwards:

$$\begin{split} g^{\star} \tau \langle \mathsf{pr}_1, f^{\dagger} \rangle \\ &= g^{\star} (\delta \langle \mathsf{pr}_1, f \rangle)^{\dagger} \\ &= ([(T \operatorname{in}_1)g, \eta \operatorname{in}_2]^{\star} \delta \langle \mathsf{pr}_1, f \rangle)^{\dagger} \end{split} /\!\!\!/ \text{ Lemma 4} \\ \end{split}$$

$$= ([(T \operatorname{in}_1) g \operatorname{pr}_2, \eta \operatorname{in}_2(\operatorname{id} imes \operatorname{pr}_2)]^{\star})$$

$$T(\langle \mathsf{pr}_1, \mathsf{id} \rangle + \langle \mathsf{pr}_1, \mathsf{id} \rangle) \, \delta \langle \mathsf{pr}_1, f \rangle)^{\dagger} \qquad /\!\!/ (A)$$

$$= \left( \left[ (T \operatorname{in}_{1}) g \operatorname{pr}_{2}, \eta \operatorname{in}_{2} (\operatorname{id} \times \operatorname{pr}_{2}) \right]^{\star} \delta \langle \operatorname{pr}_{1}, \delta \langle \operatorname{pr}_{1}, f \rangle \rangle \right)^{\dagger} \qquad /\!\!/ (2)$$

$$= (T(\mathsf{pr}_2 + \mathsf{id}) \lfloor (T \,\mathsf{in}_1)\tau(\mathsf{id} \times g), (T \,\mathsf{in}_2)\tau(\mathsf{id} \times \eta \,\mathsf{pr}_2) \rfloor^* \\ \delta \langle \mathsf{pr}_1, \delta \langle \mathsf{pr}_1, f \rangle \rangle )^{\dagger} \qquad /\!\!/ (B)$$

$$= (T(\mathsf{pr}_2 + \mathsf{id}) [(T \operatorname{in}_1)\tau, (T \operatorname{in}_2)\tau]^* T(\mathsf{id} \times g + \mathsf{id} \times \eta \operatorname{pr}_2) \delta\langle \mathsf{pr}_1, \delta\langle \mathsf{pr}_1, f \rangle\rangle)^{\dagger}$$
 // coproducts  
$$= (T(\mathsf{pr}_2 + \mathsf{id}) [(T \operatorname{in}_1)\tau, (T \operatorname{in}_2)\tau]^*$$

$$\begin{split} & \delta \langle \mathsf{pr}_1, T(g + \eta \, \mathsf{pr}_2) \delta \langle \mathsf{pr}_1, f \rangle \rangle \big)^{\dagger} & /\!\!/ \text{ naturality of } \delta \\ &= \left( T(\mathsf{pr}_2 + \mathsf{id}) \, \delta \left( \mathsf{id} \times [T \, \mathsf{in}_1, T \, \mathsf{in}_2]^{\star} \right) \\ & \left\langle \mathsf{pr}_1, T(g + \eta \, \mathsf{pr}_2) \delta \langle \mathsf{pr}_1, f \rangle \rangle \right)^{\dagger} & /\!\!/ (C) \end{split}$$

$$= (T(\mathsf{pr}_2 + \mathsf{id}) \,\delta \,\langle \mathsf{pr}_1, [(T \,\mathsf{in}_1) \,g, (T \,\mathsf{in}_2)\eta \,\mathsf{pr}_2]^* \,\delta \,\langle \mathsf{pr}_1, f \rangle \rangle)' \quad /\!\!/ \text{ (co-)products}$$
$$= ([(T \,\mathsf{in}_1) \,g, \eta \,\mathsf{in}_2 \,\mathsf{pr}_2]^* \,\delta \,\langle \mathsf{pr}_1, f \rangle)^{\ddagger} \qquad /\!\!/ \text{ definition}$$

In step (A), we use that generally,  $k^{\star}(Th) = (kh)^{\star}$  and that

$$\left(\mathsf{pr}_2 + \mathsf{id} \times \mathsf{pr}_2\right) \left( \langle \mathsf{pr}_1, \mathsf{id} \rangle + \langle \mathsf{pr}_1, \mathsf{id} \rangle \right) = \mathsf{id} + \mathsf{id} = \mathsf{id}.$$

In step (B), we use that generally,  $(Tk)\,[h,u]^\star=(Tk\,[h,u])^\star=[(Tk)\,h,(Tk)\,u]^\star$  and that

$$T(\mathsf{pr}_{2} + \mathsf{id}) (T \mathsf{in}_{1}) \tau (\mathsf{id} \times g)$$

$$= (T \mathsf{in}_{1}) (T \mathsf{pr}_{2}) \tau (\mathsf{id} \times g) \qquad /\!\!/ \text{ coproducts}$$

$$= (T \mathsf{in}_{1}) \mathsf{pr}_{2} (\mathsf{id} \times g) \qquad /\!\!/ \text{ coherence of } \tau$$

$$= (T \mathsf{in}_{1}) g \mathsf{pr}_{2} \qquad /\!\!/ \text{ products}$$

as well as

$$\begin{split} T(\mathsf{pr}_2 + \mathsf{id}) & (T \, \mathsf{in}_2) \tau \, (\mathsf{id} \times \eta \, \mathsf{pr}_2) \\ &= (T \, \mathsf{in}_2) \tau \, (\mathsf{id} \times \eta \, \mathsf{pr}_2) & /\!\!/ \text{ coproducts} \\ &= (T \, \mathsf{in}_2) \eta \, (\mathsf{id} \times \mathsf{pr}_2) & /\!\!/ \text{ coherence of } \tau \\ &= \eta \, \mathsf{in}_2 \, (\mathsf{id} \times \mathsf{pr}_2) & /\!\!/ \text{ naturality of } \eta \end{split}$$

Finally, we justify step (C) as follows. First, we note that

$$[T \operatorname{in}_1, T \operatorname{in}_2]^* T(\tau + \tau) = \delta^* T[\operatorname{id} \times T \operatorname{in}_1, \operatorname{id} \times T \operatorname{in}_2], \tag{6}$$

as seen by the following calculation:

$$\begin{split} \delta^{\star} T[\operatorname{id} \times T \operatorname{in}_{1}, \operatorname{id} \times T \operatorname{in}_{2}] &= ((T \operatorname{dist}) \tau [\operatorname{id} \times T \operatorname{in}_{1}, \operatorname{id} \times T \operatorname{in}_{2}])^{\star} & // \operatorname{definition} \\ &= ((T \operatorname{dist}) [\tau (\operatorname{id} \times T \operatorname{in}_{1}), \tau (\operatorname{id} \times T \operatorname{in}_{2})])^{\star} & // \operatorname{coproducts} \\ &= ((T \operatorname{dist}) [T (\operatorname{id} \times \operatorname{in}_{1}), T (\operatorname{id} \times \operatorname{in}_{2})] (\tau + \tau))^{\star} & // \operatorname{naturality} \operatorname{of} \tau \\ &= ([T \operatorname{in}_{1}, T \operatorname{in}_{2}] (\tau + \tau))^{\star} & // \operatorname{distributivity} \\ &= [T \operatorname{in}_{1}, T \operatorname{in}_{2}]^{\star} T (\tau + \tau), \end{split}$$

Using (6), we now calculate

$$\begin{split} [(T \operatorname{in}_1) \tau, (T \operatorname{in}_2) \tau]^* \delta \\ &= [T \operatorname{in}_1, T \operatorname{in}_2]^* T(\tau + \tau) T \operatorname{dist} \tau \qquad /\!\!/ \text{ definition} \\ &= \delta^* T[\operatorname{id} \times T \operatorname{in}_1, \operatorname{id} \times T \operatorname{in}_2] T \operatorname{dist} \tau \qquad /\!\!/ (6) \end{split}$$

$= \delta^{\star} T(id \times [T in_1, T in_2]) \tau$	// distributivity
$= (T\operatorname{dist})\tau^{\star}T(\operatorname{id}\times[T\operatorname{in}_1,T\operatorname{in}_2])\tau$	$/\!\!/$ definition
$= (T\operatorname{dist})\tau (\operatorname{id} \times [T\operatorname{in}_1, T\operatorname{in}_2]^{\star})$	// coherence of $\tau$
$= \delta \left( id \times [T in_1, T in_2]^\star \right)$	$/\!\!/$ definition

as used in (C).

• (codiagonal) We have to show that

$$(T[\mathsf{id},\mathsf{in}_2]f)^{\ddagger} = f^{\ddagger\ddagger}$$

for  $f: W \times X \to_{12,2} T((Y + X) + X)$ . We have the following straightforward identity (which we prove after the main argument) between two morphisms from  $W \times T((Y + X) + X)$  to  $T(W \times Y + W \times X)$ :

$$\delta(\mathsf{id} \times T[\mathsf{id}, \mathsf{in}_2]) = T[\mathsf{dist}, \mathsf{in}_2] \,\delta. \tag{7}$$

Using this equation, we obtain on the one hand, using *codiagonal* for  $(-)^{\dagger}$ :

$(T[id,in_2]f)^{\ddagger}$	
$= \left(T(pr_2 + id)\delta\langlepr_1,T[id,in_2]f\rangle\right)^\dagger$	$/\!\!/$ definition
$= (T(pr_2 + id)\delta(id \times T[id,in_2])\langlepr_1,f\rangle)^\dagger$	$/\!\!/$ products
$= (T(pr_2 + id)  T[dist, in_2]  \delta  \langle pr_1, f \rangle)^\dagger$	∥ (7)
$= (T[(pr_2 + id) \operatorname{dist}, in_2] \delta \langle pr_1, f \rangle)^\dagger$	$/\!\!/$ coproducts
$= \left(T[id,in_2]T((pr_2+id)dist+id)\delta\langlepr_1,f\rangle\right)^\dagger$	$/\!\!/$ coproducts
$= (T((pr_2 + id)  dist + id) \delta \langle pr_1, f \rangle)^{\dagger \dagger}$	// codiagonal

and on the other hand:

$$\begin{split} f^{\ddagger\ddagger} &= (T(\mathsf{pr}_2 + \mathsf{id}) \,\delta \,\langle \mathsf{pr}_1, f^{\ddagger} \rangle)^{\dagger} & // \text{ definition} \\ &= (T(\mathsf{pr}_2 + \mathsf{id}) \,(T \,\mathsf{dist}) \,\tau \,\langle \mathsf{pr}_1, f^{\ddagger} \rangle)^{\dagger} & // \,\text{ defin. of } \delta \\ &= (T(\mathsf{pr}_2 + \mathsf{id}) \,(T \,\mathsf{dist}) \,(\delta \,\langle \mathsf{pr}_1, f \rangle)^{\dagger})^{\dagger} & // \,\text{ Lemma 4} \\ &= (T((\mathsf{pr}_2 + \mathsf{id}) \,\mathsf{dist} + \mathsf{id}) \,\delta \,\langle \mathsf{pr}_1, f \rangle)^{\dagger\dagger}. & // \,\text{ naturality} \end{split}$$

It remains to prove (7): We have

$\delta \left( id \times T[id, in_2] \right)$	
$= (T \operatorname{dist}) \tau \left( \operatorname{id} \times T[\operatorname{id}, \operatorname{in}_2] \right)$	$/\!\!/$ definition
$= (T \operatorname{dist}) T(\operatorname{id} \times [\operatorname{id}, \operatorname{in}_2]) \tau$	// naturality of $\tau$
$= (T\operatorname{dist}) T([\operatorname{id}\times\operatorname{id},\operatorname{id}\times\operatorname{in}_2]) (T\operatorname{dist})  \tau$	// distributivity
$= T[dist,dist(id\timesin_2)]\delta$	$/\!\!/$ coproducts, definition
$= T[dist,in_2]\delta$	// distributivity

• (uniformity) For  $f: W \times X \to_2 T(Y + X)$ ,  $g: W \times Z \to_2 T(Y + Z)$ , and  $h: W \times Z \to X$ , the premise of the uniformity law expands by the definition of the structure of  $\mathbb{C} / W$  to the equation

$$f \langle \mathsf{pr}_1, h \rangle = T(\mathsf{pr}_2 + h) \,\delta\langle \mathsf{pr}_1, g \rangle. \tag{8}$$

Then we derive the conclusion of the uniformity law,

$$f^{\ddagger} \langle \mathsf{pr}_1, h \rangle = (T(\mathsf{pr}_2 + \mathsf{id}) \,\delta \,\langle \mathsf{pr}_1, f \rangle)^{\dagger} \,\langle \mathsf{pr}_1, h \rangle = (T(\mathsf{pr}_2 + \mathsf{id}) \,\delta \,\langle \mathsf{pr}_1, g \rangle)^{\dagger} = g^{\ddagger},$$

using the definition of  $(-)^{\ddagger}$  and uniformity of  $(-)^{\dagger}$ , whose premise is verified as follows:

• (strength) Again, we go via Theorem 6. By the above, the guarded monad  $\mathbf{T}^{W \times V}$  on  $\mathbf{C} / / W \times V$  is pre-iterative and satisfies uniformity. The same thus transfers to the isomorphic guarded monad  $(\mathbf{T}^W)^V$  on  $(\mathbf{C} / / W) / / V$ . Moreover, the embedding  $\mathbf{C} / / W \rightarrow (\mathbf{C} / / W) / / V$  corresponds to  $\mathbf{C} / / \operatorname{pr}_1$  under the isomorphism  $(\mathbf{C} / / W) / / V \cong \mathbf{C} / / W \times V$ , and thus preserves iteration by item 3. By Theorem 6.4, it follows that iteration on  $(\mathbf{T}^W)^V$  is strong iteration on  $\mathbf{T}^W$ . Moreover, again by the above,  $(\mathbf{T}^W)^V$  satisfies naturality. By Theorem 6.5, it follows that  $\mathbf{T}^W$  satisfies strength.

5. Suppose that **T** is guarded iterative, hence guarded Elgot. By the previous clause we know that given  $f: W \times X \to_2 T(Y + X)$ ,  $f^{\ddagger}$  satisfies the fixpoint law; by unfolding the definitions of the coproduct and monad structures on  $\mathbf{C} /\!\!/ W$ , we obtain  $f^{\ddagger} = [\eta \operatorname{pr}_2, f^{\ddagger}]^* \delta \langle \operatorname{pr}_1, f \rangle$ . We are left to show that this equation is satisfied by  $f^{\ddagger}$  uniquely. Indeed, suppose that for some  $g: W \times X \to T(Y + X)$ ,  $g = [\eta \operatorname{pr}_2, g]^* \delta \langle \operatorname{pr}_1, f \rangle$ . Hence  $g = [\eta, g]^* T(\operatorname{pr}_2 + \operatorname{id}) \delta \langle \operatorname{pr}_1, f \rangle$ , and therefore  $g = (T(\operatorname{pr}_2 + \operatorname{id}) \delta \langle \operatorname{pr}_1, f \rangle)^{\ddagger}$ , using the fact that **T** is guarded iterative; but the right hand side is just the definition (4) of  $f^{\ddagger}$ .

The proof of the converse statements then runs as follows:

*Proof (Theorem 6).* 1. Most of the claim is immediate from the known fact that giving a lifting of a monad to the Kleisli category of a comonad is equivalent to giving a comonad-over-monad distributive law [42]; that is, for each W we have a distributive law  $\tau_{W,-}$  of  $W \times (-)$  over **T**, defined as

$$(\tau_{W,X} \colon W \times TX \to T(W \times X)) = T^W j_X, \tag{9}$$

where the  $\mathbb{C} /\!\!/ W$ -morphism  $T^W j_X \colon TX \to T(W \times X)$  is converted into a  $\mathbb{C}$ -morphism  $W \times TX \to T(W \times X)$ , and this construction is inverse to the construction of a lifting of  $\mathbb{T}$  from a strength of  $\mathbb{T}$  given in the proof of Theorem 5.1. Explicitly, this means that throughout the remainder of the proof, we can assume that strength and lifting relate to each other via Equation (9) above and the description of  $\mathbb{T}^W$  in the proof of Theorem 5.1. In particular,

$$T^W f = (Tf)\,\tau\tag{10}$$

for  $f: X \to Y$  in  $\mathbb{C}/\!\!/ W$  (i.e.  $f: W \times X \to Y$  in  $\mathbb{C}$ ). Of course,  $\tau_{W,X}$  will serve as the strength; it remains only to verify those axioms that do not already feature among the properties of  $\tau_{W,-}$  as a distributive law (cf. [7]) – that is, we need to verify naturality of  $\tau_{W,X}$  in W and compatibility with the associator, which both involve two different instances  $W \times (-), V \times (-)$  of the product comonad.

Naturality in W: Let  $k: V \to W$ ; we have to show that

$$(T^W j^W_X) \left( k \times T \mathsf{id}_X \right) = T(k \times \mathsf{id}_X) \left( T^V j^V_X \right)$$

in  $\mathbf{C}$ , where we have decorated the unit j of the co-Kleisli adjunction with additional superscripts to indicate the relevant simple slice. We calculate as follows:

$$\begin{aligned} (T^W j_X^W) \left(k \times T \operatorname{id}_X\right) &= (\mathbb{C} /\!\!/ k) \left(T^W j_X^W\right) & /\!\!/ \text{ definition} \\ &= T^V ((\mathbb{C} /\!\!/ k) j_X^W) & /\!\!/ \text{ coherence} \\ &= T^V (j_X^W (k \times \operatorname{id}_X)) & /\!\!/ \text{ definition of } \mathbb{C} /\!\!/ k \\ &= T^V (k \times \operatorname{id}_X) & /\!\!/ j_X^W \text{ is id in } \mathbb{C} \\ &= T^V (J^V (k \times \operatorname{id}_X) \circ^V j_X^V) & /\!\!/ (5) \\ &= T^V (J^V (k \times \operatorname{id}_X)) \circ^V T^V j_X^V & /\!\!/ \text{ functoriality} \\ &= J^V (T (k \times \operatorname{id}_X)) \circ^V T^V j_X^V & /\!\!/ \text{ extension} \\ &= T (k \times \operatorname{id}_X) (T^V j_X^V). & /\!\!/ \text{ definitions of } J^V, \circ^V \end{aligned}$$

Compatibility with the associator: Eliding the actual associator  $V \times (W \times X) \cong (V \times W) \times X$ , we have to show that the diagram

commutes. Since each  $\tau_{U,-}$  is a distributive law, it is compatible with the comultiplication of  $U \times (-)$ , which is  $\Delta_U \times \operatorname{id}_X : U \times X \to U \times U \times X$ ; explicitly,

all diagrams

$$U \times TX \xrightarrow{\Delta_U \times \mathrm{id}_{TX}} U \times U \times TX$$

$$\downarrow^{\mathrm{id}_U \times \tau_{U,X}} U \times T(U \times X)$$

$$\downarrow^{\tau_{U,U \times X}} T(U \times X) \xrightarrow{T(\Delta_U \times \mathrm{id}_X)} T(U \times U \times X)$$

$$(12)$$

commute. We apply this to  $U = V \times W$  in the following calculation proving commutation of (11), using moreover naturality of  $\tau$  in both variables:

$$T_{V \times W,X} = T(\mathsf{pr}_1 \times \mathsf{pr}_2 \times \mathsf{id}_X)T(\Delta_{V \times W} \times \mathsf{id}_X)\tau_{V \times W,X}$$

$$= T(\mathsf{pr}_1 \times \mathsf{pr}_2 \times \mathsf{id}_X)\tau_{U,U \times X}(\mathsf{id}_U \times \tau_{U,X})(\Delta_U \times \mathsf{id}_{TX}) \qquad // (12)$$

$$= \tau_{V,W \times X}(\mathsf{pr}_1 \times T(\mathsf{pr}_2 \times \mathsf{id}_X))(\mathsf{id}_U \times \tau_{U,X})(\Delta_U \times \mathsf{id}_{TX}) \qquad // \text{ naturality}$$

$$= \tau_{V,W \times X}(\mathsf{pr}_1 \times T(\mathsf{pr}_2 \times \mathsf{id}_X)\tau_{U,X})\langle\mathsf{pr}_1,\mathsf{id}_U \times \tau_X\rangle$$

$$= \tau_{V,W \times X}(\mathsf{pr}_1 \times \tau_{W,X}(\mathsf{pr}_2 \times \mathsf{id}_TX))\langle\mathsf{pr}_1,\mathsf{id}_U \times \tau_X\rangle \qquad // \text{ naturality}$$

$$= \tau_{V,W \times X}(\mathsf{id}_V \times \tau_{W,X}).$$

2. Let  $f: W \times X \to_{\sigma} TY$  in **C**. Then  $J^{W}f: W \times X \to T^{W}Y$  is  $\sigma$ -guarded in **C** // W since  $J^{W}$  preserves guardedness. By (5),  $f = (J^{W}f) \circ^{W} j_{X}: X \to T^{W}Y$  in **C** // W, so  $f: X \to_{\sigma} T^{W}Y$  in **C** // W by (cdm).

3. We have to show that the guardedness structure on **T** satisfies the axiom (str). So let  $\sigma: Y' \hookrightarrow Y$ , with complement  $\sigma': Y'' \hookrightarrow Y$ , and let  $f: X \to_{\sigma} TY$  in **C**. We have to show that  $\tau(\operatorname{id}_W \times f): W \times X \to_{\operatorname{id}_W \times \sigma} T(W \times X)$ . By (9) and the definition of composition in **C** // W,  $\tau(\operatorname{id}_W \times f)$  is the morphism

$$(T^W j) \circ^W (J^W f) \colon X \to T^W (W \times Y)$$

in  $\mathbb{C} /\!\!/ W$ . By Theorem 5.2,  $J^W$  preserves guardedness, so we have  $J^W f: X \to_{\sigma} T^W(W \times Y)$ . Since  $W \times Y$  is a coproduct of  $W \times Y'$  and  $W \times Y''$ , and j then has the form j' + j'' with  $j': Y' \to W \times Y'$ ,  $j'': Y'' \to W \times Y''$ , it follows by (cmp) that  $(T^W j) \circ^W (J^W f)$  is  $\mathrm{id}_W \times \sigma$ -guarded. By the definition of guardedness in  $\mathbb{C} /\!\!/ W$ , the required guardedness of  $\tau(\mathrm{id}_W \times f)$  follows.

4. We denote iteration in  $\mathbb{C} / W$  by  $(-)^{\ddagger}$ , and show that the equality (4) holds. Let  $f: W \times X \to T(Y + X)$  in  $\mathbb{C}$ , i.e.  $f: X \to T^W(Y + X)$  in  $\mathbb{C} / W$ . The square

$$\begin{array}{c|c} X & \xrightarrow{j_X} & W \times X \\ f \downarrow & & \downarrow^{J^W(T(\mathsf{pr}_2 + \mathsf{id})\delta\langle \mathsf{pr}_1, f \rangle)} \\ T^W(Y + X) & \xrightarrow{T^W(\mathsf{id} + j_X)} & T^W(Y + W \times X) \end{array}$$

commutes in  $\mathbb{C} /\!\!/ W$ : By (5), the upper right composite equals  $T(\mathrm{pr}_2 + \mathrm{id})\delta\langle \mathrm{pr}_1, f \rangle \colon W \times X \to T(Y + W \times X)$  in  $\mathbb{C}$ , which is precisely the term obtained by unfolding the definition of the structure of  $\mathbb{C} /\!\!/ W$  in terms of that of  $\mathbb{C}$  in the lower left composite, in particular using (10). By uniformity in  $\mathbb{C} /\!\!/ W$ , it follows that

$$\begin{split} f^{\ddagger} &= (J^{W}(T(\mathsf{pr}_{2} + \mathsf{id})\delta\langle\mathsf{pr}_{1}, f\rangle))^{\ddagger} \circ^{W} j \\ &= J^{W}((T(\mathsf{pr}_{2} + \mathsf{id})\delta\langle\mathsf{pr}_{1}, f\rangle)^{\dagger}) \circ^{W} j \qquad /\!\!/ J^{W} \text{ preserves iteration} \\ &= (T(\mathsf{pr}_{2} + \mathsf{id})\delta\langle\mathsf{pr}_{1}, f\rangle)^{\dagger}. \qquad /\!\!/ (5) \end{split}$$

5. Let  $f: X \to_2 T(Y + X)$ . Like in the proof of Claim 3, we have that the left-hand side of the *strength* law for f is written within  $\mathbb{C} / / W$  as  $(T^W j) \circ^W (J^W f^{\dagger}): X \to T^W (W \times Y)$  with j as above, which we rewrite using preservation of iteration by  $J^W$  and naturality in  $\mathbb{C} / / W$  as

$$\begin{split} (T^W j) \circ^W (J^W f^{\dagger}) &= (T^W j) \circ^W (J^W f)^{\ddagger} \\ &= (T^W (j + \operatorname{id}_X) \circ^W (J^W f))^{\ddagger} \end{split}$$

with  $(-)^{\ddagger}$  denoting iteration in  $\mathbb{C} / W$  and all further data, including + and identities, read in  $\mathbb{C} / W$  as well. Expanding definitions, we have

$$T^{W}(j + id_{X}) \circ^{W} (J^{W}f)$$
  
=  $T(id_{W \times Y} + pr_{2})\delta\langle pr_{1}, f pr_{2} \rangle$   
=  $T(id_{W \times Y} + pr_{2})\delta(id_{W} \times f)$ 

in **C**. Since  $(-)^{\ddagger}$  is assumed to be strong iteration, we further have

$$\begin{split} (T^W(j + \mathrm{id}_X) \circ^W (J^W f))^{\ddagger} \\ &= (T(\mathrm{id}_{W \times Y} + \mathrm{pr}_2)\delta(\mathrm{id}_W \times f))^{\ddagger} \\ &= (T(\mathrm{pr}_2 + \mathrm{id}_{W \times X})\delta\langle \mathrm{pr}_1, T(\mathrm{id}_{W \times Y} + \mathrm{pr}_2)\delta(\mathrm{id}_W \times f)\rangle)^{\dagger} \\ &= (\delta(\mathrm{id}_W \times f))^{\dagger}, \end{split}$$

which is the right-hand side of the *strength* law.

# 3. A Metalanguage for Guarded Iteration

We proceed to define a variant of fine-grain call-by-value [29] following the ideas from [14] on *labelled iteration*. For our purposes we extend the standard setup by allowing a custom signature of operations  $\Sigma$ , but restrict the expressiveness of the language being defined slightly, mainly by excluding function spaces for the moment. The latter require some additional treatment, and we return to this point in Section 5. We fix a supply Base of *base types* and define (composite) *types A*, *B* by the grammar

$$A, B, \dots := C \mid 0 \mid 1 \mid A + B \mid A \times B \qquad (C \in \mathsf{Base}) \tag{13}$$

The signature  $\Sigma$  consists of two disjoint parts: a value signature  $\Sigma_v$  containing signature symbols of the form  $f: A \to B$ , and an effect signature  $\Sigma_c$  containing signature symbols of the form  $f: A \to B[C]$ . While the former symbols represent pure functions, the latter capture morphisms of type  $A \to_2 T(B + C)$ ; in particular they carry side-effects from T. The term language over these data is given in Fig. 4. We use a syntax inspired by Haskell's do-notation [40]. The metalanguage features two kinds of judgements:

$$\Gamma \vdash_{\mathsf{v}} v : A$$
 and  $\Delta \mid \Gamma \vdash_{\mathsf{c}} p : A$  (14)

for values and computations, respectively. These involve two kinds of contexts:  $\Gamma$  denotes the usual context of typed variables x: A, and  $\Delta$  denotes the context of typed exceptions  $e: E^{\alpha}$  with E being a type from (13) and  $\alpha$  being a tag from the two-element set  $\{g, u\}$  to distinguish the exceptions raised in a guarded context (g) from those raised in an unguarded context (u) of the program code. Let us denote by  $|\Delta|$  the list of pairs e: E obtained from an exception context  $\Delta$  by removing the g and u tags. Variable and exception names are drawn from the same infinite stock of symbols; they are required to occur non-repetitively in  $\Gamma$  and in  $\Delta$  separately, but the same symbol may occur in  $\Gamma$  and in  $\Delta$  at the same time.

**Notation 7.** As usual, we use the dash (-) to denote a fresh variable in binding expressions, e.g. do  $- \leftarrow p$ ; q, and use the standard conventions of shortening do  $- \leftarrow p$ ; q to do p; q and do  $x \leftarrow p$ ; (do  $y \leftarrow q$ ; r) to do  $x \leftarrow p$ ;  $y \leftarrow q$ ; r. Moreover, we encode the if-then-else construct if b then p else q as case b of inl  $- \mapsto p$ ; inr  $- \mapsto q$ , and also use the notation

f(v) & p for gcase f(v) of inl  $x \mapsto \text{init } x$ ; inr –  $\mapsto p$ 

whenever  $f: X \to 0[1] \in \Sigma_c$ .

The language constructs relating to products, coproducts, and the monad structure are standard (except maybe init, which forms unique morphisms from the null type 0 into any type A) and should be largely self-explanatory. The key features of our metalanguage, discussed next, concern algebraic operations on the one hand, and exception-based iteration on the other hand.

Algebraic operations via Generic effects. The signature symbols  $f: A \to B[0]$ from  $\Sigma_c$  have Kleisli morphisms  $A \to TB$  as their intended semantics, specifically, if A = n and B = m, with n and m being identified with the corresponding n-fold and m-fold coproducts of 1, the respective morphisms  $n \to Tm$ dually correspond to algebraic operations, i.e. certain natural transformations  $T^m \to T^n$ , as elaborated by Plotkin and Power [41]. In context of this duality the Kleisli morphisms of type  $n \to Tm$  are also called generic effects. Hence we regard  $\Sigma_c$  as a stock of generic effects declared to be available to the language. The respective algebraic operations thus become automatically available – for a brief example consider the binary algebraic operation of nondeterministic choice  $\oplus: T^2 \to T^1$ , which is modelled by a generic effect toss:  $1 \to T2$  as follows:

$$p \oplus q = \operatorname{do} c \leftarrow \operatorname{toss}$$
; case c of inl –  $\mapsto p$ ; inr –  $\mapsto q$ .



Figure 4: Term formation rules for values (top) and computations (bottom).

Exception raising. Following [14], we involve an exception raising/handling mechanism for organizing loops (we make the connection to exceptions more explicit, in particular, we use the term 'exceptions' and not 'labels', as the underlying semantics does indeed accurately match the standard exception semantics). Note that the design of the syntax presented here deviates slightly from the conference version [19]. We allow raising of a standard unguarded exception  $e: E^{u}$  with raise<sub>e</sub> q. More importantly, guarded exceptions  $e: E^{g}$  can be arbitrarily introduced into the context by the typing rule for ret, in which  $\Delta$  is completely unspecified. The guarded case command

gcase 
$$f(v)$$
 of inl  $x \mapsto p$ ; inr  $y \mapsto q$ .

then works as follows: The f(v) part acts as a guard partitioning the control flow into the left (unguarded) part in which a computation p is executed, and the right (guarded) part, in which execution continues with q. Since  $\Delta'$  need not agree with  $\Delta$  on guardedness tags, the exceptions occurring in q and therefore recorded in  $\Delta'$  may be promoted from unguarded to guarded.

The guarded case operator gcase also allows us to expose the guarded part of an operation to the result; i.e. for  $f: A \to B[C] \in \Sigma_c$  we can take f(v) to be an abbreviation for

gcase 
$$f(v)$$
 of inl  $x \mapsto$  ret inl  $x$ ; inr  $y \mapsto$  ret inr  $y$ 

and then derive a typing rule

$$\frac{f \colon A \to B[C] \in \Sigma_c \qquad \Gamma \vdash_{\mathbf{v}} v \colon A}{\Delta \mid \Gamma \vdash_{\mathbf{c}} f(v) \colon B + C}$$

This is particularly useful for performing operations without considering their guardedness properties, e.g. the final *print* in Fig. 1.

(Iterated) exception handling. The syntax for exception handling via handle e in p with q is meant to be understood as follows: p is a program possibly raising the exception e and q is a handling term for it. This can be compared to the richer exception handling syntax of Benton and Kennedy [3] whose construct try  $x \leftarrow p$  in q unless  $\{e \mapsto r\}_{e \in E}$  we can encode as:

do 
$$z \leftarrow$$
 handle  $e$  in (do  $x \leftarrow p$ ; ret inl  $x$ ) with (do  $y \leftarrow r$ ; ret inr  $y$ );  
case  $z$  of inl  $x \mapsto q$ ; inr  $y \mapsto$  ret  $y$ 

where p, q and r come from the judgements

 $\Delta, e: E^{\mathsf{u}} \mid \Gamma \vdash_{\mathsf{c}} p: A, \qquad \Delta \mid \Gamma, x: A \vdash_{\mathsf{c}} q: B, \qquad \Delta \mid \Gamma, e: E \vdash_{\mathsf{c}} r: B,$ 

and the idea is to capture the following behaviour: unless p raises exception  $e: E^{u}$ , the result is bound to x and passed to q (which may itself raise e), and otherwise the exception is handled by r. An analogous encoding is already discussed in [3] where the richer syntax is advocated and motivated by tasks in compiler optimization, but since these considerations are not relevant for our present developments, we stick to the minimalist syntax as above.

Note that we restrict to handling unguarded exceptions only; since all exceptions are introduced as unguarded ones, and promoted to guarded exceptions only for the purpose of iteration, this clearly suffices.

The idea of the new construct handleit e = p in q is to handle the exception in q recursively using q itself as the handling term, so that if q raises e, handling continues repetitively. The value p is substituted into q to initialize the iteration. For handleit, it is crucial that the exception comes from a guarded context, as required by the relevant typing rule.

**Example 8.** We illustrate a type derivation process in Fig. 5, using the example in Fig. 1 from the introduction. Due to the page width limitations the complete derivation tree is cut into five pieces with the curved arrows indicating how conclusions are further used as premises of subsequent derivations; additionally, we indicate by dots '...' the repeated program fragments taken from the premises.



Figure 5: Typing derivation for the example in Fig. 1.

#### 4. Generic Denotational Semantics

We proceed to give a denotational semantics of the guarded metalanguage assuming the following:

- a distributive category **C** (with initial objects);
- a strong guarded pre-iterative monad **T** on **C**.

Supposing that every base type  $A \in \mathsf{Base}$  is interpreted as an object  $\underline{A}$  in  $|\mathbf{C}|$ , we define  $\underline{A}$  for types A (see (13)) inductively by

$$\underline{0} = \emptyset, \qquad \underline{1} = 1, \qquad A + B = \underline{A} + \underline{B}, \qquad \underline{A \times B} = \underline{A} \times \underline{B}.$$

To every  $f: A \to B \in \Sigma_v$  we associate an interpretation  $\llbracket f \rrbracket \in \operatorname{Hom}(\underline{A}, \underline{B})$  in **C** and to every  $f: A \to B[C] \in \Sigma_c$  an interpretation  $\llbracket f \rrbracket \in \operatorname{Hom}_{\operatorname{in}_2}(\underline{A}, T(\underline{B} + \underline{C}))$ . Based on these we define the semantics of the term language from Fig. 4. The semantics of a value judgment  $\Gamma \vdash_{\mathsf{v}} p: A$  is a morphism  $\llbracket \Gamma \vdash_{\mathsf{v}} p: A \rrbracket \in \operatorname{Hom}(\underline{\Gamma}, \underline{A})$ , and the semantics of a computation judgment  $\Delta \mid \Gamma \vdash_{\mathsf{c}} p: A$  is a morphism  $\llbracket \Delta \mid \Gamma \vdash_{\mathsf{c}} p: A \rrbracket \in \operatorname{Hom}_{! + \sigma_\Delta}(\underline{\Gamma}, T(\underline{A} + \underline{\Delta}))$  where

$$\underline{\Gamma} = \underline{A_1} \times \ldots \times \underline{A_n} \qquad \text{for } \Gamma = (x_1 \colon A_1, \ldots, x_n \colon A_n)$$
$$\underline{\Delta} = E_1 + \ldots + E_m \qquad \text{for } \Delta = (e_1 \colon E_1^{\alpha_1}, \ldots, e_m \colon E_m^{\alpha_m})$$

and  $\sigma_{\Delta} : \underline{\Delta}' \hookrightarrow \underline{\Delta}$  is the summand induced by removal of unguarded exceptions  $e : E^{\mathsf{u}}$  from  $\Delta$  with  $\Delta'$  denoting the result.

$$\begin{split} \llbracket \Gamma \vdash_{\mathsf{v}} v : A \rrbracket &= h : \underline{\Gamma} \to \underline{A} \\ \llbracket f : A \to B[C] \rrbracket = g : \underline{A} \to_2 T(\underline{B} + \underline{C}) \\ \llbracket \Delta \mid \Gamma, x : B \vdash_{\mathsf{c}} p : D \rrbracket = u : \underline{\Gamma} \times \underline{B} \to_{!+\sigma_{\Delta}} T(\underline{D} + \underline{\Delta}) \\ \hline \llbracket \Delta' \mid \Gamma, y : C \vdash_{\mathsf{c}} q : D \rrbracket = w : \underline{\Gamma} \times \underline{C} \to T(\underline{D} + \underline{\Delta}) \\ \hline \llbracket \Delta \mid \Gamma \vdash_{\mathsf{c}} \mathsf{gcase} f(v) \mathsf{ of inl } x \mapsto p; \mathsf{ inr } y \mapsto q : D \rrbracket = \\ \underline{\Gamma} \xrightarrow{\delta(\mathsf{id}, gh)} T(\underline{\Gamma} \times \underline{B} + \underline{\Gamma} \times \underline{C}) \xrightarrow{[u,w]^*} T(\underline{D} + \underline{\Delta}) \end{split}$$

$$\begin{split} & [\![\Gamma \vdash_{\vee} p \colon A \times B]\!] = g \colon \underline{\Gamma} \to \underline{A} \times \underline{B} \\ & (\mathbf{prod}) \quad \frac{[\![\Delta \mid \Gamma, x \colon A, y \colon B \vdash_{\mathsf{c}} q \colon C]\!] = h \colon \underline{\Gamma} \times \underline{A} \times \underline{B} \to_{!+\sigma_{\Delta}} T(\underline{C} + \underline{\Delta})}{[\![\Delta \mid \Gamma \vdash_{\mathsf{c}} \mathsf{case} \ p \ \mathsf{of} \ \langle x, y \rangle \mapsto q \colon C]\!] = h \langle \mathsf{id}_{\underline{\Gamma}}, g \rangle \colon \underline{\Gamma} \to T(\underline{C} + \underline{\Delta})} \end{split}$$

(ret) 
$$\frac{\llbracket \Gamma \vdash_{\mathsf{v}} t \colon A \rrbracket = g \colon \underline{\Gamma} \to \underline{A}}{\llbracket \Delta \mid \Gamma \vdash_{\mathsf{c}} \mathsf{ret} t \colon A \rrbracket = \eta \mathsf{in}_{1} g \colon \underline{\Gamma} \to T(\underline{A} + \underline{\Delta})$$

$$\begin{split} & \begin{bmatrix} \Delta \mid \Gamma \vdash_{\mathsf{c}} p \colon A \end{bmatrix} = g \colon \underline{\Gamma} \to_{!+\sigma_{\Delta}} T(\underline{A} + \underline{\Delta}) \\ & \\ (\mathbf{do}) \quad \underbrace{\llbracket \Delta \mid \Gamma, x \colon A \vdash_{\mathsf{c}} q \colon B \rrbracket = h \colon \underline{\Gamma} \times \underline{A} \to_{!+\sigma_{\Delta}} T(\underline{B} + \underline{\Delta})}_{\llbracket \Delta \mid \Gamma \vdash_{\mathsf{c}} \mathsf{do} x \leftarrow p; q \rrbracket = [h, \eta \mathsf{in}_2 \mathsf{pr}_2]^* \delta\langle \mathsf{id}_{\underline{\Gamma}}, g \rangle \colon \underline{\Gamma} \to T(\underline{B} + \underline{\Delta}) \end{split}$$

(init) 
$$\frac{\llbracket \Gamma \vdash_{\mathsf{v}} t : 0 \rrbracket = g : \underline{\Gamma} \to \emptyset}{\llbracket \Delta \mid \Gamma \vdash_{\mathsf{c}} \mathsf{init} t : A \rrbracket = ! g : \underline{\Gamma} \to T(\underline{A} + \underline{\Delta})}$$

$$\begin{split} & \left[\!\left[\Gamma \vdash_{\mathbf{v}} p \colon A + B\right]\!\right] = g \colon \underline{\Gamma} \to \underline{A} + \underline{B} \\ & \left[\!\left[\Delta \mid \Gamma, x \colon A \vdash_{\mathbf{c}} q \colon C\right]\!\right] = h \colon \underline{\Gamma} \times \underline{A} \to_{! + \sigma_{\Delta}} T(\underline{C} + \underline{\Delta}) \\ & \underbrace{\left[\!\left[\Delta \mid \Gamma, y \colon B \vdash_{\mathbf{c}} r \colon C\right]\!\right] = u \colon \underline{\Gamma} \times \underline{B} \to_{! + \sigma_{\Delta}} T(\underline{C} + \underline{\Delta})}_{\left[\!\left[\Delta \mid \Gamma \vdash_{\mathbf{c}} \mathsf{case} \ p \ \mathsf{of} \ \mathsf{inl} \ x \mapsto q; \ \mathsf{inr} \ y \mapsto r \colon C\right]\!\right] = \\ & \left[\!\left[h, u\right] \mathsf{dist} \langle \mathsf{id}_{\underline{\Gamma}}, g \rangle \colon \underline{\Gamma} \to T(\underline{C} + \underline{\Delta}) \end{split}$$

(raise)  $\frac{\llbracket \Gamma \vdash_{\mathsf{v}} q \colon E \rrbracket = g \colon \underline{\Gamma} \to \underline{E}}{\llbracket \Delta \mid \Gamma \vdash_{\mathsf{c}} \mathsf{raise}_{e} q \colon D \rrbracket = \eta \mathsf{in}_{2} \mathsf{in}_{e} g \colon \underline{\Gamma} \to T(\underline{D} + \underline{\Delta})}$ 

(handle)  

$$\begin{bmatrix} \Delta, e: E^{\mathsf{u}} \mid \Gamma \vdash_{\mathsf{c}} p: A \end{bmatrix} = g: \underline{\Gamma} \rightarrow_{!+(\sigma_{\Delta}+!)} T(\underline{A} + (\underline{\Delta} + \underline{E})) \\
\begin{bmatrix} \Delta \mid \Gamma, e: E \vdash_{\mathsf{c}} q: A \end{bmatrix} = h: \underline{\Gamma} \times \underline{E} \rightarrow_{!+\sigma_{\Delta}} T(\underline{A} + \underline{\Delta}) \\
\begin{bmatrix} \Delta \mid \Gamma \vdash_{\mathsf{c}} \text{ handle } e \text{ in } p \text{ with } q: A \end{bmatrix} = \\
\begin{bmatrix} \underline{\Gamma} \xrightarrow{T(\mathsf{id}_{\underline{\Gamma} \times \underline{A}} + \mathsf{dist})\delta(\mathsf{id}_{\underline{\Gamma}}, g)} \\
\underline{\Gamma} \xrightarrow{T(\underline{\Gamma} \times \underline{A} + \mathsf{dist})\delta(\mathsf{id}_{\underline{\Gamma}}, g)} \\
\end{bmatrix} T(\underline{\Gamma} \times \underline{A} + (\underline{\Gamma} \times \underline{\Delta} + \underline{\Gamma} \times \underline{E})) \\
\underline{\Gamma} \xrightarrow{[\eta \text{ in}_{1} \text{ pr}_{2}, [\eta \text{ in}_{2} \text{ pr}_{2}, h]]^{\star}} T(\underline{A} + \underline{\Delta})$$

$$\begin{split} & [\![\Gamma \vdash_{\vee} v : E]\!] = g \colon \underline{\Gamma} \to \underline{E} \\ \text{(iter)} \quad & [\![\Delta, e \colon E^{\mathsf{g}} \mid \Gamma, e \colon E \vdash_{\mathsf{c}} q \colon A]\!] = h \colon \underline{\Gamma} \times \underline{E} \to_{! + (\sigma_{\Delta} + \mathsf{id})} T(\underline{A} + (\underline{\Delta} + \underline{E})) \\ & [\![\Delta \mid \Gamma \vdash_{\mathsf{c}} \mathsf{handleit} \ e = v \text{ in } q \colon A]\!] = ((T \operatorname{assoc}) h)^{\ddagger} \langle \mathsf{id}_{\underline{\Gamma}}, g \rangle \colon \underline{\Gamma} \to T(\underline{A} + \underline{\Delta}) \end{split}$$

Figure 6: Denotational semantics.

The semantic assignments for computation judgments are given in Fig. 6 (we skip the obvious standard rules for values) where  $in_e: \underline{E} \to \underline{\Delta}$  is the obvious coproduct injection of  $\underline{E}$  to  $\underline{\Delta}$  identified by e, assoc is the associativity isomorphism  $X + (Y + Z) \cong (X + Y) + Z$ , and  $(-)^{\ddagger}$  is the strong iteration operator from (4). The correctness of our semantic assignments is established by the following claim:

**Proposition 9.** For every rule in Fig. 4, assuming the premises, the morphism in the conclusion is  $(!+\sigma_{\Delta})$ -guarded.

Proof. First note that each  $f: X \to_{!+\sigma_{\Delta}} T(\underline{A} + \underline{\Delta})$  is isomorphic to some  $\hat{f}: X \to_{!+id} T((\underline{A} + \underline{\Delta}_{u}) + \underline{\Delta}_{g})$ , where  $\Delta \cong \Delta_{u} + \Delta_{g}$  separates unguarded from guarded exceptions. Consider the rule (handle) in detail. By regarding g and h as morphisms in  $\mathbb{C} /\!\!/ \underline{\Gamma}$ , we reformulate the goal as follows: assuming  $g: 1 \to_{!+(\sigma_{\Delta}+!)} T(\underline{A} + ((\underline{\Delta}_{u} + \underline{\Delta}_{g}) + \underline{E}))$  and  $h: \underline{E} \to_{!+\sigma_{\Delta}} T(\underline{A} + (\underline{\Delta}_{u} + \underline{\Delta}_{g}))$ , show that  $[\eta \operatorname{in}_{1}, [\eta \operatorname{in}_{2}, h]]^{*}g: 1 \to_{!+\sigma_{\Delta}} T(\underline{A} + (\underline{\Delta}_{u} + \underline{\Delta}_{g}))$ . Let w.l.o.g.  $\sigma_{\Delta} = \operatorname{in}_{2}: \underline{\Delta}_{g} \to \underline{\Delta}_{u} + \underline{\Delta}_{g}$ . We obtain

 $T\left[\operatorname{in}_{1}\operatorname{in}_{1}\operatorname{in}_{1},\left[\operatorname{in}_{1}\operatorname{in}_{2}+\operatorname{id},\operatorname{in}_{1}\operatorname{in}_{2}\right]\right]g\colon 1 \to_{\operatorname{in}_{2}} T\left(\left((\underline{A}+\underline{\Delta}_{u})+E\right)+\underline{\Delta}_{g}\right)$ 

by (iso), and

$$[\eta(\mathsf{id} + \mathsf{in}_1), h]: (\underline{A} + \underline{\Delta}_{\mathsf{u}}) + E \to_{!+\sigma_{\Delta}} T(\underline{A} + (\underline{\Delta}_{\mathsf{u}} + \underline{\Delta}_{\mathsf{g}}))$$

by (trv) and (sum). Then by (cmp),

$$\begin{split} & \left[ \left[ \eta(\mathsf{id} + \mathsf{in}_1), h \right], \eta \; \mathsf{in}_2 \; \mathsf{in}_2 \right]^{\star} \\ & T \left[ \mathsf{in}_1 \; \mathsf{in}_1 \; \mathsf{in}_1, \left[ \mathsf{in}_1 \; \mathsf{in}_2 + \mathsf{id}, \mathsf{in}_1 \; \mathsf{in}_2 \right] \right] g : 1 \to_{!+\sigma_{\Lambda}} T(\underline{A} + (\underline{\Delta}_{\mathsf{u}} + \underline{\Delta}_{\mathsf{g}})), \end{split}$$

which further reduces down to the goal.

For (prod), (ret), (case) and (init), the verification is straightforward by the axioms of guardedness in **C**. For (gcase) and (do), we proceed analogously to (handle) using the axioms of guardedness in  $\mathbb{C} /\!\!/ \Gamma$  and Theorem 5. Strong iteration as figuring in (iter) satisfies the fixpoint law by Theorem 5, and the problem in question amounts to verifying that  $f^{\dagger}: X \rightarrow_{\sigma} TY$  whenever  $f: X \rightarrow_{\sigma+id} T(Y + X)$ . This is already shown in [22], using only the fixpoint law.

### 5. Functional Types

In order to interpret functional types in fine-grain call-by-value, it normally suffices to assume existence of *Kleisli exponentials*, i.e. objects  $TB^A$  such that  $\operatorname{Hom}(C, TB^A)$  and  $\operatorname{Hom}(C \times A, TB)$  are naturally isomorphic, or equivalently that all presheaves  $\operatorname{Hom}(-\times A, TB): \mathbb{C}^{\operatorname{op}} \to \operatorname{Set}$  are representable. In order to add functional types to our metalanguage we additionally need to assume that all presheaves  $\operatorname{Hom}_{\sigma}(-, TA): \mathbb{C}^{\operatorname{op}} \to \operatorname{Set}$  are representable, i.e. for every A and  $\sigma: A' \hookrightarrow A$  there is  $A_{\sigma} \in |\mathbb{C}|$  such that

$$\xi : \operatorname{Hom}(X, A_{\sigma}) \cong \operatorname{Hom}_{\sigma}(X, TA)$$
(15)

naturally in X. By the Yoneda lemma, this requirement is equivalent to the following.

**Definition 10** (Greatest  $\sigma$ -algebra). Given  $\sigma: A' \hookrightarrow A$ , a pair  $(A_{\sigma}, \iota_{\sigma})$  consisting of an object  $A_{\sigma} \in |\mathbf{C}|$  and a morphism  $\iota_{\sigma}: A_{\sigma} \to_{\sigma} TA$  is called a *greatest*  $\sigma$ -algebra if for every  $f: X \to_{\sigma} TA$  there is a unique  $\hat{f}: X \to A_{\sigma}$  with the property that  $f = \iota_{\sigma} \hat{f}$ .

By the usual arguments,  $(A_{\sigma}, \iota_{\sigma})$  is defined uniquely up to isomorphism. The connection between  $\iota_{\sigma}$  and  $\xi$  in (15) is as follows:  $\iota_{\sigma} = \xi(\operatorname{id}: A_{\sigma} \to A_{\sigma})$  and  $\xi(f: X \to A_{\sigma}) = A_{\sigma}$ 

It immediately follows by definition that  $\iota_{\sigma}$  is a monomorphism. The name ' $\sigma$ -algebra' for  $(A, \iota_{\sigma})$  is justified as follows.

**Proposition 11.** Suppose that  $(A, \iota_{\sigma})$  is a greatest  $\sigma$ -algebra. Then there is a unique  $\alpha_{\sigma} \colon TA_{\sigma} \to A_{\sigma}$  such that  $\iota_{\sigma} \alpha_{\sigma} = \iota_{\sigma}^{\star}$ . The pair  $(A_{\sigma}, \alpha_{\sigma})$  is a **T**subalgebra of  $(TA, \mu)$ .

*Proof.* Since  $\iota_{\sigma}^{\star}: TA_{\sigma} \to TA$  is the Kleisli composite of  $\iota_{\sigma}: A_{\sigma} \to_{\sigma} TA$  and id:  $TA_{\sigma} \to TA_{\sigma}, \iota_{\sigma}^{\star}$  is  $\sigma$ -guarded by (**cmp**), so we obtain  $\alpha_{\sigma}$  such that  $\iota_{\sigma}\alpha_{\sigma} = \iota_{\sigma}^{\star}$ by the universal property of  $(A_{\sigma}, \iota_{\sigma})$ . Since  $\iota_{\sigma}^{\star} = \mu_A(T\iota_{\sigma})$ , it follows that  $\iota_{\sigma}: (A_{\sigma}, \alpha_{\sigma}) \to (A, \mu_A)$  is a morphism of functor algebras. Since monad algebras are closed under taking functor subalgebras and  $\iota_{\sigma}$  is monic as observed above, it follows that  $(A_{\sigma}, \alpha_{\sigma})$  is a **T**-subalgebra of  $(A, \mu_A)$ .

**Proposition 12.** 1. Suppose that a greatest  $\sigma$ -algebra  $(A_{\sigma}, \iota_{\sigma})$  exists. Then

- (a)  $\iota_{\sigma}$  is the greatest element in the class of all  $\sigma$ -guarded subobjects of TA;
- (b) for every regular epic  $e: X \to Y$  and every morphism  $f: Y \to TA$ ,  $f e: X \to_{\sigma} TA$  implies that  $f: Y \to_{\sigma} TA$ .
- 2. Assuming that every morphism in **C** admits a factorization into a regular epic and a monic, the converse of (1) is true: If (a) and (b) hold for  $(A_{\sigma}, \iota_{\sigma})$ , then  $(A_{\sigma}, \iota_{\sigma})$  is a greatest  $\sigma$ -algebra.

*Proof.* 1.: Part 1a is immediate; we show 1b. Given a regular epic  $e: X \to Y$  and a morphism  $f: Y \to TA$  such that  $f e: X \to_{\sigma} TA$ , consider the diagram



where e is the coequalizer of h and g, and w exists uniquely by the universal property of  $\iota_{\sigma}$ . Since  $\iota_{\sigma} w h = f e h = f e g = \iota_{\sigma} w g$  and  $\iota_{\sigma}$  is monic, w h = w g. Hence, there is  $u: Y \to A_{\sigma}$  such that w = u e. Therefore we have  $\iota_{\sigma} u e =$ 



Figure 7: Syntax (top) and semantics (bottom) of functional types.

 $\iota_{\sigma} w = f e$ . Since e is epi, this implies  $f = \iota_{\sigma} u$ . Since  $\iota_{\sigma}$  is  $\sigma$ -guarded, so is f by (cmp).

2.: Let  $f: X \to_{\sigma} TA$ , with factorization f = me into a mono m and a regular epi e. By 1b, m is  $\sigma$ -guarded; by 1a, it follows that m, and hence f, factor through  $\iota_{\sigma}$ , necessarily uniquely since  $\iota_{\sigma}$  is monic.

**Example 13.** Let **T** be a strong monad on a distributive category **C** and let  $\Sigma: \mathbf{C} \to \mathbf{C}$  be an endofunctor such that all the fixpoints  $T_{\Sigma}X = \nu\gamma$ .  $T(X + \Sigma\gamma)$  exist. These extend to a strong monad  $\mathbf{T}_{\Sigma}$ , called the *generalized coalgebraic* resumption monad transform of **T** [22]. Moreover,  $\mathbf{T}_{\Sigma}$  is guarded iterative with  $f: X \to_{\sigma} T_{\Sigma}A$  iff out  $f: X \to T(A + \Sigma T_{\Sigma}A)$  factors as  $T(\bar{\sigma} + \mathrm{id})g$  for some  $g: X \to T(A' + \Sigma T_{\Sigma}A)$ . Suppose that coproduct injections in **C** are monic and T preserves monics. Then for every  $A \in |\mathbf{C}|$  and  $\sigma$  there is at most one g such that out  $f = T(\bar{\sigma} + \mathrm{id})g$ . This entails an isomorphism

$$\operatorname{Hom}(X, T(A' + \Sigma T_{\Sigma}A)) \cong \operatorname{Hom}_{\sigma}(X, T_{\Sigma}A)$$

obviously natural in X, from which we obtain by comparison with (15) that  $A_{\sigma} = T(A' + \Sigma T_{\Sigma} A).$ 

**Example 14.** Let  $\sigma: A'' \hookrightarrow A$ , whose complement is  $\overline{\sigma}: A' \hookrightarrow A$  and let us revisit Example 3.

- 1.  $T = \nu \gamma . \mathcal{P}_{\omega}(-+\operatorname{Act} \times \gamma)$  is an instance of Example 13, and thus  $A_{\sigma} = \mathcal{P}_{\omega}(A' + \operatorname{Act} \times TA)$ .
- 2. For  $T = \nu \gamma$ .  $\mathcal{P}_{\omega_1}(-+\operatorname{Act} \times \gamma)$  under total guardedness,  $A_{\sigma} = TA$  independently of  $\sigma$ . For the other notion of guardedness on  $\mathbf{T}$ ,  $A_{\sigma}$  is constructed in analogy to Clause 1.
- 3. For  $T = \mathcal{P}$  being totally guarded, again  $A_{\sigma} = \mathcal{P}A$ .
- 4. For  $T = \mathcal{P}(\mathsf{Act}^* \times -)$ , it follows that  $A_{\sigma} = \mathcal{P}(\mathsf{Act}^* \times A' + \mathsf{Act}^+ \times A'')$ .
- 5. Finally, for  $T = \mathcal{P}^+$ , it follows by definition that  $A_{\sigma} = \mathcal{P}A' \times \mathcal{P}^+A''$ .

Assuming that greatest  $\sigma$ -algebras exist, we complement our metalanguage with functional types  $A \to_{\Delta} B$  where the index  $\Delta$  serves to store information about (guarded) exceptions of the curried function. Formally, these types are interpreted as  $\underline{A} \to_{\Delta} \underline{B} = \underline{A} \to (\underline{B} + \underline{\Delta})_{!+\sigma_{\Delta}}$ . In the term language, this is reflected by the introduction of  $\lambda$ -abstraction and application, with syntax and semantics as shown in Fig. 7, where  $\xi$  is the isomorphism from (15).

### 6. Operational Semantics and Adequacy

We proceed to complement our denotational semantics from Sections 4 and 5 with a big-step operational semantics. Following Geron and Levy [14], we choose the simplest concrete monad  $\mathbf{T}$  sensibly illustrating all the main features and model it operationally. In [14] this is the maybe monad TX = X + 1 on Set. which suffices to give a sensible account of total iteration. The +1 part is necessary for modelling divergence. Since total iteration is subsumed by guarded iteration, we could formulate an adequate operational semantics over this monad too. To that end we would need to assume that the only operation  $f: A \to B[C]$ in  $\Sigma_c$  with  $C \neq 0$  is a distinguished element  $tick: 1 \rightarrow 0[1]$  whose denotation is the unit of the monad (regarded as totally guarded). However, total iteration is only a degenerate instance of guarded iteration, and here we therefore replace X + 1 with the guarded iterative monad freely generated by an operation put:  $\mathbb{N} \to 0[1]$  of outputting a natural number (say, to console), explicitly (on Set):  $TX = (X \times \mathbb{N}^*) \cup \mathbb{N}^{\omega}$ . More abstractly, TX is the final  $(X + \mathbb{N} \times -)$ coalgebra. The denotations in TX are of two types: pairs  $(x, \tau) \in X \times \mathbb{N}^*$  of a value x and a finite trace  $\tau$  of outputs (for terminating iteration) and infinite traces  $\pi \in \mathbb{N}^{\omega}$  of outputs (for non-terminating iteration).

We fix  $TX = (X \times \mathbb{N}^*) \cup \mathbb{N}^{\omega}$  for the rest of the section. Let us spell out the details of the structure of **T**, which is in fact an instance of Example 13 under  $T = \text{Id}, \Sigma = \mathbb{N} \times (-)$ . The unit of **T** sends x to  $(x, \langle \rangle)$ . Given  $f: X \to TY$ , we have

$$f^{\star}(x,\tau) = \begin{cases} (y,\tau + \tau') & \text{if } f(x) = (y,\tau'), \\ \tau + \pi & \text{if } f(x) = \pi, \end{cases} \qquad f^{\star}(\pi) = \pi.$$

for  $x \in X$ ,  $\tau \in \mathbb{N}^{\star}$ ,  $\pi \in \mathbb{N}^{\omega}$  with # denoting concatenation of a finite trace with a possibly infinite one. Guardedness for **T** is defined as follows:  $f: X \to_2$  $(Y + Z) \times \mathbb{N}^{\star} \cup \mathbb{N}^{\omega}$  if for every  $x \in X$ , either  $f(x) \in \mathbb{N}^{\omega}$  or  $f(x) = (\operatorname{in}_1 y, \tau)$  for some  $y \in Y$ ,  $\tau \in \mathbb{N}^{\star}$  or  $f(x) = (\operatorname{in}_2 z, \tau)$  for some  $z \in Z$ ,  $\tau \in \mathbb{N}^+$ . Finally, given  $f: X \to_2 T(Y + X)$ ,

$$f^{\dagger}(x) = \begin{cases} (y, \tau_1 + \dots + \tau_n) & \text{if } f(x) = (\text{in}_2 x_1, \tau_1), \dots, f(x_n) = (\text{in}_1 y, \tau_n), \\ \tau_1 + \dots + \tau_{n-1} + \pi & \text{if } f(x) = (\text{in}_2 x_1, \tau_1), \dots, f(x_n) = \pi, \\ \tau_1 + \dots & \text{if } f(x) = (\text{in}_2 x_1, \tau_1), \dots \end{cases}$$

where the first clause addresses the situation when iteration finishes after finitely many steps, the second one addresses the situation when we hit divergence Values, Computations, Terminals:

$$\begin{array}{l} v, w \coloneqq x \mid \star \mid zero \mid succ v \mid \mathsf{inl} v \mid \mathsf{inr} v \mid \langle v, w \rangle \mid \lambda x. p \\ p, q \coloneqq \mathsf{ret} v \mid pred(v) \mid put(v) \mid \mathsf{raise}_x v \mid \mathsf{gcase} \ put(v) \ \mathsf{of} \ \mathsf{inl} - \mapsto p; \ \mathsf{inr} \ x \mapsto q \\ \mid \mathsf{case} \ v \ \mathsf{of} \ \langle x, y \rangle \mapsto p \mid \mathsf{init} \ v \mid \mathsf{case} \ v \ \mathsf{of} \ \mathsf{inl} \ x \mapsto p; \ \mathsf{inr} \ y \mapsto q \\ \mid vw \mid \mathsf{do} \ x \leftarrow p; \ q \mid \mathsf{handle} \ x \ \mathsf{in} \ p \ \mathsf{with} \ q \mid \mathsf{handleit} \ y = v \ \mathsf{in} \ p \\ t \coloneqq \mathsf{ret} v, \tau \mid \mathsf{raise}_x v, \tau \mid \pi \qquad (\tau \in \mathbb{N}^*, \pi \in \mathbb{N}^{\omega}) \end{array}$$

Rules:

$q[\star/x]\Downarrow u, au$	
gcase $put(v)$ of inl – $\mapsto p$ ; inr $x \mapsto q \Downarrow u, \langle v \rangle$	$v\rangle + \tau$ raise <sub>x</sub> $v \Downarrow$ raise <sub>x</sub> $v, \langle \rangle$
$q[\star/x]\Downarrow\pi$	$p[v/x] \Downarrow t$
$gcaseput(v)ofinl{\operatorname{-}}\mapsto p;inrx\mapsto q\Downarrow$	$\overline{\langle v \rangle + \pi} \qquad \overline{(\lambda x. p) v \Downarrow t}$
$\overline{pred(zero)\Downarrow retinl\star,\langle\rangle}$	$\overline{pred(succ(v))\Downarrow retin_2v,\langle\rangle}$
$\frac{p[v/x] \Downarrow t}{case inl  v \text{ of inl}  x \mapsto p;  inr  y \mapsto q \Downarrow t} \qquad c$	$\frac{q[w/y] \Downarrow t}{\text{case inr } w \text{ of inl } x \mapsto p; \text{ inr } y \mapsto q \Downarrow t}$
$\frac{p \Downarrow ret v, \tau \qquad q[v/x] \Downarrow u, \tau'}{\operatorname{do} x \leftarrow p; \ q \Downarrow u, \tau + \tau'}$	$\frac{p \Downarrow \operatorname{ret} v, \tau \qquad q[v/x] \Downarrow \pi}{\operatorname{do} x \leftarrow p; \ q \Downarrow \tau + \pi}$
$\frac{p \Downarrow raise_x v, \tau}{do  x \leftarrow p;  q \Downarrow raise_x v, \tau} \qquad \frac{p \Downarrow \pi}{do  x \leftarrow p;  q \Downarrow \tau}$	$\frac{p \Downarrow \operatorname{ret} v, \tau}{\operatorname{handle} x \text{ in } p \text{ with } q \Downarrow \operatorname{ret} v, \tau}$
$\frac{p \Downarrow raise_y v, \tau}{handle \ x \ in \ p \ with \ q \Downarrow raise_y v, \tau}  (x \neq y)$	$\frac{p \Downarrow raise_x v, \tau}{handle \ x \text{ in } p \text{ with } q \Downarrow u, \tau + \tau'}$
$\frac{p\Downarrow \pi}{\text{handle }x \text{ in }p \text{ with }q\Downarrow \pi}$	$\frac{p \Downarrow raise_x  v, \tau  q[v/x] \Downarrow \pi}{handle \; x \; in \; p \; with \; q \Downarrow \tau + \pi}$
$\frac{1}{\operatorname{ret} v \Downarrow \operatorname{ret} v, \langle \rangle} \qquad \frac{v_0 = v  q[v_0/x] \Downarrow \operatorname{raise}_x}{\operatorname{handleit} x = 1}$	$\begin{array}{cccc} v_1, \tau_1 & \dots & q[v_{n-1}/x] \Downarrow u, \tau_n \\ \hline v \text{ in } q \Downarrow u, \tau_1 + \dots + \tau_n \end{array}$
$q[v/x, w/y] \Downarrow t \qquad v_0 = v \qquad q[v_0$	$[x]$ $\Downarrow$ raise $x v_1, \tau_1  \dots  q[v_{n-1}/x] \Downarrow \pi$
$case\langle v,w\rangleof\langle x,y\rangle\mapsto q\Downarrow t\qquad\qquadhandleit$	$x = v \text{ in } q \Downarrow \tau_1 + \dots + \tau_{n-1} + \pi$
$\frac{v_0 = v \qquad q[v_0/x] \Downarrow raise_x v_1, \tau_1 \qquad q[v_1/x]}{handleit \ x = v \text{ in } q \Downarrow \tau_1 + \tau_2}$	$ \begin{array}{ccc}   \Downarrow raise_x v_2, \tau_2 & \dots \\ 2^+ \cdots & (\forall i.  \tau_i \neq \langle \rangle) \end{array} $

Figure 8: Operational semantics.

witnessed by some  $x_n \in X$  reachable after finitely many iterations, and the third clause addresses the remaining situation of divergence via unfolding the loop infinitely many times. In the latter case, the guardedness assumption for f

is crucial, as it ensures that each  $\tau_i$  is nonempty, and therefore the resulting trace  $\tau_1 + \tau_2 + \cdots$  is indeed infinite.

Operationally, guardedness in the above sense is modelled by cutting the control flow with the *put* command, which is the only command contributing to the traces. Concretely, let  $Base = \{\mathbb{N}\}, \Sigma_v = \{zero: 1 \to \mathbb{N}, succ: \mathbb{N} \to \mathbb{N}\}$  and  $\Sigma_c = \{pred: \mathbb{N} \to (1 + \mathbb{N})[0], put: \mathbb{N} \to 0[1]\}$  (note that while *pred* does not cause any side effects, it does perform a computation and therefore needs to be in  $\Sigma_c$  rather than  $\Sigma_v$ ). The operational semantics over these data is given in Fig. 8, where pred(v) is a shortcut for gcase pred(v) of inl  $x \mapsto ret x$ ; inr  $y \mapsto$  init y, and similarly for put(v). The judgement  $p \Downarrow t$  relates programs p with *terminals* t, which can consist of either a finite trace  $\tau$  together with a result value ret v or an exceptional value raise xv, or an infinite trace  $\pi$ . The traces correspond to the natural numbers written explicitly using the operation put.

In Fig. 9 we give a full account of the denotational semantics in an appropriate set-based notation for the concrete choice of the monad **T** as above. We omit contexts and types and moreover we index the semantic brackets with a *valuation*  $\rho$  sending each variable x: A from the context  $\Gamma$  to a corresponding element of the set <u>A</u>. That is, we assume the following equations

$$[\![\Gamma \vdash_{\mathsf{v}} p \colon A]\!]_{\rho} = [\![\Gamma \vdash_{\mathsf{v}} p \colon A]\!]_{\rho}, \qquad [\![\Delta \mid \Gamma \vdash_{\mathsf{c}} p \colon A]\!]_{\rho} = [\![\Delta \mid \Gamma \vdash_{\mathsf{c}} p \colon A]\!]_{\rho}.$$

(that is, composition with  $\rho$  is written as indexing by  $\rho$ .)

As usual, we have a substitution lemma saying that substitution of terms can be replaced by calculating values of terms and correspondingly updating the valuation. We write substitution in postfix notation, and assume the standard notion of capture-avoiding substitution.

**Lemma 15** (Substitution Lemma). Let  $\sigma$  be a substitution sending each variable  $x_i: A_i$  from the context  $\Gamma$  to a term  $\Gamma' \vdash_{\vee} v_i: A_i$ , and let  $\rho$  be a valuation for the variables in  $\Gamma'$ . Then

$$\llbracket \Gamma' \vdash_{\mathsf{v}} v\sigma \colon A \rrbracket_{\rho} = \llbracket \Gamma \vdash_{\mathsf{v}} v \colon A \rrbracket_{\varrho}, \qquad \llbracket \Delta \mid \Gamma' \vdash_{\mathsf{c}} p\sigma \colon A \rrbracket_{\rho} = \llbracket \Delta \mid \Gamma \vdash_{\mathsf{c}} p \colon A \rrbracket_{\varrho}.$$

where the valuation  $\underline{\sigma}$  sends each  $x_i$  to  $\llbracket \Gamma' \vdash_{\vee} v_i \colon A_i \rrbracket_{\rho}$ .

*Proof.* Straightforward induction over the term structure.

We now can state the main result of this section as follows.

**Theorem 16** (Soundness and Adequacy). Let  $\Delta \mid - \vdash_{c} p$ : B. Then

- 1.  $p \Downarrow \operatorname{ret} v, \tau \operatorname{iff} [\![\Delta \mid \vdash_{\mathsf{c}} p : B]\!] = (\operatorname{in}_1[\![v]\!], \tau) \in (B + \Delta) \times \mathbb{N}^{\star}.$
- 2.  $p \Downarrow \operatorname{raise}_x v, \tau$  and  $x \colon E^{\mathsf{g}}$  is in  $\Delta$  iff  $\llbracket \Delta \mid \vdash_{\mathsf{c}} p \colon B \rrbracket = (\operatorname{in}_2 \operatorname{in}_x v, \tau) \in (B + \Delta) \times \mathbb{N}^+$ .
- 3.  $p \Downarrow \operatorname{raise}_{x} v, \tau$  and  $x \colon E^{\mathsf{u}}$  is in  $\Delta$  iff  $\llbracket \Delta \mid \vdash_{\mathsf{c}} p \colon B \rrbracket = (\operatorname{in}_{2} \operatorname{in}_{x} v, \tau) \in (B + \Delta) \times \mathbb{N}^{\star}$ .
- 4.  $p \Downarrow \pi$  iff  $\llbracket \Delta \mid \vdash_{\mathsf{c}} p \colon B \rrbracket = \pi \in \mathbb{N}^{\omega}$ .

$$\begin{split} \llbracket x \rrbracket_{\rho} &= \rho(x) & \llbracket zero \rrbracket_{\rho} &= 0 & \llbracket \operatorname{inl} v \rrbracket_{\rho} &= \operatorname{in}_{1} \llbracket v \rrbracket_{\rho} \\ \llbracket \star \rrbracket_{\rho} &= \star & \llbracket succ \, u \rrbracket_{\rho} &= \llbracket u \rrbracket_{\rho} + 1 & \llbracket \operatorname{inr} v \rrbracket_{\rho} &= \operatorname{in}_{2} \llbracket v \rrbracket_{\rho} \\ \llbracket \langle v, w \rangle \rrbracket_{\rho} &= \langle \llbracket v \rrbracket_{\rho}, \llbracket w \rrbracket_{\rho} \rangle & \llbracket \lambda x. \, p \rrbracket_{\rho} &= \xi^{-1}(\lambda a. \llbracket p \rrbracket_{\rho \lceil a/x \rceil}) & \llbracket vw \rrbracket_{\rho} &= \xi(\llbracket v \rrbracket_{\rho})(\llbracket w \rrbracket_{\rho}) \end{split}$$

 $\llbracket pred(v) \rrbracket_{\rho} = \begin{cases} (\operatorname{in}_{1} \operatorname{in}_{1} \star, \langle \rangle) & \text{ if } \llbracket v \rrbracket_{\rho} = 0\\ (\operatorname{in}_{1} \operatorname{in}_{2} n, \langle \rangle) & \text{ if } \llbracket v \rrbracket_{\rho} = n+1 \end{cases}$ 

 $\llbracket \operatorname{\mathsf{ret}} v \rrbracket_{\rho} = (\operatorname{\mathsf{in}}_1 \llbracket v \rrbracket_{\rho}, \langle \rangle) \qquad \qquad \llbracket \operatorname{\mathsf{raise}}_x v \rrbracket_{\rho} = (\operatorname{\mathsf{in}}_2 \operatorname{\mathsf{in}}_x \llbracket v \rrbracket_{\rho}, \langle \rangle)$ 

$$\begin{split} \llbracket \mathsf{case} \ v \ \mathsf{of} \ \langle x, y \rangle &\mapsto p \rrbracket_{\rho} = \llbracket p \rrbracket_{[\rho, u/x, w/y]} \quad \text{where} \quad \llbracket v \rrbracket_{\rho} = \langle u, w \rangle \\ \llbracket \mathsf{g}\mathsf{case} \ put(v) \ \mathsf{of} \ \mathsf{inl} - &\mapsto p; \ \mathsf{inr} \ x \mapsto q \rrbracket_{\rho} = \begin{cases} (t, \langle \llbracket v \rrbracket_{\rho} \rangle + \tau) & \text{if} \ \llbracket q \rrbracket_{\rho[\star/x]} = (t, \tau) \\ (\langle \llbracket v \rrbracket_{\rho} \rangle + \pi) & \text{if} \ \llbracket q \rrbracket_{\rho[\star/x]} = \pi \end{cases} \\ \\ \llbracket \mathsf{case} \ v \ \mathsf{of} \ \mathsf{inl} \ x \mapsto p; \ \mathsf{inr} \ y \mapsto q \rrbracket_{\rho} = \begin{cases} \llbracket p \rrbracket_{\rho[u/x]} & \text{if} \ \llbracket v \rrbracket_{\rho} = \mathsf{in}_1 \ u \\ \llbracket q \rrbracket_{\rho[w/y]} & \text{if} \ \llbracket v \rrbracket_{\rho} = \mathsf{in}_2 \ w \end{cases} \end{split}$$

$$\llbracket \operatorname{handleit} x = v \text{ in } q \rrbracket_{\rho} = \begin{cases} (t, \tau + \tau') & \operatorname{if } \llbracket p \rrbracket_{\rho} = (\operatorname{in1} a, \tau) \text{ and } \llbracket q \rrbracket_{\rho} [a/x] = (t, \tau') \\ \tau + \pi & \operatorname{if } \llbracket p \rrbracket_{\rho} = (\operatorname{in1} a, \tau) \text{ and } \llbracket q \rrbracket_{\rho} [a/x] = \pi \\ (\operatorname{in2} b, \tau) & \operatorname{if } \llbracket p \rrbracket_{\rho} = (\operatorname{in2} b, \tau) \\ \pi & \operatorname{if } \llbracket p \rrbracket_{\rho} = \pi \end{cases}$$

$$\llbracket \operatorname{handle} x \text{ in } p \text{ with } q \rrbracket_{\rho} = \begin{cases} (\operatorname{in1} t, \tau) & \operatorname{if } \llbracket p \rrbracket_{\rho} = (\operatorname{in1} t, \tau) \\ (\operatorname{in2} \operatorname{in} t, \tau) & \operatorname{if } \llbracket p \rrbracket_{\rho} = (\operatorname{in2} \operatorname{in} t, \tau) \text{ and } x \neq e \\ (b, \tau + \tau') & \operatorname{if } \llbracket p \rrbracket_{\rho} = (\operatorname{in2} \operatorname{in} x, \tau) \text{ and } \llbracket q \rrbracket_{\rho} [a/x] = (b, \tau') \\ \tau + \pi & \operatorname{if } \llbracket p \rrbracket_{\rho} = (\operatorname{in2} \operatorname{inx} a, \tau) \text{ and } \llbracket q \rrbracket_{\rho} [a/x] = \pi \\ \pi & \operatorname{if } \llbracket p \rrbracket_{\rho} = \pi \end{cases}$$

$$\llbracket \operatorname{handleit} x = v \text{ in } q \rrbracket_{\rho} = \begin{cases} (\operatorname{in1} w, \tau_0 + \cdots + \tau_k) & \operatorname{if } v_0 = \llbracket v \rrbracket_{\rho}, \\ (\operatorname{in2} \operatorname{in} w, \tau_0 + \cdots + \tau_k) & \operatorname{if } v_0 = \llbracket v \rrbracket_{\rho}, \\ \llbracket q \rrbracket_{[\rho, v_i/x]} = (\operatorname{in2} \operatorname{inx} v_{i+1}, \tau_i), \\ \operatorname{and } \llbracket q \rrbracket_{[\rho, v_k/x]} = (\operatorname{in2} \operatorname{iny} w, \tau_k) \\ \tau_0 + \cdots + \tau_{k-1} + \pi & \operatorname{if } v_0 = \llbracket v \rrbracket_{\rho}, \\ \llbracket q \rrbracket_{[\rho, v_i/x]} = (\operatorname{in2} \operatorname{inx} v_{i+1}, \tau_i), \\ \operatorname{and } \llbracket q \rrbracket_{[\rho, v_k/x]} = \pi \\ \tau_0 + \tau_1 + \cdots & \operatorname{if } v_0 = \llbracket v \rrbracket_{\rho}, \\ \llbracket q \rrbracket_{[\rho, v_i/x]} = (\operatorname{in2} \operatorname{inx} v_{i+1}, \tau_i), \\ \operatorname{and } \llbracket q \rrbracket_{[\rho, v_k/x]} = \pi \\ \tau_0 + \tau_1 + \cdots & \operatorname{if } v_0 = \llbracket v \rrbracket_{\rho}, \\ \llbracket q \rrbracket_{[\rho, v_i/x]} = (\operatorname{in2} \operatorname{inx} v_{i+1}, \tau_i), \\ \operatorname{and } \llbracket q \rrbracket_{[\rho, v_i/x]} = \pi \\ \operatorname{if } v_0 = \operatorname{in2} \operatorname{inx} v_{i+1}, \tau_i), \\ \operatorname{and } v_0 = \llbracket v \rrbracket_{\rho}, \\ \operatorname{if } v_0 = \operatorname{in2} \operatorname{inx} v_{i+1}, \tau_i), \\ \operatorname{in2} \operatorname{in2}$$

Figure 9: Denotational semantics over  $TX = (X \times \mathbb{N}^{\star}) \cup \mathbb{N}^{\omega}$ .

Each clause of Theorem 16 is an iff-statement in which the left-to-right direction stands for soundness and the right-to-left direction stands for adequacy. This view of soundness and adequacy in not entirely standard and we compare it to the more established one. Suppose that we give a big-step semantics to a deterministic language in a system where every computation p either provably evaluates to some value v via  $p \Downarrow v$ , indicated by writing  $p \Downarrow$ , or  $p \Downarrow v$  is not provable for any v, indicated by writing  $p \uparrow$ . Denotationally, the former situation corresponds to  $[\![p]\!] = v$  and the latter to  $[\![p]\!] = \bot$  for a designated divergence constant  $\perp$ . Soundness then means that  $p \Downarrow v$  implies  $[\![p]\!] = v$ , while adequacy means that  $p \uparrow$ implies  $[p] = \bot$ . Equivalently, by contraposition, adequacy amounts to the implication from  $[p] \neq \bot$  to  $p \Downarrow$ . Now,  $[p] \neq \bot$  is the same as  $\llbracket p \rrbracket = v$  for some value v and  $p \Downarrow$  means that  $p \Downarrow w$  for a possibly different value w. Using soundness of the denotational semantics, we obtain w = v; thus, adequacy amounts to the implication from  $\llbracket p \rrbracket = v$  to  $p \Downarrow v$ , i.e. the perfect converse of soundness. We argue that the obtained reformulation of adequacy is advantageous in two respects: it does not hinge on contraposition, which is equivalent to excluded middle, and it does not depend on the presence of only one type of divergence  $\perp$  – e.g. in our present semantics there are as many types of divergence as infinite traces.

We prove Theorem 16 analogously to [14] by showing a stronger type-indexed property used as an induction invariant in the style of Tait [43]. Specifically, let us define a predicate  $\mathfrak{P}$  over all terms that are typable with empty variable context (-) by induction over their return types as follows:

- if  $\vdash_{\mathsf{v}} v \colon 1$  or  $\vdash_{\mathsf{v}} v \colon \mathbb{N}$  then  $\mathfrak{P}(v)$ ;
- if  $\vdash_{\mathsf{v}} v \colon A$  then  $\mathfrak{P}(\mathsf{inl} v)$  if  $\mathfrak{P}(v)$ ;
- if  $\vdash_{\mathsf{v}} v \colon A$  then  $\mathfrak{P}(\mathsf{inr} v)$  if  $\mathfrak{P}(v)$ ;
- if  $-\vdash_{\mathsf{v}} v \colon A$  and  $-\vdash_{\mathsf{v}} w \colon B$  then  $\mathfrak{P}(\langle v, w \rangle)$  if  $\mathfrak{P}(v)$  and  $\mathfrak{P}(w)$ ;
- if  $\vdash_{\mathsf{v}} \lambda x. p: A \to_{\Delta} B$  then  $\mathfrak{P}(\lambda x. p)$  if  $\mathfrak{P}(v)$  implies  $\mathfrak{P}(p[v/x])$  for all  $\vdash_{\mathsf{v}} v: A$ ;
- if  $\Delta \mid \vdash_{c} p \colon A$  then  $\mathfrak{P}(p)$  if one of the following clauses applies
  - 1.  $\llbracket \Delta \mid \vdash_{\mathsf{c}} p \colon A \rrbracket = (\mathsf{in}_1 \llbracket \vdash_{\mathsf{v}} v \colon A \rrbracket, \tau), \mathfrak{P}(v), \text{ and } p \Downarrow \mathsf{ret} v, \tau \text{ with } \tau \in \mathbb{N}^*;$
  - 2.  $\llbracket \Delta \mid \vdash_{\mathsf{c}} p : A \rrbracket = (\operatorname{in}_{2} \operatorname{in}_{x} \llbracket \vdash_{\mathsf{v}} v : A \rrbracket, \tau), \mathfrak{P}(v) \text{ and } p \Downarrow \operatorname{raise}_{x} v, \tau$ with  $\tau \in \mathbb{N}^{+}$  and  $x : E^{\mathsf{g}} \text{ in } \Delta;$
  - 3.  $\llbracket \Delta \mid \vdash_{\mathsf{c}} p : A \rrbracket = (\operatorname{in}_{2} \operatorname{in}_{x} \llbracket \vdash_{\mathsf{v}} v : A \rrbracket, \tau), \mathfrak{P}(v) \text{ and } p \Downarrow \operatorname{raise}_{x} v, \tau$ with  $\tau \in \mathbb{N}^{\star}$  and  $x : E^{\mathsf{u}}$  in  $\Delta$ ;
  - 4.  $\llbracket \Delta \mid \vdash_{\mathsf{c}} p \colon A \rrbracket = \pi \text{ and } p \Downarrow \pi \text{ with } \pi \in \mathbb{N}^{\omega}.$

Our main technical task is to prove the following lemma:

- **Lemma 17.** 1. Whenever  $x_1: B_1, \ldots, x_n: B_n \vdash_{\mathsf{v}} v: A \text{ and } \vdash_{\mathsf{v}} w_i: B_i$ such that  $\mathfrak{P}(w_i)$  for  $i = 1, \ldots, n$ , then  $\mathfrak{P}(v[w_1/x_1, \ldots, w_n/x_n])$ .
  - 2. Whenever  $\Delta \mid x_1 \colon B_1, \ldots, x_n \colon B_n \vdash_{\mathsf{c}} p \colon A \text{ and } \vdash_{\mathsf{v}} w_i \colon B_i \text{ such that } \mathfrak{P}(w_i) \text{ for } i = 1, \ldots, n, \text{ then } \mathfrak{P}(p[w_1/x_1, \ldots, w_n/x_n]).$

Using Lemma 17, Theorem 16 is obtained straightforwardly:

Proof of Theorem 16. Lemma 17 implies that  $\mathfrak{P}$  is totally true on all closed value and computation terms, and thus we are done by definition of  $\mathfrak{P}$ .

Proof of Lemma 17. We proceed by induction over the structure of values and computations. We write  $\sigma = [w_1/x_1, \ldots, w_n/x_n]$ . During the proof, we make extensive use of the substitution lemma (Lemma 15) without notice. Consider the value terms.

•  $v = x_i$ : since  $x_i \sigma = w_i$ ,  $\mathfrak{P}(v\sigma)$  holds by assumption;

• for v of type 1 or N, i.e.  $v = \star$ , v = zero, v = succ u,  $\mathfrak{P}(v\sigma)$  holds by definition;

- for  $v = \operatorname{inl} u$  or  $v = \operatorname{inr} u$ ,  $\mathfrak{P}(v\sigma)$  reduces to  $\mathfrak{P}(u\sigma)$  by induction;
- for  $v = \langle u, w \rangle$ ,  $\mathfrak{P}(v\sigma)$  reduces to  $\mathfrak{P}(u\sigma)$  and  $\mathfrak{P}(w\sigma)$  by induction;

• if  $v = \lambda x. p$  then we need to show that for every  $- \vdash_{\mathsf{v}} u$ : A satisfying  $\mathfrak{P}$ ,  $\mathfrak{P}(p\sigma[u/x])$  is true, and the latter follows by induction.

Next, we analyse computation terms.

• If  $p = \operatorname{ret} v$  then we are done straightforwardly by induction.

• If p = pred(v) then  $v\sigma$  can either be zero or succ v'. In both cases,  $p \Downarrow \operatorname{ret} u, \langle \rangle$  for some u, and  $\llbracket \Delta \mid - \vdash_{\mathsf{c}} pred(v\sigma) \colon \mathbb{N} \rrbracket = (\operatorname{in}_{1} u, \langle \rangle)$ , so the first clause from the definition of  $\mathfrak{P}$  applies.

• With  $p = \mathsf{raise}_x v$  we are done immediately by induction.

• If  $p = \operatorname{gcase} put(v)$  of  $\operatorname{inl} - \mapsto q$ ;  $\operatorname{inr} x \mapsto r$  then by induction  $\mathfrak{P}(r[\star/x]\sigma)$ . The latter must follow from one of the four clauses in the definition of  $\mathfrak{P}$ . E.g. if it follows from the first clause then  $\llbracket \Delta \mid \Gamma \vdash_{\mathsf{c}} r[\star/x]\sigma \colon A \rrbracket = (\operatorname{in}_{1} w, \tau)$ and  $r\sigma[\star/x] \Downarrow \operatorname{ret} w, \tau$ , hence  $p\sigma \Downarrow \operatorname{ret} w, \langle v \rangle + \tau$ ,  $\llbracket \Delta \mid \Gamma \vdash_{\mathsf{c}} p[\star/x]\sigma \colon A \rrbracket = (\operatorname{in}_{1} w, \tau)$  $(\operatorname{in}_{1} w, \langle v \rangle + \tau)$ , and therefore  $\mathfrak{P}(p\sigma)$ , again by the Clause 1 in the definition of  $\mathfrak{P}$ . The remaining three alternatives are checked analogously.

• Let  $p = \operatorname{case} v$  of  $\langle x, y \rangle \mapsto q$  and let  $v\sigma = \langle u, w \rangle$ . By induction,  $\mathfrak{P}(q\sigma[u/x, w/y])$  and further analysis runs analogously to the previous case.

• Let  $p = \operatorname{case} v$  of  $\operatorname{inl} x \mapsto q$ ;  $\operatorname{inr} y \mapsto r$ . Since  $v\sigma$  is either of the form  $\operatorname{inl} w$  or of the form  $\operatorname{inr} u$ , by induction, in the corresponding cases either  $\mathfrak{P}(q[w/x])$  or  $\mathfrak{P}(r[u/y])$ . Each of these cases is analyzed analogously to the previous two clauses.

• If p = init v then  $v\sigma$  must have 0 as the return type, but there are no values of this type. Therefore,  $\mathfrak{P}(p\sigma)$  is vacuously true.

• Let  $p = (\lambda x. q) w$ . Assuming that  $x\sigma = x$ , note that  $p\sigma = (\lambda x. q\sigma) w\sigma$ . By induction,  $\mathfrak{P}(w\sigma)$ , and thus, in turn, also by induction,  $\mathfrak{P}(q\sigma[w\sigma/x])$ . Now, on the one hand

$$\begin{bmatrix} \Delta \mid - \vdash_{\mathsf{c}} p\sigma \colon A \end{bmatrix}$$
  
=  $\llbracket \Delta \mid - \vdash_{\mathsf{c}} (\lambda x. q\sigma) w\sigma \colon A \rrbracket$ 

$$= \xi \big( \xi^{-1} (\lambda a. \llbracket \Delta \mid x \colon B \vdash_{\mathsf{c}} q\sigma \colon A \rrbracket_{[a/x]}) \big) \llbracket \Delta \mid - \vdash_{\mathsf{v}} w\sigma \colon B \rrbracket$$
$$= \llbracket \Delta \mid x \colon B \vdash_{\mathsf{c}} q\sigma \colon A \rrbracket_{\llbracket [\llbracket \Delta \mid - \vdash_{\mathsf{v}} w\sigma \colon B \rrbracket/x]}$$
$$= \llbracket \Delta \mid - \vdash_{\mathsf{c}} q\sigma \llbracket w\sigma/x \rrbracket \colon A \rrbracket,$$

and on the other hand  $p\sigma$  and  $q\sigma[w\sigma/x]$  reduce to the same terminal. Therefore  $\mathfrak{P}(p\sigma)$  is equivalent to  $\mathfrak{P}(q\sigma[w\sigma/x])$ , i.e. true.

•  $p = \operatorname{do} x \leftarrow q$ ; r. By induction hypothesis,  $\mathfrak{P}(q\sigma)$ . Depending on how q reduces, we have the following cases to cover.

•  $q\sigma \Downarrow \operatorname{ret} v, \tau, \llbracket \Delta \Vdash \vdash_{\mathsf{c}} q\sigma \colon B \rrbracket = (\operatorname{in}_{\mathbb{I}}\llbracket v \rrbracket, \tau) \text{ and } \mathfrak{P}(v)$ . By induction,  $\mathfrak{P}(r\sigma[v/x])$ . Observe that either  $r\sigma[v/x] \Downarrow t, \tau'$  and  $p\sigma \Downarrow t, \tau + \tau'$  or  $r\sigma[v/x] \Downarrow \pi$  and  $p\sigma \Downarrow \tau + \pi$  for suitable  $t, \pi, \tau'$  and analogously, either  $\llbracket \Delta \Vdash \vdash_{\mathsf{c}} r\sigma[v/x] \colon A \rrbracket = (t, \tau')$  and  $\llbracket \Delta \Vdash \vdash_{\mathsf{c}} p\sigma \colon A \rrbracket = (t, \tau + \tau')$  or  $\llbracket \Delta \vdash \vdash_{\mathsf{c}} r\sigma[v/x] \colon A \rrbracket = \pi$  and  $\llbracket \Delta \vdash \vdash_{\mathsf{c}} p\sigma \colon A \rrbracket = \tau + \pi$ . By further case distinction over the Clauses 1-4 in the definition of  $\mathfrak{P}$ , we conclude that  $\mathfrak{P}(p\sigma)$  is equivalent to  $\mathfrak{P}(r\sigma[v/x])$  and therefore true.

•  $q\sigma \Downarrow \operatorname{raise}_e v, \tau, \llbracket \Delta \mid - \vdash_{\mathsf{c}} q\sigma \colon A \rrbracket = (\operatorname{in}_2 \operatorname{in}_e v, \tau) \text{ and } \mathfrak{P}(v)$ . This case is analysed completely analogously to the previous one.

•  $q\sigma \Downarrow \pi$ . By the respective operational semantic rule, also  $p\sigma \Downarrow \pi : A$ . Also, by definition,  $\llbracket \Delta \mid - \vdash_{\mathsf{c}} p\sigma \rrbracket = \pi$ , hence  $\mathfrak{P}(v)$  follows from Clause 4 in the definition of  $\mathfrak{P}(v)$ .

• p = handle x in q with r. Again, we have multiple subcases, which are analogous to the cases for  $p = \text{do } x \leftarrow q$ ; r, as considered previously, with an important distinction that we now have to process the exception context  $\Delta$ .

•  $q\sigma \Downarrow \operatorname{ret} v, \tau$ ,  $\llbracket\Delta, x \colon E^{\mathsf{u}} \mid - \vdash_{\mathsf{c}} q\sigma \colon A\rrbracket = (\operatorname{in}_1\llbracket v\rrbracket, \tau)$  and  $\mathfrak{P}(v)$ . By the derivation rule, we obtain  $p\sigma \Downarrow \operatorname{ret} v, \tau$  and by definition,  $\llbracket\Delta \mid - \vdash_{\mathsf{c}} p\sigma \colon A\rrbracket = (\operatorname{in}_1\llbracket v\rrbracket, \tau)$ , whence  $\mathfrak{P}(p\sigma)$  holds by Clause 1 in the definition of  $\mathfrak{P}$ .

•  $q\sigma \Downarrow \operatorname{raise}_{e} v, \tau, \llbracket\Delta, x: E^{\mathsf{u}} \vdash - \vdash_{\mathsf{c}} q\sigma: A\rrbracket = (\operatorname{in}_{2} \operatorname{in}_{e}\llbracket v\rrbracket, \tau) \text{ and } \mathfrak{P}(v).$ If  $e \neq x$ , then from the respective operational semantic rule we know that  $p\sigma \Downarrow \operatorname{raise}_{e} v, \tau$ . Moreover,  $\llbracket\Delta \vdash - \vdash_{\mathsf{c}} p\sigma: A\rrbracket = (\operatorname{in}_{2} \operatorname{in}_{e}\llbracket v\rrbracket, \tau)$ , and hence Clause 3 of the definition  $\mathfrak{P}$  can be applied to obtain  $\mathfrak{P}(p\sigma)$ . Let us proceed under the assumption that e = x. Then either  $r\sigma[v/x] \Downarrow \operatorname{ret} v', \tau'$ , or  $r\sigma[v/x] \Downarrow \operatorname{raise}_{e'} v', \tau'$  or  $r\sigma[v/x] \Downarrow \pi$ , and therefore, respectively, either  $p\sigma \Downarrow \operatorname{ret} v', \tau + \tau'$ , or  $p\sigma \Downarrow \operatorname{raise}_{e'} v', \tau + \tau'$ , or  $p\sigma \Downarrow \tau + \pi$ . Since by induction  $\mathfrak{P}(r\sigma[v/x])$ , in the respective cases we obtain that  $\llbracket\Delta \vdash - \vdash_{\mathsf{c}} p\sigma: A\rrbracket$  is either  $(\operatorname{in}_{1}\llbracket v \rrbracket, \tau + \tau')$  or  $(\operatorname{in}_{2} \operatorname{in}_{e'}\llbracket v \rrbracket, \tau + \tau')$  or  $\tau + \pi$ . Now,  $\mathfrak{P}(p\sigma)$  follows by further analysis into the Clauses 1-4 in the definition of  $\mathfrak{P}$ .

•  $q\sigma \Downarrow \pi$ . Analogous to the case for do.

• p = (handleit x = v in q). Let  $v_0 = \llbracket v \rrbracket_{\sigma}$  and consider the sequence  $v_0, \ldots$  formed as follows:  $\llbracket \Delta, x \colon E^{\mathsf{g}} \mid - \vdash_{\mathsf{c}} q\sigma[v_i/x] \colon A \rrbracket = (\inf_2 \inf_x \llbracket v_{i+1} \rrbracket, \tau_i)$ . This sequence can either be infinite or terminate according to three different scenarios. Depending on this we proceed by case distinction.

• Suppose that the sequence  $v_0, \ldots$  is infinite. Then, by induction  $\mathfrak{P}(q\sigma[v_i/x])$  for every *i* and therefore also  $q\sigma[v_i/x] \Downarrow$  raise  $v_{i+1}, \tau_{i+1}$  by Clause 2 in the definition of  $\mathfrak{P}$  where each  $\tau_i$  is from  $\mathbb{N}^+$ . Now  $p\sigma \Downarrow \tau_1 + \tau_2 + \ldots$  and we are done by Clause 4 in the definition of  $\mathfrak{P}$ .

• Suppose that the sequence  $v_0, \ldots$  ends with  $v_k$  such that  $\llbracket\Delta, x \colon E^{\mathsf{g}} \mid - \vdash_{\mathsf{c}} q\sigma[v_k/x] \colon A \rrbracket = (\mathsf{in}_2 \mathsf{in}_z \llbracket w \rrbracket, \tau_k)$  or  $\llbracket\Delta, x \colon E^{\mathsf{g}} \mid - \vdash_{\mathsf{c}} q\sigma[v_k/x] \colon A \rrbracket = (\mathsf{in}_1 \llbracket w \rrbracket, \tau_k)$ . By induction,  $\mathfrak{P}(q\sigma[v_i/x])$  for every  $i \leq k$  and therefore  $q\sigma[v_k/x] \Downarrow$  raise<sub>z</sub>  $w, \tau_k$  or  $z \neq x$  or  $q\sigma[v_k/x] \Downarrow$  ret  $w, \tau_k$ . By the same considerations as in the previous clause, we obtain  $\mathfrak{P}(p\sigma)$ .

• The case of the sequence  $v_0, \ldots$  ending with  $v_k$  such that  $[\![\Delta, x: E^g | - \vdash_c q\sigma[v_k/x]: A]\!] = \pi$  is handled analogously to the above.

# 7. Conclusions and Further Work

We have instantiated the notion of abstract guardedness [22, 20] to a multivariable setting in the form of a metalanguage for guarded iteration, which incorporates both monad-based encapsulation of side-effects [34] and the fine-grain call-by-value paradigm [29]. As a side product, this has additionally resulted in a semantically justified unification of (guarded) iteration and exception handling, extending previous work by Geron and Levy [14]. Our denotational semantics is generic, and is parametrized by two orthogonal features: a notion of computation, given as a strong monad, and a notion of axiomatic guardedness, which serves to support guarded iteration. The notion of guardedness can range from vacuous guardedness (inducing trivial iteration, which unfolds at most once) to total guardedness (supported by monads equipped with a total iteration operator, specifically Elgot monads); the latter case covers classical denotational semantics, since any monad in a category of domains is Elgot [21].

In contrast, our (big-step) operational semantics is specific and addresses a concrete guarded iterative monad on **Set**, for which we have proved a soundness and adequacy result. This discrepancy in the status of operational and denotational semantics is related to the phenomenon that operational semantics generally appears to be harder to generalize than denotational semantics. For one example, we note that operational semantics needs to be completely reframed in a constructive setting, where it must arguably be understood coinductively rather than inductively [39].

In future research, we thus aim to use our present work for developing operational accounts of computational phenomena from their denotational models. One prospective example is suggested by the mentioned work of Nakata and Uustalu [39], who give a coinductive big-step trace semantics for a whilelanguage. We conjecture that this work has an implicit guarded iterative monad **T**<sub>R</sub> under the hood, for which guardedness cannot be defined using the standard argument based on a final coalgebra structure of the monad because the objects  $T_R X$  are not final coalgebras. The relevant notion of guardedness is thus to be identified. More generally, we regard the generic denotational semantics for our metalanguage as a guiding principle for identifying semantic structures underlying computational phenomena found in the wild, most importantly those that resist standard treatment, e.g. via domain theory. A recent example of such identification is *hybrid computation*, where the iterative behaviour can be organized in the form of Elgot iteration on a suitable *hybrid monad*, and the notion of guardedness naturally corresponds to *progressiveness* of computations in time [17].

Another direction for further research on generic soundness and adequacy theorems is motivated by previous work on operational semantics for languages parametrized by algebraic effects [41, 25], which provide syntactic access to generic notions of side effect. We will pay particular attention to the tension between iteration and general recursion, of which iteration is conventionally viewed as a light-weight counterpart. As the case of hybrid computation indicates, in some models it is not quite clear what general recursion can mean, and even formalizing total (unguarded) iteration presents considerable difficulties. Nevertheless, we will explore connections between guarded iteration and guarded recursion (in the sense of previous work [20]) whenever the latter can be identified. Standardly, iteration is expressible as a combination of recursion and second order types. We plan to explore conditions under which this connection generalizes to the guarded setting. As a basis for the prospective "metalanguage for guarded recursion" we plan to use Levy's call-by-push-value as the most natural candidate [27], into which fine-grain call-by-value embeds. In view of this fact, our task can be seen as the task of enriching this embedding with respective guarded fixpoints on both sides.

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#### References

- S. Abramsky. Intensionality, definability and computation. In A. Baltag and S. Smets, editors, Johan van Benthem on Logic and Information Dynamics, volume 5 of Outstanding Contributions to Logic, pages 121–142. Springer, 2014. doi:10.1007/978-3-319-06025-5\_5.
- [2] A. W. Appel, P.-A. Melliès, C. D. Richards, and J. Vouillon. A very modal model of a modern, major, general type system. In *Proceedings of the 34th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL 2007)*, pages 109–122. Association for Computing Machinery, 2007. doi:10.1145/1190216.1190235.
- [3] N. Benton and A. Kennedy. Exceptional syntax. J. Funct. Prog., 11(4):395–410, 2001. doi:10.1017/S0956796801004099.
- J. Bergstra, A. Ponse, and S. Smolka, editors. Handbook of Process Algebra. Elsevier, 2001. doi:10.1016/b978-0-444-82830-9.x5017-6.

- [5] L. Birkedal, R. E. Møgelberg, J. Schwinghammer, and K. Støvring. First steps in synthetic guarded domain theory: Step-indexing in the topos of trees. Log. Methods Comput. Sci., 8(4), 2012. doi:10.2168/LMCS-8(4: 1)2012.
- S. Bloom and Z. Esik. Iteration theories: The equational logic of iterative processes. Springer, 1993. doi:10.1007/978-3-642-78034-9.
- [7] S. Brookes and K. Van Stone. Monads and comonads in intensional semantics. Technical Report CMU-CS-93-140, Carnegie-Mellon University Pittsburgh, Dept. of Comp. Sci., 1993. https://apps.dtic.mil/dtic/ tr/fulltext/u2/a266522.pdf.
- [8] A. Carboni, S. Lack, and R. Walters. Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, 84(2):145–158, 1993. doi:0022-4049(93)90035-R.
- [9] P. Cenciarelli and E. Moggi. A syntactic approach to modularity in denotational semantics. Presented at the 5th Conference on Category Theory and Computer Science, CTCS 1993. URL: https://person.dibris.unige. it/moggi-eugenio/ftp/mod-sem.pdf.
- [10] R. Clouston, A. Bizjak, H. B. Grathwohl, and L. Birkedal. The guarded lambda-calculus: Programming and reasoning with guarded recursion for coinductive types. *Logical Methods in Computer Science*, 12(3), 2016. doi: 10.2168/LMCS-12(3:7)2016.
- [11] J. R. B. Cockett. Introduction to distributive categories. Mathematical Structures in Computer Science, 3(3):277–307, 1993. doi:10.1017/ S0960129500000232.
- [12] T. Coquand. Infinite objects in type theory. In H. Barendregt and T. Nipkow, editors, *Types for Proofs and Programs, International Work-shop (TYPES 1993)*, volume 806 of *Lecture Notes in Computer Science*, pages 62–78. Springer, 1993. doi:10.1007/3-540-58085-9\_72.
- [13] M. Escardó. A metric model of PCF. Presented at the Workshop on Realizability Semantics and Applications, Trento, Italy, June 30-July 1, 1999. URL: https://www.cs.bham.ac.uk/~mhe/papers/metricpcf.pdf.
- [14] B. Geron and P. B. Levy. Iteration and labelled iteration. In Mathematical Foundations of Programming Semantics (MFPS XXXII), volume 325 of Electronic Notes in Theoret. Comput. Sci., pages 127–146. Elsevier, 2016. doi:10.1016/j.entcs.2016.09.035.
- [15] E. Giménez. Codifying guarded definitions with recursive schemes. In P. Dybjer, B. Nordström, and J. Smith, editors, *Types for Proofs and Programs (TYPES 1994)*, volume 996 of *Lecture Notes in Computer Science*, pages 39–59. Springer, 1995. doi:10.1007/3-540-60579-7\_3.

- [16] E. Giménez. Structural recursive definitions in type theory. In K. G. Larsen, S. Skyum, and G. Winskel, editors, Automata, Languages and Programming, 25th International Colloquium (ICALP 1998), volume 1443 of Lecture Notes in Computer Science, pages 397–408. Springer, 1998. doi:10.1007/BFb0055070.
- [17] S. Goncharov, J. Jakob, and R. Neves. A Semantics for Hybrid Iteration. In S. Schewe and L. Zhang, editors, 29th International Conference on Concurrency Theory (CONCUR 2018), volume 118 of Leibniz International Proceedings in Informatics (LIPIcs), pages 22:1–22:17. Dagstuhl Publishing, 2018. doi:10.4230/LIPIcs.CONCUR.2018.22.
- [18] S. Goncharov, S. Milius, and C. Rauch. Complete Elgot monads and coalgebraic resumptions. In *Mathematical Foundations of Programming Semantics (MFPS XXXII)*, volume 325 of *Electronic Notes in Theoret. Comput. Sci.*, pages 147–168. Elsevier, 2016. doi:10.1016/j.entcs.2016.09.036.
- [19] S. Goncharov, C. Rauch, and L. Schröder. A metalanguage for guarded iteration. In B. Fischer and T. Uustalu, editors, *Theoretical Aspects of Computing (ICTAC 2018) – 15th International Colloquium*, volume 11187 of *Lecture Notes in Comput. Sci.*, pages 191–210. Springer, 2018. doi: 978-3-030-02508-3.
- [20] S. Goncharov and L. Schröder. Guarded traced categories. In C. Baier and U. D. Lago, editors, Foundations of Software Science and Computation Structures (FoSSaCS 2018) – 21st International Conference, volume 10803 of Lecture Notes in Comput. Sci., pages 313–330. Springer, 2018. doi: 10.1007/978-3-319-89366-2\_17.
- [21] S. Goncharov, L. Schröder, C. Rauch, and J. Jakob. Unguarded recursion on coinductive resumptions. *Log. Methods Comput. Sci.*, 14(3), 2018. doi: 10.23638/LMCS-14(3:10)2018.
- [22] S. Goncharov, L. Schröder, C. Rauch, and M. Piróg. Unifying guarded and unguarded iteration. In J. Esparza and A. Murawski, editors, Foundations of Software Science and Computation Structures (FoSSaCS 2017) – 20th International Conference, volume 10203 of Lecture Notes in Comput. Sci., pages 517–533. Springer, 2017. doi:10.1007/978-3-662-54458-7\_30.
- [23] P. Hancock and A. Setzer. Guarded induction and weakly final coalgebras in dependent type theory. In L. Crosilla and P. Schuster, editors, From Sets and Types to Topology and Analysis. Towards Practicable Foundations for Constructive Mathematics, volume 48 of Oxford Logic Guides, pages 115– 134. Clarendon Press, 2005. doi:10.1093/acprof:oso/9780198566519. 003.007.
- [24] B. Jacobs. Categorical Logic and Type Theory, volume 141 of Studies in Logic and the Foundations of Mathematics. North Holland, 1999. URL: https://www.sciencedirect.com/book/9780444501707.

- [25] P. Johann, A. Simpson, and J. Voigtländer. A generic operational metatheory for algebraic effects. In *Logic in Computer Science (LICS 2010)*, pages 209–218. IEEE Computer Society, 2010. doi:10.1109/LICS.2010.29.
- [26] K. Krishnaswami and N. Benton. Ultrametric semantics of reactive programs. In *Logic in Computer Science (LICS 2011)*, pages 257–266. IEEE Computer Society, 2011. doi:10.1109/LICS.2011.38.
- [27] P. B. Levy. Call-by-push-value: A subsuming paradigm. In J.-Y. Girard, editor, *Typed Lambda Calculi and Applications (TLCA 1999)*, volume 1581 of *Lecture Notes in Computer Science*, pages 228–242. Springer, 1999. doi: 10.1007/3-540-48959-2\_17.
- [28] P. B. Levy and S. Goncharov. Coinductive resumption monads: Guarded iterative and guarded elgot. In M. Roggenbach and A. Sokolova, editors, 8th Conference on Algebra and Coalgebra in Computer Science (CALCO 2019), volume 139 of Leibniz International Proceedings in Informatics (LIPIcs), pages 13:1-13:17. Dagstuhl Publishing, 2019. doi: 10.4230/LIPIcs.CALC0.2019.13.
- [29] P. B. Levy, J. Power, and H. Thielecke. Modelling environments in call-byvalue programming languages. *Information and Computation*, 185(2):182– 210, 2003. doi:10.1016/S0890-5401(03)00088-9.
- [30] S. Mac Lane. Categories for the Working Mathematician, Second Edition, volume 5 of Graduate Texts in Mathematics. Springer, 1998. doi:10.1007/ 978-1-4757-4721-8.
- [31] S. Milius. Completely iterative algebras and completely iterative monads. Inf. Comput., 196(1):1–41, 2005. doi:10.1016/j.ic.2004.05.003.
- [32] S. Milius and T. Litak. Guard your daggers and traces: Properties of guarded (co-)recursion. Fund. Inform., 150(3-4):407-449, 2017. doi:10. 3233/FI-2017-1475.
- [33] R. Milner. Communication and concurrency. Prentice-Hall, 1989.
- [34] E. Moggi. A modular approach to denotational semantics. In D. H. Pitt, P.-L. Curien, S. Abramsky, A. M. Pitts, A. Poigné, and D. E. Rydeheard, editors, *Category Theory and Computer Science (CTCS 1991) – 4th International Conference*, volume 530 of *Lecture Notes in Comput. Sci.*, pages 138–139. Springer, 1991. doi:10.1007/BFb0013462.
- [35] H. Nakano. A modality for recursion. In Logic in Computer Science (LICS 2000), pages 255–266. IEEE Computer Society, 2000. doi:10.1109/LICS. 2000.855774.
- [36] K. Nakata. Resumption-based big-step and small-step interpreters for while with interactive I/O. In O. Danvy and C. Shan, editors, *Proceedings IFIP* Working Conference on Domain-Specific Languages (DSL 2011), volume 66

of *Electronic Proceedings in Theoret. Comput. Sci.*, pages 226–235. Open Publishing Association, 2011. doi:10.4204/EPTCS.66.12.

- [37] K. Nakata and T. Uustalu. A Hoare logic for the coinductive trace-based big-step semantics of while. In A. D. Gordon, editor, *Programming Languages and Systems*, 19th European Symposium on Programming, (ESOP 2010), volume 6012 of Lecture Notes in Computer Science, pages 488–506. Springer, 2010. doi:10.1007/978-3-642-11957-6\_26.
- [38] K. Nakata and T. Uustalu. Resumptions, weak bisimilarity and big-step semantics for while with interactive I/O: an exercise in mixed inductioncoinduction. In L. Aceto and P. Sobocinski, editors, *Proceedings 7th Work*shop on Structural Operational Semantics (SOS 2010), volume 32 of Electronic Proceedings in Theoret. Comput. Sci., pages 57–75. Open Publishing Association, 2010. doi:10.4204/EPTCS.32.5.
- [39] K. Nakata and T. Uustalu. A Hoare logic for the coinductive trace-based big-step semantics of While. Log. Methods Comput. Sci., 11(1), 2015. doi: 10.2168/LMCS-11(1:1)2015.
- [40] S. Peyton Jones, J. Hughes, L. Augustsson, D. Barton, B. Boutel, W. Burton, J. Fasel, K. Hammond, R. Hinze, P. Hudak, T. Johnsson, M. Jones, J. Launchbury, E. Meijer, J. Peterson, A. Reid, C. Runciman, and P. Wadler. Haskell 98: A non-strict, purely functional language, 1999.
- [41] G. Plotkin and J. Power. Adequacy for algebraic effects. In F. Honsell and M. Miculan, editors, *Foundations of Software Science and Computation Structures (FoSSaCS 2001)*, volume 2030 of *Lecture Notes in Computer Science*, pages 1–24. Springer, 2001. doi:10.1007/3-540-45315-6\_1.
- [42] J. Power and H. Watanabe. Combining a monad and a comonad. *Theoret. Comput. Sci.*, 280(1-2):137-162, 2002. doi:10.1016/S0304-3975(01) 00024-X.
- [43] W. W. Tait. Intensional interpretations of functionals of finite type I. J. Symbolic Logic, 32(2):198-212, 06 1967. doi:10.2307/2271658.
- [44] G. Winskel. The Formal Semantics of Programming Languages. MIT Press, Cambridge, Massachusetts, 1993. doi:10.7551/mitpress/3054. 001.0001.