# Finding Cheeger Cuts in Hypergraphs via Heat Equation

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#### Abstract

Cheeger's inequality states that a tightly connected subset can be extracted from a graph G using an eigenvector of the normalized Laplacian associated with G. More specifically, we can compute a subset with conductance  $O(\sqrt{\phi_G})$ , where  $\phi_G$  is the minimum conductance of a set in G.

It has recently been shown that Cheeger's inequality can be extended to hypergraphs. However, as the normalized Laplacian of a hypergraph is no longer a matrix, we can only approximate its eigenvectors; this causes a loss in the conductance of the obtained subset. To address this problem, we here consider the heat equation on hypergraphs, which is a differential equation exploiting the normalized Laplacian. We show that the heat equation has a unique global solution and that we can extract a subset with conductance  $\sqrt{\phi_G}$  from the solution under a mild condition. An analogous result also holds for directed graphs.

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### **1** Introduction

The goal of spectral clustering of graphs is to extract tightly connected communities from a given weighted graph G = (V, E, w), where  $w: E \to \mathbb{R}_+$  is a weight function, using eigenvectors of matrices associated with G. One of the most fundamental results in this area is Cheeger's inequality, which relates the secondsmallest eigenvalue of the normalized Laplacian of G and the conductance of G. Here, the *(random-walk) normalized Laplacian* of G is defined as  $\mathcal{L}_G = I - A_G D_G^{-1}$ , where  $A_G \in \mathbb{R}^{V \times V}$  and  $D_G \in \mathbb{R}^{V \times V}$ are the (weighted) adjacency matrix and the (weighted) degree matrix, respectively, of G, that is,  $D_G$  is a diagonal matrix with the (v, v)-th element for  $v \in V$  being the (weighted) degree  $d_G(v) := \sum_{e \in E | v \in e} w(e)$ of v. Note that all eigenvalues of  $\mathcal{L}_G$  are non-negative and the smallest eigenvalue is always zero, as  $\mathcal{L}_G(D_G \mathbf{1}) = \mathbf{0}$ , where  $\mathbf{1}$  is the all-one vector and  $\mathbf{0}$  is the zero vector. The *conductance* of a set  $\emptyset \subsetneq S \subsetneq V$ is defined as

$$\phi_G(S) := \frac{\sum_{e \in \partial_G(S)} w(e)}{\min\{\operatorname{vol}_G(S), \operatorname{vol}_G(V \setminus S)\}}$$

where  $\partial_G(S)$  is the set of edges between S and  $V \setminus S$ , and  $\operatorname{vol}_G(S) := \sum_{v \in S} d_G(v)$  is the volume of S. Intuitively, smaller  $\phi_G(S)$  corresponds to more tightly connected S. The conductance of G is the minimum conductance of a set in G; that is,  $\phi_G := \min_{\emptyset \subseteq S \subseteq V} \phi_G(S)$ . Then, Cheeger's inequality [2, 3] states that

$$\frac{\lambda_G}{2} \le \phi_G \le \sqrt{2\lambda_G},\tag{1}$$

where  $\lambda_G \in \mathbb{R}_+$  is the second-smallest eigenvalue of  $\mathcal{L}_G$ . The second inequality of (1) is algorithmic in the sense that we can compute a set  $\emptyset \subsetneq S \subsetneq V$  with conductance of at most  $\sqrt{2\lambda_G} = O(\sqrt{\phi_G})$ , which is called a *Cheeger cut*, in polynomial time from an eigenvector corresponding to  $\lambda_G$ . Moreover, Cheeger's inequality is tight in the sense that computing a set with conductance  $o(\sqrt{\phi_G})$  is NP-hard [16], assuming the small set expansion hypothesis (SSEH) [15].

Several attempts to extend Cheeger's inequality to hypergraphs have been made. To explain the known results, we first extend the concepts of conductance and the normalized Laplacian to hypergraphs. Let G = (V, E, w) be a weighted hypergraph, where  $w : E \to \mathbb{R}_+$  is a weight function. The *(weighted) degree* of a vertex  $v \in V$  is  $d_G(v) := \sum_{e \in E | v \in e} w(e)$ . For a vertex set  $\emptyset \subsetneq S \subsetneq V$ , the *conductance* of S is defined as

$$\phi_G(S) := \frac{\sum_{e \in \partial_G(S)} w(e)}{\min\{\operatorname{vol}_G(S), \operatorname{vol}_G(V \setminus S)\}}$$

where  $\partial_G(S)$  is the set of hyperedges intersecting both S and  $V \setminus S$ , and  $\operatorname{vol}_G(S)$  has the same definition as usual graph. The *conductance* of G is defined as  $\phi_G := \min_{\emptyset \subseteq S \subseteq V} \phi_G(S)$ .

The normalized Laplacian  $\mathcal{L}_G \colon \mathbb{R}^V \to 2^{\mathbb{R}^{V_1}}$  of a hypergraph G [4, 20] is multi-valued and no longer linear (see Section 2 for a detailed definition). In the simplest setting that the hypergraph G is unweighted and d-regular, that is, every vertex has degree d, and the elements of the given vector  $\boldsymbol{x} \in \mathbb{R}^V$  are pairwise distinct, the  $\mathcal{L}_G$  acts as follows: We create an undirected graph  $G_{\boldsymbol{x}}$  on V from G by adding for each hyperedge  $e \in E$  an undirected edge uv, where  $u = \operatorname{argmin}_{w \in e} \boldsymbol{x}(w)$  and  $v = \operatorname{argmax}_{w \in e} \boldsymbol{x}(w)$ , then return  $\mathcal{L}_{G_{\boldsymbol{x}}} \boldsymbol{x}$ .

<sup>&</sup>lt;sup>1</sup>We note that the range of the Laplacian defined in [4] is  $\mathbb{R}^V$  instead of  $2^{\mathbb{R}^V}$ . They chose the value of  $\mathcal{L}_G(\boldsymbol{x})$  so that it satisfies a necessary condition that the heat equation (HE;  $\boldsymbol{s}$ ) has a solution, which makes it unique. Hence as long as we consider the solution to (HE;  $\boldsymbol{s}$ ), our Laplacian behaves as that defined in [4]. Nevertheless, we keep the range  $2^{\mathbb{R}^V}$  for a general treatment of the heat equation using the theory of monotone operators. See Section 5 for more details.

When  $\mathcal{L}_G(v) \ni \lambda v$  holds for  $\lambda \in \mathbb{R}$  and  $v \neq 0$ , we can state that  $\lambda$  and v are an *eigenvalue* and an *eigenvector*, respectively, of  $\mathcal{L}_G$ . As with the graph case, all eigenvalues of  $\mathcal{L}_G$  are non-negative and the first eigenvalue is zero as  $\mathcal{L}_G(D1) = 0$  holds. Moreover, the second-smallest eigenvalue  $\lambda_G \in \mathbb{R}_+$  exists. Cheeger's inequality for hypergraphs [4, 20] states that

$$\frac{\lambda_G}{2} \le \phi_G \le 2\sqrt{\lambda_G}.\tag{2}$$

Again, the second inequality is algorithmic: If we can compute an eigenvector corresponding to  $\lambda_G$ , we can obtain a Cheeger cut; that is, a set  $\emptyset \subseteq S \subseteq V$  with  $\phi_G(S) = O(\sqrt{\phi_G})$ , in polynomial time. Unlike the undirected graph case, however, only an  $O(\log n)$ -approximation algorithm is available for computing  $\lambda_G$  [20]. Further, this approximation ratio is tight under the SSEH [4]. Hence, the following natural question arises: Can we compute a Cheeger cut without computing  $\lambda_G$  and applying Cheeger's inequality on the corresponding eigenvector?

To answer this question, we consider the following differential equation called the *heat equation* [4]:

$$\frac{d\boldsymbol{\rho}_t}{dt} \in -\mathcal{L}_G(\boldsymbol{\rho}_t) \quad \text{and} \quad \boldsymbol{\rho}_0 = \boldsymbol{s},$$
 (HE;  $\boldsymbol{s}$ )

where  $s \in \mathbb{R}^V$  is an initial vector. Intuitively, we gradually diffuse values (or *heat*) on vertices along hyperedges so that the maximum and minimum values in each hyperedge become closer. We can show that (HE; s) always has a unique global solution for  $t \ge 0^2$  using the theory of monotone operators and evolution equations [12], [14] (see Section 5 for details), and let  $\rho_t^s \in \mathbb{R}^V$  be the solution at time  $t \ge 0$ . In particular,  $\rho_0^s = s$  holds. In addition, if  $\sum_{v \in V} s(v) = 1$ , we can show that  $\sum_{v \in V} \rho_t^s(v) = 1$  holds for any  $t \ge 0$ , and that  $\rho_t^s$  converges to  $\pi \in \mathbb{R}^V$  as  $t \to \infty$  when G is connected, where  $\pi(v) := d_G(v)/\operatorname{vol}(V)$ (see [4, Theorem 3.4]). Throughout this paper, we assume that hypergraph G is connected.

For a vector  $\boldsymbol{x} \in \mathbb{R}^V$ , let sweep $(\boldsymbol{x})$  denote the set of all *sweep sets* with respect to  $\boldsymbol{x}$ ; that is, sets of the form either  $\{v \in V \mid \boldsymbol{x}(v) \geq \tau\}$  or  $\{v \in V \mid \boldsymbol{x}(v) \leq \tau\}$ , for some  $\tau \in \mathbb{R}$ . We want to show that the conductance of the sweep set of a vector obtained from the heat equation is small. To this end, for  $T \geq 0$ , we introduce a key quantity in our analysis:

$$g_v(T) = -\frac{d}{dt} \log \|\boldsymbol{\rho}_t^{\boldsymbol{\pi}_v} - \boldsymbol{\pi}\|_{D^{-1}}^2 \bigg|_{t=T},$$

where  $\|\boldsymbol{x}\|_{D^{-1}}^2 := \boldsymbol{x}^\top D^{-1} \boldsymbol{x}$ , which quantifies how fast the heat converges to the limit, that is,  $\boldsymbol{\pi}$ . We can show that  $g_v(T)$  is twice the Rayleigh quotient of  $D^{-1/2}(\boldsymbol{\rho}_T^{\boldsymbol{\pi}_v} - \boldsymbol{\pi})$  with respect to the *normalized Laplacian*  $\boldsymbol{x} \mapsto D^{-1/2} \mathcal{L}_G(D^{-1/2}(\boldsymbol{x}))$ . This fact, combined with Cheeger's inequality for hypergraphs, implies that  $g_v(T)$  captures the minimum conductance of a sweep set obtained from  $\boldsymbol{\rho}_T^{\boldsymbol{\pi}_v}$ .

**Theorem 1.** Let G = (V, E, w) be a weighted hypergraph and  $\emptyset \subsetneq S \subsetneq V$  be a set. For any t > 0, we have

$$g_v(T) \ge (\kappa_{T,t}^v)^2$$

where  $\kappa_{T,t}^v := \min\{\phi_G(S') \mid \xi \in [T,t], S' \in \operatorname{sweep}(\rho_{\xi}^{\pi_v})\}$  and  $\pi_v \in \mathbb{R}^V$  is a vector for which  $\pi_v(v) = 1$ and  $\pi_v(u) = 0$  for  $u \neq v$ .

Let  $u_2(G_{v,T})$  be an eigenvector corresponding to the second smallest eigenvalue  $\lambda_2(G_{v,T})$  of the normalized Laplacian  $\mathcal{L}_{G_{v,T}}$ . Then, by using Cheeger's inequality for undirected graphs, we have the following corollary:

<sup>&</sup>lt;sup>2</sup>Previous works [4, 20] only guaranteed that it has a local solution for  $0 \le t \le T_0$  for some  $T_0 > 0$ .

Algorithm 1: Finding Cheeger Cuts via Heat Equation Input : Hypergraph G = (V, E, w) and t > T > 0Output:  $S \subseteq V$ 1 Select an arbitrary  $v \in V$ ; 2 Solve (HE;  $\pi_v$ ) to obtain  $\rho_{\xi}^{\pi_v}$  for  $\xi \in [T, t]$ ; 3  $S_{\text{out}} \leftarrow \operatorname{argmin}_{S \in \bigcup_{\xi \in [T,t]} \operatorname{sweep}(\rho_{\xi}^{\pi_v})} \phi_G(S)$ ; 4 return  $S_{\text{out}}$ 

**Corollary 2.** Assume that  $\langle u_2(G_{v,T}), \rho_T^{\pi_v} \rangle_{D^{-1}} \neq 0$  holds. Then, we have

$$4\phi_G h_v(T) \ge (\kappa_{T,t}^v)^2$$

where  $h_v(T) = g_v(T)/\lambda_2(G_{v,T})$ .

We can show that  $h_v(T)$  is close to 1 when T is large. Hence, Corollary 2 implies that, when T is sufficiently large, under the assumption in the statement, we can obtain a set  $\emptyset \subseteq S \subseteq V$  such that  $\phi_G(S) = O(\sqrt{\phi_G})$ , thereby avoiding the problem of computing the second smallest eigenvalue  $\lambda_G$  of the hypergraph normalized Laplacian  $\mathcal{L}_G$ . Algorithm 1 gives a pseudocode of our algorithm.

Although we cannot solve the differential equation (HE; s) exactly in polynomial time, we can efficiently simulate it by discretizing time using, e.g., the Euler method or the Runge-Kutta method. Indeed these methods have already been used in practice [19]. Alternatively, we can use difference approximation, developed in the theory of monotone operators and evolution equations [14], to obtain the following:

**Theorem 3.** Let G = (V, E, w) be a weighted hypergraph and  $v \in V$ , and let  $T \ge 1$  and  $\lambda \in (0, 1)$ . Then, we can compute (a concise representation) of a solution  $\{\rho_t^\lambda\}_{0 \le t \le T}$  such that  $\|\rho_t^{\pi_v} - \rho_t^\lambda\|_{D^{-1}} = O(\sqrt{\lambda T})$  for every  $0 \le t \le T$ , in time polynomial in  $1/\lambda$ , T, and  $\sum_{e \in E} |e|$ .

### **1.1 Directed graphs**

We briefly discuss directed graphs here, as we can show analogues of Theorem 1, Corollary 2, and Theorem 3 for such graphs with almost the same proof.

For a directed graph G = (V, E, w), the *degree* of a vertex  $v \in V$  is  $d_G(v) = \sum_{e \in E | v \in e} w(e)$  and the *volume* of a set  $S \subseteq V$  is  $vol_G(S) = \sum_{v \in S} d_G(v)$ . Note that we do not distinguish out-going and in-coming edges when calculating degrees. Then, the *conductance* of a set  $\emptyset \subsetneq S \subsetneq V$  is defined as

$$\phi_G(S) := \frac{\min\{\sum_{e \in \partial_G^+(S)} w(e), \sum_{e \in \partial_G^-(S)} w(e)\}}{\min\{\operatorname{vol}_G(S), \operatorname{vol}_G(V \setminus S)\}},$$

where  $\partial_G^+(S)$  and  $\partial_G^-(S)$  are the sets of edges leaving and entering S, respectively. Then, the *conductance* of G is  $\phi_G := \min_{\emptyset \subseteq S \subseteq V} \phi_G(S)$ . Note that  $\phi_G = 0$  when G is a directed acyclic graph.

Yoshida [19] introduced the notion of a Laplacian for directed graphs and derived Cheeger's inequality, which relates  $\phi_G$  and the second-smallest eigenvalue  $\lambda_G$  of the normalized Laplacian of G. As with the hypergraph case, computing  $\lambda_G$  is problematic, and we can apply an analogue of Theorem 1 to obtain a set of small conductance without computing  $\lambda_G$ . In this paper, we focus on hypergraphs for simplicity of exposition.

### **1.2** Sketch of proof

Chung [7] presented analogues of Theorem 1 and Corollary 2 for usual undirected graphs. Here, we review her proofs of these analogue, because our proofs of Theorem 1 and Corollary 2 extend them partially.

For the undirected graph case, we consider the following single-valued differential equation:

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ho}_t}{dt} = -\mathcal{L}_Goldsymbol{
ho}_t \quad ext{and} \quad oldsymbol{
ho}_0 = s.$$

This differential equation has a unique global solution  $\rho_t^s = \exp(-t\mathcal{L}_G)s$ . We define a function  $f^s \colon \mathbb{R}_+ \to \mathbb{R}$  as

$$f^{s}(t) = \| \boldsymbol{\rho}_{t/2}^{s} - \boldsymbol{\pi} \|_{D^{-1}}^{2}.$$

When G is connected,  $\rho_t^s$  converges to  $\pi$  as  $t \to \infty$  irrespective of s; hence,  $f^s$  measures the difference between  $\rho_{t/2}^s$  and its unique stationary distribution  $\pi$ . For a set  $S \subseteq V$ , we define  $\pi_S \in \mathbb{R}^V$  as  $\pi_S(v) = d(v)/\operatorname{vol}(S)$  if  $v \in S$  and  $\pi_S(v) = 0$  otherwise. Then, we can show the inequalities

$$\exp(-O(\phi(S)t)) \le f^{\pi_S}(t) \le \exp\left(-\Omega\left((\kappa_t^{\pi_S})^2 t\right)\right),\tag{3}$$

for every  $S \subseteq V$ , where  $\kappa_t^{\pi_S}$  is the minimum conductance of a sweep set with respect to the vector  $(\boldsymbol{\rho}_t^{\pi_S}(v)/d(v))_{v\in V}$ . From the closed solution of  $\boldsymbol{\rho}_t^s$ , we observe that  $\boldsymbol{\rho}_{t/2}^{\pi_S} = \sum_{v\in S} \frac{d(v)}{\operatorname{vol}(S)} \boldsymbol{\rho}_{t/2}^{\pi_v}$ . Then, we have

$$\exp(-O(\phi(S)t)) \le f^{\pi_S}(t) = \|\boldsymbol{\rho}_{t/2}^{\pi_S} - \pi\|_{D^{-1}}^2 \le \left(\sum_{v \in S} \frac{d(v)}{\operatorname{vol}(S)} \|\boldsymbol{\rho}_{t/2}^{\pi_v} - \pi\|_{D^{-1}}\right)^2$$
(by triangle inequality)  
(by triangle inequality)

$$\leq \max_{v \in S} \|\boldsymbol{\rho}_{t/2}^{\boldsymbol{\pi}_v} - \boldsymbol{\pi}\|_{D^{-1}}^2 = \max_{v \in S} f^{\boldsymbol{\pi}_v}(t) \leq \max_{v \in S} \exp\left(-\Omega\left((\kappa_t^{\boldsymbol{\pi}_v})^2 t\right)\right).$$

Taking the logarithm yields the desired result.

The main obstacle to extending the above argument to hypergraphs is that  $\rho_t$  does not have a closed-form solution as  $\mathcal{L}_G$  is no longer a linear operator and single-valued. To overcome this obstacle, we observe that there exists the sequence  $t_0 = 0 < t_1 < t_2 < \cdots$  such that  $\mathcal{L}_G$  can be regarded as a linear operator  $\mathcal{L}_i$  in each interval  $[t_i, t_{i+1})$ . Here,  $\mathcal{L}_i$  is the normalized Laplacian of a graph constructed from the hypergraph G and the vector  $\rho_{t_i}$ . Then, we can show a counterpart of the second inequality of (3) for each  $f_i^s : \mathbb{R}_+ \to \mathbb{R}$  defined as  $f_i^s(\Delta) = \|\rho_{t_i+\Delta/2}^s - \pi\|_{D^{-1}}^2$ , which is sufficient for our analysis. (We will use another equivalent definition for  $f_i^s$  for convenience. See Section 4 for details.)

Another obstacle is that the triangle inequality applied in the above argument is not true in general, because  $\rho_{t/2}^{\pi_S}$  may not generally be equal to  $\sum_{v \in S} \frac{d(v)}{\operatorname{vol}(S)} \rho_{t/2}^{\pi_v}$  for the hypergraph case. Due to this obstacle, it is hard to obtain a counterpart of the first inequality of (3). To overcome this problem, using the fact that the logarithmic derivative  $g_v(t)$  is monotonically non-increasing and considering t > T for T > 0, we obtain a non-trivial lower bound  $\exp(-O(g_v(T)(t-T)))$  of the square of norm  $\|\rho_t^{\pi_v} - \pi\|_{D^{-1}}^2$ . Then, we show that  $g_v(T)$  goes to an eigenvalue of the normalized Laplacian  $\mathcal{L}_{G_{v,T}}$  as T becomes larger. Hence, if  $g_v(T)$  is close to  $\lambda_2(G_{v,T})$ , by using the Cheeger inequality (1) for graphs and the relation  $\phi_{G_{v,T}} \leq \phi_G$ , we obtain a counterpart of the first inequality of (3).

#### **1.3 Related work**

As noted above, an analogue of Theorem 1 for usual graphs has been presented by Chung [7]. However, as the normalized Laplacian  $\mathcal{L}_G = I - A_G D_G^{-1}$  is a matrix for the graph case, that analysis is much simpler than that presented herein. Kloster and Gleich [11] have presented a deterministic algorithm that approximately simulates the heat equation for graphs. Hence, they extracted a tightly connected subset by considering a local part of the graph only.

The concept of the Laplacian for hypergraphs has been implicitly employed in semi-supervised learning on hypergraphs in the form  $\mathbf{x}^{\top} L_G(\mathbf{x})$ , where  $\mathcal{L}_G(\mathbf{x}) = L_G(D_G^{-1}\mathbf{x})$  [10, 21]. This concept was then formally presented by Chan et al. [4] at a later time. Subsequently, the Laplacian concept was further generalized to handle submodular transformations [13, 20]; this development encompasses Laplacians for graphs, hypergraphs [4], directed graphs [19], and directed hypergraphs [6]. On our work here, we need precise description of undirected graphs  $\tilde{G}_i$  introduced below. To achieve this, we borrow some results in [6, Sections 3 and 4].

Finally, we note that another type of Laplacian for hypergraphs, which essentially replaces each hyperedge with a clique, has been used in the literature [1, 17]. We stress that that Laplacian differs from the Laplacian for hypergraphs studied in this work.

#### 1.4 Organization

The remainder of this paper is organized as follows. In Section 2, we introduce the basic concepts used throughout this paper. In Section 3, we show some basic facts on the heat equation (3). In Section 4, we prove Theorem 1. We show that (HE; s) has a unique global solution in Section 5. A proof of Theorem 3 is given in Section 6.

### 2 Preliminaries

For a vector  $\boldsymbol{x} \in \mathbb{R}^{V}$  and a set  $S \subseteq V$ , let  $\boldsymbol{x}(S) = \sum_{v \in S} \boldsymbol{x}(v)$ . For a vector  $\boldsymbol{x} \in \mathbb{R}^{V}$  and a positive semidefinite matrix  $A \in \mathbb{R}^{V \times V}$ , we define  $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{A} = \boldsymbol{x}^{\top} A \boldsymbol{y}$  and  $\|\boldsymbol{x}\|_{A} = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{A}} = \sqrt{\boldsymbol{x}^{\top} A \boldsymbol{x}}$ .

Let G = (V, E, w) be a hypergraph. We omit the subscript G from notations such as  $A_G$  when it is clear from the context. For a set  $S \subseteq V$ , let  $\mathbf{1}_S \in \mathbb{R}^V$  denote the characteristic vector of S, that is,  $\mathbf{1}_S(v) = 1$  if  $v \in S$  and  $\mathbf{1}_S(v) = 0$  otherwise. When S = V or  $S = \{v\}$ , we simply write  $\mathbf{1}$  and  $\mathbf{1}_v$ , respectively. For a set  $S \subseteq V$ , we define a vector  $\pi_S \in \mathbb{R}^V$  as  $\pi_S(v) = \frac{d_G(v)}{\operatorname{vol}_G(S)}$  if  $v \in S$  and  $\pi_S(v) = 0$  otherwise. When S = V or  $S = \{v\}$ , we simply write  $\pi$  and  $\pi_v$ , respectively. For a vector  $\rho \in \mathbb{R}^V$ , we write  $\rho/d_G$  to denote a vector with  $(\rho/d_G)(v) = \rho(v)/d_G(v)$  for each  $v \in V$ .

### 2.1 Normalized Laplacian for hypergraphs

We define (random-walk) normalized Laplacian for hypergraphs precisely. Let G = (V, E, w) be a hypergraph. For each hyperedge  $e \in E$ , we define a polytope  $B_e = \operatorname{conv}(\{\mathbf{1}_u - \mathbf{1}_v \mid u, v \in e\})$ , where  $\operatorname{conv}(S)$  denotes the convex hull of  $S \subseteq \mathbb{R}^V$ . Then, the Laplacian  $L_G \colon \mathbb{R}^V \to 2^{\mathbb{R}^V}$  of G is defined as

$$L_G(\boldsymbol{x}) = \left\{ \sum_{e \in E} w(e) \boldsymbol{b}_e \boldsymbol{b}_e^\top \boldsymbol{x} \mid \boldsymbol{b}_e \in \operatorname*{argmax}_{\boldsymbol{b} \in B_e} \boldsymbol{b}^\top \boldsymbol{x} \right\},\tag{4}$$

and the *normalized Laplacian* is defined as  $\mathcal{L}_G \colon \boldsymbol{x} \mapsto L_G(D_G^{-1}\boldsymbol{x})$ .

We can write  $L_G(\mathbf{x})$  more explicitly as follows. For each hyperedge  $e \in E$ , let  $S_e = \operatorname{argmax}_{v \in e} \mathbf{x}(v)$ and  $I_e = \operatorname{argmin}_{v \in e} \mathbf{x}(v)$ . Let  $E' = \{uv \mid e \in E, u \in S_e, v \in I_e\} \cup \{vv \mid v \in V\}$ . Then, we arbitrarily define a function  $w'_e : E' \to \mathbb{R}_+$  so that  $w'_e(uv) > 0$  only if  $u \in S_e$  and  $v \in I_e$  and we have  $\sum_{u \in S_e, v \in I_e} w'_e(uv) = w(e)$ . Then, we construct a graph G' = (V, E', w'), where w'(uv) = $\sum_{e \in E \mid u \in S_e, v \in I_e} w'_e(uv)$  for each  $uv \in E'$  and  $w'(vv) = d_G(v) - \sum_{e \in E' \mid v \in e} w'(e)$  for each  $v \in V$ . Note that  $d_G(v) = d_{G'}(v)$  for every  $v \in V$ . Let  $\mathcal{G}(G, \mathbf{x})$  be the set of graphs constructed this way. Then, we have  $L_G(\mathbf{x}) = \{L_G'\mathbf{x} \mid G' \in \mathcal{G}(G, \mathbf{x})\}$ .

We can understand Laplacian for hypergraphs in terms of submodular functions. Let  $F_e: 2^V \to \{0, 1\}$ be the cut function associated with a hyperedge  $e \in E$ , that is,  $F_e(S) = 1$  if and only if  $S \cap e \neq \emptyset$  and  $(V \setminus S) \cap e \neq \emptyset$ . It is known that  $F_e$  is submodular, that is,  $F_e(S) + F_e(T) \ge F_e(S \cap T) + F_e(S \cup T)$ holds for every  $S, T \subseteq V$ . Then,  $B_e$  is the base polytope of  $F_e$  and  $b_e$  in (4) is chosen so that  $b_e^{\top} x = f_e(x)$ , where  $f_e: \mathbb{R}^V \to \mathbb{R}$  is the Lovász extension of  $F_e$ . See [9] for detailed definitions of these notions.

When G = (V, E, w) is a usual graph, its Laplacian  $L_G \in \mathbb{R}^{V \times V}$  and (random-walk) normalized Laplacian  $\mathcal{L}_G \in \mathbb{R}^{V \times V}$  are defined as  $D_G - A_G$  and  $I_G - A_G D_G^{-1}$ , respectively. Indeed, this coincides with (4) when we regard G as a hypergraph with each hyperedge having size two.

### **3 Properties of Solutions to Heat Equation**

We review some facts on the heat equation (HE; s). We say that  $\{\rho_t\}_{t\geq 0}$  is a *solution* of (HE; s) if  $\rho_t$  is absolutely continuous with respect to t (hence  $\rho_t$  is differentiable at almost all t) and  $\rho_0 = s$  and satisfies  $\frac{d}{dt}\rho_t \in -\mathcal{L}_G(\rho_t)$  for almost all  $t \geq 0$ . As we see in Section 5, the heat equation (HE; s) always has a unique global solution. Also as we mentioned, when G is connected,  $\rho_t$  converges to  $\pi$  as  $t \to \infty$  for any  $s \in \mathbb{R}^V$  with  $\sum_{v \in V} s(v) = 1$ .

We consider the heat equation (HE; s) on a hypergraph G = (V, E, w) with an initial vector  $s \in \mathbb{R}^V$ and let  $\{\rho_t^s\}_{t\geq 0}$  be its unique solution. Let  $\mu_t^s = D^{-1}\rho_t^s$ . Then, there is an ordered equivalence relation  $(\sigma^*, \succ)$  on V consistent with  $\{d^k \mu_t^s/dt^k\}_k$  introduced in [6, Section 3.1], i.e., for  $u, v \in V$ ,  $u \sim_{\sigma^*} v$  if all higher (right) derivatives of  $\mu_t^s(u)$  and  $\mu_t^s(v)$  at t = 0 are equal and for two  $\sigma^*$ -equivalence classes U and  $U', U \succ U'$  if there is an integer  $l \in \mathbb{Z}_+$  such that for  $u \in U$  and  $u' \in U'$ , the following hold:

$$\frac{d^{k}\boldsymbol{\mu}_{t}^{s}}{dt^{k}}\Big|_{t=0}(u) = \frac{d^{k}\boldsymbol{\mu}_{t}^{s}}{dt^{k}}\Big|_{t=0}(u') \text{ for } k = 0, \dots, l-1, \text{ and } \frac{d^{l}\boldsymbol{\mu}_{t}^{s}}{dt^{l}}\Big|_{t=0}(u) > \frac{d^{l}\boldsymbol{\mu}_{t}^{s}}{dt^{l}}\Big|_{t=0}(u').$$

We define  $\succeq$  as  $\succ$  or =. We divide V by the equivalence relation  $\sigma^*$  as  $V = \bigsqcup_{k=1}^m U_k$  so that  $U_k \succ U_{k+1}$  for every k. For  $v \in V$ , let  $[v]_{\sigma^*}$  be the equivalence class including v.

Let  $\boldsymbol{x} = \boldsymbol{\mu}_0^{\boldsymbol{s}} = D^{-1}\boldsymbol{s}$ . For  $e \in E$ , we recall  $S_e = S_e(\boldsymbol{x}) = \operatorname{argmax}_{v \in e} \boldsymbol{x}(v)$  and  $I_e = I_e(\boldsymbol{x}) = \operatorname{argmin}_{v \in e} \boldsymbol{x}(v)$ . Then, we define  $S_e^{\sigma^*} = S_e^{\sigma^*}(\boldsymbol{x})$  and  $I_e^{\sigma^*} = I_e^{\sigma^*}(\boldsymbol{x})$  as

$$\begin{aligned} S_e^{\sigma^*}(\boldsymbol{x}) &= \{ u \in S_e(\boldsymbol{x}) \mid [u]_{\sigma^*} \succeq [v]_{\sigma^*} \text{ for any } v \in S_e(\boldsymbol{x}) \} \\ I_e^{\sigma^*}(\boldsymbol{x}) &= \{ u \in I_e(\boldsymbol{x}) \mid [u]_{\sigma^*} \preceq [v]_{\sigma^*} \text{ for any } v \in I_e(\boldsymbol{x}) \}. \end{aligned}$$

We set  $\widetilde{V}$  as a complete system of representatives  $\{u_1, u_2, \ldots, u_m\} \subseteq V$   $(u_k \in U_k)$  and set  $\widetilde{E} = \{u_k u_l \subset U_k\}$ 

 $\widetilde{V} \mid k, l = 1 \dots m$ . We define the weights  $\widetilde{w}$  on  $\widetilde{E}$  as

$$\widetilde{w}(u_k u_l) = \sum_{\substack{e \in E, \ S_e^{\sigma^*} \cap U_k \neq \emptyset \\ I_e^{\sigma^*} \cap U_l \neq \emptyset}} w_e + \sum_{\substack{e \in E, \ S_e^{\sigma^*} \cap U_l \neq \emptyset \\ I_e^{\sigma^*} \cap U_k \neq \emptyset}} w_e \quad \text{for } k \neq l,$$
$$\widetilde{w}(u_k u_k) = \sum_{\substack{v \in U_k \\ v \in U_k}} d_G(v) - \sum_{\substack{l=1, \dots, m, \\ l \neq k}} \widetilde{w}(u_k u_l),$$

Then, the triple  $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{w})$  can be regarded as a weighted undirected graph. Let  $d_{\widetilde{G}}(u_k) = \sum_{v \in U_k} d_G(v)$ ,  $D_{\widetilde{G}} = \operatorname{diag}(d_{\widetilde{G}}(u_k)) \in \mathbb{R}^{m \times m}$ ,  $\widetilde{s} = \left(\sum_{u \in U_k} s(u)\right)_k \in \mathbb{R}^m$ , and  $\widetilde{x} = (x(u_k))_k \in \mathbb{R}^m$ . Then, we have  $\widetilde{s} = \widetilde{D}\widetilde{x}$  by using the equivalence relation  $\sigma^*$ . Then for  $\widetilde{\rho}_t^s = \left(\sum_{u \in U_k} \rho_t^s(u)\right)_k \in \mathbb{R}^m$ , the following holds. The proof is deferred to Appendix A.

**Theorem 4.**  $\tilde{\rho}_t^s$  is a unique solution of the following heat equation:

$$\frac{d\widetilde{\rho}_t}{dt} = -\mathcal{L}_{\widetilde{G}}\widetilde{\rho}_t, \ \widetilde{\rho}_0 = \widetilde{s}.$$

until when a next tie occurs for  $\mu_t^s$ , i.e., if we retake the ordered equivalence relation  $\sigma^*$  consistent with  $\{d^k \mu_t^s/dt^k\}_k$  at t, either  $S_e^{\sigma^*}(\mu_t^s)$  or  $I_e^{\sigma^*}(\mu_t^s)$  changes for some e. Here,  $\mathcal{L}_{\widetilde{G}}$  is the graph normalized Laplacian of  $\widetilde{G}$ . Moreover, the solution of this heat equation  $\widetilde{\rho}_t^s$  determines  $\rho_t^s$  for such t.

Then, there is a time sequence  $t_0 = 0 < t_1 < t_2 < \cdots$  such that there is a weighted graph  $G_i = (\widetilde{V}_i, \widetilde{E}_i, \widetilde{w}_i)$  for each  $i \in \mathbb{Z}_+$  such that the heat equation on the interval  $[t_i, t_{i+1})$  satisfies

$$\frac{d\widetilde{\boldsymbol{\rho}}_t}{dt} = -\mathcal{L}_i \widetilde{\boldsymbol{\rho}}_t$$

where  $\tilde{\rho}_t := \left(\sum_{u \in U_k^i} \rho_t(u)\right)_k$  for equivalence classes  $\{U_k^i\}_k$  by equivalence relation  $\sigma^*$  consistent with  $\{d^k \boldsymbol{\mu}_t^s/dt^k\}_k$  at  $t = t_i$ , and  $\mathcal{L}_i$  is the normalized Laplacian associated with  $\tilde{G}_i$ . Hence, we can write the solution  $\tilde{\rho}_{i,\Delta} := \left(\sum_{u \in U_k^i} \rho_{t_i+\Delta}(u)\right)_k$  for  $\Delta \in [0, t_{i+1} - t_i)$  as

$$\widetilde{\rho}_{i,\Delta} := H_{i,\Delta} \widetilde{\rho}_{t_i}, \quad \text{where } H_{i,\Delta} := e^{-\Delta \mathcal{L}_i} = \sum_{n=0}^{\infty} \frac{(-\Delta)^n \mathcal{L}_i^n}{n!}.$$
(5)

Although  $\tilde{\rho}_{i,\Delta}$  was originally defined for  $\Delta \in [0, t_{i+1} - t_i)$ , we can extend it to any  $\Delta \ge 0$  by using (5). When we want to stress the initial vector, we write  $\rho_t^s, \rho_{i,\Delta}^s, \tilde{\rho}_t^s, \tilde{\rho}_{i,\Delta}^s$ , etc.

In what follows, we assume that for the initial vector s and t > 0, there is an integer  $n \in \mathbb{Z}_+$  and a sequence  $0 = t_0 < t_1 < \cdots < t_n < T$  satisfying the following condition: On each interval  $[t_i, t_{i+1}]$ ,  $i = 0, 1, \ldots, n-1$ , and  $[t_n, T]$ , the solution  $\rho_t^s$  of heat equation (HE; s) can be obtained by the solution of the heat equation on the weighted graph  $\tilde{G}_i$   $(i = 0, 1, \ldots, n)$  as above. We assume this only for simplicity of exposition. Indeed, if the above condition does not hold, the sequence  $\{t_i\}_i$  converges to some  $T_0 < \infty$ . Then, the existence of the global solution  $\rho_t$  shown in Section 5 implies that  $\rho_{T_0}$  is well defined, and hence another sequence  $\{t'_i\}_i$  starts from  $T_0$  again, we can continue this process until we reach T. It is not hard to generalize our argument for such a case.

### 4 **Proof of Theorem 1**

In this section, we prove Theorem 1. Missing proofs are found in Appendix B.

Consider the heat equation (HE; *s*). We borrow notations from Section 3. For each  $i \in \mathbb{Z}_+$ , we define a function  $f_i \colon \mathbb{R}_+ \to \mathbb{R}$  as

$$f_i(\Delta) := \widetilde{\rho}_{i,0}^{\top} D_{\widetilde{G}_i}^{-1} \left( \widetilde{\rho}_{i,\Delta} - \widetilde{\pi}^i \right),$$

where  $\widetilde{\pi}^i = \left(\sum_{u \in U_k^i} \pi(u)\right)_k = \left(d_{\widetilde{G}_i}(u_k^i)/\operatorname{vol}\widetilde{V}_i\right)_k$ . When we wish to stress the initial vector  $s \in \mathbb{R}^V$ , we write  $f_i^s$ . As the following proposition implies, the value of  $f_i(\Delta)$  indicates the difference between  $\widetilde{\rho}_{i,\Delta/2}$  and the stationary distribution  $\widetilde{\pi}^i$  on  $\widetilde{G}_i$ .

**Proposition 5.** For any initial vector  $s \in \mathbb{R}^V$ ,  $i \in \mathbb{Z}_+$ , and  $\Delta \ge 0$ , we have

$$f_i(\Delta) = \|\widetilde{\rho}_{i,\Delta/2} - \widetilde{\pi}^i\|_{D_{\widetilde{G}_i}^{-1}}^2 = \sum_{v \in \widetilde{V}_i} \left(\frac{\widetilde{\rho}_{i,\Delta/2}(v)}{d_{\widetilde{G}_i}(v)} - \frac{1}{\operatorname{vol}(\widetilde{V}_i)}\right)^2 d_{\widetilde{G}_i}(v) \ge 0.$$

Proof. We have

$$\widetilde{\boldsymbol{\rho}}_{i,0}^{\top} D_{\widetilde{G}_i}^{-1}(\widetilde{\boldsymbol{\rho}}_{i,\Delta} - \widetilde{\boldsymbol{\pi}}^i) = \| D_{\widetilde{G}_i}^{-1/2} (H_{i,\Delta/2} - \widetilde{\boldsymbol{\pi}}^i \mathbf{1}^{\top}) \widetilde{\boldsymbol{\rho}}_{i,0} \|^2 = \| D_{\widetilde{G}_i}^{-1/2} (\widetilde{\boldsymbol{\rho}}_{i,\Delta/2} - \widetilde{\boldsymbol{\pi}}^i) \|^2.$$

The following lemma shows the compatibility of norms between vectors on G and  $\tilde{G}_i$ .

**Lemma 6.** For  $t = t_i + \Delta$ ,  $0 \le \Delta \le t_{i+1} - t_i$ , we have  $\|\widetilde{\rho}_{i,\Delta} - \widetilde{\pi}^i\|_{D_{\tilde{G}_i}^{-1}} = \|\rho_t - \pi\|_{D^{-1}}$ .

Theorem 1 is obtained by bounding  $f_i(\Delta)$  from above and below. To obtain an upper bound, for  $0 \le T \le t$ , we define

$$\begin{split} \widetilde{\kappa}_{i,I} &= \min \left\{ \phi_{\widetilde{G}_{i}}(S) \; \left| \; \xi \in I, \, S \in \operatorname{sweep}\left(\frac{\widetilde{\rho}_{i,\xi}}{d_{\widetilde{G}_{i}}}\right) \right\} \quad (i \in \mathbb{Z}_{+}, I \subset [0, t_{i+1} - t_{i}]) \\ \widetilde{\kappa}_{i} &= \widetilde{\kappa}_{i,[0,t_{i+1} - t_{i}]} \quad (i \in \mathbb{Z}_{+}), \\ \widetilde{\kappa}_{T,t} &= \min \left\{ \min_{j=i_{0}+1, \dots, i_{1}-1} \widetilde{\kappa}_{j}, \, \widetilde{\kappa}_{i_{0},[T-t_{i_{0}}, t_{i_{0}+1}]}, \widetilde{\kappa}_{i_{1},[0,t-t_{i_{1}}]} \right\}, \\ \text{where } i_{0}, i_{1} \in \mathbb{Z}_{+} \text{ are such that } T \in [t_{i_{0}}, t_{i_{0}+1}) \text{ and } t \in [t_{i_{1}}, t_{i_{1}+1}). \end{split}$$

Again, when we wish to stress the initial vector  $\pi_v \in \mathbb{R}^V$ , we write  $\tilde{\kappa}_{i,I}^v$ , etc. In the following lemma, we present an upper bound on a quotient of norms of heat when the initial vector s is  $\pi_v$  for some  $v \in V$ .

**Lemma 7.** For any  $t \ge T \ge 0$ , the following inequality holds:

$$\frac{\|\boldsymbol{\rho}_t^{\boldsymbol{\pi}_v} - \boldsymbol{\pi}\|_{D^{-1}}^2}{\|\boldsymbol{\rho}_T^{\boldsymbol{\pi}_v} - \boldsymbol{\pi}\|_{D^{-1}}^2} \le \exp(-(\widetilde{\kappa}_{T,t}^v)^2(t-T)).$$

Next, we consider a lower bound on the squared norm of the heat with the initial vector  $\pi_v$ . Let  $T \ge 0$  and set

$$g_{v}(T) = -\frac{d}{dt} \log \left\| \boldsymbol{\rho}_{t}^{\boldsymbol{\pi}_{v}} - \boldsymbol{\pi} \right\|_{D^{-1}}^{2} \bigg|_{t=T} = -\frac{\frac{d}{dt} \left\| \boldsymbol{\rho}_{t}^{\boldsymbol{\pi}_{v}} - \boldsymbol{\pi} \right\|_{D^{-1}}^{2} |_{t=T}}{\left\| \boldsymbol{\rho}_{T}^{\boldsymbol{\pi}_{v}} - \boldsymbol{\pi} \right\|_{D^{-1}}^{2}} = 2\frac{\langle \boldsymbol{\rho}_{T}^{\boldsymbol{\pi}_{v}}, \mathcal{L}(\boldsymbol{\rho}_{T}^{\boldsymbol{\pi}_{v}}) \rangle_{D^{-1}}}{\left\| \boldsymbol{\rho}_{T}^{\boldsymbol{\pi}_{v}} - \boldsymbol{\pi} \right\|_{D^{-1}}^{2}}.$$

Then, the following inequality holds:

**Lemma 8.** For any  $t \ge T$ , the following inequality holds:

$$\frac{\|\boldsymbol{\rho}_t^{\boldsymbol{\pi}_v} - \boldsymbol{\pi}\|_{D^{-1}}^2}{\|\boldsymbol{\rho}_T^{\boldsymbol{\pi}_v} - \boldsymbol{\pi}\|_{D^{-1}}^2} \ge \exp\left(-g_v(T)(t-T)\right).$$

Based on these lemmas, we obtain the following:

**Theorem 9.** Let G = (V, E, w) be a hypergraph, and  $v \in V$  and  $t \ge T \ge 0$ . Then, we have

$$g_v(T) \ge (\widetilde{\kappa}_{T,t}^v)^2.$$

We remark that the  $g_v(T)$  is close to an eigenvalue of normalized Laplacian of an undirected graph in the following sense: For  $\rho_t^{\pi_v}$ , as in Section 2.1, there is a graph  $G_{v,t} = G_{\rho_t^{\pi_v}} = (V, E_{v,t}, w_{v,t})$  such that  $\mathcal{L}_{G_{v,t}}\rho_t^{\pi_v} \in \mathcal{L}_G(\rho_t^{\pi_v})$ . We remark that if  $t \in [t_i, t_{i+1})$ , the graph  $\tilde{G}_i$  introduced in Section 3 is obtained by contracting  $G_{v,t}$ . We fix  $T \ge 0$  and consider a small  $\Delta > 0$ . Let  $t = T + \Delta$ . Then,  $\rho_t^{\pi_v}$  can be written by

$$\boldsymbol{\rho}_t^{\boldsymbol{\pi}_v} = \sum_{j=1}^n a_j e^{-\lambda_j (G_{v,T})(T+\Delta)} \boldsymbol{u}_j(G_{v,T}),$$

for some  $a_j \in \mathbb{R}$ . Here,  $0 = \lambda_1(G_{v,T}) \leq \lambda_2(G_{v,T}) \leq \cdots \leq \lambda_n(G_{v,T}) \leq 2$  are the eigenvalues of  $\mathcal{L}_{G_{v,T}}$ and  $u_1(G_{v,T}), \ldots, u_n(G_{v,T})$  are these eigenvectors such that  $\{D^{-1/2}u_1(G_{v,T}), \ldots, D^{-1/2}u_n(G_{v,T})\}$  is orthonormal. We set  $j_0$  as

$$j_0 = \min\{j \mid j \ge 2, a_j \ne 0\}.$$

Then, the  $g_v(T)$  can be rephrased as

$$g_{v}(T) = 2 \frac{\sum_{j=j_{0}}^{n} a_{j}^{2} \lambda_{j}(G_{v,T}) e^{-2\lambda_{j}(G_{v,T})T}}{\sum_{j=j_{0}}^{n} a_{j}^{2} e^{-2\lambda_{j}(G_{v,T})T}}$$
  
=  $2\lambda_{j_{0}}(G_{v,T}) \left( \frac{1 + (a_{j_{0}+1}/a_{j_{0}})^{2} (\lambda_{j_{0}+1}(G_{v,T})/\lambda_{j_{0}}(G_{v,T})) e^{-2(\lambda_{j_{0}+1}(G_{v,T})-\lambda_{j_{0}}(G_{v,T}))T} + \cdots}{1 + (a_{j_{0}+1}/a_{j_{0}})^{2} e^{-2(\lambda_{j_{0}+1}(G_{v,T})-\lambda_{j_{0}}(G_{v,T}))T} + \cdots} \right).$ 

We define  $h_v(T)$  so that  $g_v(T) = 2\lambda_{j_0}(G_{v,T})h_v(T)$ . Then,  $h_v(T)$  goes to 1 as T increases. Hence,  $h_v(T)$  is close to 1 for large T. If  $j_0 = 2$  (equivalent to  $\langle u_2(G_{v,T}), \rho_T^{\pi_v} \rangle_{D^{-1}} \neq 0$ ), we can find a nearly Cheeger cut:

**Corollary 10.** Notation is the same as above. We assume that  $\langle u_2(G_{v,T}), \rho_T^{\pi_v} \rangle_{D^{-1}} \neq 0$ . Then, we have the following inequality:

$$4\phi_G h_v(T) \ge (\widetilde{\kappa}_{T,t}^v)^2.$$

*Proof.* By the assumption  $j_0 = 2$ , Theorem 9, and Cheeger's inequality for graphs (1), we have

$$g_v(T) = 2\lambda_2(G_{v,T})h_v(T) \le 4\phi_{G_{v,T}}h_v(T).$$

It is easy to see that  $\phi_{G_{v,T}}(S) \leq \phi_G(S)$  holds for any  $S \subset V$ . This completes the proof.

To deduce Theorem 1 and Corollary 2, we need to show a relation  $\widetilde{\kappa}_{T,t}^v$  with  $\kappa_{T,t}^v$ . The following relates the conductance of a sweep set in a hypergraph G and that in a graph  $\widetilde{G}$ .

**Lemma 11.** Let G = (V, E, w) be a hypergraph,  $x \in \mathbb{R}^V$  be a vector, a be a real number, and  $\sigma^*$  be the ordered equivalence relation compatible with x in the sense of [6, Section 3], i.e., u and v are  $\sigma^*$ -equivalent if and only if x(u) = x(v). Let  $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{w})$  be the weighted graph defined as in Section 3 with this equivalent relation  $\sigma^*$ . If  $S^a \subseteq V$  (resp.,  $\widetilde{S}^a \subseteq \widetilde{V}$ ) is the sweep set on G (resp.,  $\widetilde{G}$ ) with  $x(u) \ge a$ , then  $\phi_G(S^a) = \phi_{\widetilde{G}}(\widetilde{S}^a)$  holds.

*Proof of Theorem 1 and Corollary 2.* As  $\tilde{\kappa}_{T,t}^v = \kappa_{T,t}^v$  by Lemma 11, we see that Theorem 9 and Corollary 10 imply Theorem 1 and Corollary 2, respectively.

## 5 Existence and Uniqueness of Solution

In this section, we show the existence and uniqueness of a solution to the heat equation (HE; s) using the theory of monotone operators. We refer the interested reader to the books by Miyadera [14] and Showalter [18] for a detailed description of this topic.

We begin by introducing some definitions. Let  $X = (X, \langle \cdot, \cdot \rangle)$  be a Hilbert space,  $\|\cdot\|$  be the norm defined from the inner product, and  $A: X \to 2^X$  be a multi-valued operator on X. Let  $R(A) \subseteq X$  be the *range* of A. We often identify A with the *graph* of A; that is,  $\{(x, y) \mid x \in X, y \in A(x)\} \subseteq X \times X$ .

**Definition 12.** An operator  $A: X \to 2^X$  is monotone (or accretive) if, for any  $x, x' \in X$  and  $y \in A(x), y' \in A(x')$ , we have

$$\langle y - y', x - x' \rangle \ge 0.$$

When -A is monotone, A is called dissipative.

**Definition 13.** A monotone operator  $A: X \to 2^X$  is maximal if A is maximal as a graph of the monotone operator on X; i.e., if there is a monotone operator  $B: X \to 2^X$  with  $A(x) \subseteq B(x)$  for any  $x \in X$ . Then we have A = B.

To show that the heat equation (HE; s) has a unique global solution, by the theory of monotone operators, it is sufficient to show that  $\mathcal{L}_G : \mathbb{R}^V \to 2^{\mathbb{R}^V}$  is a maximal monotone operator. In our case, the Hilbert space is  $X = \mathbb{R}^V$  equipped with the inner product  $\langle \cdot, \cdot \rangle_{D^{-1}}$  for  $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^V$ .

**Lemma 14.** The operator  $\mathcal{L}_G$  is monotone.

*Proof.* For any  $\boldsymbol{x} \in \mathbb{R}^V$  and  $\boldsymbol{y} \in \mathcal{L}_G(\boldsymbol{x})$ , we can write

$$\boldsymbol{y} = BWB^{\top}D^{-1}\boldsymbol{x} = \sum_{e \in E} w(e)\boldsymbol{b}_e \boldsymbol{b}_e^{\top} \overline{\boldsymbol{x}}_e$$

where  $\overline{\boldsymbol{x}} = D^{-1}\boldsymbol{x}$ . Further,  $W \in \mathbb{R}^{E \times E}$  is a diagonal matrix with the (e, e)-th entry being w(e).  $B = (\boldsymbol{b}_e)_{e \in E}$  is a matrix with column vectors  $\boldsymbol{b}_e \in \mathbb{R}^V$ , for which

$$oldsymbol{b}_e \in rgmax_{oldsymbol{b}_e} \langle oldsymbol{b}, \overline{oldsymbol{x}} 
angle.$$

We use this to show monotonicity. For  $x_1, x_2 \in \mathbb{R}^V$  and  $y_1 \in \mathcal{L}_G(x_1), y_2 \in \mathcal{L}_G(x_2)$ , we have

$$\boldsymbol{y}_1 = B_1 W B_1^\top \overline{\boldsymbol{x}}_1, \quad \boldsymbol{y}_2 = B_2 W B_2^\top \overline{\boldsymbol{x}}_2.$$

Then, we have

$$\langle \boldsymbol{y}_{1} - \boldsymbol{y}_{2}, \boldsymbol{x}_{1} - \boldsymbol{x}_{2} \rangle_{D^{-1}} = \langle \boldsymbol{y}_{1}, \boldsymbol{x}_{1} \rangle_{D^{-1}} + \langle \boldsymbol{y}_{2}, \boldsymbol{x}_{2} \rangle_{D^{-1}} - \langle \boldsymbol{y}_{2}, \boldsymbol{x}_{1} \rangle_{D^{-1}} - \langle \boldsymbol{y}_{1}, \boldsymbol{x}_{2} \rangle_{D^{-1}}$$

$$= \|B_{1}^{\top} \overline{\boldsymbol{x}}_{1}\|_{W}^{2} + \|B_{2}^{\top} \overline{\boldsymbol{x}}_{2}\|_{W}^{2} - \overline{\boldsymbol{x}}_{2}^{\top} B_{2} W B_{2}^{\top} \overline{\boldsymbol{x}}_{1} - \overline{\boldsymbol{x}}_{1}^{\top} B_{1} W B_{1}^{\top} \overline{\boldsymbol{x}}_{2}$$

$$\geq \|B_{1}^{\top} \overline{\boldsymbol{x}}_{1}\|_{W}^{2} + \|B_{2}^{\top} \overline{\boldsymbol{x}}_{2}\|_{W}^{2} - \overline{\boldsymbol{x}}_{2}^{\top} B_{2} W B_{1}^{\top} \overline{\boldsymbol{x}}_{1} - \overline{\boldsymbol{x}}_{1}^{\top} B_{1} W B_{2}^{\top} \overline{\boldsymbol{x}}_{2}$$

$$= \|B_{1}^{\top} \overline{\boldsymbol{x}}_{1} - B_{2}^{\top} \overline{\boldsymbol{x}}_{2}\|_{W}^{2} \geq 0.$$

**Lemma 15.** The operator  $\mathcal{L}_G$  is maximal.

*Proof.* By [18, IV.1. Proposition 1.6], it is sufficient to show that  $R(I + \mathcal{L}_G) = \mathbb{R}^V$ . This condition means that, for any  $b \in \mathbb{R}^V$ , the equation  $x + \mathcal{L}_G(x) \ni b$  has a solution x in  $\mathbb{R}^V$ . In a previous work [8, Section 3.1], an equivalent condition of the existence of the solution to  $\mathcal{L}_G(x) \ni b$  was given. By a similar argument, we can give an equivalent condition for  $x + \mathcal{L}_G(x) \ni b$  and show the existence of the solution to  $x + \mathcal{L}_G(x) \ni b$ .

We obtain the following corollary using the theory of nonlinear semigroup:

**Corollary 16.** *The heat equation* (HE; *s*) *has a unique global solution.* 

*Proof.* Immediate from Lemmas 14 and 15. See [18, IV, Proposition 3.1] for details.

#### 

### 6 Computation and Error Analysis of Difference Approximation

In this section, we prove Theorem 3. In what follows, we fix a hypergraph  $G = (V, E, w), v \in V, T \ge 1$ , and  $\lambda \in (0, 1)$ .

We first review the construction of difference approximation  $\rho_t^{\lambda}$  given in [14, Section 5.3]. By the condition (5.27) in [14] and the maximality of  $\mathcal{L}_G$ , for any  $x \in \mathbb{R}^V$ , there is a real number  $\mu$  satisfying the following conditions:

$$\begin{cases} 0 < \mu \le \lambda, \\ \boldsymbol{x}_{\mu} \in \mathbb{R}^{V}, \ \boldsymbol{y}_{\mu} \in -\mathcal{L}_{G}(\boldsymbol{x}_{\mu}), \\ \|\boldsymbol{x}_{\mu} - \boldsymbol{x} - \mu \boldsymbol{y}_{\mu}\|_{D^{-1}} < \mu \lambda. \end{cases}$$
(6)

We define  $\mu(\boldsymbol{x})$  as the least upper bound on  $\mu$  satisfying (6). We consider an initial vector  $\boldsymbol{x}_0 \in \mathbb{R}^V$ . Then, there is  $h_1 \in \mathbb{R}$  such that  $\mu(\boldsymbol{x}_0)/2 < h_1 \leq \lambda$  and there are  $\boldsymbol{x}_1 \in \mathbb{R}^V$  and  $\boldsymbol{y}_1 \in -\mathcal{L}_G(\boldsymbol{x}_1)$  satisfying  $\|\boldsymbol{x}_1 - \boldsymbol{x}_0 - h_1\boldsymbol{y}_1\|_{D^{-1}} < h_1\lambda$ . By repeating this argument, we can take sequences  $\{h_k\}, \{\boldsymbol{x}_k\}$ , and  $\{\boldsymbol{y}_k\}$  for  $k = 1, 2, \ldots$  satisfying the following conditions:

- 1.  $\mu(\boldsymbol{x}_{k-1})/2 < h_k \leq \lambda$ ,
- 2.  $\|\boldsymbol{x}_k \boldsymbol{x}_{k-1} h_k \boldsymbol{y}_k\|_{D^{-1}} < h_k \lambda.$

Let  $t_k = \sum_{j=1}^k h_j$ . Then, it is easy to show that  $\{t_k\}$ ,  $\{x_k\}$ , and  $\{y_k\}$  satisfy the following conditions for  $\{t_k^{\lambda}\}, \{x_k^{\lambda}\}, \{x_k^{\lambda}\}, \{x_k^{\lambda}\}$  and  $\{y_k^{\lambda}\}$ :

1. 
$$0 = t_0^{\lambda} < t_1^{\lambda} < \dots < t_k^{\lambda} < \dots$$
 with  $\lim_{k \to \infty} t_k^{\lambda} = \infty$ ,  
2.  $t_k^{\lambda} - t_{k-1}^{\lambda} < \lambda$   $(k = 1, 2, \dots)$ ,  
3.  $\|\boldsymbol{x}_k^{\lambda} - \boldsymbol{x}_{k-1}^{\lambda} - (t_k^{\lambda} - t_{k-1}^{\lambda})\boldsymbol{y}_k^{\lambda}\|_{D^{-1}} < \lambda(t_k^{\lambda} - t_{k-1}^{\lambda})$   $(k = 1, 2, \dots)$ .

Then, the function  $\rho_t^{\lambda}$  was defined by

$$\boldsymbol{\rho}_t^{\lambda} = \begin{cases} \boldsymbol{x}_0 & \text{if } t = 0, \\ \boldsymbol{x}_k^{\lambda} & \text{if } t \in (t_k^{\lambda}, t_{k+1}^{\lambda}] \cap (0, T]. \end{cases}$$
(7)

Theorem 3 follows from Lemmas 17 and 18 below.

**Lemma 17.** We can compute (a concise representation) of  $\{\rho_t^{\lambda}\}_{0 \le t \le T}$  for every  $0 \le t \le T$  in time polynomial in  $1/\lambda$ , T, and  $\sum_{e \in E} |e|$ .

*Proof.* From the construction of  $\rho_t^{\lambda}$ , it suffices to compute  $x_k^{\lambda}$  until  $t_k \geq T$ . Note that we can obtain  $x_k^{\lambda}$ from  $x_{k-1}^{\lambda}$  by solving the equation

$$\boldsymbol{x} - \boldsymbol{x}_{k-1}^{\lambda} \in -\lambda \mathcal{L}_G(\boldsymbol{x}),$$
 (8)

because, then, we can set  $h_k = \lambda$  and  $x_k^{\lambda}$  to be the obtained solution.

Let  $\overline{x} = D^{-1}x$  for any  $x \in \mathbb{R}^V$ . Then, solving (8) is equivalent to solving

$$D\overline{\boldsymbol{x}} - D\overline{\boldsymbol{x}}_{k-1}^{\lambda} \in -\lambda L_G(\overline{\boldsymbol{x}}).$$
(9)

By an argument similar to [8, Section 3.1], solving (9) is equivalent to computing the following proximal operator

$$\operatorname{prox}(\overline{\boldsymbol{x}}_{k-1}^{\lambda}) := \operatorname{argmin}_{\overline{\boldsymbol{x}} \in \mathbb{R}^{V}} \left( \frac{\lambda}{2} \sum_{e \in E} w(e) f_{e}(\overline{\boldsymbol{x}})^{2} + \frac{1}{2} \|\overline{\boldsymbol{x}} - \overline{\boldsymbol{x}}_{k-1}^{\lambda}\|_{D}^{2} \right),$$
(10)

which can be computed in time polynomial in  $\sum_{e \in E} |\mathcal{V}_e|$ , where  $\mathcal{V}_e$  is the set of extreme points of  $\boldsymbol{b}_e$  [8, Theorem D.1 (i)]. As  $\mathcal{V}_e \leq |e|^2$ , we can compute  $\boldsymbol{x}_k^{\lambda} = D \operatorname{prox}(\overline{\boldsymbol{x}}_{k-1}^{\lambda})$  in time polynomial in  $\sum_{e \in E} |e|$ .

As  $h_k = \lambda$ , we need to compute  $x_k^{\lambda}$  for  $k \leq \lceil T/k \rceil$ . Hence, the total time complexity is polynomial in  $1/\lambda$ , T, and  $\sum_{e \in E} |e|$ . 

**Lemma 18.** We have  $\|\boldsymbol{\rho}_t^{\lambda} - \boldsymbol{\rho}_t^{\boldsymbol{\pi}_v}\|_{D^{-1}} = O(\sqrt{\lambda T}).$ 

*Proof.* Let  $|||\mathcal{L}_G(\boldsymbol{x})||| = \inf\{||\boldsymbol{y}||_{D^{-1}} | \boldsymbol{y} \in \mathcal{L}_G(\boldsymbol{x})\}$ . We set  $N_{\lambda} \in \mathbb{Z}_+$  as  $t_{N_{\lambda}}^{\lambda} < T \leq t_{N_{\lambda}}^{\lambda}$ ,  $|\Delta_{\lambda}| = \max\{t_k^{\lambda} - t_{k-1}^{\lambda}; k = 1, 2, \dots, N_{\lambda}\}$  and  $\mathcal{E}_{\lambda} = \sum_{k=1}^{N_{\lambda}} ||\mathcal{E}_k^{\lambda}||_{D^{-1}} (t_k^{\lambda} - t_{k-1}^{\lambda})$ , where  $\mathcal{E}_k^{\lambda}$  is defined as

$$\boldsymbol{\mathcal{E}}_{k}^{\lambda} = \frac{\boldsymbol{x}_{k}^{\lambda} - \boldsymbol{x}_{k-1}^{\lambda}}{t_{k}^{\lambda} - t_{k-1}^{\lambda}} - \boldsymbol{y}_{k}^{\lambda} \quad (k = 1, 2, \dots).$$

Then, by the equation (5.20) of [14] instantiated with  $\omega_0 = 0, t = s, x_p = x$ , we have

$$\|\boldsymbol{\rho}_t^{\lambda} - \boldsymbol{\rho}_t^{\mu}\|_{D^{-1}} \leq \mathcal{E}_{\lambda} + \mathcal{E}_{\mu} + \left(\left(|\Delta_{\lambda}| + |\Delta_{\mu}|\right)^2 + |\Delta_{\lambda}|(t + |\Delta_{\lambda}|) + |\Delta_{\mu}|(t + |\Delta_{\mu}|)\right)^{\frac{1}{2}} \times ||\mathcal{L}_G(\boldsymbol{\pi}_v)|||$$

for  $t \in [0,T]$  and  $\mu > 0$ . The condition 3 for  $\{t_k^{\lambda}\}, \{x_k^{\lambda}\}$ , and  $\{y_k^{\lambda}\}$  implies  $\|\mathcal{E}_k^{\lambda}\|_{D^{-1}} < \lambda$ . Hence,  $\mathcal{E}_{\lambda} < \lambda t_{N_{\lambda}}^{\lambda} < \lambda(T+\lambda)$  as  $t_{N_{\lambda}-1}^{\lambda} < T \leq t_{N_{\lambda}}^{\lambda}$ . Therefore by taking limit  $\mu \to 0+$ , we have

$$\|\boldsymbol{\rho}_t^{\lambda} - \boldsymbol{\rho}_t^{\boldsymbol{\pi}_v}\|_{D^{-1}} < \lambda(T+\lambda) + \sqrt{\lambda^2 + \lambda(t+\lambda)} |||\mathcal{L}_G(\boldsymbol{\pi}_v)||| = O(\sqrt{\lambda T}).$$

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### A Proof of Theorem 4

*Proof.* By Corollary 16, for any initial vector s, there exists a unique solution  $\rho_t^s$  of (HE; s). Let  $\mu_t^s = D^{-1}\rho_t^s$ . By [6, §.3 and §.4], we can compute any higher right derivatives  $\frac{d^n \mu_t^s}{dt^n}|_{t=0}$ . Let  $(\sigma^*, \succ)$  be the lexicographical ordered equivalence relation on V consistent with  $\{d^n \mu_t^s/dt^n|_{t=0}\}_n$ .

For each  $e \in E$ , let  $S_e$ ,  $I_e$ ,  $S_e^{\sigma^*}$ , and  $I_e^{\sigma^*}$  be subsets introduced in §.2.1 and §.2.2. Let G' = (V', E', w') be the undirected weighted graph with respect to  $\mu_t^s$  as in §.2.1. We remark that for each  $e \in E$ ,  $w'_e(uv) \neq 0$  only if (u, v) or (v, u) is in  $S_e^{\sigma^*} \times I_e^{\sigma^*}$ , because any vertex in  $S_e \setminus S_e^{\sigma^*}$  (resp.  $I_e \setminus I_e^{\sigma^*}$ ) will leave  $S_e$  (resp.  $I_e$ ) after infinitesimal time. We take T > 0 such that if we retake ordered equivalence relation  $(\sigma^*, \succ)$  consistent with  $\{d^n \mu_t^s / dt^n |_{t=t'}\}_n$  at  $t' \in [0, T]$ , for  $e \in E$ ,  $S_e^{\sigma^*}$  and  $I_e^{\sigma^*}$  do not change. Then, for any  $t \in [0, T]$ , we have

$$\begin{split} -L_{G}(\boldsymbol{\rho}_{t}^{s}) &= -(I - A_{G}D^{-1})(\boldsymbol{\rho}_{t}^{s}) \ni -\boldsymbol{\rho}_{t}^{s} + A_{\boldsymbol{\mu}_{t}^{s}}\boldsymbol{\mu}_{t}^{s} \\ &= -\boldsymbol{\rho}_{t}^{s} + \left(\sum_{v \in V} w'(uv)\boldsymbol{\mu}_{t}^{s}(v)\right)_{u} \\ &= \left(-d_{G}(u)\boldsymbol{\mu}_{t}^{s}(u) + \sum_{v \in V} w'(uv)\boldsymbol{\mu}_{t}^{s}(v)\right)_{u} \\ &= \left(-\left(\sum_{v \in V} w'(uv)\right)\boldsymbol{\mu}_{t}^{s}(u) + \sum_{v \in V} w'(uv)\boldsymbol{\mu}_{t}^{s}(v)\right)_{u} \\ &= \left(-\left(\sum_{\substack{v \in V, \\ v \neq u}} w'(uv)\right)\boldsymbol{\mu}_{t}^{s}(u) + \sum_{\substack{v \in V, \\ v \neq u}} w'(uv)\boldsymbol{\mu}_{t}^{s}(v)\right)_{u} \\ &= \left(-\left(\sum_{\substack{v \in V, \\ v \neq u}} w'(uv)\right)(\boldsymbol{\mu}_{t}^{s}(u) - \boldsymbol{\mu}_{t}^{s}(v))\right)_{u} \end{split}$$

$$= \left(-\sum_{e \in E} \left(\sum_{v \in V, \\ v \neq u} w'_e(uv)\right) (\boldsymbol{\mu}^{\boldsymbol{s}}_t(u) - \boldsymbol{\mu}^{\boldsymbol{s}}_t(v))\right)_u.$$

If  $u \in S_e^{\sigma^*}$ , then  $w_e'(uv) \neq 0$  holds only if  $v \in I_e^{\sigma^*}$ . Hence, we have

$$\left(\sum_{\substack{v \in V, \\ v \neq u}} w'_e(uv)\right) (\boldsymbol{\mu}^{\boldsymbol{s}}_t(u) - \boldsymbol{\mu}^{\boldsymbol{s}}_t(v)) = \left(\sum_{v \in I_e^{\sigma^*}} w'_e(uv)\right) \Delta_e(\boldsymbol{\mu}^{\boldsymbol{s}}_t),$$

where  $\Delta_e(\boldsymbol{\mu}_t^s) = \max_{u,v \in e}(\boldsymbol{\mu}_t^s(u) - \boldsymbol{\mu}_t^s(v))$ . On the other hand, if  $u \in I_e^{\sigma^*}$ , then  $w'_e(uv) \neq 0$  holds only if  $v \in S_e^{\sigma^*}$ . Hence, we have

$$\left(\sum_{\substack{v \in V, \\ v \neq u}} w'_e(uv)\right) (\boldsymbol{\mu}^{\boldsymbol{s}}_t(u) - \boldsymbol{\mu}^{\boldsymbol{s}}_t(v)) = -\left(\sum_{v \in S_e^{\sigma^*}} w'_e(uv)\right) \Delta_e(\boldsymbol{\mu}^{\boldsymbol{s}}_t).$$

Using these equalities, we obtain

$$\left(-\sum_{\substack{e \in E \\ v \neq u}} \left(\sum_{\substack{v \in V, \\ v \neq u}} w'_e(uv)\right) (\boldsymbol{\mu}^s_t(u) - \boldsymbol{\mu}^s_t(v))\right)_u$$
$$= \left(-\sum_{\substack{e \in E, \\ u \in S_e^{\sigma^*}}} \left(\sum_{v \in I_e^{\sigma^*}} w'_e(uv)\right) \Delta_e(\boldsymbol{\mu}^s_t) + \sum_{\substack{e \in E, \\ u \in I_e^{\sigma^*}}} \left(\sum_{v \in S_e^{\sigma^*}} w'_e(uv)\right) \Delta_e(\boldsymbol{\mu}^s_t) + \sum_{\substack{e \in E, \\ u \in I_e^{\sigma^*}}} \left(\sum_{v \in S_e^{\sigma^*}} w'_e(uv)\right) \Delta_e(\boldsymbol{\mu}^s_t) + \sum_{\substack{e \in E, \\ u \in I_e^{\sigma^*}}} \left(\sum_{v \in S_e^{\sigma^*}} w'_e(uv)\right) \Delta_e(\boldsymbol{\mu}^s_t) + \sum_{\substack{e \in E, \\ u \in I_e^{\sigma^*}}} \left(\sum_{v \in I_e^{\sigma^*}} w'_e(uv)\right) \Delta_e(\boldsymbol{\mu}^s_t) + \sum_{\substack{e \in E, \\ u \in I_e^{\sigma^*}}} \left(\sum_{v \in I_e^{\sigma^*}} w'_e(uv)\right) \Delta_e(\boldsymbol{\mu}^s_t) + \sum_{\substack{e \in E, \\ u \in I_e^{\sigma^*}}} \left(\sum_{v \in I_e^{\sigma^*}} w'_e(uv)\right) \Delta_e(\boldsymbol{\mu}^s_t) + \sum_{\substack{e \in E, \\ u \in I_e^{\sigma^*}}} \left(\sum_{v \in I_e^{\sigma^*}} w'_e(uv)\right) \Delta_e(\boldsymbol{\mu}^s_t) + \sum_{\substack{e \in E, \\ u \in I_e^{\sigma^*}}} w'_e(uv) + \sum_{\substack{e \in E, \\ u \in I$$

Let  $C[\sigma^*] = \{U_1, \ldots, U_m\}$  be the family of  $\sigma^*$ -equivalence classes such that  $U_k \succ U_{i+1}$  and we fix  $u_k \in U_k$  for each *i*. For  $u \in U_k$ , we note  $[u]_{\sigma^*} = U_k$ .

We sum up the entries of the above vector along  $U_k$ . Then, we have

$$\sum_{u \in U_k} \left( -\sum_{\substack{e \in E, \\ u \in S_e^{\sigma^*}}} \left( \sum_{v \in I_e^{\sigma^*}} w'_e(uv) \right) \Delta_e(\boldsymbol{\mu}_t^{\boldsymbol{s}}) + \sum_{\substack{e \in E, \\ u \in I_e^{\sigma^*}}} \left( \sum_{v \in S_e^{\sigma^*}} w'_e(uv) \right) \Delta_e(\boldsymbol{\mu}_t^{\boldsymbol{s}}) \right)$$
$$= -\sum_{\substack{e \in E, \\ S_e^{\sigma^*} \cap U_k \neq \emptyset}} \left( \sum_{\substack{u \in S_e^{\sigma^*} \\ v \in I_e^{\sigma^*}}} w'_e(uv) \right) \Delta_e(\boldsymbol{\mu}_t^{\boldsymbol{s}}) + \sum_{\substack{e \in E, \\ I_e^{\sigma^*} \cap U_k \neq \emptyset}} \left( \sum_{\substack{u \in I_e^{\sigma^*} \\ v \in S_e^{\sigma^*}}} w'_e(uv) \right) \Delta_e(\boldsymbol{\mu}_t^{\boldsymbol{s}})$$
$$= -\sum_{\substack{e \in E, \\ S_e^{\sigma^*} \cap U_k \neq \emptyset}} w_e \Delta_e(\boldsymbol{\mu}_t^{\boldsymbol{s}}) + \sum_{\substack{e \in E, \\ I_e^{\sigma^*} \cap U_k \neq \emptyset}} w_e \Delta_e(\boldsymbol{\mu}_t^{\boldsymbol{s}}).$$

We remark that the last form is independent of the choice of  $w'_e(uv)$ . Now, if  $S_e^{\sigma^*} \cap U_k \neq \emptyset$ ,  $\Delta_e(\mu_t^s) = \mu_t^s(u_k) - \mu_t^s(u_l)$  for some l > k such that  $I_e^{\sigma^*} \cap U_l \neq \emptyset$ . Similarly, if  $I_e^{\sigma^*} \cap U_k \neq \emptyset$ ,  $\Delta_e(\mu_t^s) = 0$ 

 $\mu_t^s(u_l) - \mu_t^s(u_k)$  for some l < k such that  $S_e^{\sigma^*} \cap U_l \neq \emptyset$ . Hence the sum becomes

$$- \sum_{\substack{e \in E, \\ S_e^{\sigma^*} \cap U_k \neq \emptyset}} w_e \Delta_e(\mu_t^s) + \sum_{\substack{e \in E, \\ I_e^{\sigma^*} \cap U_k \neq \emptyset}} w_e \Delta_e(\mu_t^s)$$

$$= - \sum_{\substack{e \in E, \\ S_e^{\sigma^*} \cap U_k \neq \emptyset}} \sum_{\substack{l \neq k \\ I_e^{\sigma^*} \cap U_l \neq \emptyset}} w_e(\mu_t^s(u_k) - \mu_t^s(u_l)) + \sum_{\substack{e \in E, \\ I_e^{\sigma^*} \cap U_k \neq \emptyset}} \sum_{\substack{l \neq k \\ S_e^{\sigma^*} \cap U_l \neq \emptyset}} w_e(\mu_t^s(u_k) - \mu_t^s(u_l)) - \sum_{l \neq k} \sum_{\substack{e \in E, I_e^{\sigma^*} \cap U_k \neq \emptyset \\ S_e^{\sigma^*} \cap U_l \neq \emptyset}} w_e(\mu_t^s(u_k) - \mu_t^s(u_l))$$

$$= - \sum_{l \neq k} a_{kl}(\mu_t^s(u_k) - \mu_t^s(u_l)) - \sum_{l \neq k} a_{lk}(\mu_t^s(u_k) - \mu_t^s(u_l)),$$

$$= - \left(\sum_{l \neq k} (a_{kl} + a_{lk})\right) \mu_t^s(u_k) + \sum_{l \neq k} (a_{kl} + a_{lk})\mu_t^s(u_l),$$

where

$$a_{kl} = \sum_{e \in E, S_e^{\sigma^*} \cap U_k \neq \emptyset \atop I_e^{\sigma^*} \cap U_l \neq \emptyset} w_e$$

We set

$$\widetilde{w}(u_k u_l) = a_{kl} + a_{lk} \text{ for } k \neq l \text{ and } \widetilde{w}(u_k u_k) = d_{\widetilde{G}}(u_k) - \sum_{l \neq k} \widetilde{w}(u_k u_l),$$

where  $d_{\tilde{G}}(u_k) = \sum_{i \in U_k} d_G(u)$ . For  $\rho_t^s$ , we set  $\tilde{\rho}_t^s = \left(\sum_{u \in U_k} \rho_t^s(u)\right)_k \in \mathbb{R}^m$ . Then, we have  $\tilde{\rho}_t^s = \left(\left(\sum_{u \in U_k} d_G(u)\right) \mu_t^s(u_k)\right)_k = \left(d_{\tilde{G}}(u_k) \mu_t^s(u_k)\right)_k$ .

Let  $D_{\tilde{G}} = \operatorname{diag}(d_{\tilde{G}}(u_k))$  and  $\widetilde{\boldsymbol{\mu}_t^s} := (\boldsymbol{\mu}_t^s(u_k))_k = D_{\tilde{G}}^{-1} \widetilde{\boldsymbol{\rho}}_t^s \in \mathbb{R}^m$ . Then, we have

$$\begin{split} \left( \sum_{u \in U_k} (-\boldsymbol{\rho}_t^{\boldsymbol{s}}(u) + (A_{\boldsymbol{\mu}_t^{\boldsymbol{s}}} \boldsymbol{\mu}_t^{\boldsymbol{s}})(u)) \right)_k \\ &= \left( -\left( \sum_{l \neq k} \widetilde{w}(u_k u_l) \right) \boldsymbol{\mu}_t^{\boldsymbol{s}}(u_k) + \sum_{l \neq k} \widetilde{w}(u_k u_l) \boldsymbol{\mu}_t^{\boldsymbol{s}}(u_l) \right)_k \\ &= -\widetilde{\boldsymbol{\rho}}_t^{\boldsymbol{s}} + \widetilde{\boldsymbol{\rho}}_t^{\boldsymbol{s}} + \left( -\left( \sum_{l \neq k} \widetilde{w}(u_k u_l) \right) \boldsymbol{\mu}_t^{\boldsymbol{s}}(u_k) + \sum_{l \neq k} \widetilde{w}(u_k u_l) \boldsymbol{\mu}_t^{\boldsymbol{s}}(u_l) \right)_k \\ &= -\widetilde{\boldsymbol{\rho}}_t^{\boldsymbol{s}} + \left( \left( d_{\tilde{G}}(u_k) - \sum_{l \neq k} \widetilde{w}(u_k u_l) \right) \boldsymbol{\mu}_t^{\boldsymbol{s}}(u_k) + \sum_{l \neq k} \widetilde{w}(u_k u_l) \boldsymbol{\mu}_t^{\boldsymbol{s}}(u_l) \right)_k \\ &= -\widetilde{\boldsymbol{\rho}}_t^{\boldsymbol{s}} + \left( \widetilde{w}(u_k u_l) \right)_{k,l} \widetilde{\boldsymbol{\mu}}_t^{\boldsymbol{s}} = -(I - (\widetilde{w}(u_k u_l))_{k,l} D_{\tilde{G}}^{-1}) \widetilde{\boldsymbol{\rho}}_t^{\boldsymbol{s}}. \end{split}$$

 $D_G^{-1} \rho_t^s$ . By the definition of  $\sigma^*$ ,  $\mu_t^s(u) = \mu_t^s(v)$  if  $u \sim_{\sigma^*} v$  until the next tie occurs. We remark that  $\rho_t^s$  until the next tie occurs is determined by  $\mu_t^s(u_k)$ , i = 1, ..., m. Also  $\frac{d\mu_t^s(u)}{dt} = \frac{d\mu_t^s(v)}{dt}$  holds for such u, v and t. Hence, we have

$$\sum_{u \in U_k} \frac{d\boldsymbol{\rho}_t^s}{dt}(u) = d_{\tilde{G}}(u_k) \frac{d\boldsymbol{\mu}_t^s}{dt}(u_k)$$

Let  $\tilde{\rho}_t^s = \left(\sum_{u \in U_k} \rho_t^s(u)\right)_k \in \mathbb{R}^m$ , and  $\tilde{\mu}_t^s := D_{\tilde{G}}^{-1} \tilde{\rho}_t^s$ . By the argument above,  $\tilde{\rho}_t^s$  is the unique solution of the heat equation

$$\frac{d\widetilde{\rho}_t}{dt} = -\mathcal{L}_{\widetilde{G}}\widetilde{\rho}_t, \quad \widetilde{\rho}_0 = \widetilde{s}.$$
(11)

This solution  $\tilde{\rho}_t^s$  determines  $\tilde{\mu}_t^s$ , and hence  $\mu_t^s$ . If  $u \in U_k$ , then

$$\boldsymbol{\rho}_t^{\boldsymbol{s}}(u) = d_G(u)\boldsymbol{\mu}_t^{\boldsymbol{s}}(u_k)$$

holds. Hence, we can recover  $\rho_t^s$  from the heat equation (11).

#### B **Proofs of Section 4**

#### **B.1 Useful lemmas**

In this section, we derive several inequalities on  $f_i$  that will be useful later. Note that the proofs are deferred to Section B. We define  $\mathcal{R}_i : \mathbb{R}^{\widetilde{V}_i} \to \mathbb{R}$  as

$$\mathcal{R}_{i}(\boldsymbol{x}) = \frac{\boldsymbol{x}^{\top} L_{\widetilde{G}_{i}} \boldsymbol{x}}{\|\boldsymbol{x}\|_{D_{\widetilde{G}_{i}}}^{2}} = \frac{\sum_{uv \in \widetilde{E}_{i}} (\boldsymbol{x}(u) - \boldsymbol{x}(v))^{2} \widetilde{w}_{i}(uv)}{\sum_{v \in \widetilde{V}_{i}} \boldsymbol{x}(v)^{2} d_{\widetilde{G}_{i}}(v)}.$$
(12)

**Lemma 19.** For any  $i \in \mathbb{Z}_+$ , we have

$$\frac{d}{d\Delta}\log f_i(\Delta) = \frac{\widetilde{\rho}_{i,0}^{\top} D_{\widetilde{G}_i}^{-1} \frac{d}{d\Delta} \widetilde{\rho}_{i,\Delta}}{\widetilde{\rho}_{i,0}^{\top} D_{\widetilde{G}_i}^{-1} (\widetilde{\rho}_{i,\Delta} - \widetilde{\pi}^i)} = -\mathcal{R}_i \left( \frac{\widetilde{\rho}_{i,\Delta/2}}{d_{\widetilde{G}_i}} - \frac{1}{\operatorname{vol}(\widetilde{V}_i)} \right).$$

*Proof.* We first prove the following lemma:

**Claim 20.** For any  $i \in \mathbb{Z}_+$  and  $\Delta \ge 0$ , we have

$$\widetilde{\rho}_{i,0}^{\top} D_{\widetilde{G}_{i}}^{-1} \frac{d\widetilde{\rho}_{i,\Delta}}{d\Delta} = -\left(D_{\widetilde{G}_{i}}^{-1} \widetilde{\rho}_{i,\Delta/2}\right)^{\top} \left(D_{\widetilde{G}_{i}} - A_{\widetilde{G}_{i}}\right) \left(D_{\widetilde{G}_{i}}^{-1} \widetilde{\rho}_{i,\Delta/2}\right)$$
$$= -\sum_{uv \in \widetilde{E}_{i}} \left(\frac{\widetilde{\rho}_{i,\Delta/2}(u)}{d_{\widetilde{G}_{i}}(u)} - \frac{\widetilde{\rho}_{i,\Delta/2}(v)}{d_{\widetilde{G}_{i}}(v)}\right)^{2} \widetilde{w}_{i}(uv) \leq 0,$$

where  $A_{\widetilde{G}_i}$  is the adjacency matrix of  $\widetilde{G}_i$ .

Proof. We have

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$$\begin{split} \widetilde{\boldsymbol{\rho}}_{i,0}^{\top} D_{\tilde{G}_{i}}^{-1} \frac{d\boldsymbol{\rho}_{i,\Delta}}{d\Delta} &= -\widetilde{\boldsymbol{\rho}}_{i,0}^{\top} D_{\tilde{G}_{i}}^{-1} H_{i,\Delta} \mathcal{L}_{i} \widetilde{\boldsymbol{\rho}}_{i,0} = -\widetilde{\boldsymbol{\rho}}_{i,0}^{\top} D_{\tilde{G}_{i}}^{-1} H_{i,\Delta/2} \mathcal{L}_{i} \widetilde{\boldsymbol{\rho}}_{i,0} \quad \text{(by } H_{i,\Delta} = H_{i,\Delta/2} H_{i,\Delta/2} ) \\ &= -\widetilde{\boldsymbol{\rho}}_{i,0}^{\top} (H_{i,\Delta/2})^{\top} D_{\tilde{G}_{i}}^{-1} (D_{\tilde{G}_{i}} - A_{\tilde{G}_{i}}) D_{\tilde{G}_{i}}^{-1} H_{i,\Delta/2} \widetilde{\boldsymbol{\rho}}_{i,0} \qquad \text{(by } D_{\tilde{G}_{i}} H_{i,\Delta/2} = (H_{i,\Delta/2})^{\top} D_{\tilde{G}_{i}} ) \\ &= - (D_{\tilde{G}_{i}}^{-1} \widetilde{\boldsymbol{\rho}}_{i,\Delta/2})^{\top} (D_{\tilde{G}_{i}} - A_{\tilde{G}_{i}}) (D_{\tilde{G}_{i}}^{-1} \widetilde{\boldsymbol{\rho}}_{i,\Delta/2}). \end{split}$$

The second equality in the statement is obtained through a direct calculation.

We are now ready to prove Lemma 19. The first equality is obtained through direct calculation and the second equality follows from Proposition 5 and Lemma 20.  $\hfill \Box$ 

**Lemma 21.** For any  $i \in \mathbb{Z}_+$ , we have

$$\frac{d^2}{d\Delta^2}\log f_i(\Delta) \ge 0.$$

*Proof.* By Lemma 19, we have

$$-\frac{d^2}{d\Delta^2}(-\log(\widetilde{\rho}_{i,0}^{\top}D_{\tilde{G}_i}^{-1}(\widetilde{\rho}_{i,\Delta}-\widetilde{\pi}^i))) = \frac{d}{d\Delta} \left(-\frac{\widetilde{\rho}_{i,0}^{\top}D_{\tilde{G}_i}^{-1}\frac{d}{d\Delta}\widetilde{\rho}_{i,\Delta}}{\widetilde{\rho}_{i,0}^{\top}D_{\tilde{G}_i}^{-1}(\widetilde{\rho}_{i,\Delta}-\widetilde{\pi}^i)}\right)$$
$$= \frac{d}{d\Delta} \left(\frac{\widetilde{\rho}_{i,0}^{\top}D_{\tilde{G}_i}^{-1}\mathcal{L}_i\widetilde{\rho}_{i,\Delta}}{\widetilde{\rho}_{i,0}^{\top}D_{\tilde{G}_i}^{-1}(\widetilde{\rho}_{i,\Delta}-\widetilde{\pi}^i)}\right) = \frac{\widetilde{\rho}_{i,0}^{\top}D_{\tilde{G}_i}^{-1}(\widetilde{\rho}_{i,\Delta}-\widetilde{\pi}^i)\widetilde{\rho}_{i,0}^{\top}D_{\tilde{G}_i}^{-1}\mathcal{L}_i^2\widetilde{\rho}_{i,\Delta} - (\widetilde{\rho}_{i,0}^{\top}D_{\tilde{G}_i}^{-1}\mathcal{L}_i\widetilde{\rho}_{i,\Delta})^2}{(\widetilde{\rho}_{i,0}^{\top}D_{\tilde{G}_i}^{-1}(\widetilde{\rho}_{i,\Delta}-\widetilde{\pi}^i))^2}.$$

It is sufficient to check the positivity of the numerator. Note that the numerator can be written as

$$(\widetilde{\boldsymbol{\rho}}_{i,0}^{\top} D_{\widetilde{G}_{i}}^{-1} (\widetilde{\boldsymbol{\rho}}_{i,\Delta} - \widetilde{\boldsymbol{\pi}}^{i})) (\widetilde{\boldsymbol{\rho}}_{i,0}^{\top} D_{\widetilde{G}_{i}}^{-1} \mathcal{L}_{i}^{2} \widetilde{\boldsymbol{\rho}}_{i,\Delta}) - (\widetilde{\boldsymbol{\rho}}_{i,0}^{\top} D_{\widetilde{G}_{i}}^{-1} \mathcal{L}_{i} \widetilde{\boldsymbol{\rho}}_{i,\Delta})^{2}.$$

$$(13)$$

The first factor of the first term of (13) is

$$\widetilde{\boldsymbol{\rho}}_{i,0}^{\top} D_{\widetilde{G}_i}^{-1} (\widetilde{\boldsymbol{\rho}}_{i,\Delta} - \widetilde{\boldsymbol{\pi}}^i) = \| D_{\widetilde{G}_i}^{-1/2} (\widetilde{\boldsymbol{\rho}}_{i,\Delta/2} - \widetilde{\boldsymbol{\pi}}^i) \|^2$$

by Proposition 5. The second factor of the first term of (13) is

$$\begin{split} \widetilde{\rho}_{i,0}^{\top} D_{\tilde{G}_{i}}^{-1} \mathcal{L}_{i}^{2} \widetilde{\rho}_{i,\Delta} &= \widetilde{\rho}_{i,0}^{\top} D_{\tilde{G}_{i}}^{-1} (I - A_{\tilde{G}_{i}} D_{\tilde{G}_{i}}^{-1})^{2} H_{i,\Delta} \widetilde{\rho}_{i,0} \\ &= \widetilde{\rho}_{i,0}^{\top} D_{\tilde{G}_{i}}^{-1} D_{\tilde{G}_{i}} (H_{i,\Delta/2})^{\top} D_{\tilde{G}_{i}}^{-1} (I - A_{\tilde{G}_{i}} D_{\tilde{G}_{i}}^{-1})^{2} H_{i,\Delta/2} \widetilde{\rho}_{i,0} \\ &= \widetilde{\rho}_{i,0}^{\top} (H_{i,\Delta/2})^{\top} D_{\tilde{G}_{i}}^{-1} (D_{\tilde{G}_{i}} - A_{\tilde{G}_{i}}) D_{\tilde{G}_{i}}^{-1} (D_{\tilde{G}_{i}} - A_{\tilde{G}_{i}}) D_{\tilde{G}_{i}}^{-1} H_{i,\Delta/2} \widetilde{\rho}_{i,0} \\ &= \| D_{\tilde{G}_{i}}^{-1/2} (D_{\tilde{G}_{i}} - A_{\tilde{G}_{i}}) D_{\tilde{G}_{i}}^{-1} H_{i,\Delta/2} \widetilde{\rho}_{i,0} \|^{2} \\ &= \| D_{\tilde{G}_{i}}^{-1/2} (I - A_{\tilde{G}_{i}} D_{\tilde{G}_{i}}^{-1}) \widetilde{\rho}_{i,\Delta/2} \|^{2} \\ &= \| D_{\tilde{G}_{i}}^{-1/2} \mathcal{L}_{i} \widetilde{\rho}_{i,\Delta/2} \|^{2}. \end{split}$$

The second term of (13) is

$$\widetilde{\boldsymbol{\rho}}_{i,0}^{\top} D_{\tilde{G}_{i}}^{-1} \mathcal{L}_{i} \widetilde{\boldsymbol{\rho}}_{i,\Delta} = \widetilde{\boldsymbol{\rho}}_{i,0}^{\top} D_{\tilde{G}_{i}}^{-1} (I - A_{\tilde{G}_{i}} D_{\tilde{G}_{i}}^{-1}) H_{i,\Delta} \widetilde{\boldsymbol{\rho}}_{i,0}$$

$$= \widetilde{\boldsymbol{\rho}}_{i,0}^{\top} (H_{i,\Delta/2})^{\top} D_{\tilde{G}_{i}}^{-1} (I - A_{\tilde{G}_{i}} D_{\tilde{G}_{i}}^{-1}) H_{i,\Delta/2} \widetilde{\boldsymbol{\rho}}_{i,0}$$

$$= \widetilde{\boldsymbol{\rho}}_{i,\Delta/2}^{\top} D_{\tilde{G}_{i}}^{-1} (I - A_{\tilde{G}_{i}} D_{\tilde{G}_{i}}^{-1}) \widetilde{\boldsymbol{\rho}}_{i,\Delta/2} = \widetilde{\boldsymbol{\rho}}_{i,\Delta/2}^{\top} D_{\tilde{G}_{i}}^{-1} \mathcal{L}_{i} \widetilde{\boldsymbol{\rho}}_{i,\Delta/2}.$$
(14)

We can rephrase (14) as the inner product of the vectors  $D_{\tilde{G}_i}^{-1/2} \mathcal{L}_i \widetilde{\rho}_{i,\Delta/2}$  and  $D_{\tilde{G}_i}^{-1/2} (\widetilde{\rho}_{i,\Delta/2} - \widetilde{\pi}^i)$ , as follows:

$$(D_{\tilde{G}_{i}}^{-1/2} \mathcal{L}_{i} \widetilde{\rho}_{i,\Delta/2})^{\top} D_{\tilde{G}_{i}}^{-1/2} (\widetilde{\rho}_{i,\Delta/2} - \widetilde{\pi}^{i}) = \widetilde{\rho}_{i,\Delta/2}^{\top} \mathcal{L}_{i}^{\top} D_{\tilde{G}_{i}}^{-1/2} D_{\tilde{G}_{i}}^{-1/2} (\widetilde{\rho}_{i,\Delta/2} - \widetilde{\pi}^{i})$$
$$= \widetilde{\rho}_{i,\Delta/2}^{\top} \mathcal{L}_{i}^{\top} D_{\tilde{G}_{i}}^{-1} \widetilde{\rho}_{i,\Delta/2} - \widetilde{\rho}_{i,\Delta/2}^{\top} \mathcal{L}_{i}^{\top} D_{\tilde{G}_{i}}^{-1} \widetilde{\pi}^{i} = \widetilde{\rho}_{i,\Delta/2}^{\top} \mathcal{L}_{i}^{\top} D_{\tilde{G}_{i}}^{-1} \widetilde{\rho}_{i,\Delta/2},$$

where the last equality follows from

$$\widetilde{\boldsymbol{\rho}}_{i,\Delta/2}^{\top} \mathcal{L}_{i}^{\top} D_{\tilde{G}_{i}}^{-1} \widetilde{\boldsymbol{\pi}}^{i} = \widetilde{\boldsymbol{\rho}}_{i,\Delta/2}^{\top} \mathcal{L}_{i}^{\top} \frac{1}{\operatorname{vol}(V_{i})} \mathbf{1}$$

and  $\mathcal{L}_i^{\top} \mathbf{1} = D_{\tilde{G}_i}^{-1} (D_{\tilde{G}_i} - A_{\tilde{G}_i}) \mathbf{1} = D_{\tilde{G}_i}^{-1} \mathbf{0} = \mathbf{0}.$ Hence, we have

$$(13) = \|D_{\tilde{G}_{i}}^{-1/2} \mathcal{L}_{i} \widetilde{\rho}_{i,\Delta/2}\|^{2} \cdot \|D_{\tilde{G}_{i}}^{-1/2} (\widetilde{\rho}_{i,\Delta/2} - \widetilde{\pi}^{i})\|^{2} - \left( (D_{\tilde{G}_{i}}^{-1/2} \mathcal{L}_{i} \widetilde{\rho}_{i,\Delta/2})^{\top} D_{\tilde{G}_{i}}^{-1/2} (\widetilde{\rho}_{i,\Delta/2} - \widetilde{\pi}^{i}) \right)^{2} \ge 0,$$

where the last inequality follows from the Cauchy-Schwarz inequality.

### **B.2** Proof of Lemma 6

Proof. We recall that

$$\widetilde{\boldsymbol{\rho}}_{i,\Delta/2}(u_k^i) = \sum_{v \in U_k^i} \boldsymbol{\rho}_{i,\Delta/2}(v) = \left(\sum_{v \in U_k^i} d_G(v)\right) \boldsymbol{\mu}_{i,\Delta/2}(u_k^i).$$

Hence, we obtain

$$\begin{split} \|\widetilde{\rho}_{i,\Delta/2} - \widetilde{\pi}^{i}\|_{D_{\tilde{G}_{i}}^{-1}}^{2} &= (\widetilde{\rho}_{i,\Delta/2} - \widetilde{\pi}^{i})^{\top} D_{\tilde{G}_{i}}^{-1} (\widetilde{\rho}_{i,\Delta/2} - \widetilde{\pi}^{i}) \\ &= \sum_{k=1}^{m_{i}} \frac{1}{d_{\tilde{G}_{i}}(u_{k}^{i})} (\widetilde{\rho}_{i,\Delta/2}(u_{k}^{i}) - \widetilde{\pi}^{i}(u_{k}^{i}))^{2} \\ &= \sum_{k=1}^{m_{i}} \frac{1}{d_{\tilde{G}_{i}}(u_{k}^{i})} \left( \sum_{u \in U_{k}^{i}} \rho_{i,\Delta/2}(u) - \frac{d_{\tilde{G}_{i}}(u_{k}^{i})}{\operatorname{vol}(\widetilde{V}_{i})} \right)^{2} \\ &= \sum_{k=1}^{m_{i}} d_{\tilde{G}_{i}}(u_{k}^{i}) \left( \mu_{i,\Delta/2}(u_{k}^{i}) - \frac{1}{\operatorname{vol}(V)} \right)^{2}. \end{split}$$

On the other hand, the norm on G becomes the following:

$$\begin{split} \| \boldsymbol{\rho}_{i,\Delta/2} - \boldsymbol{\pi}^{i} \|_{D^{-1}}^{2} &= \sum_{u \in V} \frac{1}{d_{G}(u)} (\boldsymbol{\rho}_{i,\Delta/2}(u) - \boldsymbol{\pi}(u))^{2} \\ &= \sum_{k=1}^{m_{i}} \sum_{u \in U_{k}^{i}} \frac{1}{d_{G}(u)} \left( \boldsymbol{\rho}_{i,\Delta/2}(u) - \frac{d_{G}(u)}{\operatorname{vol}(V)} \right)^{2} \\ &= \sum_{k=1}^{m_{i}} \sum_{u \in U_{k}^{i}} d_{G}(u) \left( \boldsymbol{\mu}_{i,\Delta/2}(u_{k}^{i}) - \frac{1}{\operatorname{vol}(V)} \right)^{2} \\ &= \sum_{k=1}^{m_{i}} d_{\tilde{G}_{i}}(u_{k}) \left( \boldsymbol{\mu}_{i,\Delta/2}(u_{k}^{i}) - \frac{1}{\operatorname{vol}(V)} \right)^{2}. \end{split}$$

### **B.3** Proof of Lemma 7

We first derive a lower bound on the log derivative of  $f_i(\Delta)$ .

**Lemma 22.** For any  $i \in \mathbb{Z}_+$  and  $\Delta \ge 0$ , we have

$$-\frac{d}{d\Delta}\log f_i(\Delta) \ge \frac{\widetilde{\kappa}_{i,\Delta/2}^2}{2}.$$

Proof of Lemma 22. By Lemma 19, we have

$$-\frac{d}{d\Delta}\log f_i(\Delta) = \mathcal{R}_i\left(\frac{\widetilde{\rho}_{i,\Delta/2}}{d_{\widetilde{G}_i}} - \frac{1}{\operatorname{vol}(\widetilde{V}_i)}\right).$$

Then, by applying Cheeger's inequality on the vector  $\widetilde{
ho}_{i,\Delta/2}/d_{\widetilde{G}_i}$ , we obtain

$$\max_{c \in \mathbb{R}} \mathcal{R}_i \left( \frac{\widetilde{\rho}_{i,\Delta/2}}{d_{\widetilde{G}_i}} - c \right) \ge \frac{\widetilde{\kappa}_{i,\Delta/2}^2}{2}.$$

Hence, it suffices to show that the left hand side (LHS) attains the maximum value when  $c = 1/\text{vol}(\widetilde{V}_i)$ . Let  $\varphi \colon \mathbb{R} \to \mathbb{R}$  be the denominator of the LHS (recall (12)) as a function of c. Then,

$$\varphi'(c) = -2\sum_{v\in\widetilde{V}_i} \left(\frac{\widetilde{\rho}_{i,\Delta/2}(v)}{d_{\widetilde{G}_i}(v)} - c\right) d_{\widetilde{G}_i}(v).$$

Hence  $\varphi'(c) = 0$  yields

$$\sum_{v\in \widetilde{V}_i}\widetilde{\rho}_{i,\Delta/2}(v)-\left(\sum_{v\in \widetilde{V}_i}d_{\widetilde{G}_i}(v)\right)c=0,$$

which implies  $c = 1/\mathrm{vol}(\widetilde{V}_i)$  attains the minimum of  $\varphi$ .

Proof of Lemma 7. We are now ready to prove Lemma 7. By Lemma 22, we have

$$\log f_{i_1}(2(t-t_{i_1})) - \log f_{i_1}(0) \leq -\widetilde{\kappa}_{i_1,[0,\Delta]}^2(t-t_{i_1}),$$
  
$$\log f_j(2(t_{j+1}-t_j)) - \log f_j(0) \leq -\widetilde{\kappa}_j^2(t_{j+1}-t_j) \quad (j=0,\ldots,i-1),$$
  
$$\log f_{i_0}(2(t_{i_0+1}-t_{i_0})) - \log f_{i_0}(2T-2t_{i_0}) \leq -\widetilde{\kappa}_{i_0,[T-t_{i_0},t_{i_0}+1-t_{i_0}]}^2(t_{i_0+1}-T),$$

Hence, we have

$$\begin{split} f_{i_1}(t-t_{i_1}) &\leq f_{i_1}(0) \exp\left(-\widetilde{\kappa}_{i_1,[0,t-t_{i_1}]}^2(t-t_{i_1})\right) = \|\widetilde{\rho}_{i_1,0} - \widetilde{\pi}^{i_1}\|_{D_{\tilde{G}_{i_1}}^{-1}}^2 \exp\left(-\widetilde{\kappa}_{i_1,[0,t-t_{i_1}]}^2(t-t_{i_1})\right) \\ &= \|\widetilde{\rho}_{i_1-1,t_{i_1}-t_{(i_1-1)}} - \widetilde{\pi}^{(i_1-1)}\|_{D_{\tilde{G}_{(i_1-1)}}^{-1}}^2 \exp\left(-\widetilde{\kappa}_{i_1,[0,t-t_{i_1}]}^2(t-t_{i_1})\right) \\ &= f_{(i_1-1)}(2(t_{i_1} - t_{(i_1-1)})) \exp\left(-\widetilde{\kappa}_{i_1,[0,t-t_{i_1}]}^2(t-t_{i_1})\right) \\ &\leq f_{(i_1-1)}(0) \left(-\widetilde{\kappa}_{i_1,[0,t-t_{i_1}]}^2(t-t_{i_1}) - \widetilde{\kappa}_{(i_1-1)}^2(t_{i_1} - t_{(i_1-1)})\right)\right) \leq \cdots \\ &\leq f_{i_0}(2T - 2t_{i_0}) \exp\left(-\widetilde{\kappa}_{i_1,[0,t-t_{i_1}]}^2(t-t_{i_1}) - \sum_{j=i_0+1}^{i_1-1} \widetilde{\kappa}_j^2(t_{j+1} - t_j) - \widetilde{\kappa}_{i_0,[T-t_{i_0},t_{i_0+1}-t_{i_0}]}^2(t_{i_0+1} - T)\right) \\ &\leq \|\rho_T^{\pi_v} - \pi\|_{D^{-1}}^2 \exp(-\widetilde{\kappa}_{T,t}^v(t-T)). \qquad \Box$$

### **B.4** Proof of Lemma 8

*Proof.* As in [5, Lemma 4.11, 3], the derivative of the Rayleigh quotient  $\langle \rho_t^{\pi_v}, \mathcal{L}\rho_t^{\pi_v} \rangle_{D^{-1}} / \|\rho_t^{\pi_v} - \pi\|_{D^{-1}}^2$  is non-positive, hence this does not increase about t. By this monotonicity, we have

$$-\frac{d}{dt}\log\|\boldsymbol{\rho}_t^{\boldsymbol{\pi}_v}-\boldsymbol{\pi}\|_{D^{-1}}^2 = 2\frac{\langle \boldsymbol{\rho}_t^{\boldsymbol{\pi}_v}, \mathcal{L}\boldsymbol{\rho}_t^{\boldsymbol{\pi}_v}\rangle_{D^{-1}}}{\|\boldsymbol{\rho}_t^{\boldsymbol{\pi}_v}-\boldsymbol{\pi}\|_{D^{-1}}^2} \le 2\frac{\langle \boldsymbol{\rho}_T^{\boldsymbol{\pi}_v}, \mathcal{L}\boldsymbol{\rho}_T^{\boldsymbol{\pi}_v}\rangle_{D^{-1}}}{\|\boldsymbol{\rho}_T^{\boldsymbol{\pi}_v}-\boldsymbol{\pi}\|_{D^{-1}}^2} = g_v(T).$$

By integrating this on [T, t], we obtain the claimed inequality.

### **B.5** Proof of Lemma 11

*Proof.* Let  $\{U_1, U_2, \ldots, U_m\} \subseteq 2^V$  be the  $\sigma^*$ -equivalence classes such that  $U_k \succ U_{k+1}$   $(k = 1, \ldots, m-1)$ , i.e., for any  $u \in U_k, v \in U_{k+1}, \mathbf{x}(u) > \mathbf{x}(v)$ . Then, the sweep set S can be written by

$$S^a = S_i := U_1 \cup \cdots \cup U_i$$

for a certain integer i. We recall that the conductance of this S on G is

$$\phi_G(S_i) = \frac{\sum_{\substack{e \in E, e \cap S_i \neq \emptyset \\ e \cap V \setminus S_i \neq \emptyset}} w_e}{\min\{\operatorname{vol}(S_i), \operatorname{vol}(V \setminus S_i)\}}.$$

Now,  $\widetilde{S}^a$  is equal to  $\widetilde{S}_i = \{u_1, u_2, \dots, u_i\}$  for same *i*. Then, the conductance  $\phi_{\widetilde{G}}(\widetilde{S}_i)$  is

$$\phi_{\widetilde{G}}(\widetilde{S}_i) = \frac{\sum_{\substack{uv \in \widetilde{E}, uv \cap \widetilde{S}_i \neq \emptyset}} \widetilde{w}(uv)}{\min\{\operatorname{vol}(\widetilde{S}_i), \operatorname{vol}(\widetilde{V} \setminus \widetilde{S}_i)\}}$$

By simple calculation, we can show that the denominators are equal. We check the equality of the numerators here.

$$\sum_{\substack{uv \in \tilde{E}, uv \cap \tilde{S}_i \neq \emptyset \\ uv \cap \tilde{V} \setminus \tilde{S}_i \neq \emptyset}} \widetilde{w}(uv) = \sum_{j \le i} \sum_{k \ge i+1} \widetilde{w}(u_j u_k)$$
$$= \sum_{j \le i} \sum_{k \ge i+1} \left( \sum_{\substack{e \in E, S_e^{\sigma^*} \cap U_j \neq \emptyset \\ I_e^{\sigma^*} \cap U_k \neq \emptyset}} w_e + \sum_{\substack{e \in E, S_e^{\sigma^*} \cap U_j \neq \emptyset \\ I_e^{\sigma^*} \cap U_k \neq \emptyset}} w_e \right)$$
$$= \sum_{\substack{e \in E, S_e^{\sigma^*} \cap S_i \neq \emptyset \\ I_e^{\sigma^*} \cap V \setminus S_i \neq \emptyset}} w_e = \sum_{\substack{e \in E, e \cap S_i \neq \emptyset \\ e \cap V \setminus S_i \neq \emptyset}} w_e.$$

Hence, the numerators are also the same.