

# On Additive Approximate Submodularity

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## Abstract

A real-valued set function is (*additively*) *approximately submodular* if it satisfies the submodularity conditions with an additive error. Approximate submodularity arises in many settings, especially in machine learning, where the function evaluation might not be exact. In this paper we study how close such approximately submodular functions are to truly submodular functions.

We show that an approximately submodular function defined on a ground set of  $n$  elements is  $O(n^2)$  pointwise-close to a submodular function. This result also provides an algorithmic tool that can be used to adapt existing submodular optimization algorithms to approximately submodular functions. To complement, we show an  $\Omega(\sqrt{n})$  lower bound on the distance to submodularity.

These results stand in contrast to the case of approximate modularity, where the distance to modularity is a constant, and approximate convexity, where the distance to convexity is logarithmic.

## 1 Introduction

The study of submodular functions is classical. A real-valued set function is said to be submodular if it satisfies the so-called “diminishing returns property”, i.e., the incremental value of adding a new element to the set decreases as the set becomes larger. Submodularity surfaces naturally in many situations and has turned out to be a key concept in combinatorial optimization. The well-known greedy algorithm for submodular maximization, owing to its simplicity, has found important applications in many subfields of computer science including approximation algorithms, machine learning, natural language processing, and game theory. Submodularity also has profound relationships to both convex and concave functions. See the books by [Lov83, Fuj05, Sch03] for more details of the connection and applications of these concepts, as well as the monograph by [Bac13] and the references in [KSG08, KC10] for developments in machine learning that rely on submodularity.

In this paper we develop and study the notion of approximate submodularity. Informally, a real-valued set function is approximately submodular if the definition of submodularity is additively relaxed, for example, the incremental value of adding a new element to the set either decreases, or *increases up to a small additive error*, as the set becomes larger.

**Motivation.** There are many reasons to study approximate submodularity. Firstly, various definitions of approximately submodular functions have been investigated in the machine learning community. A powerful notion called *submodularity ratio* [DK11, KCZ<sup>+</sup>14], which captures how much more is the value of adding a large subset compared to the value of adding its individual elements, has been studied to understand why greedy algorithms perform well even with correlations. *Approximate submodularity*, where the submodular inequalities can hold to within an additive error, was considered in [KC10, KSG08] to handle failures and uncertainties in models; while their definition is analogous to ours, importantly, they also require the function to be monotone and non-negative.

The aim of these works is to extend the scope of the standard greedy algorithm and make it more applicable to real-world settings where the function is not exactly submodular but only weakly satisfies the submodularity property. Some of these results can be used to find an approximate maximum of a set function (which is close to submodularity in some sense) only if it is also non-negative and monotonically increasing. It is less clear how to use them to optimize approximately submodular functions in other ways, e.g., minimization, maximization under constraints, etc, when non-negativity and monotonicity are not satisfied.

Other papers dealing with approximate submodularity [HS16, SH18] assume explicitly that the input set function is to within a multiplicative constant of an (exactly) submodular function. While these papers provide useful algorithms for such functions, they do not address the basic question of how far are “approximately submodular” functions to submodularity, the main focus of our work.

Secondly, approximate submodularity is a natural computational and mathematical notion. For example, approximate submodularity has been studied in property testing [PRR03, SV14], along with related properties of monotonicity and modularity. In the continuous setting, approximate convexity has been studied in functional analysis [Cho84], where the stability of a functional equation associated with the convex function has been investigated by many researchers over the last few decades. In contrast, much less is known about the properties of approximate submodularity.

The stronger and simpler notion of approximate modularity, where the function satisfies the set additive equation to an additive approximation, has been studied both in the mathematics and theoretical computer science communities. A classical result of [KR83] showed that an approximately modular function is constant-close to a truly modular function; see [BPR13, FFTC17] for recent improvements to the constant. Progress on approximate modularity might be seen as a first step towards progress on the approximate submodularity problem. However the latter poses substantial technical challenges and it is unclear how much, if any, of the machinery developed for approximate modularity can be adapted to the approximate submodularity case.

**Our contributions.** In this paper we study the concept of approximate submodularity, with the goal of understanding the following question:

*Given a set function that satisfies an approximate submodularity property, how close can it be to a truly submodular function in the  $\ell_\infty$ -sense?*

Let  $n$  be the number of elements in the universe. Recall that a function  $f : 2^{[n]} \rightarrow \mathbb{R}$  is defined to be submodular if it satisfies  $f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$ , for each pairs of sets  $S, T \subseteq [n]$ . We relax those inequalities so that each of them allows a small additive error, and we study upper and lower bounds on the distance of such approximately submodular functions to their closest submodular functions.

Our main positive result is that if a function satisfies submodularity in an approximate manner, then it is no more than  $O(n^2)$ -close to being submodular<sup>1</sup>. This upper bound is constructive in that it is given through an algorithmic filter, which returns the value of a ( $O(n^2)$ -close) submodular function on the generic set, in time  $\text{poly}(n)$ . This filter can then be used as a black-box by generic submodular algorithms. We also conduct some experiments with this filter and discuss the results. As for the lower bound, we show that there is a function that satisfies submodularity approximately but is  $\Omega(\sqrt{n})$ -far from every submodular function.

These findings are interesting for two main reasons. First, they show that approximate submodularity, which is at polynomial distance to submodularity, is very different from approximate modularity, which is at constant distance to modularity [KR83, FFTC17]. More intriguingly, it is also different from approximate convexity since if a function is approximately convex, its distance to convexity is only logarithmic [Cho84]. It is to be noted that, even though submodularity is sometimes viewed as a discrete analog of convexity, when it comes to approximate notions, their properties vastly differ.

Secondly, for approximately submodular functions that do not satisfy the joint conditions of being both monotone and non-negative, our construction, that creates a submodular function that is point-wise close to the given function, gives the first optimization technique with any provable guarantee.

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<sup>1</sup>We will later detail the role of the additive error in the closeness bound.

We also study the distance of approximate submodular functions to submodularity, in the case where the set of constraints that are guaranteed to hold to within an additive error is restricted to, e.g., diminishing returns, and “second-order derivatives” constraints.

**Overview of techniques.** To prove the  $O(n^2)$  upper bound, we give a *filter* that is able to transform the value (at an arbitrary set) of an approximately submodular function into the value of a submodular function at that same set. The function is not created explicitly beforehand (such an approach would take exponential time) but instead is defined incrementally and can be very efficiently computed as queries are presented to the filter. The filter can then be used to seamlessly apply algorithms for submodular functions to approximately submodular functions.

The lower bound is the most technically involved of our results. It is based on three steps. First, we obtain a general lower bound on the distance to submodularity for a class of “block”-functions. Then, we propose an approximately submodular function that maximizes this lower bound on the distance to submodularity. We end up with a lower bound of  $\Omega(\sqrt{n})$  on the distance to submodularity of functions that are approximately submodular.

**Other notions.** We relate our notion to other existing notions of approximate submodularity.

*Marginal gain.* Exact submodularity can equivalently be defined in terms of diminishing marginal gains. In Section 6, we consider the  $\epsilon$ -approximate marginal gain property. We observe that the algorithmic filter in the main body of the paper, and its additive  $O(\epsilon n^2)$  approximation, directly carry over to the case of  $\epsilon$ -approximate marginal gain. We also give a lower bound on the distance of functions satisfying the  $\epsilon$ -approximate marginal gain property.

*Submodularity ratio [DK11].* Submodularity ratio and approximate submodularity are very different, and seem to be unrelated notions. Indeed, consider the following examples.

Let  $f : [2] \rightarrow \mathbb{R}$  be such that  $f(\emptyset) = 0$ ,  $f(\{1\}) = f(\{2\}) = \epsilon^{-2} - \epsilon^{-1}$  and  $f(\{1, 2\}) = 2\epsilon^{-2}$ , for a small  $\epsilon > 0$ . Then, the submodularity ratio of  $f$  is  $1 - \epsilon$ . (Recall that the submodularity ratio of submodular functions is at least 1. Thus, in terms of submodularity ratio,  $f$  is as close to submodularity as a non-submodular function can be.) On the other hand, the distance of  $f$  to submodularity according to our definition (i.e., its additive distance to submodularity) is  $\Omega(1/\epsilon)$ , i.e., it is very large. Conversely, let  $f : [2] \rightarrow \mathbb{R}$  be s.t.  $f(\emptyset) = f(\{1\}) = f(\{2\}) = 0$ ,  $f(\{1, 2\}) = \epsilon$ . Then, the submodularity ratio of  $f$  is 0 (i.e., it is as bad as possible) but  $f$  is at  $\ell_\infty$  distance at most  $\epsilon$  from submodularity (i.e., it is close). Tightening the relationship for special cases looks intriguing.

*Horel–Singer notion [HS16].* If  $f$  satisfies the Horel–Singer property with factor  $(1 + \epsilon)$ , it is easy to show that  $f$  is  $O(\epsilon \max_S |f(S)|)$ -approximately submodular; no better bound is possible, in general.

**Roadmap.** In Section 2, we discuss related work. In Section 3 we introduce the notation. In Section 4, we provide our algorithmic filter, which can also be used to upper bound the distance to submodularity of approximately submodular functions. In Section 5, we prove our lower bound on the distance to submodularity of functions that are approximately submodular. In Section 6 we consider different classes of constraints. In Section 7, we detail the experimental results we obtained through our algorithmic filter.

## 2 Related work

Approximate notions of submodularity have recently been studied in cases where the diversity, or valuation, measures are themselves being estimated from the data, and hence the submodularity constraints hold only approximately. While modeling the effectiveness of greedy feature selection, [DK11] defined an approximate version of submodularity that characterizes the multiplicative factor of incremental gain obtained by adding  $k$  items, versus their union, to a ground set. Additively approximate submodular functions were used by [KSG08, KC10] to model sensor placements. Motivated by the robustness of welfare guarantees, [RTV17] relax the typical assumption of submodularity on valuation functions to one where they are only pointwise multiplicatively close to submodular functions.

Our additive definition is aligned with the notion of an approximately convex function, defined first by [HU52] as part of the study of stability of functional equations. Approximate convex functions are  $O(\log n)$ -

close to convex functions, where the domain is  $\mathbb{R}^n$  [HU52, Cho84]. Unfortunately, the proof techniques for showing closeness to convexity depend heavily on the fact that the function domain itself is either a Banach space or convex. Belloni et al. [BLNR15] studied methods to optimize such functions using random walk-based techniques.

The work of [KR83] showed that an approximately modular function is 44.5-close to a modular function, using inequalities created via a “split and merge” strategy; this constant was improved to 35 by [BPR13]. Recently, these bounds were improved by [FFTC17] to 12.65, using a number of novel ideas including some from [CDDK15]. Unfortunately, the strategies in [KR83, CDDK15, BPR13, FFTC17] depend crucially on the two-sided nature of the approximate modular inequalities (i.e., the fact that one can lower and upper bound  $f(A) + f(B)$  in terms of  $f(A \cup B) + f(A \cap B)$  plus some additive error) and seem hard to be adapted to the submodular setting.

Seshadri and Vondrak [SV14] studied the testability of submodularity property in terms of the number of queries needed to decide whether the function be modified on a small fraction of the inputs to make it submodular; such a definition is different from the multiplicative or additive definitions mentioned above. Goemans et al. [GHIM09] and Balcan and Harvey [BH11] gave sampling based algorithms to learn submodular function to a multiplicative factor. In the approximate modularity case, [CDDK15] and [FFTC17] give randomized and deterministic algorithms respectively for learning the closest modular function via queries.

### 3 Preliminaries

Let  $[n] = \{1, \dots, n\}$ . We will denote subsets of  $[n]$  as  $A, B, S, T, \dots$ . We consider the set of functions  $2^{[n]} \rightarrow \mathbb{R}$ , and denote them by  $f(\cdot), g(\cdot), \dots$ . We begin by defining the *distance* between two set functions.

**Definition 1** (Distance). *The distance between functions  $f, g : 2^{[n]} \rightarrow \mathbb{R}$  is defined as  $\max_{S \subseteq [n]} |f(S) - g(S)|$ .*

Recall that a set function is *submodular* if, for each  $A, B \subseteq [n]$ , it holds that [Sch03]  $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ . Formally, there are three equivalent natural ways of defining submodular functions [Sch03]. Each of them asks for the submodular rule to be satisfied on some class  $\mathcal{C} \subseteq \binom{[n]}{2}$  of pairs of sets:

- (i) The *full* constraints set:  $\mathcal{C}^{\text{full}} = \{\{A, B\} \mid \min(|A|, |B|) \geq |A \cap B| + 1\}$ , e.g.,  $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ , for each  $A \not\subseteq B \not\subseteq A$ .
- (ii) The *diminishing returns* constraints set:  $\mathcal{C}^{\text{dimin}} = \{\{A, B\} \mid \min(|A|, |B|) = |A \cap B| + 1\}$ , e.g.,  $f(A \cup \{c\}) - f(A) \geq f(B \cup \{c\}) - f(B)$  for each  $A \subset B$  and for each  $c \notin B$ .
- (iii) The *cross-second order derivatives* constraints set:  $\mathcal{C}^{\text{cross}} = \{\{A, B\} \mid |A| = |B| = |A \cap B| + 1\}$ , e.g.,  $f(A \cup \{a_1\}) + f(A \cup \{a_2\}) \geq f(A \cup \{a_1, a_2\}) + f(A)$ , for each  $A \subseteq [n]$  and for each  $\{a_1, a_2\} \in \binom{[n] \setminus A}{2}$ .

We observe that the three sets of constraints range from the full set  $\mathcal{C}^{\text{full}}$  of constraints<sup>2</sup> to the smallest set  $\mathcal{C}^{\text{cross}}$  of constraints guaranteeing submodularity.<sup>3</sup>

We introduce the following notion.

**Definition 2** (Approximate  $\mathcal{C}$ -submodularity). *We say that a function  $f : 2^{[n]} \rightarrow \mathbb{R}$  is  $\epsilon$ -approximately  $\mathcal{C}$ -submodular, if it satisfies the constraints*

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) - \epsilon,$$

for each  $\{A, B\} \in \mathcal{C}$ .

<sup>2</sup>All constraints induced by pairs in  $\binom{[n]}{2} \setminus \mathcal{C}^{\text{full}}$  are satisfied by any function and we have thus excluded them from the definition.

<sup>3</sup>One can show that each set of constraints  $\mathcal{C}$  that is not a superset of  $\mathcal{C}^{\text{cross}}$  does not guarantee a finite distance from submodularity *even if* all the submodular constraints induced by the pairs in  $\mathcal{C}$  hold with no error. Thus,  $\mathcal{C}^{\text{cross}}$  is the unique minimal set of constraints guaranteeing a finite distance to submodularity.

The distance of a function  $f$  to submodularity is the distance of  $f$  to its closest submodular function. A function  $f$  is  $\alpha$ -far from submodularity if its distance to submodularity is at least  $\alpha$ , and is  $\alpha$ -close to submodularity otherwise. Note that the ratio of the distance of  $f$  to submodularity and  $\epsilon$  remains invariant under scaling. Hence, by scaling, we can take  $\epsilon = 1$  without loss of generality; when convenient, we refer to a 1-approximately  $\mathcal{C}$ -submodular function as simply *approximately submodular*.

For  $x \in \{\text{full}, \text{dimin}, \text{cross}\}$ , we use  $\epsilon^x = \epsilon^x(f)$  to denote the minimum  $\epsilon \geq 0$  for which the function  $f : 2^{[n]} \rightarrow \mathbb{R}$  is  $\epsilon$ -approximately  $\mathcal{C}^x$ -submodular.

Clearly, for any function  $f$  it holds  $\epsilon^{\text{full}} \geq \epsilon^{\text{dimin}} \geq \epsilon^{\text{cross}}$ , since  $\mathcal{C}^{\text{cross}} \subseteq \mathcal{C}^{\text{dimin}} \subseteq \mathcal{C}^{\text{full}}$ .

## 4 The upper bound

We show that approximately  $\mathcal{C}^{\text{cross}}$ -submodular functions are  $O(n^2)$ -close to submodular functions.

**Theorem 3.** *For any  $\epsilon$ -approximately  $\mathcal{C}^{\text{cross}}$ -submodular function, there is a submodular function at distance at most  $(1/8) \cdot \lfloor n^2/2 \rfloor \cdot \epsilon$ .*

Given an  $\epsilon$ -approximately submodular function  $f : 2^{[n]} \rightarrow \mathbb{R}$ , we define the function  $g_{f,\epsilon} : 2^{[n]} \rightarrow \mathbb{R}$ ,

$$g_{f,\epsilon}(S) = f(S) + \epsilon \cdot \left( \frac{1}{8} \left\lfloor \frac{n^2}{2} \right\rfloor - \frac{1}{2} \left( |S| - \frac{n}{2} \right)^2 \right).$$

We will prove Theorem 3 by showing that  $g_{f,\epsilon}$  is a submodular function at distance at most  $\epsilon \cdot n^2/16$  from  $f$ . We begin by bounding the distance to  $f$ .

**Lemma 4.** *For each  $S \subseteq [n]$ , it holds  $|f(S) - g_{f,\epsilon}(S)| \leq \epsilon \cdot (1/8) \cdot \lfloor n^2/2 \rfloor \leq \epsilon \cdot n^2/16$ .*

*Proof.* We have

$$f(S) - g_{f,\epsilon}(S) \leq \epsilon \left( \frac{1}{2} \left( \frac{n}{2} \right)^2 - \frac{n^2 + [n \text{ is odd}]}{16} \right) = \epsilon \left( \frac{n^2 - [n \text{ is odd}]}{16} \right) = \frac{\epsilon}{8} \left\lfloor \frac{n^2}{2} \right\rfloor,$$

and

$$f(S) - g_{f,\epsilon}(S) \geq \epsilon \left( \frac{1}{2} \left( \frac{[n \text{ is odd}]}{2} \right)^2 - \frac{n^2 + [n \text{ is odd}]}{16} \right) = \epsilon \left( \frac{[n \text{ is odd}] - n^2}{16} \right) = -\frac{\epsilon}{8} \left\lfloor \frac{n^2}{2} \right\rfloor.$$

It follows that  $|f(S) - g_{f,\epsilon}(S)| \leq \epsilon \cdot (1/8) \cdot \lfloor n^2/2 \rfloor \leq \epsilon \cdot n^2/16$ .  $\square$

We next prove that if  $f$  is  $\epsilon$ -approximately  $\mathcal{C}^{\text{cross}}$ -submodular, then the function  $g_{f,\epsilon}$  is submodular.

**Lemma 5.** *If  $f$  is  $\epsilon$ -approximately  $\mathcal{C}^{\text{cross}}$ -submodular, then the function  $g_{f,\epsilon}$  is submodular.*

*Proof.* To prove the submodularity of  $g_{f,\epsilon}$ , it is sufficient to prove that for each pairs of sets  $A, B$ , such that there exists  $c$  satisfying  $c = |A| = |B|$ ,  $|A \cap B| = c - 1$  and  $|A \cup B| = c + 1$ , it holds  $g_{f,\epsilon}(A) + g_{f,\epsilon}(B) \geq g_{f,\epsilon}(A \cap B) + g_{f,\epsilon}(A \cup B)$ . That is, it is sufficient to prove that the  $\mathcal{C}^{\text{cross}}$  constraints — the *second-order derivatives* constraints — are satisfied [Sch03]. By the  $\epsilon$ -approximate  $\mathcal{C}^{\text{cross}}$ -submodularity of  $f$ , we have  $f(A) + f(B) - f(A \cap B) - f(A \cup B) \geq -\epsilon$ . Then, using the definition of  $g_{f,\epsilon}$ , we have

$$\begin{aligned} & g_{f,\epsilon}(A) + g_{f,\epsilon}(B) - g_{f,\epsilon}(A \cap B) - g_{f,\epsilon}(A \cup B) \\ &= f(A) + f(B) - f(A \cap B) - f(A \cup B) - \frac{\epsilon}{2} \cdot \left( \left( c - \frac{n}{2} \right)^2 + \left( c - \frac{n}{2} \right)^2 - \left( c - 1 - \frac{n}{2} \right)^2 - \left( c + 1 - \frac{n}{2} \right)^2 \right) \\ &\geq -\epsilon - \frac{\epsilon}{2} \cdot \left( 2 \left( c - \frac{n}{2} \right)^2 - \left( c - 1 - \frac{n}{2} \right)^2 - \left( c + 1 - \frac{n}{2} \right)^2 \right) \\ &= -\epsilon - \frac{\epsilon}{2} \cdot \left( 2c^2 - (c - 1)^2 - (c + 1)^2 - n(2c - (c + 1) - (c - 1)) \right) = -\epsilon + \epsilon = 0. \end{aligned}$$

Thus, the function  $g_{f,\epsilon}$  is submodular.  $\square$

Notice that through the use of the *filter*  $g_{f,\epsilon}$ , we can adapt any algorithm operating on submodular functions to work with the function  $f$ : whenever the algorithm asks for the value associated to the generic set  $S$ , we query  $f(S)$  and return  $g_{f,\epsilon}(S)$  according to its definition.

For instance, we could obtain in polynomial time an  $(\epsilon \cdot n^2/16)$ -additive approximation of the minimum value of an  $\epsilon$ -approximately submodular function  $f$ , by running any of the classical algorithms [Cun85, GLS81] for submodular minimization on the filter  $g_{f,\epsilon}$ . Or, assuming that  $f$  is uniformly larger than  $\epsilon \cdot n^2/4$  (i.e.,  $f(S) \geq \epsilon \cdot n^2/4, \forall S$ ), one can find the set  $S$  whose value  $O(1)$ -multiplicatively-approximates the maximum value of  $f$ , by using, e.g., [BFNS15] on  $g_{f,\epsilon}$ .

We point out that [IB12, Lemma 3.1, Lemma 3.2] can be used to show an  $O(n^2 \cdot \epsilon^{\text{dimin}})$  upper bound on the distance to submodularity. Since  $\epsilon^{\text{cross}} \leq \epsilon^{\text{dimin}}$ , our upper bound of  $O(n^2 \cdot \epsilon^{\text{cross}})$  is never worse. In fact, there are simple functions for which  $\epsilon^{\text{cross}} \leq \frac{\epsilon^{\text{dimin}}}{n-1}$ ; for these functions, our bound is stronger by a factor  $\Theta(n)$ . Take, for instance,  $f(A) = |A|^2$ . The function  $f$  has  $\epsilon^{\text{cross}} = 2$ ; indeed, the generic cross-derivative constraint reduces to  $2(|A|+1)^2 \geq (|A|+2)^2 + |A|^2 - \epsilon^{\text{cross}}$ , so that  $\epsilon^{\text{cross}} = 2 = (|A|+2)^2 + |A|^2 - 2(|A|+1)^2$ . On the other hand, for the same  $f$ ,  $\epsilon^{\text{dimin}} \geq 2n-2$ ; indeed, the diminishing returns constraint for  $A = \emptyset, B = [n-1]$  and  $c = n$ , reduces to  $1 + (n-1)^2 \geq n^2 + 0 - \epsilon'$ , entailing  $\epsilon' \geq 2n-2$ .

## 5 A lower bound

In this section we present a lower bound on the distance to submodularity of approximately  $\mathcal{C}^{\text{full}}$ -submodular. By  $\mathcal{C}^{\text{cross}} \subseteq \mathcal{C}^{\text{dimin}} \subseteq \mathcal{C}^{\text{full}}$ , our lower bound directly carries over to the other two sets of constraints (in Section 6, we present stronger lower bounds for these other sets of approximate submodularity constraints.) Without loss of generality, we assume  $\epsilon = 1$ .

The main technical novelty here is a lower bound on the distance to submodularity of a special class of functions that are based on a partition of the underlying set of items.

The first construction, which mostly serves illustrative purposes, is easy to state. Consider a partition  $[n] = S_1 \cup \dots \cup S_{\sqrt{n}}$ , where, say,  $|S_i| = \sqrt{n}$  and the function  $f(S) = \log_2(\max_{i \in [\sqrt{n}]} |S \cap S_i|)$ , for  $|S| \geq 1$  and  $f(\emptyset) = 0$ . It is not hard to show that the function  $f$  is approximately submodular — our main technical lemma entails that this  $f$  is  $\Omega(\log n)$ -far from any submodular function.

Our strongest construction is based on a generalization of the proof that the above function is  $\Omega(\log n)$ -far from any submodular function. Here is a brief overview. First, we show a general statement (Lemma 7) that can be used to lower bound the distance to submodularity of an arbitrary “block” function, i.e., a function whose value on a set  $S$  depends on the sizes of the intersections of  $S$  with the parts of a partition of the ground set  $[n]$ . Next, we propose a specific partition into  $k$  parts, and a specific function  $f_k$  (Definition 9) that (i) is approximately submodular (Lemma 10), and that (ii) pushes the lower bound on the distance given by Lemma 7 to  $\sqrt{n/8} - O(1)$  (Lemma 11).

Before delving into the proofs, we give a brief intuition on the function  $f_k$ . This function is chosen to be very far from submodularity while still being approximately submodular. Loosely speaking,  $f_k$ , on part of its domain, is the sum of two functions that are very far from submodularity: if  $S$  is the input set, then the first function is the maximum of the intersection sizes between  $S$  and the blocks<sup>4</sup> and the second function is the number of empty intersections of  $S$  and the blocks<sup>5</sup>. (Let us also mention the following interpretation of the two functions. Suppose that, for an arbitrary set  $S$ , we let the vector  $x_S \in \mathbb{R}^k$  contain in its  $i$ th position the size of the intersection of  $S$  and the  $i$ th block. Then, the first function evaluated at  $S$  is equal to the  $\ell_\infty$ -norm of  $x_S$ , while the second function at  $S$  is equal to the opposite of the  $\ell_0$ -norm of  $x_S$ , plus a fixed term.)

<sup>4</sup>This function can be easily shown to be far from submodularity on the full domain. Pick two blocks of sizes  $c$  and  $d$  where  $2 \leq c \leq d$ . Let the set  $S$  contain half of the first block and half of the second, and let the set  $T$  contain the same half of the first block and the other half of the second. Then, the max-sizes function can be easily shown to be off by  $c/2$  on the  $\{S, T\}$  submodular constraint.

<sup>5</sup>For this function, suppose that there are  $k$  blocks of size at least 2 each plus possibly some other blocks of size 1. Let  $S$  be composed of one element from each of the  $k$  blocks of size at least 2. Let  $T$  have the same property but let the two sets satisfy  $S \cap T = \emptyset$ . It is easy to observe that the empty-intersections function is off by  $k$  on the  $\{S, T\}$  submodular constraint.

While these two functions are very far from submodularity on their full domain, they are approximately submodular on a restricted domain: we define the backbone class to contain a set if and only if it intersects each block, bar at most one exception, in at most one element. The two functions above become approximately submodular on the generic pair  $\{S, T\}$  if  $S \cup T$  belongs to the backbone.

The function  $f_k$  that we will use to prove the lower bound is (essentially) equal to the average of the two functions mentioned above, whenever the input set is part of the backbone; for other sets  $S$ ,  $f_k(S)$  will be chosen so to make it easy to prove that each approximately submodular constraint involving  $S$  is satisfied. We show that (i)  $f_k$  is approximately submodular everywhere and (ii) the constant submodularity error introduced in the backbone pushes  $f_k$  to an  $\Omega(\sqrt{n})$  distance to submodularity.

We begin the technical part of this section by giving a formal definition of block functions:

**Definition 6** ( $(t_1, \dots, t_k)$ -block functions). *A function  $f : 2^{[n]} \rightarrow \mathbb{R}$  is a  $(t_1, \dots, t_k)$ -block function if there exists a partition of  $[n]$  into blocks  $S_1, \dots, S_k$ , with  $|S_i| = t_i$  for  $i \in [k]$ , and there exists a function  $F : \{0, 1, \dots, t_1\} \times \dots \times \{0, 1, \dots, t_k\} \rightarrow \mathbb{R}$  such that, for each  $S \subseteq [n]$ , it holds that*

$$f(S) = F(|S \cap S_1|, \dots, |S \cap S_k|).$$

We say that  $F$  is the cardinality representation of  $f$ .

We introduce a shortcut to represent the value of a cardinality representation: we will use  $F(\overbrace{1, \dots, 1}^{i-1}, t_i, \overbrace{0, \dots, 0}^{k-i})$ , to denote the value of  $F$  with an input sequence composed of  $i-1$  leading 1's,  $k-i$  trailing 0's, and a single  $t_i$  in between; we will also use  $F(\overbrace{1, \dots, 1}^k)$  and  $F(\overbrace{0, \dots, 0}^k)$ , to denote the values of  $F$  with, respectively, an input composed of  $k$  distinct 1's, and an input composed of  $k$  distinct 0's.

**Lemma 7.** *Let  $f : 2^{[\sum_{i=1}^k t_i]} \rightarrow \mathbb{R}$  be a  $(t_1, \dots, t_k)$ -block function, and let  $F$  be its cardinality representation. Then, for each submodular function  $g : 2^{[\sum_{i=1}^k t_i]} \rightarrow \mathbb{R}$ , there exists  $S \subseteq [\sum_{i=1}^k t_i]$  such that  $|f(S) - g(S)| \geq \nu(f)$ , where*

$$\nu(f) = \frac{\prod_{j=1}^k \frac{t_j-1}{t_j}}{2} \cdot F(\overbrace{0, \dots, 0}^k) + \sum_{i=1}^k \left( \frac{\prod_{j=i+1}^k \frac{t_j-1}{t_j}}{2 \cdot t_i} \cdot F(\overbrace{1, \dots, 1}^{i-1}, t_i, \overbrace{0, \dots, 0}^{k-i}) \right) - \frac{1}{2} \cdot F(\overbrace{1, \dots, 1}^k).$$

*Proof.* Given any submodular function  $g : 2^{[n]} \rightarrow \mathbb{R}$ , let  $X = X(g) = \max_{S \subseteq [n]} |f(S) - g(S)|$ . Therefore, for each  $S \subseteq [n]$ , we have  $-X \leq f(S) - g(S) \leq X$ . We will prove that  $X \geq \nu(f)$ ; the main claim will then follow.

Suppose that the blocks of the  $(t_1, \dots, t_k)$ -block function  $f$  are  $S_1, \dots, S_k$ , with  $|S_i| = t_i$  for  $i \in [k]$ . Define the following sums for  $s = 1, \dots, k$ :

$$\sigma_s = \sum_{\underbrace{i_1 \in S_1 \ i_2 \in S_2 \ \dots \ i_s \in S_s}_s} g(\{i_1, i_2, \dots, i_s\}),$$

and let  $\sigma_0 = g(\emptyset)$ . Observe that

$$\begin{aligned} \sigma_k &\leq \sum_{i_1 \in S_1} \sum_{i_2 \in S_2} \dots \sum_{i_k \in S_k} (f(\{i_1, i_2, \dots, i_k\}) + X) \\ &= \sum_{i_1 \in S_1} \sum_{i_2 \in S_2} \dots \sum_{i_k \in S_k} \left( F(\overbrace{1, \dots, 1}^k) + X \right) = \left( \prod_{j=1}^k t_j \right) \cdot \left( F(\overbrace{1, \dots, 1}^k) + X \right), \end{aligned}$$

where the inequality follows from the definition of  $X$ , and the last equation follows from  $|S_i| = t_i$  for  $i \in [k]$ .

For each  $s \geq 1$ ,  $\sigma_s$  satisfies:

$$\begin{aligned}
\sigma_s &= \sum_{i_1 \in S_1} \sum_{i_2 \in S_2} \cdots \sum_{i_{s-1} \in S_{s-1}} \left( \sum_{i_s \in S_s} g(\{i_1, i_2, \dots, i_{s-1}\} \cup \{i_s\}) \right) \\
&\geq \sum_{i_1 \in S_1} \sum_{i_2 \in S_2} \cdots \sum_{i_{s-1} \in S_{s-1}} (g(\{i_1, i_2, \dots, i_{s-1}\} \cup S_s) + (t_s - 1) \cdot g(\{i_1, i_2, \dots, i_{s-1}\})) \\
&\geq \sum_{i_1 \in S_1} \sum_{i_2 \in S_2} \cdots \sum_{i_{s-1} \in S_{s-1}} (f(\{i_1, i_2, \dots, i_{s-1}\} \cup S_s) - X + (t_s - 1) \cdot g(\{i_1, i_2, \dots, i_{s-1}\})) \\
&= \sum_{i_1 \in S_1} \sum_{i_2 \in S_2} \cdots \sum_{i_{s-1} \in S_{s-1}} \left( F(\overbrace{1, \dots, 1}^{s-1}, \overbrace{0, \dots, 0}^{k-s}, t_s) - X + (t_s - 1) \cdot g(\{i_1, i_2, \dots, i_{s-1}\}) \right) \\
&= \left( \prod_{j=1}^{s-1} t_j \right) \cdot \left( F(\overbrace{1, \dots, 1}^{s-1}, \overbrace{0, \dots, 0}^{k-s}, t_s) - X \right) + (t_s - 1) \sum_{i_1 \in S_1} \sum_{i_2 \in S_2} \cdots \sum_{i_{s-1} \in S_{s-1}} g(\{i_1, i_2, \dots, i_{s-1}\}) \\
&= \left( \prod_{j=1}^{s-1} t_j \right) \cdot \left( F(\overbrace{1, \dots, 1}^{s-1}, \overbrace{0, \dots, 0}^{k-s}, t_s) - X \right) + (t_s - 1) \sigma_{s-1}.
\end{aligned}$$

By expanding this sum we get:

$$\begin{aligned}
\sigma_k &\geq \sum_{i=0}^{k-1} \left( \left( \prod_{j=1}^i t_j \right) \left( \prod_{j=i+2}^k (t_j - 1) \right) \cdot \left( F(\overbrace{1, \dots, 1}^i, t_{i+1}, \overbrace{0, \dots, 0}^{k-1-i}) - X \right) \right) + \left( \prod_{j=1}^k (t_j - 1) \right) g(\emptyset) \\
&\geq \sum_{i=0}^{k-1} \left( \left( \prod_{j=1}^i t_j \right) \left( \prod_{j=i+2}^k (t_j - 1) \right) \cdot \left( F(\overbrace{1, \dots, 1}^i, t_{i+1}, \overbrace{0, \dots, 0}^{k-1-i}) - X \right) \right) \\
&\quad + \left( \prod_{j=1}^k (t_j - 1) \right) \left( F(\overbrace{0, \dots, 0}^k) - X \right).
\end{aligned}$$

By chaining this lower bound on  $\sigma_k$  with the previously proven upper bound on  $\sigma_k$ , we get:

$$\begin{aligned}
\left( \prod_{j=1}^k t_j \right) \left( F(\overbrace{1, \dots, 1}^k) + X \right) &\geq \sum_{i=0}^{k-1} \left( \left( \prod_{j=1}^i t_j \right) \left( \prod_{j=i+2}^k (t_j - 1) \right) \left( F(\overbrace{1, \dots, 1}^i, t_{i+1}, \overbrace{0, \dots, 0}^{k-1-i}) - X \right) \right) + \\
&\quad \left( \prod_{j=1}^k (t_j - 1) \right) \left( F(\overbrace{0, \dots, 0}^k) - X \right) \\
F(\overbrace{1, \dots, 1}^k) + X &\geq \sum_{i=0}^{k-1} \left( \left( \frac{1}{t_{i+1}} \cdot \prod_{j=i+2}^k \left( 1 - \frac{1}{t_j} \right) \right) \cdot \left( F(\overbrace{1, \dots, 1}^i, t_{i+1}, \overbrace{0, \dots, 0}^{k-1-i}) - X \right) \right) + \\
&\quad \left( \prod_{j=1}^k \left( 1 - \frac{1}{t_j} \right) \right) \left( F(\overbrace{0, \dots, 0}^k) - X \right).
\end{aligned}$$

We move the  $X$  terms to the LHS, and the  $F(1, \dots, 1)$  term to the RHS, to get:

$$\begin{aligned} & \left( 1 + \prod_{j=1}^k \left( 1 - \frac{1}{t_j} \right) + \sum_{i=0}^{k-1} \left( \frac{1}{t_{i+1}} \cdot \prod_{j=i+2}^k \left( 1 - \frac{1}{t_j} \right) \right) \right) \cdot X \geq \\ & \sum_{i=0}^{k-1} \left( \left( \frac{1}{t_{i+1}} \cdot \prod_{j=i+2}^k \left( 1 - \frac{1}{t_j} \right) \right) \cdot F(\overbrace{1, \dots, 1}^i, t_{i+1}, \overbrace{0, \dots, 0}^{k-1-i}) \right) \\ & + \left( \prod_{j=1}^k \left( 1 - \frac{1}{t_j} \right) \right) F(\overbrace{0, \dots, 0}^k) - F(\overbrace{1, \dots, 1}^k). \end{aligned}$$

Now, if we define  $p_j = 1 - \frac{1}{t_j}$ , we have  $0 \leq p_j \leq 1$ . Suppose that we flip the mutually independent coins  $C_k, C_{k-1}, \dots, C_1$  in order, stopping when we either get tails, or right after having flipped coin  $C_1$ . Let  $p_j$  be the heads probability of coin  $C_j$ . Then, the probability of stopping right after having flipped the coin of index  $i$ ,  $2 \leq i \leq k$ , is equal to  $(1 - p_i) \cdot \prod_{j=i+1}^k p_j$ . The probability of stopping after having flipped the coin of index 1 is, instead,  $(1 - p_1) \cdot \prod_{j=2}^k p_j + \prod_{j=1}^k p_j$ . Since these events partition the probability space, we have

$$\prod_{j=1}^k p_j + \sum_{i=1}^k \left( (1 - p_i) \cdot \prod_{j=i+1}^k p_j \right) = 1.$$

Going back to our inequality, we then have

$$\prod_{j=1}^k \left( 1 - \frac{1}{t_j} \right) + \sum_{i=0}^{k-1} \left( \frac{1}{t_{i+1}} \cdot \prod_{j=i+2}^k \left( 1 - \frac{1}{t_j} \right) \right) = 1.$$

The inequality then reduces to

$$\begin{aligned} 2X & \geq \sum_{i=0}^{k-1} \left( \left( \frac{1}{t_{i+1}} \cdot \prod_{j=i+2}^k \left( 1 - \frac{1}{t_j} \right) \right) \cdot F(\overbrace{1, \dots, 1}^i, t_{i+1}, \overbrace{0, \dots, 0}^{k-1-i}) \right) \\ & + \left( \prod_{j=1}^k \left( 1 - \frac{1}{t_j} \right) \right) F(\overbrace{0, \dots, 0}^k) - F(\overbrace{1, \dots, 1}^k), \end{aligned}$$

and the proof is concluded.  $\square$

We observe that Lemma 7 directly gives a lower bound of  $\Omega(\log n)$  on the distance to submodularity of the  $(\sqrt{n}, \dots, \sqrt{n})$ -block function  $f(S) = \log_2(\max_{i \in [\sqrt{n}]} |S \cap S_i|)$  for  $|S| \geq 1$ , and  $f(\emptyset) = 0$ .

Our main approximately  $\mathcal{C}^{\text{full}}$ -submodular block function  $f_k$ , instead, will be a function over  $k$  blocks of sizes  $2, 3, \dots, k+1$ . We will prove that its distance to submodularity is at least  $\nu(f_k) = \frac{k}{4}$ . As our next step, we consider a special case of Lemma 7 dealing with  $(2, 3, \dots, k+1)$ -block functions.

**Corollary 8.** *Let  $f : 2^{[n_k]} \rightarrow \mathbb{R}$  with  $n_k = \frac{(k+3)k}{2}$  be a  $(2, 3, \dots, k+1)$ -block function and let  $F$  be its cardinality representation. Then, for each submodular function  $g : 2^{[n_k]} \rightarrow \mathbb{R}$ , there exists  $S \subseteq [n_k]$  such that  $|f(S) - g(S)| \geq \nu(f)$ , where*

$$\nu(f) = \frac{F(\overbrace{0, \dots, 0}^k) + \sum_{i=1}^k F(\overbrace{1, \dots, 1}^{i-1}, i+1, \overbrace{0, \dots, 0}^{k-i})}{2 \cdot (k+1)} - \frac{F(\overbrace{1, \dots, 1}^k)}{2}.$$

*Proof.* We apply Lemma 7, with  $t_i = i + 1$ , for  $i = 1, \dots, k$ , to get that

$$\begin{aligned} 2\nu(f) &= \sum_{i=1}^k \left( \frac{\prod_{j=i+1}^k \left(1 - \frac{1}{t_j}\right)}{t_i} \cdot F(\overbrace{1, \dots, 1}^{i-1}, t_i, \overbrace{0, \dots, 0}^{k-i}) \right) + \left( \prod_{j=1}^k \left(1 - \frac{1}{t_j}\right) \right) F(\overbrace{0, \dots, 0}^k) - F(\overbrace{1, \dots, 1}^k) \\ &= \sum_{i=1}^k \left( \frac{\prod_{j=i+1}^k \frac{t_j-1}{t_j}}{t_i} \cdot F(\overbrace{1, \dots, 1}^{i-1}, t_i, \overbrace{0, \dots, 0}^{k-i}) \right) + \left( \prod_{j=1}^k \frac{t_j-1}{t_j} \right) F(\overbrace{0, \dots, 0}^k) - F(\overbrace{1, \dots, 1}^k). \end{aligned}$$

We now substitute the values of  $t_i$  and  $t_j$ , so to get:

$$\begin{aligned} 2\nu(f) &= \sum_{i=1}^k \left( \frac{\prod_{j=i+1}^k \frac{j}{j+1}}{i+1} \cdot F(\overbrace{1, \dots, 1}^{i-1}, i+1, \overbrace{0, \dots, 0}^{k-i}) \right) + \left( \prod_{j=1}^k \frac{j}{j+1} \right) F(\overbrace{0, \dots, 0}^k) - F(\overbrace{1, \dots, 1}^k) \\ &= \sum_{i=1}^k \left( \frac{\frac{i+1}{k+1}}{i+1} \cdot F(\overbrace{1, \dots, 1}^{i-1}, i+1, \overbrace{0, \dots, 0}^{k-i}) \right) + \frac{1}{k+1} \cdot F(\overbrace{0, \dots, 0}^k) - F(\overbrace{1, \dots, 1}^k). \end{aligned}$$

Thus, finally,

$$2\nu(f) = \frac{\sum_{i=1}^k F(\overbrace{1, \dots, 1}^{i-1}, i+1, \overbrace{0, \dots, 0}^{k-i}) + F(\overbrace{0, \dots, 0}^k)}{k+1} - F(\overbrace{1, \dots, 1}^k). \quad \square$$

We now describe our approximately submodular block function  $f_k$ .

**Definition 9.** Let  $k \geq 1$  be an integer and let  $n_k = \frac{(k+3)k}{2}$ . Consider a partition of  $[n_k]$  into  $k$  pairwise disjoint blocks  $S_1, \dots, S_k$ , satisfying  $|S_i| = t_i = i + 1$  for  $i = 1, \dots, k$ . Let the backbone  $\mathcal{B}$  of the partition  $S_1, \dots, S_k$  be the class of sets

$$\mathcal{B} = \{S \mid S \subseteq [n_k] \text{ and at most one } i \in [k] \text{ satisfies } |S \cap S_i| \geq 2\}.$$

Given  $S \subseteq [n_k]$ , let  $Z_S = \sum_{i=1}^k [S \cap S_i = \emptyset]$ , and let  $M_S = \max_{i=1}^k |S \cap S_i|$ . The function  $f_k : 2^{[n_k]} \rightarrow \mathbb{R}$  is defined to be:

$$f_k(S) = \begin{cases} \frac{M_S + Z_S - [S \neq \emptyset]}{2} & \text{if } S \in \mathcal{B}, \\ -(k+2)^{|S|} & \text{otherwise.} \end{cases}$$

Observe that  $f_k(S)$  is a function of the sequence  $(|S \cap S_1|, |S \cap S_2|, \dots, |S \cap S_k|)$ : it follows that  $f_k$  is a  $(2, 3, \dots, k+1)$ -block function. The non-negative part of  $f_k$  (i.e.,  $f_k$  restricted to its backbone  $\mathcal{B}$ ) can be interpreted as the “structure” of the function. We will apply Corollary 8 to that part in order to prove the lower bound on the distance of  $f_k$  to submodularity. The negative part of  $f_k$  is irrelevant for the lower bound; it was chosen in order to simplify the proof of  $f_k$ 's approximate submodularity.

**Lemma 10.** The function  $f_k$  is approximately  $\mathcal{C}^{\text{full}}$ -submodular.

*Proof.* We aim to prove that, for each  $\{A, B\} \subseteq [n]$ , such that  $A \not\subseteq B$  and  $B \not\subseteq A$ , it holds  $f_k(A) + f_k(B) \geq f_k(A \cup B) + f_k(A \cap B) - 1$ . Observe that, then,  $A, B \neq \emptyset$  and  $A \cup B \neq \emptyset$ . We consider two cases.

First, we assume that  $A \cup B \in \mathcal{B}$ . Then, since for each  $S \in \mathcal{B}$  and for each  $T \subseteq S$  we have  $T \in \mathcal{B}$ , it must be that  $A, B, A \cap B \in \mathcal{B}$ , as well. We now have,

$$2 \cdot (f_k(A) + f_k(B)) = M_A + M_B + Z_A + Z_B - [A \neq \emptyset] - [B \neq \emptyset] = M_A + M_B + Z_A + Z_B - 2.$$

Moreover,

$$2 \cdot f_k(A \cup B) = M_{A \cup B} + Z_{A \cup B} - [A \cup B \neq \emptyset] = M_{A \cup B} + Z_{A \cup B} - 1,$$

and

$$2 \cdot f_k(A \cap B) = M_{A \cap B} + Z_{A \cap B} - [A \cap B \neq \emptyset].$$

Thus,

$$2 \cdot (f_k(A \cup B) + f_k(A \cap B)) = M_{A \cup B} + M_{A \cap B} + Z_{A \cup B} + Z_{A \cap B} - 1 - [A \cap B \neq \emptyset].$$

Proving that  $f_k(A) + f_k(B) \geq f_k(A \cup B) + f_k(A \cap B) - 1$  is equivalent to proving that  $2 \cdot (f_k(A) + f_k(B)) \geq 2 \cdot (f_k(A \cup B) + f_k(A \cap B)) - 2$  which, after substituting the values of  $2 \cdot (f_k(A) + f_k(B))$  and  $2 \cdot (f_k(A \cup B) + f_k(A \cap B))$ , becomes:

$$(M_A + M_B + Z_A + Z_B - 2) \geq (M_{A \cup B} + M_{A \cap B} + Z_{A \cup B} + Z_{A \cap B} - 1 - [A \cap B \neq \emptyset]) - 2,$$

or, equivalently,

$$(M_A + M_B - M_{A \cup B} - M_{A \cap B} + [A \cap B \neq \emptyset]) + (Z_A + Z_B - Z_{A \cup B} - Z_{A \cap B} + 1) \geq 0.$$

We will prove the latter inequality by proving the two inequalities  $Z_A + Z_B \geq Z_{A \cup B} + Z_{A \cap B} - 1$  and  $M_A + M_B \geq M_{A \cup B} + M_{A \cap B} - [A \cap B \neq \emptyset]$ :

- First, we consider the  $Z$ 's inequality. We will show that there exists at most one block  $S_j$  where the equation

$$[A \cap S_j = \emptyset] + [B \cap S_j = \emptyset] = [(A \cup B) \cap S_j = \emptyset] + [(A \cap B) \cap S_j = \emptyset]$$

does not hold and, in that block, the inequality

$$[A \cap S_j = \emptyset] + [B \cap S_j = \emptyset] \geq [(A \cup B) \cap S_j = \emptyset] + [(A \cap B) \cap S_j = \emptyset] - 1$$

must hold. These two observations imply that  $Z_A + Z_B \geq Z_{A \cup B} + Z_{A \cap B} - 1$ .

Consider any block  $S_j$ . We first prove that if  $|(A \cup B) \cap S_j| \leq 1$ , then the equation  $[A \cap S_j = \emptyset] + [B \cap S_j = \emptyset] = [(A \cup B) \cap S_j = \emptyset] + [(A \cap B) \cap S_j = \emptyset]$  holds. If  $|(A \cup B) \cap S_j| = 0$ , then the equation holds trivially. If  $|(A \cup B) \cap S_j| = 1$ , then either (i)  $|(A \cap B) \cap S_j| = 1$ , in which case  $A \cap S_j = B \cap S_j = (A \cup B) \cap S_j = (A \cap B) \cap S_j$ , so the equality holds or (ii)  $|(A \cap B) \cap S_j| = 0$ , in which case  $\{|A \cap S_j|, |B \cap S_j|\} = \{0, 1\} = \{|(A \cup B) \cap S_j|, |(A \cap B) \cap S_j|\}$ , so the equation holds yet again.

Second, we consider the case where  $|(A \cup B) \cap S_j| \geq 2$ . Observe that by  $A \cup B \in \mathcal{B}$ , there can exist at most one  $j \in [k]$  for which the latter inequality holds. For such a  $j$ , we have

$$\begin{aligned} [A \cap S_j = \emptyset] + [B \cap S_j = \emptyset] &\geq 0 \geq [(A \cap B) \cap S_j = \emptyset] - 1 \\ &= [(A \cup B) \cap S_j = \emptyset] + [(A \cap B) \cap S_j = \emptyset] - 1. \end{aligned}$$

- Second, we consider the  $M$ 's inequality. Let  $u$  be such that  $|(A \cup B) \cap S_u| = M_{A \cup B}$  and let  $i$  be such that  $|(A \cap B) \cap S_i| = M_{A \cap B}$ . We have that  $M_{A \cup B} = |A \cap S_u| + |B \cap S_u| - |(A \cap B) \cap S_u| \leq M_A + M_B - |(A \cap B) \cap S_u|$ .

Now, if  $M_{A \cap B} \geq 2$ , it must hold  $i = u$  by  $A \cup B \in \mathcal{B}$ . Thus, if  $M_{A \cap B} \geq 2$ , we have

$$M_A + M_B \geq M_{A \cup B} + |(A \cap B) \cap S_u| = M_{A \cup B} + |(A \cap B) \cap S_i| = M_{A \cup B} + M_{A \cap B}.$$

Otherwise,  $M_{A \cap B} \leq 1$ . In this case, we have  $M_{A \cap B} = [A \cap B \neq \emptyset]$ . Thus,

$$M_A + M_B \geq M_{A \cup B} + |(A \cap B) \cap S_u| \geq M_{A \cup B} = M_{A \cup B} + M_{A \cap B} - [A \cap B \neq \emptyset].$$

Therefore, the claim has been proved for each  $A, B$  such that  $A \cup B \in \mathcal{B}$ .

We next consider the case  $A \cup B \notin \mathcal{B}$ . Observe that, in general,  $-(k+2)^{|S|} \leq f_k(S) \leq k$ . Thus,  $f_k(A) + f_k(B) \geq -2 \cdot (k+2)^{\max(|A|, |B|)}$ . Moreover,  $f_k(A \cap B) \leq k$  and  $f_k(A \cup B) = -(k+2)^{|A \cup B|} \leq -(k+2)^{\max(|A|, |B|)+1}$ . Then,

$$\begin{aligned}
f_k(A) + f_k(B) &\geq -2 \cdot (k+2)^{\max(|A|, |B|)} \\
&= -(k+2)^{\max(|A|, |B|)+1} + \left( (k+2)^{\max(|A|, |B|)+1} - 2 \cdot (k+2)^{\max(|A|, |B|)} \right) \\
&\geq f_k(A \cup B) + \left( (k+2)^{\max(|A|, |B|)+1} - 2 \cdot (k+2)^{\max(|A|, |B|)} \right) \\
&= f_k(A \cup B) + (k+2-2) \cdot (k+2)^{\max(|A|, |B|)} \\
&\geq f_k(A \cup B) + k \geq f_k(A \cup B) + f_k(A \cap B). \quad \square
\end{aligned}$$

The following Lemma proves the claimed lower bound on the distance to submodularity of  $f_k$ .

**Lemma 11.** *The distance of  $f_k$  to submodularity is at least  $\nu(f_k) = \frac{k}{4}$ .*

*Proof.* By applying Corollary 8, we obtain that the distance of  $f_k$  to submodularity is at least:

$$\nu(f_k) = \frac{\frac{k}{2} + \sum_{i=1}^k \frac{(i+1)+(k-i)-1}{2}}{2 \cdot (k+1)} - \frac{1}{2} \cdot 0 = \frac{\frac{k}{2} + \sum_{i=1}^k \frac{k}{2}}{2 \cdot (k+1)} = \frac{k \cdot (k+1)}{4 \cdot (k+1)} = \frac{k}{4}. \quad \square$$

We then have the lower bound for the submodular case.

**Theorem 12.** *There is an infinite sequence of increasing integers  $1 < n_1 < n_2 < \dots$  such that, for each  $k \geq 1$ , there exists an approximately  $\mathcal{C}^{\text{full}}$ -submodular function  $f_k : 2^{[n_k]} \rightarrow \mathbb{R}$  whose distance to submodularity is larger than  $\sqrt{n_k/8} - 3/8$ .*

*Proof.* Given a  $k \geq 1$ , pick the  $f_k$  of Definition 9. Then, by Lemma 10,  $f_k$  is approximately  $\mathcal{C}^{\text{full}}$ -submodular. The size of the ground set of  $f_k$  is equal to  $n_k = \frac{(k+3)k}{2}$ . By the AM-GM inequality, we have  $(k+3/2)^2 > (k+3)k$ . Thus,  $n_k < \frac{1}{2} \cdot (k+3/2)^2$  and, therefore,  $\sqrt{\frac{n_k}{8}} < \frac{k}{4} + \frac{3}{8}$ . By Lemma 11, the distance of  $f_k$  to submodularity is at least  $\nu(f_k) = k/4$ , and by the latter inequality  $\nu(f_k) > \sqrt{\frac{n_k}{8}} - \frac{3}{8}$ .  $\square$

Note that the lower bound also holds if we restrict the functions to be non-negative. If we define  $g_k$  as  $g_k(S) = f_k(S) - \min_T f_k(T)$ , we have that (i)  $g_k$  is non-negative, that (ii)  $g_k$  retains the same approximate submodularity property of  $f_k$ , and that (iii)  $g_k$  and  $f_k$  are at the same distance to submodularity.

## 6 Lower Bounds for Other Classes of Constraints

In this section, we give lower bounds on the distance to submodularity of functions that are approximately submodular with various other classes of constraints.

Observe that the polytope of approximately  $\mathcal{C}^{\text{full}}$ -submodular functions is contained in the polytope of approximately  $\mathcal{C}^{\text{dimin}}$ -submodular functions, which is contained in the polytope of  $\mathcal{C}^{\text{cross}}$ -submodular functions.

### 6.1 An $\Omega(n)$ lower bound for approximately $\mathcal{C}^{\text{dimin}}$ -submodular functions

In this section we obtain an easy lower bound for the  $\mathcal{C}^{\text{dimin}}$ -submodular case.

**Theorem 13.** *For each odd  $n \geq 1$ , there exists an approximately  $\mathcal{C}^{\text{dimin}}$ -submodular function  $f : 2^{[n]} \rightarrow \mathbb{R}$  whose distance to submodularity is at least  $(n-1)/8$ .*

*Proof.* Let the function  $f : 2^{[n]} \rightarrow \mathbb{R}$  be defined as  $f(S) = \max\left(0, |S| - \frac{n-1}{2}\right)$ , for each  $S \subseteq [n]$ .

First, consider any  $\{A, B\} \in \mathcal{C}^{\text{dimin}}$ . Wlog, let  $a = |A| \leq |B| = b$ ; then  $|A \cap B| = a - 1$  and  $|A \cup B| = b + 1$ . We have,

$$f(A) - f(A \cap B) = \max\left(0, a - \frac{n-1}{2}\right) - \max\left(0, a - 1 - \frac{n-1}{2}\right) \geq 0,$$

and

$$f(B) - f(A \cup B) = \max\left(0, b - \frac{n-1}{2}\right) - \max\left(0, b + 1 - \frac{n-1}{2}\right) \geq -1.$$

Thus,  $f(A) + f(B) \geq f(A \cap B) + f(A \cup B) - 1$ , for each  $\{A, B\} \in \mathcal{C}^{\text{dimin}}$ . The function  $f$  is therefore approximately  $\mathcal{C}^{\text{dimin}}$ -submodular.

We will now show that for any submodular function  $g$ , there is a subset  $S \subseteq [n]$  such that  $|f(S) - g(S)| \geq (n-1)/8$ . Let  $A = \left[\frac{n-1}{2}\right]$ . Then,  $f(A) + f([n] \setminus A) = 0 + 1 = 1$ . Moreover,  $f([n]) + f(\emptyset) = \frac{n+1}{2} + 0 = \frac{n+1}{2}$ . Thus, for any submodular function  $g$ ,

$$\begin{aligned} \sum_{S \in \{A, [n] \setminus A, [n], \emptyset\}} |f(S) - g(S)| &\geq \sum_{S \in \{[n], \emptyset\}} (f(S) - g(S)) - \sum_{S \in \{A, [n] \setminus A\}} (f(S) - g(S)) \\ &= (f([n]) + f(\emptyset) - f(A) - f([n] \setminus A)) + (g(A) + g([n] \setminus A)) - (g([n]) + g(\emptyset)) \\ &\geq f([n]) + f(\emptyset) - f(A) - f([n] \setminus A) = \frac{n+1}{2} - 1, \end{aligned}$$

where the last inequality follows from the submodularity of  $g$ . Hence, the distance to submodularity of  $f$  is at least  $\frac{1}{4} \cdot \left(\frac{n+1}{2} - 1\right) = \frac{n-1}{8}$ .  $\square$

## 6.2 A $\Omega(n^2)$ lower bound for approximately $\mathcal{C}^{\text{cross}}$ -submodular functions

Finally, we show a lower bound for the  $\mathcal{C}^{\text{cross}}$  case that matches exactly the upper bound given by our algorithmic filter.

**Theorem 14.** *For any  $n \geq 1$ , there exists an approximately  $\mathcal{C}^{\text{cross}}$ -submodular function  $f : 2^{[n]} \rightarrow \mathbb{R}$  whose distance to submodularity is at least  $(1/8) \cdot \lfloor n^2/2 \rfloor \geq (n^2 - 1)/16$ .*

*Proof.* We define  $f$  to be:

$$f(S) = \frac{(n - 2 \cdot |S|)^2}{8}.$$

We will first lower bound the distance of  $f$  to submodularity. For  $A = \lfloor n/2 \rfloor, B = [n] \setminus A$ , we have

$$f(A) + f(B) = \frac{(n - 2 \cdot \lfloor n/2 \rfloor)^2 + (n - 2 \cdot \lceil n/2 \rceil)^2}{8} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{4} & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, we have

$$f(A \cup B) + f(A \cap B) = f([n]) + f(\emptyset) = \frac{2 \cdot n^2}{8} = \frac{n^2}{4}.$$

Thus,  $(f(A \cup B) + f(A \cap B)) - (f(A) + f(B)) = \frac{n^2 - \lfloor n \text{ is odd} \rfloor}{4} = \nu$ . It follows that, if the maximum additive absolute change to the values of  $f$  is less than  $\nu/4$ , the resulting function will not be submodular. Thus,  $f$  is at distance at least  $\nu/4 = \frac{n^2 - \lfloor n \text{ is odd} \rfloor}{16}$  from submodularity.

Now, consider any  $\{A, B\} \in \mathcal{C}^{\text{cross}}$ . There must exist  $a, b \notin A \cap B, a \neq b$ , such that  $A = (A \cap B) \cup \{a\}$  and  $B = (A \cap B) \cup \{b\}$ . Let  $i = |A \cap B|$ . We then have:

$$(f(A \cup B) + f(A \cap B)) - (f(A) + f(B)) = \frac{(n - 2(i+2))^2 + (n - 2i)^2 - 2(n - 2(i+1))^2}{8} = 1,$$

thus,  $f$  satisfies the approximate  $\mathcal{C}^{\text{cross}}$ -submodularity condition.  $\square$

## 7 Experiments

In this section we describe the experiments with our filter. We sampled approximately submodular (cut) functions, and compared the results obtained by running maximization algorithms on the original approximately submodular function  $f$ , and on the submodular function  $g_{f,\epsilon}$  obtained with our filter.

Recall that the standard algorithm for maximizing monotone submodular functions is the greedy algorithm [NW81]. An interesting property of our filter is the following: for any function  $f$ , and for any value of  $\epsilon$ , the execution of the greedy algorithm on the filtered function  $g_{f,\epsilon}$  is the same as the execution of the greedy algorithm on the original function  $f$ . Indeed, for each set  $S$  and each element  $x \notin S$ , the quantities  $g_{f,\epsilon}(S) - f(S)$  and  $g_{f,\epsilon}(S \cup \{x\}) - f(S \cup \{x\})$  are independent of  $x$  and hence greedy makes the same choice for  $f(\cdot)$  and  $g_{f,\epsilon}(\cdot)$  at each step. Hence, in our experiments, we focus on algorithms for non-monotone submodular functions, more specifically, the cut function. Recall that, given an input graph  $G(V, E)$ , the cut function  $\text{cut}_G(\cdot)$  is defined as:  $\forall S \subseteq V, \text{cut}_G(S) = |E(S, V \setminus S)|$ , i.e., the number of edges from  $S$  to  $V \setminus S$  in  $G$ .

**Local search algorithms.** In this setting, our approximate submodular function are chosen as follows: we sample a graph  $G$  from the Erdős-Rényi model with  $n = 100$  nodes and  $p = 0.5$ . Then, the function  $f$  is defined as:  $\forall S \subseteq V, f(S) = \text{cut}_G(S) + Z_S$  where  $Z_S$  is a random variable sampled iid from  $N(\mu = 0, \sigma^2)$ ; we considered  $\sigma^2 \in \{np/2, np\}$ .

Given access to  $f(\cdot)$ , the algorithm first estimates  $\epsilon$  by sampling pairs of sets. The pair  $\{S_i, T_i\}$  was sampled as follows. We first sampled two integers  $n_1$  and  $n_2$  uniformly independently and uniformly at random from  $[n - 1]$ .  $S_i$  (resp.  $T_i$ ) is then created by sampling  $n_1$  (resp.  $n_2$ ) elements from the ground set  $V$  without replacement. We then estimate  $\epsilon$  as  $\epsilon = \max(0, \max_i [f(S_i \cup T_i) + f(S_i \cap T_i) - f(S_i) - f(T_i)])$ . Note that the true  $\epsilon$  can be obtained by taking the maximum over all set pairs. All experiments were done on a standard i5 desktop.

With this estimated  $\epsilon$ , the maximization algorithm uses a local search procedure [FMV11] for the function  $g_{f,\epsilon}$ . The table below notes the results for  $\sigma^2 = np/2$  and  $\sigma^2 = np$ . For each value of  $\sigma$ , for 50 times (each time, with different random graph and noise realizations), we ran both algorithms. Denote the output set of local search on  $f(\cdot)$  as  $S_f$ , and the output set of local search on  $g_{f,\epsilon}(\cdot)$  as  $S_g$ . We calculate the ratio  $\frac{f(S_g)}{f(S_f)}$  for 50 instantiations of the random graph, and of the noise  $Z$ , and then report the minimum, mean, and standard deviation of the ratios.

Local search using the filtered function never returns a set of lesser value than local search using the original function. In a large number of runs, the sets returned are actually same. When they differ, the filtered function yields a set of higher value than the original function. This observation is consistent across the values of  $n$  and  $p$  that we have experimented with.

**Double greedy algorithm.** In the second set of experiments we used the randomized double greedy (RDG) algorithm [BFNS15] for maximization. In this setup, we created the submodular function as follows: first a stochastic block model random graph in 1000 nodes is created, with partition size as  $[100, 900]$  and the block probability matrix =  $[[0.1, 0.8], [0.8, 0.1]]$ . Again, we consider two situations where noise is added with  $\sigma^2$  to be 500 and 1000 respectively.

The  $\epsilon$  is estimated as described above and the RDG algorithm is run on the filtered function. We summarize the results in the following table. For both the algorithms, we observe that the filtered function yields a better solution than the unfiltered one.

**Comparison against exact solution.** In the above experiments, the ground-set size is large and it is computationally infeasible to obtain the exact maximum. We also ran the experiments with  $n = 20$  and the graph generated as per the description above and with  $\sigma = np/2$ , and using an exhaustive search for maximization. The (min, median, s.d.) ratios obtained over 10 iterations are reported in the Table 1. As per the results, the set found using the filtered function has a value which is at least 91% of the true maximum;

		$\sigma^2 = np/2$	$\sigma^2 = np$
Local search	min(ratio)	1	1
	avg(ratio)	2.981	2.174
	median (ratio)	2.376	2.506
Randomized	min(ratio)	0.924	1.922
	double avg(ratio)	2.282	3.763
	greedy median(ratio)	1.947	3.313
Exhaustive	min(ratio)	0.918	-
	avg(ratio)	0.945	-
	median (ratio)	0.936	-

Table 1: Results for local search, double greedy and exhaustive search algorithms with our filter.

		$c = 160$	$c = 80$
RDG	min(ratio)	0.831	0.893
	avg(ratio)	0.983	1.01
	median(ratio)	0.99	1.02

Table 2: Results for RDG with our filter on the BitCoin graph.

the low standard deviation (0.017) that we observed demonstrates the robustness of this statement to the errors in  $\epsilon$ -estimation.

**Summary.** To summarize, the experiments demonstrate the using the proposed filter gives empirically a good solution, often better than running the approximation algorithms on the unfiltered approximately submodular function. Even when exhaustive search is being used, the solution from the filtered function is quite close to the optimal. This observation holds across the different  $\epsilon$ -values that we tested.

## 7.1 Experiments on real data

We experimented with the BitCoin trust graph (5000 nodes) obtained from SNAP repository <sup>6</sup>. Here, the nodes represent users and the edge  $(i, j)$  represents the trust value assigned by user  $i$  to user  $j$ . We considered the graph as an undirected graph, the weight of every edge is an integer in  $[-10, 10]$  (the weight of an undirected edge is an average of the two directed edges, if both existed). We considered the cut-function on this graph and we created an approximately submodular function in the same way as before, i.e.,  $f(S) = \text{cut}_G(S) + Z_S$  where  $Z_S$  is defined as follows:  $Z_S = c$  with probability 0.5 and  $Z_S = -c$  else. (Observe that the noise model is additive and thus guarantees that  $f$  is approximately submodular; additive noise models capture the fact that, in practice, one can only estimate the weight of a cut by sampling the weight of its edges and, thus, one is bound to an additive error in the estimation. We chose the simplest additive model for our experiments.) Note that  $f(S)$  is a  $(4c)$ -approximately submodular and is neither monotone nor non-negative. We follow the same procedure for estimating  $\epsilon$  as described in Section 7, run randomized double greedy (RDG) and evaluate the result in a similar fashion as before.

Table 2 presents the minimum, mean, and median of the ratios of the solutions obtained by RDG on actual  $f(\cdot)$  and that applied on the filtered  $g_{f,\epsilon}(\cdot)$ . Notice that unlike the result on synthetic graphs, the solution obtained from the filtered  $g_{f,\epsilon}(\cdot)$  can sometimes be a little worse than the one obtained from the unfiltered  $f(\cdot)$ . However, note that the average ratio is always larger than than 0.98, that the ratio is a function of the noise model and, while filtering does provide a provable guarantee on the solution, no guarantee is available on the solution obtained by RDG on the unfiltered  $f(\cdot)$ .

<sup>6</sup><https://snap.stanford.edu/data/soc-sign-bitcoin-otc.html>

## 8 Conclusions

In this paper we have taken a first step in studying the distance to submodularity of approximately submodular functions. There are many open questions that stand out: first, what is the tight bound on the distance to submodularity of approximately submodular functions? We have proved that it is polynomial in  $n$  (whereas, in the approximately modular case it is a constant independent of  $n$ ), bounded between  $\Omega(\sqrt{n})$  and  $O(n^2)$ . There are also some algorithmic open questions. What type of optimization can we perform on approximately submodular functions? We have shown that it is possible to compute, in polynomial time, an additive approximation of the minimum value of these functions. Can one obtain a better approximation in polynomial time? And, with which approximation guarantees can approximately submodular functions be optimized under cardinality constraints, and under matroid constraints?

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