Syntactic View of Sigma-Tau Generation of Permutations

Wojciech Rytter and Wiktor Zuba

Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Warsaw, Poland
[rytter,w.zuba]@mimuw.edu.pl

Abstract

We give a syntactic view of the Sawada-Williams (σ, τ) -generation of permutations. The corresponding sequence of $\sigma\tau$ -operations, of length n!-1 is shown to be highly compressible: it has $\mathcal{O}(n^2\log n)$ bit description. Using this compact description we design fast algorithms for ranking and unranking permutations.

1 Introduction

We consider permutations of the set $\{1, 2, ..., n\}$, called here *n*-permutations. For an *n*-permutation $\pi = (a_1, ..., a_n)$ denote:

$$\sigma(\pi) = (a_2, a_3, ..., a_n, a_1), \ \tau(\pi) = (a_2, a_1, a_3, ..., a_n).$$

In their classical book on combinatorial algorithms Nijenhuis and Wilf asked in 1975 if all n-permutations can be generated, each exactly once, using in each iteration a single operation σ or τ . This difficult problem was open for more than 40 years. Very recently Sawada and Williams presented an algorithmic solution at the conference SODA'2018. In this paper we give new insights into their algorithm by looking at the generation from syntactic point of view.

Usually in a generation of combinatorial objects of size n we have a starting object and some set Σ of very local operations. Next object results by applying an operation from Σ , the generation is efficient iff each local operation uses small memory and time. Usually the sequence of generated objects is exponential w.r.t. n. From a syntactic point of view the generation globally can be seen as a very large word in the alphabet Σ describing the sequence of operations. It is called the *syntactic sequence* of the generation. Its textual properties can help to understand better the generation and to design efficient ranking and unranking. Such syntactic approach was used for example by Ruskey and Williams in generation of (n-1)-permutations of an n-set in [3].

Here we are interested whether the syntactic sequence is highly compressible. We consider compression in terms of Straight-Line Programs (SLP, in short), which represent large words by recurrences, see [4], using operations of concatenation. We construct SLP with $\mathcal{O}(n^2)$ recurrences, which has $\mathcal{O}(n^2 \log n)$ bit description.

The syntactic sequence for some generations is highly compressible and for others is not. For example in case of reflected binary Gray code of rank n each

local operation is the position of the changed bit. Here $\Sigma = \{1, 2, ..., n\}$ and the syntactic sequence T(n) is described by the short SLP of only $\mathcal{O}(n)$ size: $T_1 = 1$; T(k) = T(k-1), k, T(k-1) for $2 \le k \le n$.

In case of de Bruijn words of length n each operation corresponds to a single letter appended at the end. However in this case the syntactic sequence is not highly compressible though the sequence can be iteratively computed in a very simple way, see [7]. In this paper we consider the syntactic sequence SEQ_n (over alphabet $\Sigma = \{\sigma, \tau\}$) of Sawada-Williams $\sigma\tau$ -generation of permutations presented in [6, 5]. An SLP of size $\mathcal{O}(n^2)$ describing SEQ_n is given in this paper. The $\sigma\tau$ -generation of n-permutations by Sawada and Williams can be seen as a Hamiltonian path $\mathsf{SW}(n)$ in the Cayley graph \mathcal{G}_n . The nodes of this graph are permutations and the edges correspond to operations σ and τ .

We assume that (simple) arithmetic operations used in the paper are computable in constant time.

Our results. We show:

1. SEQ_n can be represented by the straight-line program of $\mathcal{O}(n^2)$ size:

•
$$\mathbf{W}_0 = \sigma$$
, $\mathbf{W}_k = \tau \cdot \prod_{i=1}^{n-2} \sigma^i \mathbf{W}_{\Delta(k,i)} \gamma_{n-2-i}$
for $1 \le k < n-3$;

•
$$\mathbf{V}_n = \gamma_{n-3} \cdot \prod_{i=2}^{n-3} \sigma^i \mathbf{W}_{\Delta(n-3,i)} \gamma_{n-2-i} \cdot \sigma^{n-1};$$

•
$$SEQ_n = \gamma_1^{n-2} \sigma^2 (\mathbf{V}_n \tau)^{n-2} \mathbf{V}_n$$
.

where
$$\Delta(k,i) = \min(k-1, n-2-i)$$
 and $\gamma_k = \sigma^k \tau$.

- 2. **Ranking:** using compact description of SEQ_n the number of steps (the rank of the permutation) needed to obtain a given permutation from a starting one can be computed in time $\mathcal{O}(n\sqrt{\log n})$ using inversion-vectors of permutations.
- 3. Unranking: again using SEQ_n the *t*-th permutation generated by SEQ_n can be computed in $\mathcal{O}(n\frac{\log n}{\log\log n})$ time.

2 Preliminaries

Denote by $cycle(\pi)$ all permutations cyclically equivalent to π . Sawada and Williams introduced an ingenious concept of a seed: a *shortened permutation* representing a group of (n-1) cycles. Informally it represents a set of permutations which are cyclically equivalent modulo one fixed element, which can appear in any place.

Let \oplus denote a modified addition modulo n-1, where $n-1 \oplus 1=1$. It gives a cyclic order of elements $\{1,...,n-1\}$. We write $a \ominus 1=b$ iff $b \oplus 1=a$. Formally a *seed* is a (n-1) tuple of distinct elements of $\{1,2,...,n\}$ of the form $\psi=(a_1,a_2,...,a_{n-1})$, such that $a_1=n$ and $(a_1,a_2 \oplus 1,a_2,...,a_{n-1})$ is a permutation. The element $x=mis(\psi)=a_2 \oplus 1$ is called a *missing* element.

Denote by $perms(\psi)$ the set of all n-permutations resulting by making a single insertion of x into any position in ψ , and making cyclic shifts. The sets $perms(\psi)$ are called packages, the seed ψ is the identifier of its package $perms(\psi)$. One of the main tricks in the Sawada-Williams construction is the requirement that the missing element equals $a_2 \oplus 1$. In particular this implies the following:

Observation 1. A given n-permutation belongs to one or two packages. We can find identifiers of these packages in linear time.

The algorithm of Sawada and Williams starts with a construction of a large and a small cycle (covering together the whole graph). The graph consisting of these two cycle is denote here by \mathcal{R}_n . The small cycle is very simple. Once \mathcal{R}_n is constructed the Hamiltonian path is very easy: In each cycle one τ -edge is removed (the cycles become simple paths), then the cycles are connected by adding one edge to \mathcal{R}_n .

2.1Structure of seed graphs

First we introduce seed-graphs. Define the seed-graph of the seed ψ , denoted here by SeedGraph(ψ) (denoted by $Ham(\psi)$ in [6]), as the graph consisting of edges implied by the seed ψ . The set of nodes consists of $perms(\psi)$, the set of edges consists of almost all σ -edges between these nodes (except the edges of the form $(*, x, *, ..., *) \rightarrow (x, *, ..., *, *)$, but the set of τ -edges consists only of the edges of the form $(*, x, *, ..., *) \rightarrow (x, *, *, ..., *)$, where x is the missing element. see Figure 1.

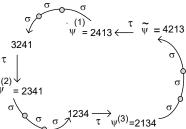


Figure 1: Structure of $SeedGraph(\psi)$, where $\psi=(4,1,3),$ $mis(\psi)=2$. For a seed $\psi=(a_1,a_2,...,a_{n-1})$ with $mis(\psi)=x$, let

$$\psi^{(n-1)} = (x, a_2, ..., a_{n-1}, a_1), \ \psi = (a_1, x, a_2, a_3, ..., a_{n-1}).$$

 $\psi^{(n-1)}=(x,a_2,...,a_{n-1},a_1),\ \widetilde{\psi}=(a_1,x,a_2,a_3,...,a_{n-1}).$ For $1\leq i\leq n-1$ denote $\psi^{(i)}=\gamma^i_{n-1}(\psi^{(n-1)}).$ In other words $\psi^{(i)},$ for n>i>0, is the word ψ right-shifted by i-1 and with x added at the beginning. Observe that: $\gamma_{n-1}(\psi^{(i)}) = \psi^{(i+1)}$ for 0 < i < n-1.

Example 2. For
$$\psi=(5,3,2,1)$$
 we have $\widetilde{\psi}=(5,4,3,2,1),\ \psi^{(1)}=(4,5,3,2,1),\ \psi^{(2)}=(4,1,5,3,2),\ \psi^{(3)}=(4,2,1,5,3),\ \psi^{(4)}=(4,3,2,1,5).$

Each $perms(\psi)$ can be sequenced easily as a simple cycle in \mathcal{G}_n . Two seeds ϕ, ψ are called neighbors iff $perms(\phi) \cap perms(\psi) \neq \emptyset$. The permutations of type $\psi^{(i)}$ play crucial role as *connecting* points between packages of neighboring seeds.

Observation 3. Two distinct seeds ϕ, ψ are neighbors iff $mis(\phi) = mis(\psi) \oplus mis(\psi)$ 1 or $mis(\psi) = mis(\phi) \oplus 1$, and after removing both $mis(\psi)$, $mis(\phi)$ from ϕ and ψ the sequences ϕ, ψ become identical.

2.2 The pseudo-tree ST_n of seeds

For a seed $\psi = a_1 a_2 ... a_{n-1}$ denote by $height(\psi)$ the maximal length k of a prefix of $a_2, a_3, ..., a_{n-1}$ such that $a_i = a_{i+1} \oplus 1$ for i = 2, 3, ..., k. For example height(94326781) = 3 (here the missing number is 5). For each two neighbors we distinguish one of them as a parent of the second one and obtain a tree-like structure called a pseudo-tree denoted by ST_n . If $height(\psi) > 1$ and $mis(\psi) = mis(\beta) \oplus 1$ we write $parent(\beta) = \psi$. Additionally if $\sigma^i(\psi^{(i)}) = \tilde{\beta}$ we write $parent(\beta) = 0$ and we say that $parent(\beta) = 0$ is the $parent(\beta) = 0$ and $parent(\beta) = 0$ we write $parent(\beta) = 0$ and we say that $parent(\beta) = 0$ is the $parent(\beta) = 0$ we write $parent(\beta) = 0$ and we say that $parent(\beta) = 0$ is the $parent(\beta) = 0$ and $parent(\beta) = 0$ we write $parent(\beta) = 0$ and $parent(\beta) = 0$ we write $parent(\beta) = 0$ and $parent(\beta) = 0$ we write $parent(\beta) = 0$ and $parent(\beta) = 0$ we write $parent(\beta) = 0$ and $parent(\beta) = 0$ we write $parent(\beta) = 0$ and $parent(\beta) = 0$ we write $parent(\beta) = 0$ and $parent(\beta) = 0$ we write $parent(\beta) = 0$ and $parent(\beta) = 0$ are $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ are $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ are $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ are $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ are $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ are $parent(\beta) = 0$ and $parent(\beta) = 0$ and $parent(\beta) = 0$ are $parent(\beta) = 0$ and $parent(\beta) = 0$ a

The function parent gives the tree-like graph of the set of seeds, it is a cycle with hanging subtrees rooted at nodes of this cycle. The set of seeds on this cycle is denoted by Hub_n . For example

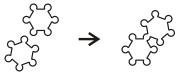
 $Hub_6 = \{(6, 5, 4, 3, 2), (6, 4, 3, 2, 1), (6, 3, 2, 1, 5), (6, 2, 1, 5, 4), (6, 1, 5, 4, 3)\}.$ Due to Lemma 5 we have:

Observation 4. If $\psi \notin Hub_n$ then <u>all</u> τ -edges of SeedGraph(ψ) are in PATH(n).

For $\psi \notin Hub_n$ let $Tree(\psi)$ be the subtree of ST_n rooted at ψ including ψ and nodes from which ψ is reachable by parent-links.

2.3 A version of Sawada-Williams algorithm

We say that an edge $u \to v$ conflicts with $u' \to v'$ iff $u = u', v \neq v'$. Non-disjoint packages ϕ, ψ can be joined into a simple cycle by removing two σ -edges conflicting with τ -edges.



By a union of graphs we mean set-theoretic union of nodes and set-theoretic union of all edges in these graphs.

Denote by \mathcal{R}_n the graph $\bigcup_{\psi} \mathsf{SeedGraph}(\psi)$ in which we removed all σ -edges conflicting with τ -edges. The τ -edges have priority here. A version of the construction of a Hamiltonian path by Sawada-Williams, denoted by $\mathsf{SW}(n)$, can be written informally as:

Algorithm Compute PATH(n);

$$\begin{split} P := \bigcup_{\psi \in SEEDS(n)} \mathsf{SeedGraph}(\psi) \\ \text{remove from } P \text{ all } \sigma\text{-edges conflicting with } \tau\text{-edges in } P \\ \pi := (n, n-1, ..., 1); \text{ add to } P \text{ the edge } \pi \to \sigma(\pi) \\ \text{remove edges } \pi \to \tau(\pi), \ \tau(\sigma(\pi)) \to \sigma(\pi) \\ \mathbf{return } P \ \{P \text{ is now a Hamiltonian path } \tau(\pi) \to^* \tau(\sigma(\pi)) \ \} \end{split}$$

Lemma 5. PATH(n) = SW(n).

Proof. To prove that the paths are the same it is enough to prove that both begin in the same place and that τ edges are used from exactly the same vertices. The particular Hamiltonian path in [6] is described in terms of a function next, which for any vertex assigns a next one on the path (by returning σ or τ). The

function next for a given permutation $\pi = (p_1, p_2, ..., p_n)$ produces a τ -edge when $p_2 = r \oplus 1$ (unless $\pi = (n, n-1, ..., 1)$ or $p_2 = n$), where r is cyclically first element after n jumping over p_2 (equal to p_3 if $n = p_1$).

The condition of being equal to $r \oplus 1$ in permutation π is exactly the same condition as being the missing element of a seed ψ such that $\pi \in perms(\psi)$ (for one of two seeds if $r = p_3 = p_2 \oplus 1$). For a permutation $\pi \in perms(\psi)$ such that $height(\psi) < n-3$ the missing element of ψ happens to be the first element of the permutation if and only if $\pi \in \bigcup_{i=1}^{n-1} \{\psi^{(i)}\}$. $\bigcup_{i=1}^{n-1} \{\psi^{(i)}\}$ is a set of those permutations from $perms(\psi) \cap bunch(\psi)$, whose ingoing edge in PATH(n) is a τ -edge. As τ -edges exchange the first two elements hence both approaches describe the same sets of edges (unless ψ belongs to the Hub_n). If the permutation belongs to the package $perms(\psi)$ for a hub seed ψ , the only difference from the previous case is that it can belong to the first part of the path of length 2n-2 (including the special permutation (n, n-1, ..., 1)).

The first 2n-2 permutations (the ones that are cyclically equivalent to (n-1,n-2,...,1) after removing element n which appears on first or second position) generate the same pattern in both constructions (alternation of σ and τ edges). All other permutations from those packages follow the same rules as the ones from packages corresponding to seeds with height $\leq n-4$, with the difference that permutations $\psi^{(i)}$ are not explicitly named (and that $\psi^{(1)}$ play a different role being the ones from the beginning of the path). In this way we have shown that SW(n) = PATH(n).

3 Compact representation of bunches of permutations

Our aim is to give a syntactic version of $\mathsf{PATH}(n)$: the sequence SEQ_n of $\sigma\tau$ -labels of $\mathsf{PATH}(n)$ represented compactly. We have to investigate more carefully the structure of seed-graphs and their interconnections. We introduce the basic components of PATH(n): groups of permutations corresponding to a subtree of seeds which are not on the cycle in the pseudo-tree. For $\psi \notin Hub_n$ define

$$bunch(\psi) = \bigcup_{\beta \in Tree(\psi)} perms(\beta) - cycle(\widetilde{\psi}) \cup \{\widetilde{\psi}, \psi^{(n-1)}\}.$$

In other words $cycle(\widetilde{\psi})$ connects $bunch(\psi)$ with the "outside world", only through $\widetilde{\psi}, \psi^{(n-1)}$.

We start with properties of local interconnection between two packages.

Lemma 6. Two seeds $\phi \neq \psi$ are neighbors iff one of them is the parent of another one. If $\phi = \operatorname{parent}(\psi)$ then $\operatorname{perms}(\phi) \cap \operatorname{perms}(\psi)$ is the σ -cycle containing both $\widetilde{\psi}$ and $\phi^{(i)}$, for some i, and has a structure as shown in Figure 2(A), where ψ is the i-th son of ϕ . If $\operatorname{height}(\phi) = k < n-3$ then $\operatorname{height}(\psi) = \Delta(k,i)$. Furthermore $\operatorname{son}(\phi,i)$ exists for all $i \in \{1,...,n-3\}$.

Proof. Assume that $\pi = (p_1, p_2, ..., p_n) \in perms(\phi)$. Without loss of generality we can assume that $p_1 = n$ (as cyclically equivalent permutations belong to the same packages). After removing any element from π the first element after n is

either p_2 or p_3 in both cases we obtain a valid seed if and only if the removed element is greater by one than that element following n. When removing element $p_2 \oplus 1$ we always receive a seed which package contains π and if we remove p_2 this is only the case if $p_2 = p_3 \oplus 1$. It means that if $\pi \in perms(\phi) \cap perms(\psi)$ then one of those seeds (denote it by ϕ) is equal to $(p_1, p_2, ..., p_n)$ (without an element $p_j = p_2 \oplus 1$) and the other (denote it by ψ) to $(p_1, p_3, ..., p_n)$. Removing both elements from π gives us the sequence obtained from ϕ and ψ after the appropriate removals.

Permutations $\widetilde{\psi} = \pi$ and $\phi^{(n-j+1)} = (p_j, p_{j+1}, ..., p_n, p_1, ..., p_{j-1})$ are cyclically equivalent. If $height(\phi) = k < n-3$ and j > k+1 then elements $p_2, p_3, ..., p_{k+1}$ form a sequence decreasing by one: a sequence with a property $p_i = p_{i+1} \oplus 1$ and $p_{k+1} \neq p_{k+2} \oplus 1$ $(p_{k+1} \neq p_{k+3} \oplus 1 \text{ if } j = k+2)$.

We obtain seed ψ by removing the element p_2 from ϕ and inserting the element $p_j = p_2 \oplus 1$ (between elements p_{j-1} and p_{j+1}). The removal shortens the decreasing sequence by 1 and the insertion can neither shorten it (as the element lands outside of the sequence) nor extend it (even if j = k+2 as for $p_j = p_{k+1} \oplus 1$ the pair (k,j) would have to be equal to (n-2,n) which is impossible outside of Hub_n). Hence

$$height(\psi) = k - 1 = \min(k - 1, j - 3) = \Delta(k, n - j + 1) = \Delta(k, i)$$

If $j \leq k+1$ then the decreasing sequence is formed by elements $p_2, p_3, ..., p_{k+2}$ (omitting element p_j). The removal of p_2 decreases the length of the sequence by 1 and insertion of p_i cut it just before that element. We have

$$height(\psi) = j - 3 = \min(k - 1, j - 3) = \Delta(k, n - j + 1) = \Delta(k, i)$$

We can remove p_2 from the seed ϕ and insert $p_2 \oplus 1$ in any of the n-3 places after p_3 obtaining a valid seed which is a son of ϕ .

Hence we know that packages of two seeds intersects if and only if they are in a parent-son relation, and that every seed of height greater than 1 has exactly n-3 seed-sons. We can also easily check which son of $parent(\psi)$ the seed ψ is or compute $son(\psi,i)$ for any $i \in \{1,...,n-3\}$.

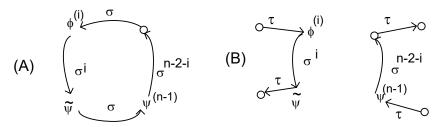


Figure 2: (A) The anatomy of $perms(\phi) \cap perms(\psi)$: the graph $SeedGraph(\psi) \cap SeedGraph(\phi)$. (B) A part of the Hamiltonian path PATH(n) after removing two conflicting σ -edges, we have that ψ is the i-th son of ϕ .

For k < n-3 and a seed ψ of height k we define \mathbf{W}_k as the sequence of labels of a sub-path in PATH(n) starting in $\widetilde{\psi}$ and ending in $\psi^{(n-1)}$. In other words it is a $\sigma\tau$ -sequence generating all n-permutations (each exactly once) of $bunch(\psi)$.

Observation 7. By Lemma 6 every seed ψ such that $1 < height(\psi) < n-3$ has exactly n-3 sons whose heights depend only on height of ψ . Hence (by induction on heights) all trees $Tree(\psi)$ are isomorphic for seeds ψ of the same height. Consequently the definition of \mathbf{W}_k is justified as it depends only on the height of ψ .

For a permutation π and a sequence α of operations σ, τ denote by $GEN(\pi, \alpha)$ the set of all permutations generated from π by following α , including π . The word \mathbf{W}_k satisfies:

 $GEN(\widetilde{\psi}, \mathbf{W}_k) = bunch(\psi) \text{ and } \mathbf{W}_k(\widetilde{\psi}) = \psi^{(n-1)}.$

In this section we give compact representation of \mathbf{W}_k

For example if $height(\psi) = 1$ then W_1 is a traversal of $perms(\psi)$ except n-2 cyclically equivalent permutations, common to $perms(\psi)$ and $perms(\phi)$, where $\phi = parent(\psi)$.

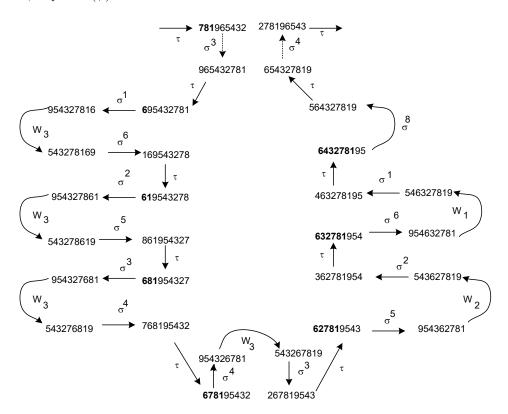


Figure 3: The structure of $bunch(\psi)$ for the seed $\psi = 95432781$. We have $parent(\psi) = \phi$, where $\phi = 96543281$. The connecting points of ψ with its parent are $\widetilde{\psi}$ and $\psi^{(n-1)}$, in other words $bunch(\psi) \cap perms(\phi) = \{\widetilde{\psi}, \psi^{(n-1)}\}$. The sequence W_4 starts in $\widetilde{\psi}$, visits all permutations in $bunch(\psi)$ and ends in $\psi^{(n-1)}$. We have: $\mathbf{W}_4 = \tau \cdot \sigma^1 \mathbf{W}_3 \gamma_6 \cdot \sigma^2 \mathbf{W}_3 \gamma_5 \cdot \sigma^3 \mathbf{W}_3 \gamma_4 \cdot \sigma^4 \mathbf{W}_3 \gamma_3 \cdot \sigma^5 \mathbf{W}_2 \gamma_2 \cdot \sigma^6 \mathbf{W}_1 \gamma_1 \cdot \gamma_8$

Recall that we denote $\gamma_k = \sigma^k \tau$

Theorem 8. For $1 \le k < n-3$ we have the following recurrences:

$$\mathbf{W}_0 = \sigma, \quad \mathbf{W}_k = \tau \cdot \prod_{i=1}^{n-2} \sigma^i \, \mathbf{W}_{\Delta(k,i)} \, \gamma_{n-2-i}$$

Proof. Assume $\psi \notin Hub_n$ is of height k, then by Lemma 6 the first, from left to right, n-k-1 children of ψ in the subtree $Tree(\psi)$ are of height k-1 and the next k-2 children are of heights k-2, k-3, ..., 1. The representative $\widetilde{\beta}_i$ of the i-th son β_i of ψ equals $\sigma^i(\psi^{(i)})$ (see Figures 3 and 4).

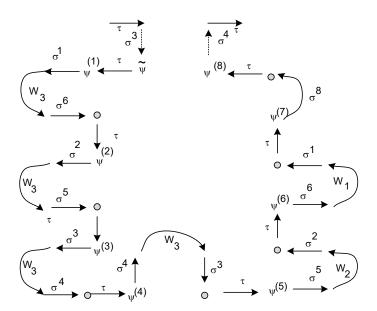


Figure 4: Schematic view of structure from Figure 3.

4 Compact representation of the whole generation

We have the following fact:

Observation 9. Assume two seeds ψ, β satisfy: $height(\psi) = k > 1$ and $\sigma^i(\psi^{(i)}) = \widetilde{\beta}$. Then if i = 1 and $\psi \in Hub_n$ then $height(\beta) = height(\psi)$.

Theorem 10. The whole $\sigma\tau$ -sequence SEQ_n starting at $\tau(n, n-1, ..., 1)$, ending at $\sigma\tau(n, n-1, ..., 1)$, and generating all n-permutations, has the following compact representation of $\mathcal{O}(n^2)$ size (together with recurrences for \mathbf{W}_k):

$$\begin{split} \mathsf{SEQ}_n \; &= \; \gamma_1^{n-2} \sigma^2 \; (\mathbf{V}_n \, \tau)^{n-2} \; \mathbf{V}_n, \; \; \text{where} \\ \mathbf{V}_n \; &= \; \gamma_{n-3} \; \cdot \; \prod_{i=2}^{n-3} \; \sigma^i \, \mathbf{W}_{\Delta(n-3,i)} \, \gamma_{n-2-i} \; \cdot \; \sigma^{n-1}. \end{split}$$

Proof. For every non-hub seed ψ we had that $GEN(\widetilde{\psi}, \mathbf{W}_k) = bunch(\psi)$, where $k = height(\psi)$. The only difference for a hub seed ϕ is that $son(\phi, 1)$ cannot be considered as part of a tree rooted at ϕ (with already defined parent-links), since $son(\phi, 1) \in Hub_n$ and this would lead to a cycle $(son(\phi, 1)$ is reachable via parent-links from ϕ). Thus to prevent this problem we define V_n as W_{n-3} with the part corresponding to the first son removed (leaving only the γ_{n-2-1} part),

and also delete the last symbol τ , as it does not appear at the end of the path (it corresponds to one of the τ -edges removed when joining two cycles into one path). Now Seq_n consists of n-1 such segments V_n (corresponding to n-1 hub seeds) joined by τ -edges (they are linked in the same way as if the previous V_n part was a son of the next one). Additionally it starts with γ_1^{n-1} -path representing the small path with the last τ -edge replaced by a σ -edge.

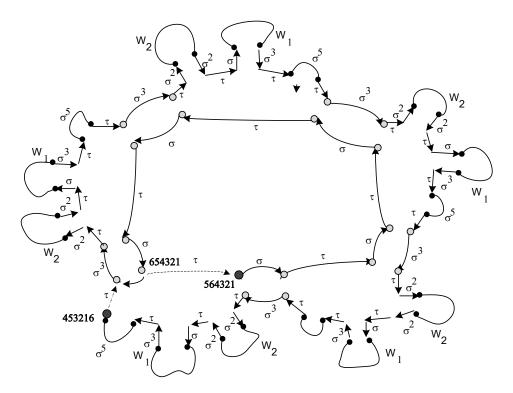


Figure 5: The compacted structure of SEQ_6 of length 720. It differs from the structure of \mathcal{R}_6 by adding one σ -edge from 654321 and removing two (dotted) τ -edges to have Hamiltonian path. We have: $SEQ_6 = (\sigma \tau)^4 \sigma^2 (\mathbf{V}_6 \tau)^4 \mathbf{V}_6$, where $\mathbf{V}_6 = \sigma^3 \tau \ \sigma^2 \mathbf{W}_2 \sigma^2 \tau \ \sigma \mathbf{W}_1 \sigma^3 \tau \ \sigma^5$. The structure is the union of graphs of 5 seeds in Hub_6 with hanging bunches. The $starting\ path$ consists of permutations from 564321 to 654321.

5 Ranking

We need some preprocessing to access later some values in constant time.

Observation 11. All the values $|W_k|$ and $\sum_{i=0}^k (|W_i| + n - 1)$ for $k \in \{0..n - 4\}$ can be computed in $\mathcal{O}(n)$ total time and accessed in $\mathcal{O}(1)$ time afterwards.

The ranks of representatives of hub seeds are easy to compute. For example for n=6 we have (see Figure 5): rank(643215)=1, rank(632154)=3, rank(621543)=5, rank(615432)=7, rank(654321)=9.

Lemma 12. For a given permutation π we can compute in time $\mathcal{O}(n)$

- (a) $rank(\pi) rank(\psi)$ if $\pi \in perms(\psi)$,
- **(b)** $rank(\pi)$ if $\pi \in perms(\psi)$ for some $\psi \in Hub_n$.

Proof.

By the *starting path* we mean the sequence on the first 2n-2 permutations of PATH(n), see Figure 5.

(a) Given a permutation $\pi \in perms(\psi)$ we define $\rho = (n, r_2, ..., r_n)$ as permutation cyclically equivalent to π and starting with n, and let j be the position such that $r_j = r_2 \oplus 1$. If $\psi = (n, r_3, ..., r_n)$, then $\rho = \widetilde{\psi}$

Now we distinguish two cases depending on the position l of n in π .

Case 1: if
$$l \le n - j + 2$$
, then $rank(\pi) = rank(\rho) - l + 1$
(it appears in the σ_{n-j+1} part of $bunch(parent(\psi))$),

Case 2: otherwise
$$rank(\pi) = |W_{height(\psi)}| + (n-1) - l + 1$$
 (it appears in the γ_{j-3} part).

If
$$\psi = (n, r_2, ..., r_{j-1}, r_{j+1}, ..., r_n)$$
, then

$$rank(\rho) = rank(\widetilde{\psi}) + SUM(height(\psi), n - j + 1)$$

Using similar arguments as before we have that:

$$rank(\pi) = rank(\rho) - l + 1$$
 if $l \le n - j + 2$ and $rank(\pi) = rank(\rho) + |W_{\Delta(height(\psi), n-j+1)}| + (n-1) - l + 1$ otherwise.

(b) If the permutation π after removing n is cyclically equivalent to (n-1, n-2, ..., 1) and n appears on the first position (π belongs to the starting path) then $rank(\pi) = 2 \cdot (n - p_2 - 1) - 1$, and if it appears on the second position then $rank(\pi) = 2 \cdot (n - p_1 - 1)$, where p_1, p_2 are the first two positions of π .

Otherwise we define ρ , j and l like in case (a) and $\psi = (n, r_2, ..., r_{j-1}, r_{j+1}, ..., r_n)$. We know that $\pi \in perms(\psi)$, and want to compute $rank(\pi)$ minus the rank of the first permutation of ψ which appears in PATH(n) with rank greater than 2n-3. That permutation is equal to

$$\mu = (r_2, ..., r_{i-1}, r_{i+1}, ..., r_n, r_2 \oplus 1, n)$$

(equal to $\sigma^2(\psi^1)$ if $\psi^{(1)}$ was defined for hub seeds in the same way as for the other ones), and has rank equal to

$$(2n-2) + |V_n\tau| \cdot (r_2 mod(n-1)) = 2n-2 + (r_2 mod(n-1)) \cdot (n(n-2)! - 2)$$

If j = n then $rank(\psi) = rank(\mu) + n - l$ else if l < n - j + 2 then

$$rank(\rho) = rank(\mu) + SUM(n-3, n-j+1) - |W_{n-4}| - 2, \ rank(\pi) = rank(\rho) - l + 1$$

Otherwise we have

$$rank(\pi) = rank(\rho) + |W_{i-3}| + (n-1) - l + 1.$$

Those two algorithms lets us rank permutations in basic cases, and allows us to reduce the main problem to a simpler one (ranking the representatives of seeds). \Box

Hence we concentrate on ranking permutations of type $\widetilde{\psi}$ (representatives of seeds). We slightly abuse notation and for a seed ψ define $rank(\psi) = rank(\widetilde{\psi})$.

For a non-hub seed ψ denote by $anchor(\psi)$ the highest non-hub ancestor ϕ of ψ and let $hub(\psi) = parent(anchor(\psi))$. Observe that the anchor ϕ is the first contacting seed with the hub, it is the first ancestor of ψ such that $perms(\phi) \cap perms(\beta) \neq \emptyset$ for some $\beta \in Hub_n$, in fact for $\beta = hub(\psi)$.

 $rank(\psi) - rank(anchor(\psi))$ for a non-hub seed ψ , can be treated as its distance from Hub_n . It happens that computing the rank of the anchor is much easier, since we have to deal only with the hub seeds. The bottleneck in ranking is computation of the distance of a seed representative from Hub_n . Define:

$$SUM(k,j) = |\tau \cdot \prod_{i=1}^{j-1} \sigma^{i} \mathbf{W}_{\Delta(k,i)} \gamma_{n-2-i}| + j.$$

Denote also by $ord(\psi)$ the position of $mis(\psi) + 1$ in ψ counting from the end of sequence ψ . For example for $\psi = (1065984312)$ we have $ord(\psi) = 5$, since $mis(\psi) = 7$ and 8 is on the 5-th position from the right.

Observation 13. If $\phi = parent(\psi) \notin Hub_n$ and ψ is the *i*-th son of ϕ then $rank(\psi) - rank(\phi) = SUM(height(\phi), i)$.

Example 14. Let $\psi = 94326781$, then $parent(\psi) = \phi = 95432781$. The path from $\widetilde{\phi} = 965432781$ to $\widetilde{\psi} = 954326781$ is

$$\tau \sigma^1 W_3 \sigma^6 \tau \sigma^2 W_3 \sigma^5 \tau \sigma^3 W_3 \sigma^4 \tau \sigma^4$$

see Figure 3. Its length equals SUM(4,4), we have: $height(\phi) = 4$, $ord(\psi) = 4$.

Observation 15. $ord(\psi) = i$ iff ψ is the *i*-th son of $parent(\psi)$.

For the parent-sequence $\psi_0 = \psi, \psi_1, ..., \psi_m = anchor(\psi_i)$ denote

$$route(\psi) = (ord(\psi_0), ord(\psi_1), ..., ord(\psi_m)).$$

For a seed $\psi = a_1 a_2 ... a_{n-1}$ define the decreasing sequence of ψ , denoted by $dec_seq(\psi)$, as the maximal sequence $a_{i_0}a_{i_1}...a_{i_m}$, where $2 = i_0 < i_1 < i_2 < ... < i_m$ such that $i_{j-1} = i_j \oplus 1$ for $0 < j \le m$. Denote $level(\psi) = n - m - 3$. The length of the parent-sequence $\psi = \psi_0, \psi_1, \psi_2, ..., \psi_r = anchor(\psi)$ from ψ to its anchor is $r = level(\psi) - 1$.

Example 16. We have: $dec_seq(96154238) = (6,5,4,3)$. Hence the path from $\psi = (96154238)$ to $anchor(\psi) = (98765423)$ is of length (9-3-3)-1=2. This path equals:

$$\psi_0 \to \psi_1 \to \psi_2 = 96154238 \to 97615423 \to 98765423.$$

We have: $ord(\psi_0) = 1$, $ord(\psi_1) = 5$, $ord(\psi_2) = 2$, route(96154238) = (1, 5, 2).

The key point is that we do not need to deal with the whole parent-sequence, including explicitly seeds on the path, which is of quadratic size (in worst-case) but it is sufficient to deal with the sequence of orders of sons, which is an implicit representation of this path of only linear size

Lemma 17. For a non-hub seed ψ we can compute $route(\psi)$ and $anchor(\psi)$ in $\mathcal{O}(n\sqrt{\log n})$ time.

Proof. We know the length of the parent sequence from ψ to its anchor, since we know $level(\psi)$. Now we use the following auxiliary problem

Inversion-Vector problem:

for a seed ψ compute for each element x the number RightSm[x] of elements smaller than x which are to the right of x in ψ .

Assume $\psi = (a_1, a_2, ..., a_{n-1})$. We introduce a new linear order

$$a_2 \prec a_2 \ominus 1 \prec a_2 \ominus 2 \prec \ldots \prec a_2 \ominus (n-2).$$

Then we compute together the numbers RightSm[z] w.r.t. linear order \prec for each element z in ψ .

Now $ord(\psi_i)$ is computed separately for each i in the following way:

$$ord(\psi_i) := RightSm[x_i + 1] + 1$$
, where $x_i = mis(\psi_i)$

The *Inversion-Vector* problem can be computed in $\mathcal{O}(n\sqrt{\log n})$ time, see [1]. Consequently the whole computation of numbers $ord(\psi_i)$ is of the same asymptotic complexity. We know that $hub(\psi) = (n, b, b \ominus 1, ..., b \ominus (n-3))$, where $b = a_2 \oplus level(\psi)$ and we know also which son of $hub(\psi)$ is $anchor(\psi)$. This knowledge allows to compute $anchor(\psi)$ within required complexity. This completes the proof.

Corollary 18. For a non-hub seed ψ the value $rank(\psi) - rank(anchor(\psi))$ can be computed in $\mathcal{O}(n\sqrt{\log n})$ time.

Proof. Let the parent-sequence from ψ to its anchor be

$$\psi = \psi_0, \psi_1, \psi_2, ..., \psi_r = anchor(\psi), \text{ where } r = level(\psi) - 1.$$

Then $rank(\psi_i) - rank(\psi_{i+1}) = SUM(height(\psi_{i+1}), ord(\psi_i))$, and

 $height(\psi_i) = \Delta(height(\psi_{i+1}), ord(\psi_i)),$ which allows us to compute in $\mathcal{O}(n)$ time:

$$rank(\psi) - rank(anchor(\psi)) = \sum_{i=m-1}^{0} \left(rank(\psi_i) - rank(\psi_{i+1}) \right)$$

Now the thesis is a consequence of Observation 11, Observation 13 and Lemma 17. This completes the proof.

Example 19. (Continuation of Example 16) For ψ from Example 16 we have:

$$rank(\psi) - rank(anchor(\psi)) = SUM(5,5) + SUM(2,1)$$

The following result follows directly from Corollary 18, Lemma 12 and Observation 11.

Theorem 20. [Ranking] For a given permutation π we can compute the rank of π in SEQ_n in time $\mathcal{O}(n\sqrt{\log n})$

6 Unranking

Denote by Perm(t) the t-th permutation in SEQ_n , and for $t < |bunch(\psi)|$ let $Perm(\psi,t) = Perm(t + rank(\widetilde{\psi}))$ (it is the t-th permutation in $bunch(\psi)$, counting from the beginning of this bunch). The following case is an easy one.

Lemma 21. If we know a seed ψ together with its rank, such that $Perm(t) \in perms(\psi)$, then we can recover Perm(t) in linear time.

Proof.

Let $\psi = (n, a_2, ..., a_{n-1})$, and $k = height(\psi)$. In linear time we find j such that $SUM(k, j) - j \le t < SUM(k, j + 1) - j - 1$.

If $SUM(k,j) < t < SUM(k,j) + |W_{\Delta(k,j)}|$, then $Perm(\psi,t)$ does not belong to $perms(\psi)$ (it belongs to $bunch(son(\psi,j))$).

If $l = SUM(k,j) - t \ge 0$, then $Perm(\psi,t)$ is equal to $(n,a_2,...,a_{n-j},a_2 \oplus 1,a_{n-j+1})$ rotated by l to the right, and if $l = t - SUM(k,j) + |W_{\Delta(k,j)}| \ge 0$, then it is the same permutation rotated by l + 1 to the left.

In this way we reduced the problem of unranking permutation outside of Hub_n to finding a package containing the permutation.

We say that a permutation π is a hub-permutation if $\pi \in perms(\psi)$ for some $\psi \in Hub_n$.

Lemma 22. We can test in O(n) time if Perm(t) is a hub-permutation.

- (a) If "yes" then we can recover Perm(t) in O(n) time.
- (b) Otherwise we can find in $\mathcal{O}(n)$ time an anchor-seed ψ together with $rank(\psi)$ such that $Perm(t) \in bunch(\psi)$.

Proof.

If t < 2n-2 then Perm(t) is equal to (n-1,...,1) rotated to the left by $\lceil \frac{t}{2} \rceil$, with n inserted on first position if x is odd, and on the second if it is even. Otherwise let $t - (2n-2) = t_1 \cdot |V_n \tau| + t_2$ (we use integer division), and let

$$\psi = (n, t_1, t_1 \ominus 1, ..., t_1 \ominus (n-3))$$

(with t_1 substituted by n-1 if equal to 0). Perm(t) belongs to $perms(\psi)$, or to $bunch(\phi)$, where $parent(\phi) = \psi$.

If $t_2 < n-2$ then Perm(t) equals to $(n, t_1, t_1 \ominus 1, ..., t_1 \ominus (n-2))$ rotated to the left by $t_2 + 1$.

In the other case in linear time we find j such that

$$SUM(n-3,j)-j \le t_2+(1+|W_{n-4}|+n-1)-(n-2) < SUM(n-3,j+1)-j-1,$$

and $l=t_2+|W_{n-4}|+2-SUM(n-3,j), \ \phi=son(\psi,j)$

If $l \leq 0$ then Perm(t) is equal to $\widetilde{\phi}$ rotated to right by -l, else if $l \geq |W_{\Delta(n-3,j)}|$, then Perm(t) is equal to $\widetilde{\phi}$ rotated to the left by $l - |W_{\Delta(n-3,j)}| + 1$. Otherwise $Perm(t) = Perm(\phi, l)$, and it is not a hub-permutation.

By using this algorithm we either already succeed in finding the right permutation, or restrict ourselves to a limited regular part of PATH(n).

For a sequence $\mathbf{b} = (b_1, b_2, ..., b_m)$ of positive integers denote

$$MaxFrac(\mathbf{b}) = \max_{i} \frac{b_{i+1}}{b_{i}}, \quad MinFrac(\mathbf{b}) = \min_{i} \frac{b_{i+1}}{b_{i}}.$$

The sequence **b** is called here D(m)-stably increasing iff

$$MinFrac(\mathbf{b}) \ge 2$$
, and $MaxFrac(\mathbf{b}) \le D(m)$.

Lemma 23.

(a) Assume we have a D(m)-stably increasing sequence **b** of length O(m). Then after linear preprocessing we can locate any integer t in the sequence **b** in

 $\mathcal{O}(\log \log D(m))$ time.

(b) The sequence $\mathbf{b} = (b_0, b_1, ..., b_{n-5})$, where $b_k = \sum_{i=0}^k (|W_i| + n - 1)$ is n-stably increasing.

Proof.

(a) Denote $B = \frac{b_m}{b_1}$, $\delta = \sqrt[m]{B}$, $Min = MinFrac(\mathbf{b})$, $Max = MaxFrac(\mathbf{b})$, and let d be a sequence such that $d[i] = \max\{j : b_j \geq b_1 \cdot \delta^i\}$.

It can be computed in linear time by scanning the sequence **b** from left to right and reporting whenever element exceeds next power of δ .

Thanks to the fact that sequence **b** is D(m)-stably increasing the maximal difference between consecutive values of a sequence d is bounded by $\log_{Min}(\delta) \le \log_{Min}(Max) \le \log_2(D(m))$.

When locating x in the sequence **b** we can compute the value

$$y = \lfloor \log_{\delta} \left(\frac{x}{b_1} \right) \rfloor = \lfloor \frac{m \cdot (\log_2 x - \log_2 b_1)}{\log_2 B} \rfloor$$

It now suffices to binary scan the part of sequence **b** between positions d[y] and d[y+1], which has a length bounded by $\log_2(D(m))$.

(b) For any $k \in \{1, ..., n-5\}$ the value of $b_k = \sum_{i=0}^{k-1} (n-1+|W_i|) + n-1 + |W_k|$ equals

$$2 \cdot \sum_{i=0}^{k-1} (n-1+|W_i|) + (n-k-2) \cdot (|W_{k-1}|+n-1) + n$$
$$> 2 \cdot \sum_{i=0}^{k-1} (n-1+|W_i|) = 2 \cdot b_{k-1}$$

We have:

$$\begin{split} \frac{b_k}{b_{k-1}} &= 2 + \frac{n + (n-k-2) \cdot (|W_{k-1}| + n - 1)}{\sum_{i=0}^{k-1} (n-1 + |W_i|)} \\ &< 2 + \frac{n + (n-k-2) \cdot (|W_{k-1}| + n - 1)}{(n-1 + |W_{k-1}|)} \\ &= n - k + \frac{n}{(|W_{k-1}| + n - 1)} \le n - k + 1 \le n \end{split}$$

Combining those two parts of the lemma allows us to locate values in the sequence $b_k = \sum_{i=0}^k (n-1+|W_i|)$ in $\mathcal{O}(\log\log n)$ time after linear preprocessing dependent only on the value of n.

Lemma 24. After linear preprocessing if we are given a height of a non-hub seed ψ , and a number $t \leq |bunch(\psi)|$ we can find the number j and height(β) of the seed-son β of ψ such that $Perm(\psi, t) \in bunch(\beta)$ in $\mathcal{O}(\log \log n)$ time if $Perm(\psi, t) \notin perms(\psi)$.

Proof. Let $k = height(\psi)$. We need j such that $SUM(k, j) - j \le t < SUM(k, j + 1) - (j + 1)$. For $j \le n - k$ we have $SUM(k, j) - j = (j - 1) \cdot (|W_{k-1}| + n - 1)$, hence if t < SUM(k, n - k) - n + k the simple division by $|W_{k-1}| + n - 1$ suffices to find the appropriate j. Otherwise we look for j such that

$$|W_k| - SUM(k, j + 1) + j + 1 < s \le |W_k| - SUM(k, j) + j$$
, where $s = |W_k| - t$.

Let
$$b_i = |W_k| - SUM(k, n-2-i) + n-2-i = (\sum_{j=0}^{i} |W_j| + n-1).$$

By Lemma 23(b) $(b_0, ..., b_{k-2})$ is n-stably increasing (it is a prefix of $(b_0, ..., b_{n-5})$ for which we made the linear preprocessing). Hence by Lemma 23(a) we can find the required j in $\mathcal{O}(\log \log n)$ time.

Moreover if $SUM(k,j) < t < SUM(k,j) + |W_{\Delta(k,j)}|$, then $Perm(\psi,t) = Perm(\beta,t-SUM(k,j))$, where $\beta = son(\psi,j)$ has height $\Delta(k,j)$. Otherwise $Perm(\psi,t) \in perms(\psi)$.

Theorem 25. [Unranking] For a given number t we can compute the t-th permutation in Sawada-Williams generation in $\mathcal{O}(n\frac{\log n}{\log\log n})$.

Proof. From Lemma 22 we either obtain the required permutation (if it is a hub-permutation) or obtain its anchor-seed ϕ and $rank(\phi)$. In the second case we know that $Perm(t) \in bunch(\phi)$ and it equals $Perm(\phi, t - rank(\widetilde{\phi}))$. Now after the linear preprocessing we apply Lemma 24 exhaustively to obtain $route(\psi)$ for a seed ψ such that $Perm(t) \in perms(\psi)$. However we do not know ψ and have to compute it.

Claim 26. If we know $anchor(\psi)$ and $route(\psi)$ then ψ can be computed in $\mathcal{O}(n\frac{\log n}{\log\log n})$ time.

Proof. We can compute the second element a_2 of ψ as $a'_2 \ominus m$ and $dec_seq(\psi)$ as $(a_2, a_2 \ominus 1, ..., a_2 \ominus (n - m - 3))$ where a'_2 is the second element of $anchor(\psi)$, and $m = |route(\psi)| - 1$. Then we use the order:

$$a_2 \prec a_2 \ominus 1 \prec a_2 \ominus 2 \prec \dots \prec a_2 \ominus (n-2).$$

We produce a linked list initialized with $dec_seq(\psi)$. For $i \in \{0, ..., m-1\}$ we want to insert $a_2 \oplus (m+1-i)$ after $ord(\psi_{m-1})$ position from the end of the current list (all the smaller elements are already in the list and we know, that after $a_2 \oplus (m+1-i)$ there are $ord(\psi_{m-1})-1$ such elements). ψ is composed of n and consecutive elements of the final list. The data structure from [2] allows us to achieve that in $\mathcal{O}(n \frac{\log n}{\log \log n})$ time.

Finally we use this claim and Lemma 21 to obtain the required permutation Perm(t).

7 Examples of ranking and unranking

We show on representative examples how the ranking and unranking algorithms are working.

Ranking. When ranking $\pi = (7, 2, 4, 1, 6, 5, 10, 9, 8, 3)$ we first find a permutation $\rho = (10, 9, 8, 3, 7, 2, 4, 1, 6, 5)$ cyclically equivalent to π and then a seed $\psi = (10, 9, 8, 3, 7, 2, 4, 6, 5)$ whose package $perms(\psi)$ contains both π and ρ (in case of two candidates for ψ we choose the parent). We have $rank(\pi) - rank(\widetilde{\psi}) = (rank(\pi) - rank(\rho)) + (rank(\rho) - rank(\widetilde{\psi})) = (|W_1| + 3) + (SUM(2, 3)) = 268$. Next we compute $route(\psi)$ by computing inversion vector of ψ . After that we compute $hub(\psi) = (10, 3, 2, 1, 9, 8, 7, 6, 5)$, and

$$\psi_2 = anchor(\psi) = son(hub(\psi), 3) = (10, 2, 1, 9, 8, 7, 4, 6, 5),$$

$$height(\psi_2) = \Delta(n-3,3) = 5, \ rank(\widetilde{\psi_2}) - rank(\widetilde{\psi_1}) = SUM(5,5) = 83246,$$

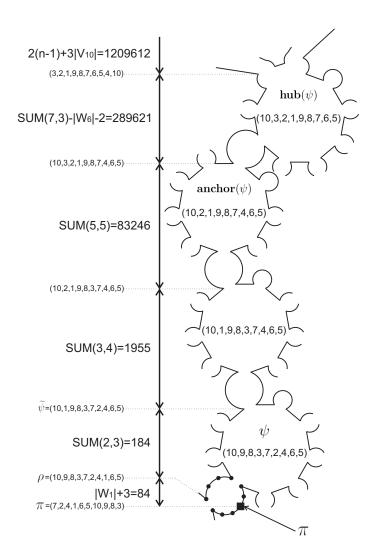


Figure 6: Illustration of ranking and unranking of $\pi = (7, 2, 4, 1, 6, 5, 10, 9, 8, 3)$. We have $\pi \in perms(\psi)$, and $route(\psi) = (3, 5, 4)$.

Now it is enough to compute $rank(\widetilde{\psi}_2)$ (knowing that $\widetilde{\psi}_2$ belongs to the hub), which again is computed in two steps – rank of the first permutation of $perms(hub(\psi))$ (outside of the starting path) is equal to $2n-2+3|V_{10}|=1209612$ and $\widetilde{\psi}_2$ occurs $SUM(7,3)-|W_6|-2=289621$ permutations later. After summing all the values our final output is:

rank(7, 2, 4, 1, 6, 5, 10, 9, 8, 3) = 1584702.

Unranking. Forget now that we already know the permutation with rank 1584702. When looking for a permutation with rank t we first check if Perm(t) is not in the starting path (t > 2n - 2) and then after subtracting 2n - 2 = 18 from t we divide it by $|V_{10}|$, to get $t_1 = 3, t_2 = 375090$. We now know, that the permutation belongs to bunch((10, 3, 2, 1, 9, 8, 7, 6, 5)).

We have

$$SUM(7,3) - |W_6| - 5 \le 375090 < SUM(7,4) - |W_6| - 6,$$

hence we know, that the permutation belongs to the $\sigma_3 W_{\Delta(7,3)} \gamma_5$ part of V_{10} . We decrease the rank by $SUM(7,3) - |W_6| - 2 = 289621$ to get 85469.

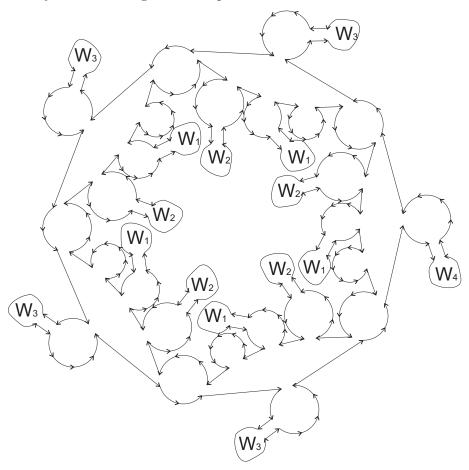
Then we descend down the seed tree by choosing the fifth son, because $SUM(5,5)-5 \le 85469 < SUM(5,6)-6$, with the remaining rank 85469-SUM(5,5)=2223. Next we go to the third son since $SUM(3,4)-4 \le 2223 < SUM(3,5)-5$, with the remaining rank equal 2223-SUM(3,4)=268.

In the next step we know that $SUM(2,3) - 3 \le 268 < SUM(2,4) - 4$, and also that $268 > SUM(2,3) + |W_1|$, hence further descent is not needed.

In this moment we came to an unknown seed ψ for which we know $route(\psi)=(3,5,4)$, and $anchor(\psi)=son((10,3,2,1,9,8,7,6,5),3)=(10,2,1,9,8,7,4,6,5)$. Using Claim 26 we recover $\psi=(10,9,8,3,7,2,4,6,5)$. Now we know that the required permutation is in $perms(\psi)$, and it equals $Perm(\psi,268)$, then we use Lemma 21 to obtain $Perm(\psi,268)=(7,2,4,1,6,5,10,9,8,3)$, and this permutation is our final output.

8 Cyclic $\sigma\tau$ -sequence

A $\sigma\tau$ -sequence of permutations is cyclic if the last permutation is equal to the first one. Sawada and Williams in [5] have given an iterative construction of a cyclic $\sigma\tau$ -sequence. They have shown how to partition the graph of permutations into two edge disjoint cycles (2-cycles cover) C', C'' of respectively *inner* and *outer* cycles. Below we give an example of this structure for n=7.



We define "switches" as permutations of the form

$$(x, n, x \oplus 1, x \oplus 2, ..., x \ominus 1)$$

for $1 \le x < n$. In other words they are cyclic shifts of the identity permutation (1, 2, 3, ..., n-1) in which n is inserted into the second position.

Observation 27. There is a one switch on the inner cycle C' and n-2 switches on the outer cycle. C''

8.1 Sawada Williams construction of Hamiltonian cycle.

The algorithm basically computes the cycles C', C'', then the switches are appropriately ordered as $\Delta_1, \Delta_2, ..., \Delta_{n-1}$. Afterwards the outgoing edges for

the switches are redirected by choosing the outgoing τ -edge (in the 2-cycle cover these were σ -edges). More explicitly we redirect:

for each i > 1 redirect $\Delta_i \overrightarrow{\tau} \sigma(\Delta_{i-1})$, additionally redirect $\Delta_1 \overrightarrow{\tau} \sigma(\Delta_{n-1})$.

8.2 Two cycles construction

Let \otimes denote a modified addition modulo n-2, where $n-1\otimes 1=2$. It gives a cyclic order of elements $\{2,...,n-1\}$, with attached element 1 $(1\otimes 1=2)$. For $a\neq 2$ we write $a\oslash 1=a-1$ and $2\oslash 1=n-1$.

Lemma 28. The outer sequence C'' is represented (after removing one edge) by the linear sequence

$$SEQ'' = (\sigma W_{n-3}\gamma_{n-3}\gamma_2)(\sigma W_{n-4}\gamma_{n-3}\gamma_2)^{n-3},$$

and the inner sequence C' is represented by the linear sequence $SEQ' = U^{n-2}$, where

$$U = \gamma_{n-4} \cdot \prod_{i=3}^{n-3} \sigma^{i} \mathbf{W}_{\Delta(n-3,i)} \gamma_{n-2-i} \cdot \gamma_{n-1};$$

Proof. First we need to prove, that for ψ outside of hub

$$GEN(\widetilde{\psi}, \mathbf{W}_k) = bunch(\psi) \text{ and } \mathbf{W}_k(\widetilde{\psi}) = \psi^{(n-1)},$$

for $bunch(\psi), \widetilde{\psi}, \psi^{(n-1)}$ and $height(\psi)$ defined as before, but after replacing \oplus with \otimes . For $k \geq 1$ the ordering of elements (which element is considered missing in the seed, and thus equal to $p_2 \oplus 1$ or $p_2 \otimes 1$) is in fact inherited from the parent seed (the first k elements after n in the permutation), thus there is no change from the proof of 8. For "k=0" (the cycle of permutations which belong to $perms(\psi)$ for just one ψ) the missing element is inserted just after n, thus there can be no further descent in the tree of seeds and at the same time there is only one permutation in the cycle for which the SW function gives a τ -edge. Every hub seed has one child which is also a hub seed. In the previous construction it was always $son(\psi, 1)$. In this construction it is $son(\psi, 2)$ as hub seeds are those of a form $(n, x, x \otimes 1, ..., x \otimes 2, 1)$. Hence the construction divides the first son from sons 3 to n-3 (and the cycle with W_0). The *outer* cycle covers the first sons of hub seeds, and the *inner* cycle covers the remaining ones. Son of each hub seed have height n-4 with one exception – seed (n, n-3, n-4, ..., 1, n-1)which is the first son of a hub seed (n, n-2, n-3, ..., 1) has height n-3. Hence the outer cycle has the stated representation (additional γ_2 represents transition to the second son – next hub seed). In the inner cycle each child of hub seeds has the same height as in the previous construction $(height(son(\psi, i)) = \Delta(n-3, i))$ and are visited in the same order. After all those children are visited there appears γ_{n-4} which represents the "return to the parent seed" (thus in this cycle hub seeds are visited in the reversed order).

8.3 Alternative path construction

Claim 29. In the path obtained with the new method the sequence representing it is equal to

$$(\sigma W_{n-3}\gamma_{n-3}\gamma_2)(\sigma W_{n-4}\gamma_{n-3}\gamma_2)^{n-4}\sigma W_{n-4}\gamma_{n-3}\sigma^3 U^{n-2}$$

(with the last τ removed) It is equal to the concatenation of representations from Lemma 28 with the ending τ removed in both representations and with σ added between them.

Lemma 30. In the new path we can rank a permutation in $\mathcal{O}(n\sqrt{\log n})$ time and unrank it in $\mathcal{O}(n\frac{\log n}{\log\log n})$.

Proof. Ranking algorithm in the previous construction contained four parts:

- 1. Counting $rank(\pi) rank(\psi)$ for $\pi \in perms(\psi)$ in $\mathcal{O}(n)$.
- 2. Computing $route(\psi)$ for a given ψ in $\mathcal{O}(n\sqrt{\log n})$.
- 3. Counting $rank(\psi) rank(anchor(\psi))$ out of $route(\psi)$ in $\mathcal{O}(n)$.
- 4. Ranking inside the hub in $\mathcal{O}(n)$.

With a few minor changes we can adjust it to the new path construction.

The first part does not really change (it is enough to know what are the missing elements in the seed and its children).

In the second part the only difference is that in the order used in inversion vector problem we must insert element 1 somewhere. As it is never a missing element, we can insert it as a minimal element (or leave the order untouched if $a_2 = 1$). The third part is identical.

The biggest difference occurs in the fourth part – we first need to determine to which cycle from the two cycle construction it belongs to. The permutation belongs to the outer cycle when it belongs to perms of two seeds ψ, ϕ , where $\phi = parent(\psi)$, and either $\psi = son(\phi, 1)$, or $(\psi = son(\psi, 2)$ and it is one of three first permutations in the cycle). In this case we must rank the permutation in relation to $\widetilde{\phi}$ (rank in $perms(\phi) + |W_{n-4}|$ (or $|W_{n-3}|$) unless it is $\widetilde{\phi}$ or $\sigma(\widetilde{\phi})$) and add

$$rank(\widetilde{\phi}) = |W_{n-3}| + n + 3 + (|W_{n-4}| + n - 2) \cdot (n - x - 2)$$

(where $x = mis(\phi)$) if $x \neq n-1$, and 1 if x = n-1.

Otherwise it belongs to the inner cycle. In this case we rank it like in the normal construction (in relation to the first permutation in $perms(\phi)$), then we subtract $|W_{n-4}| + n + 3$ (or $|W_{n-3}| + n + 3$) and add

$$(x-2)|U| + |W_{n-3}| + n + 3 + (n-3) \cdot (|W_{n-4}| + n + 2)$$

Unranking algorithm contained four parts as well:

- 1. Finding appropriate tree of seeds (or returning permutation if it belongs to hub) in $\mathcal{O}(n)$.
- 2. Computing $route(\psi)$ and $rank(\psi)$ in $\mathcal{O}(n \log \log n)$.

- 3. Obtaining ψ out of $route(\psi)$ in $\mathcal{O}(n \frac{\log n}{\log \log n})$.
- 4. Unranking in $perms(\psi)$ in $\mathcal{O}(n)$.

Parts 2,3,4 remain unchanged (the only change is the use of \otimes instead of \oplus). In the first part we first

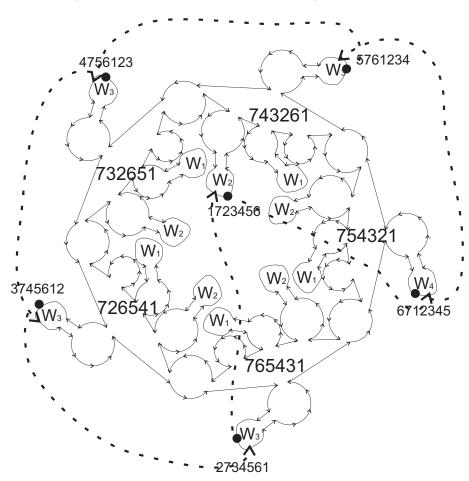
determine whether
$$t < |W_{n-3}| + n + 3 + (n-3) \cdot (|W_{n-4}| + n + 2)$$

if that is the case we check if $t < |W_{n-3}| + n + 3$ (we unrank in the part with W_{n-3}) and if that is not the case we divide $t - |W_{n-3}| - n - 2$ by $|W_{n-4}| + n + 2$ (using integer division) and unrank it in appropriate three of seeds.

If
$$t' = t - (|W_{n-3}| + n + 3 + (n-3) \cdot (|W_{n-4}| + n + 2)) \ge 0$$

we unrank t' in the inner cycle – we divide t' by |U| (using integer division) and proceed as in the previous algorithm.

8.4 Polynomial construction for the cycle



Lemma 31. There exist an SLP for Hamiltonian cycle of size $\mathcal{O}(n^3)$.

Proof. Switch $(x, n, x \oplus 1, ..., x \ominus 1) \in perms(\psi)$ for $\psi = (n, x \oplus 1, x \oplus 3, ..., x \ominus 1, x)$. For $x \neq 1$ $hub(\psi) = (n, x \ominus 1, x \ominus 2, ..., x \ominus 1, 1)$, and $anchor(\psi) = son(hub(\psi), 1)$, thus each W_{n-4} (or W_{n-3}) on the outer cycle contains one such switch. For $\psi = (n, 2, 4, ..., n-1, 1)$ $hub(\psi) = (n, n-3, n-4, ..., 2, n-1, 1)$ and $anchor(\psi) = son(hub(\psi), 3)$, thus the remaining switch belongs to one of the U parts of the inner cycle.

We can divide each such part into two – the one before the switch and the one after it. Each such part can be represented by an SLP of size $\mathcal{O}(n^2)$ as a word W_k can be divided at most once for each k and x (each other does not contain a switch, thus remain undivided).

Lemma 32. We can rank and unrank in the cycle in $\mathcal{O}(n^2 \cdot \sqrt{\log n})$.

Proof. Scheme of algorithm:

We count ranks for all the "switches" in the path in $\mathcal{O}(n^2 \cdot \sqrt{\log n})$. Then we count differences between ranks of two next "switches" (in the order of the path) and ranks of switches in the cycle (iterating through switches in the order of the cycle) in $\mathcal{O}(n)$ total time and space.

Rank:

- 1. Rank permutation in path in $\mathcal{O}(n \cdot \sqrt{\log n})$.
- 2. Find last "switch" with smaller or equal path rank, and count the difference (if rank is smaller then the rank of first switch we count everything modulo n!) in $\mathcal{O}(n)$.
- 3. Add the difference to cycle rank of that "switch".

Unrank:

1. Find last "switch" with smaller or equal cycle rank and count the difference in $\mathcal{O}(n)$.

- 2. Add the difference to path rank of that "switch".
- 3. Unrank in the path with the new value in $\mathcal{O}(n \frac{\log n}{\log \log n})$.

8.5 Efficient construction for cycle

Lemma 33. Routes of (seeds which perms contain) "switches" are always of the form (n-3, n-4, ..., 1) with 0 or 1 element erased (for example (6, 5, 3, 2, 1) for n=9) or are equal to (n-3, n-4, ..., 3). Furthermore we can compute all the ranks of "switches" on the new path in $\mathcal{O}(n)$ total time.

Proof. Each "switch" $\pi = (x, n, x \oplus 1, ..., x \ominus 1)$ belongs to perms of just one seed namely $\pi \in perms(\psi)$ for $\psi = (n, x \oplus 1, x \oplus 3, x \oplus 4, ..., x)$ (height(ψ) = 1).

When going through parent edges until reaching $hub(\psi)$ each time the missing symbol is inserted after n shifting all elements till $x \oplus 1$ by one, and erasing the element just after it (jumping over element 1). Hence each time parent edge is used we erase element closer to right by one with the exception of the one time when 1 appears just after $x \oplus 1$ for the first time. In that case the element

22

erased next is closer to the right by two. Thus each route is built of numbers decreasing by ones (sometimes with one number missing). Each time the height of the tree rises by exactly one.

When $x \neq 1$, then $anchor(\psi) = son(hub(\psi), 1)$, thus the route ends with 1. If x = n - 1 then $n - 1 \oplus 1 = 1$, thus 1 is never jumped over. As we start in a seed with tree of height 1, $route(\psi) = (n - 3, n - 4, ..., 1)$ (this is the switch which appears in the W_{n-3} part).

If 1 < x < n-2, then 1 is jumped over at the x-th step from $hub(\psi)$ resulting in $route(\psi) = (n-3,...,x+1,x-1,...,1)$.

When x = n - 2 the jump over appears inside ψ , and thus does not touch $route(\psi) = (n - 4, ..., 1)$.

When $x = 1 \ route(\psi) = (n - 3, ..., 3)$ (this is the switch from W_{n-5} in the inner cycle).

When $x \neq n-2$ π appears on the n-3-rd place of n-2-nd (last) cycle in $perms(\psi)$. When x=n-2 the jump over appears inside ψ and thus π appears on the n-4-th place of n-3-rd cycle in $perms(\psi)$.

As the heights of trees grow always by 1

$$rank(\psi) - rank(anchor(\psi)) = \sum_{i=0}^{|route(\psi)|-2} SUM(i+2, route(\psi)[i]).$$

For 1 < x < n - 3 these two values for x and x + 1 differs by

$$SUM(n-x-2,x+1) - SUM(n-x-2,x) = |W_{n-x-3}| + n.$$

Hence all the $rank(\psi) - rank(anchor(\psi))$ can be counted in $\mathcal{O}(n)$ total time (we get 4 cases - each counted in $\mathcal{O}(n)$ (see Lemma 18), plus $\mathcal{O}(n)$ time to compute the other ones by adding the differences). Each $anchor(\psi)$ is the first in a part of the sequence (U or $(\sigma W \gamma_{n-3} \gamma_2)$), thus all $rank(anchor(\psi))$ can be counted in $\mathcal{O}(n)$ total time. $rank(\pi) - rank(\psi) = (n-3) \cdot (n+1)$ (or $(n-4) \cdot (n+1)$ for x = n-2).

Theorem 34. $\sigma\tau$ -cycle from [5] has a $\mathcal{O}(n^2)$ SLP representation.

Proof. Each tree of seeds in the outer cycle (and one in the inner) is divided into two parts. Each tree is divided differently, but in each W_k the division happens either in the last W_{k-1} part, or the penultimate one, thus it is enough to divide the definitions of W_k into three parts $W_k = W_k^{(1)}(W_{k-1}\gamma_k\sigma^{n-k-1}W_{k-1})W_k^{(2)}$, and for each tree divide one of the W_{k-1} words.

More precisely for $(2 \le k \le n-3)$ let:

$$\begin{split} W_k^{(1)} &= (\tau \prod_{i=1}^{n-k-4} \sigma^i W_{\Delta(k,i)} \gamma_{n-2-i}) \sigma^{n-k-2} \\ W_k^{(2)} &= \gamma_{k-1} \sigma^{n-k} W_{k-2} W_{k-1}^{(2)} \quad (W_1^{(2)} = \tau) \\ W_k^{(3)} &= W_k^{(1)} W_{k-1} \gamma_k \sigma^{n-k-1} \qquad W_k^{(4)} &= \gamma_k \sigma^{n-k-1} W_{k-1} W_k^{(2)} \quad (= W_{k+1}^{(2)}) \end{split}$$

The division of the tree with $(x, n, x \oplus 1, ..., x \ominus 1)$ for $2 \le x \le n - 3$:

$$\begin{split} V_{n}^{(x,1)} &= W_{n-4}^{(1)} W_{n-5}^{(1)} ... W_{n-x-1}^{(1)} W_{n-x-2}^{(3)} ... W_{2}^{(3)} (\gamma_{n-1})^{n-3} \sigma^{n-3} \\ V_{n}^{(x,2)} &= \gamma_{1} W_{2}^{(2)} ... W_{n-x-2}^{(2)} W_{n-x-1}^{(4)} ... W_{n-4}^{(4)} \\ V_{n}^{(1,1)} &= W_{n-5}^{(3)} ... W_{2}^{(3)} (\gamma_{n-1})^{n-3} \sigma^{n-3} & V_{n}^{(1,2)} &= \gamma_{1} W_{2}^{(2)} ... W_{n-4}^{(2)} \\ V_{n}^{(n-2,1)} &= W_{n-4}^{(1)} ... W_{2}^{(1)} (\gamma_{n-1})^{n-4} \sigma^{n-4} & V_{n}^{(n-2,2)} &= \gamma_{2} \gamma_{n-1} W_{2}^{(4)} ... W_{n-4}^{(4)} \\ V_{n}^{(n-1,1)} &= W_{n-3}^{(3)} ... W_{2}^{(3)} (\gamma_{n-1})^{n-3} \sigma^{n-3} & V_{n}^{(n-1,2)} &= \gamma_{1} W_{2}^{(2)} ... W_{n-3}^{(2)} \\ C &= V_{n}^{(1,2)} U^{n-3} \gamma_{n-4} \sigma_{3} V_{n}^{(1,1)} \tau \prod_{i=0}^{n-3} (V_{n}^{((n-1) \otimes 2i,2)} \gamma_{n-3} \gamma_{2} \sigma^{1} V_{n}^{((n-2) \otimes 2i,1)} \tau) \end{split}$$

Theorem 35. In the cycle we can rank in $\mathcal{O}(n\sqrt{\log n})$ time and unrank in $\mathcal{O}(n\frac{\log n}{\log\log n})$ time.

Proof. We use the algorithm from 32, just count ranks of "switches" in $\mathcal{O}(n)$ total time.

References

- [1] Timothy M. Chan and Mihai Patrascu. Counting inversions, offline orthogonal range counting, and related problems. In Moses Charikar, editor, *Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA 2010, Austin, Texas, USA, January 17-19, 2010, pages 161–173. SIAM, 2010.
- [2] Paul F. Dietz. Optimal algorithms for list indexing and subset rank. In Algorithms and Data Structures, Workshop WADS '89, Ottawa, Canada, August 17-19, 1989, Proceedings, pages 39–46, 1989.
- [3] Frank Ruskey and Aaron Williams. An explicit universal cycle for the (n-1)-permutations of an n-set. ACM Trans. Algorithms, 6(3):45:1-45:12, 2010.
- [4] Wojciech Rytter. Grammar compression, LZ-encodings, and string algorithms with implicit input. In Automata, Languages and Programming: 31st International Colloquium, ICALP 2004, Turku, Finland, July 12-16, 2004. Proceedings, pages 15–27, 2004.
- [5] Joe Sawada and Aaron Williams. Solving the sigma-tau problem. URL: http://socs.uoguelph.ca/~sawada/papers/sigmaTauCycle.pdf.
- [6] Joe Sawada and Aaron Williams. A Hamilton path for the sigma-tau problem. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018, pages 568-575, 2018.
- [7] Joe Sawada, Aaron Williams, and Dennis Wong. A surprisingly simple de Bruijn sequence construction. *Discrete Mathematics*, 339(1):127–131, 2016.