Strings from linear recurrences and permutations: a Gray code

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Abstract

Each positive increasing integer sequence $\{a_n\}_{n\geq 0}$ can serve as a numeration system to represent each non-negative integer by means of suitable coefficient strings. We analyse the case of k-generalized Fibonacci sequences leading to the binary strings avoiding 1^k. We prove a bijection between the set of strings of length n and the set of permutations of $S_{n+1}(321, 312, 23\dots(k+1)1)$. Finally, basing on a known Gray code for those strings, we define a Gray code for $S_{n+1}(321, 312, 23\dots(k+1)1)$, where two consecutive permutations differ by an adjacent transposition.

Keywords: Gray code, numeration systems, k-generalized Fibonacci sequences.

1 Introduction

In [5] the authors asked for a combinatorial interpretation of the recurrence $f_{m+1} = 6f_m - f_{m-1}$, with $f_0 = 1, f_1 = 7$ (sequence M4423 of [13]). A general solution appeared in [4] where a combinatorial interpretation for the recurrences of the form $a_m = ka_{m-1} + ha_{m-2}$ and underlying some conditions on h and k, was given. In particular, by considering the sequence arising from the recurrence as a numeration system [8] able to represent each non-negative integer as strings (see next section), it is possible to completely characterize the language of these strings and give a recursive construction for such a language.

Recently ([1, 2]), a Gray code has been defined for the languages deriving from the recurrences $a_m = ka_{m-1} + ha_{m-2}$ for $k \geq h \geq 0$ and $a_m = ka_{m-1} - ha_{m-2}$ with $k > h > 0$ and h even. In this paper we continue this research considering the well known k-generalized Fibonacci sequence. A first result shows that in this case the language is the set $F^{(k)}$ of the binary strings

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avoiding k consecutive 1's (i.e. the pattern 1^k) and vice versa: each binary string avoiding 1^k is the representation of a unique non-negative integer. Moreover, by reading the strings of the set $F_n^{(k)}$ of the binary strings of $F^{(k)}$ of length n as the inversion arrays of permutations of length $n+1$, we show that the strings are in bijection with the set $S_{n+1}(321, 312, 23\ldots(k+1)1)$ of avoiding 321, 312, and $23\ldots(k+1)1$ permutations [6]. It is already known that the strings in $F_n^{(k)}$ can be listed in a Gray code order with Hamming distance [10] equal to 1 (see [15]). We show that this Gray code [9] can be transferred to $S_{n+1}(321, 312, 23\ldots(k+1)1)$ where two consecutive permutations differ only for an adjacent transposition (i.e. switching two consecutive entries).

2 Preliminaries

Given a sequence $\{a_m\}_{m\geq 0}$ of integers such that $a_0 = 1$ and $a_m < a_{m+1}$ for each $m \in \mathbb{N}$, let N be any non-negative integer. Consider the largest term a_n of the sequence such that $a_n \leq N$. More precisely, $a_n = \max\{a_m \mid a_m \leq N\}$ (for the particular case $N = 0$, see below). We divide N by a_n obtaining $N = d_n a_n + r_n$. Obviously, for the remainder r_n, it is clear that $r_n < a_n$. If we divide r_n by a_{n-1}, we get $r_n = d_{n-1}a_{n-1} + r_{n-1}$, with $r_{n-1} < a_{n-1}$. Then, iterating this procedure until the division by $a_0 = 1$ (where of course the remainder is 0), we have:

$$
\begin{array}{rcllr}
N & = & d_n a_n + r_n & \qquad\qquad & 0 \leq r_n < a_n \ , \\
r_n & = & d_{n-1}a_{n-1} + r_{n-1} & & 0 \leq r_{n-1} < a_{n-1} \ , \\
r_{n-1} & = & d_{n-2}a_{n-2} + r_{n-2} & & 0 \leq r_{n-2} < a_{n-2} \ , \\
\cdots & = & \cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots \\
\cdots & = & \cdots\cdots\cdots\cdots & & \cdots\cdots\cdots\cdots \\
r_3 & = & d_2 a_2 + r_2 & & 0 \leq r_2 < a_2 \ , \\
r_2 & = & d_1 a_1 + r_1 & & 0 \leq r_1 < a_1 \ , \\
r_1 & = & d_0 a_0 \ . & &
\end{array}
$$

The above relations imply that:

$$N = d_n a_n + d_{n-1}a_{n-1} + d_{n-2}a_{n-2} + \ldots\ldots + d_1 a_1 + d_0 a_0 \ . \tag{1}$$

Expression (1) is the representation of N in the numeration system $S = \{a_0, a_1, a_2, \ldots\ldots\}$, and the string $d_n d_{n-1}\ldots d_1 d_0$ is associated to the number

N (in what follows the term "representation" equivalently refers either to expression (1) or to its associated string). This method [8] can be applied to every non-negative integer and in the case $N = 0$, clearly, all the coefficients d_i are 0 (in other words the representation of 0 is simply the string 0). Moreover, we have

$$r_i = d_{i-1}a_{i-1} + d_{i-2}a_{i-2} + \ldots\ldots + d_1a_1 + d_0a_0 < a_i \ , \tag{2}$$

for each $i \geq 0$.

It is possible to show [8] that if $N = \sum_{i\geq 0}^{n} d_i a_i$ with

$$d_ia_i + d_{i-1}a_{i-1} + \ldots + d_1a_1 + d_0a_0 < a_{i+1} \tag{3}$$

for each $i \geq 0$, then the representation $N = \sum_{i\geq 0}^{n} d_i a_i$ is unique. For the sake of completeness, we recall the complete theorem:

Theorem 2.1 *Let $1 = a_0 < a_1 < a_2 < \ldots$ be any finite or infinite sequence of integers. Any non-negative integer N has precisely one representation in the system $S = \{a_0, a_1, a_2, ...\}$ of the form $N = \displaystyle\sum_{i\geq 0}^{n} d_i a_i$ where the d_i are non-negative integers satisfying (3).*

As an example, consider the well-known sequence of Pell numbers (sequence M1413 in [13]) $p_m = 1, 2, 5, 12, 29, \ldots$ defined by $p_0 = 1$, $p_1 = 2$, $p_m = 2p_{m-1} + p_{m-2}$. The representation of $N = 16$ is associated to the string 1020.

3 Strings from a number sequence

Given a sequence $\{a_m\}_{m\geq 0}$, for a fixed $m > 0$, we consider all the integers $\ell \in \{0, 1, 2, \ldots, a_m - 1\}$. According to the scheme of the previous section, the representations of the integers j with $a_{m-1} \leq j < a_m$ is $j = d_{m-1}a_{m-1} + d_{m-2}a_{m-2} + \ldots + d_0a_0$ (so that the associated string is $d_{m-1}d_{m-2}\ldots d_0$), while, following the same scheme, the remaining integers (i.e. the integers $0 \leq j < a_{m-1}$) have a representation with less than m digits. For example: the representation of $a_{m-1} - 1 = d_{m-2}a_{m-2} + \ldots + d_0a_0$ has $m - 1$ digits. For our purpose, we require that all the representations of the considered integers $\ell \in \{0, 1, 2, \ldots, a_m - 1\}$ have m digits, so we pad the string on the left with 0's until we have m digits: the representation of $a_{m-1} - 1$ becomes $a_{m-1} - 1 = 0a_{m-1} + d_{m-2}a_{m-2} + \ldots + d_0a_0$ (therefore, the associated string is $0d_{m-2}\ldots d_0$).

With this little adjustment, we now define the following sets:

$\mathscr{L}_0 = \{\varepsilon\}$,

$\mathscr{L}_m = \{d_{m-1} \dots\ d_0 \mid$ the string $d_{m-1} \dots\ d_0$ is the representation of an integer $\ell < a_m$ in the numeration system $\{a_n\}_{n \geq 0}$ $\}$.
Finally, we denote by $\mathscr{L}$ the language obtained by taking the union of all the sets $\mathscr{L}_m$:

$\mathscr{L} = \bigcup_{m \geq 0} \mathscr{L}_m$.

We remark that each element of $\mathscr{L}_m$ has precisely m digits, so that some string $d_{m-1} \dots\ d_0$ can have a prefix consisting of consecutive zeros. Moreover, denoting by $|A|$ the cardinality of a set A, it is $|\mathscr{L}_m| = a_m$. The a_m elements are the representations of each $\ell \in \{0, 1, \dots, a_m - 1\}$.

Referring to the sequence of Pell numbers $p_m = \{1, 2, 5, 12, 29, \dots\}$ defined in Section 2, we have:

$$\begin{aligned}
\mathscr{L}_0 &= \{\varepsilon\} \\
\mathscr{L}_1 &= \{0, 1\} \\
\mathscr{L}_2 &= \{00, 01, 10, 11, 20\} \\
\mathscr{L}_3 &= \{000, 001, 010, 011, 020, 100, 101, 110, 111, 120, 200, 201\} \\
\mathscr{L}_4 &= \{0000, 0001, 0010, 0011, 0020, 0100, 0101, 0110, 0111, 0120, 0200, 0201, 1000, \\
&\quad 1001, 1010, 1011, 1020, 1100, 1101, 1110, 1111, 1120, 1200, 1201, 2000, \\
&\quad 2001, 2010, 2011, 2020\}
\end{aligned}$$

The strings in $\mathscr{L}_2$ are, respectively, the representations of the integers $\ell \in \{0, 1, 2, 3, 4\}$. This corresponds to the case $m = 2$ where $a_m = 5$. Note that $\mathscr{L}_2$ contains exactly $a_2 = 5$ elements.

It is not difficult to realize that the alphabet of the language $\mathscr{L}$ strictly depends on the sequence $\{a_m\}_{m \geq 0}$. In general it is possible to set an upper bound for the digits d_i. From (3), we deduce $d_i a_i < a_{i+1} - \sum_{j=0}^{i-1} d_j a_j$, so that, since the numbers are all integers:

$$d_i a_i \leq a_{i+1} - 1 - \sum_{j=0}^{i-1} d_j a_j \leq a_{i+1} - 1 \ ,$$

leading to

$$d_i \leq \left\lfloor \frac{a_{i+1} - 1}{a_i} \right\rfloor \ . \tag{4}$$

Therefore, the alphabet for $\mathscr{L}_m$ is given by $\{0, 1, \ldots, s\}$ with

$$s = \max_{i=0,1,\ldots,m-1} \left\{ \left\lfloor \frac{a_{i+1} - 1}{a_i} \right\rfloor \right\} ,$$

and, denoting by Σ the alphabet for $\mathscr{L}$, we have $\Sigma = \{0, 1, \ldots, t\}$ with

$$t = \max_{i} \left\{ \left\lfloor \frac{a_{i+1} - 1}{a_i} \right\rfloor \right\} .$$

In some recent papers ([1], [2]) particular number sequences have been studied. More specifically, in [1] the recurrence

$$a_m = \begin{cases} 1 & \text{if } m = 0 \\ k & \text{if } m = 1 \\ ka_{m-1} + ha_{m-2} & \text{if } m \geq 2 \end{cases} \tag{5}$$

with $k > h > 0$ is considered. Here, the alphabet for the strings of $\mathscr{L}$ is $\{0, 1, \ldots, k\}$ and it is possible to define a Gray code for them.

In [2], the following two-termed recurrence is analysed:

$$a_m = \begin{cases} 1 & \text{if } m = 0 \\ k & \text{if } m = 1 \\ ka_{m-1} - ha_{m-2} & \text{if } m \geq 2 \end{cases} \tag{6}$$

with $k > h > 0$. In this case, the alphabet is $\{0, 1, \ldots, k - 1\}$ and if h is even it is possible to define a Gray code for the strings of $\mathscr{L}$. Moreover, recurrence (6) is equivalent to the following full history recurrence which involves only non-negative terms:

$$a_m = \begin{cases} 1 & \text{if } m = 0 \\ & \\ k & \text{if } m = 1 \\ & \\ (k-1)a_{m-1} + (k-h-1)a_{m-2} + \ldots & \\ +(k-h-1)a_1 + (k-h-1)a_0 + 1 & \text{if } m \geq 2 \end{cases} . \tag{7}$$

This is true also in the case $k = 2$ and $h = 0$ giving the one-termed recurrence

$$a_m = \begin{cases} 1 & \text{if } m = 0 \\ 2a_{m-1} & \text{if } m \geq 1 \end{cases} ,$$

equivalent to the full history recurrence

$$a_m = \begin{cases} 1 & \text{if } m = 0 \\ a_{m-1} + a_{m-2} + \ldots + a_1 + a_0 + 1 & \text{if } m \geq 1 \end{cases} \tag{8}$$

(whose general term is $a_m = 2^m$) , leading to the language $\mathscr{L} = \bigcup_{m\geq 0} \mathscr{L}_m$ where $\mathscr{L}_m$ contains binary strings of length m. Denoting by B_m the set of binary strings of length m we have $\mathscr{L}_m \subseteq B_m$, but since $|\mathscr{L}_m| = |B_m| = 2^m$, the two sets $\mathscr{L}_m$ and B_m coincide for each $m \geq 0$.

Let us introduce some notations (as in [3]) useful throughout the rest of the paper.

- If α is a symbol and L is a list or a set of strings $L = (v_1, v_2, \dots, v_s)$ or $L = \{v_1, v_2, \dots, v_s\}$, then $\alpha \cdot L = (\alpha v_1, \alpha v_2 \dots, \alpha v_s)$ (or $\alpha \cdot L = \{\alpha v_1, \alpha v_2 \dots, \alpha v_s\}$) is the list or the set obtained by left concatenating α to each string of L;
- if i and j are symbols, then $ij \cdot L$ is the list or the set obtained by left concatenating i to each string of $j \cdot L$ (or equivalently $ij \cdot L = i \cdot (j \cdot L)$);
- if L is a list or a set of strings, $\bar{L}$ is the list in the reverse order;
- if L and M are two lists, $L \circ M$ is their concatenation. For example, if $L = (v_1, v_2)$ and $M = (w_1, w_2)$, then $L \circ M = (v_1, v_2, w_1, w_2)$;
- if L is a list, then $first(L)$ is the first element of L and $last(L)$ is the last element of L.

It is known that the set B_m can be defined by:

$$B_m = \begin{cases} \{\varepsilon\}, & \text{for } m = 0 \\ 0 \cdot B_{m-1} \cup 1 \cdot B_{m-1}, & \text{for } m > 0 \end{cases}, \tag{9}$$

or equivalently by:

$$B_m = \begin{cases} \{\varepsilon\}, & \text{for } m = 0 \\ 0 \cdot B_{m-1} \cup 10 \cdot B_{m-2} \cup \dots 1^{m-2}0 \cdot B_1 \cup 1^{m-1}0 \cdot B_0 \cup 1^m, & \text{for } m > 0 \end{cases}, \tag{10}$$

where $B_i = \emptyset$ if $i < 0$.

A particular Gray code for the strings of B_m is the *local reflected lexicographical order* (*lrl-order*, see [7, 15]), derived from the well known Binary Reflected Gray code [9]. It is defined as:

$$\mathcal{C}_m = \begin{cases} \varepsilon, & \text{for } m = 0 \\ 0 \cdot \bar{\mathcal{C}}_{m-1} \circ 1 \cdot \mathcal{C}_{m-1}, & \text{for } m > 0 \end{cases}. \tag{11}$$

It is not difficult to see that $\mathcal{C}_m$ can be alternatively defined by:

$$\mathcal{C}_m = \begin{cases} \varepsilon, & \text{for } m = 0 \\ 0 \cdot \bar{\mathcal{C}}_{m-1} \circ 10 \cdot \bar{\mathcal{C}}_{m-2} \circ \ldots 1^{m-2}0 \cdot \bar{\mathcal{C}}_1 \circ 1^{m-1}0 \cdot \bar{\mathcal{C}}_0 \circ 1^m, & \text{for } m > 0 \end{cases} \tag{12}$$

where $\mathcal{C}_i = \emptyset$ if $i < 0$.

4 The case of k-genarilezed Fibonacci sequences

We now investigate on the strings derived from the well known integer sequence of k-generalized Fibonacci numbers ($k \geq 2$). The sequence can be defined as follows:

$$f_\ell^{(k)} = \begin{cases} 2^\ell & \text{if } 0 \leq \ell \leq k-1 \\ f_{\ell-1}^{(k)} + f_{\ell-2}^{(k)} + \ldots + f_{\ell-k}^{(k)} & \text{if } \ell \geq k \ . \end{cases} \tag{13}$$

We observe that this recurrence can be seen as a particular case of (7) or (8) (with suitable initial conditions) where the general term is given by the sum of the k preceding terms rather than all the terms up to the first one.

Note that usually k-generalized Fibonacci sequences are defined with different initial conditions with respect to the ones we imposed. Here, in order to agree with the hypothesis of Theorem 2.1, we need that all the terms of the sequence are different, and this can be obtained by the above initial conditions.

We indicate the set of strings of length m arising from the numeration system $\{f_\ell^{(k)}\}_{\ell \geq 0}$ by $\mathscr{L}_m^{(k)}$ and by $\mathscr{L}^{(k)}$ the language $\mathscr{L}^{(k)} = \bigcup_{m \geq 0} \mathscr{L}_m^{(k)}$.

We have the following proposition:

Proposition 4.1 *The alphabet Σ for the strings in $\mathscr{L}^{(k)}$ in the numeration system $\{f_n^{(k)}\}_{n \geq 0}$ is $\Sigma = \{0, 1\}$. Moreover, the strings avoid the pattern 1^k (i.e. each string does not contain k consecutive 1's).*

Proof. From (4) we obtain

$$d_i \leq \left\lfloor \frac{f_{i+1}^{(k)} - 1}{f_i^{(k)}} \right\rfloor = \left\lfloor \frac{f_i^{(k)} + f_{i-1}^{(k)} + \ldots + f_{i-k+1}^{(k)} - 1}{f_i^{(k)}} \right\rfloor$$
$$= \left\lfloor 1 + \frac{f_{i-1}^{(k)} + \ldots + f_{i-k+1}^{(k)} - 1}{f_i^{(k)}} \right\rfloor \ .$$

Since $\frac{f_{i-1}^{(k)} + \ldots + f_{i-k+1}^{(k)} - 1}{f_i^{(k)}} < 1$, we deduce that $\Sigma = \{0, 1\}$.

Let N be an integer and let $d_{m-1}d_{m-2}\dots d_1d_0$ be its representation in the numeration system $\{f_n^k\}_{n\geq 0}$. We prove that it avoids the pattern 1^k. Suppose ad absurdum that $d_{m-1}d_{m-2}\dots d_1d_0$ contains the pattern 1^k and let j be the first index from the left such that $d_j = d_{j-1} = \dots = d_{j-k+1} = 1$ (clearly $k-1 \leq j \leq m-1$ and, if $j < m-1$, it must be $d_{j+1} = 0$).

From our assumption we have

$$\begin{aligned} N = {} & d_{m-1}f_{m-1}^{(k)} + d_{m-2}f_{m-2}^{(k)} + \dots + d_{j+1}f_{j+1}^{(k)} \\ & + f_j^{(k)} + f_{j-1}^{(k)} + \dots + f_{j-k+1}^{(k)} \\ & + d_{j-k}f_{j-k}^{(k)} + \dots + d_1 f_1^{(k)} + d_0 f_0^{(k)} \ . \end{aligned}$$

Since $d_{j+1} = 0$ and $f_j^{(k)} + f_{j-1}^{(k)} + \dots + f_{j-k+1}^{(k)} = f_{j+1}^{(k)}$, we obtain

$$\begin{aligned} & f_j^{(k)} + f_{j-1}^{(k)} + \dots + f_{j-k+1}^{(k)} \\ & + d_{j-k}f_{j-k}^{(k)} + \dots + d_1 f_1^{(k)} + d_0 f_0^{(k)} \geq f_{j+1}^{(k)} \end{aligned}$$

against Theorem 2.1 which assures

$$\begin{aligned} & d_j f_j^{(k)} + d_{j-1}f_{j-1}^{(k)} + \dots + d_{j-k+1}f_{j-k+1}^{(k)} \\ & + d_{j-k}f_{j-k}^{(k)} + \dots + d_1 f_1^{(k)} + d_0 f_0^{(k)} < f_{j+1}^{(k)} \end{aligned}$$

Therefore the string $d_{m-1}d_{m-2}\dots d_1d_0$ avoids the pattern 1^k.

■

Denoting by $F_n^{(k)}$ the set of binary strings of length n avoiding the pattern 1^k, it is known [12] that $|F_n^{(k)}| = f_n^{(k)}$. From Proposition 4.1 we deduce that $\mathscr{L}_m^{(k)} \subseteq F_m^{(k)}$. Clearly, it is also $|\mathscr{L}_n^{(k)}| = f_n^{(k)}$. Hence, the sets $\mathscr{L}_m^{(k)}$ and $F_m^{(k)}$ coincide.

The sets $\mathscr{L}_m^{(k)}$ (and so $F_m^{(k)}$) can be defined recursively as follow:

$$\mathscr{L}_m^{(k)} = \begin{cases} B_m, & \text{for } m < k \\ 0 \cdot \mathscr{L}_{m-1}^{(k)} \cup 10 \cdot \mathscr{L}_{m-2}^{(k)} \cup \dots \cup 1^{k-1}0 \cdot \mathscr{L}_{m-k}^{(k)}, & \text{for } m \geq k \end{cases} . \tag{14}$$

The strings of $\mathscr{L}_m^{(k)}$ can be rearranged in a Gray Code $\mathcal{L}_m^{(k)}$ with the Hamming distance equal to one (see [15]):

$$\mathcal{L}_m^{(k)} = \begin{cases} \mathcal{C}_m, & \text{for } 0 \leq m < k \\ 0 \cdot \bar{\mathcal{L}}_{m-1}^{(k)} \circ 10 \cdot \bar{\mathcal{L}}_{m-2}^{(k)} \circ \ldots \circ 1^{k-1}0 \cdot \bar{\mathcal{L}}_{m-k}^{(k)}, & \text{for } m \geq k \end{cases} . \tag{15}$$

5 Permutations

Given a permutation $\pi \in S_m$, by inversion array of π we mean the array $v(\pi) = v_1 v_2 \ldots v_{m-1}$ of dimension $m-1$ whose i-th entry counts the number of entries of π at the right hand side of π_i which are smaller than π_i. Formally, we have:

Definition 5.1 *If π is a permutation of length m, the array $v(\pi) = v_1 v_2 \ldots v_{m-1}$ is the* inversion array *of π, where*

$$v_i = |\{\pi_j | \pi_j < \pi_i, j > i\}| \text{ for } i = 1, 2, \ldots, m-1 \ .$$

We now associate the permutations in $S(321, 312, 23\ldots(k+1)1)$ (with $k \geq 2$) with the strings $\mathscr{L}^{(k)}$. From Proposition 4.1 we know that a string $u \in \mathscr{L}^{(k)}$ avoids the pattern 1^k. We have the following proposition.

Proposition 5.1 *Let π be a permutation of length m and let $v(\pi)$ the inversion array of π. Then $\pi \in S_m(321, 312, 23\ldots(k+1)1)$ if and only if $v(\pi) \in \mathscr{L}_{m-1}^{(k)}$.*

Proof. We suppose $\pi \in S_m(321, 312, 23\ldots(k+1)1)$. Since π has length m, clearly $v(\pi)$ has length $m-1$, according to Definition 5.1. Since π avoids 321 and 312, then for each entry π_i there is at most only one entry π_j with $j > i$ such that $\pi_i > \pi_j$, otherwise a pattern 321 or 312 would occur. Then, either $v_i = 0$ or $v_i = 1$ (therefore the alphabet of $v(\pi)$ is $\{0, 1\}$). We now have to prove that $v(\pi)$ avoids 1^k.

Let us suppose ad absurdum that such a pattern occurs and $v_j v_{j+1} \ldots v_{j+k-1}$ be the leftmost occurrence of 1^k in $v(\pi)$ (it is $1 \leq j \leq m-k$ and $v_{j-1} = 0$ if $j \geq 1$). Since $v_j = 1$, there exists an index r such that $\pi_{j+r} < \pi_j$. It must be $r \geq k$, otherwise, being $v_{j+r} = 1$, it is $\pi_{j+r} > \pi_p$ for some $p > j+r$. But this is not possible since in this case a pattern 321 would occur in $\pi_j \pi_{j+r} \pi_p$ (and the value of v_j should be al least 2). Therefore, it is $r \geq k$.

Moreover, for each entry π_{j+i} for $i = 1, 2, \ldots, k-1$ it is $\pi_{j+i} > \pi_{j+i-1}$ otherwise a pattern 321 would occur in $\pi_{j+i-1}\pi_{j+i}\pi_{j+r}$ (and $v_{j+i-1} = 2$ since $\pi_{j+i-1} > \pi_{j+i}$ and $\pi_{j+i-1} > \pi_{j+r}$). This implies that $\pi_j < \pi_{j+1} \ldots <$

π_{j+k-1}. These entries of π together with π_{j+r} are an occurrence of $23\ldots(k+1)1$ against the hypothesis. Therefore $v(\pi)$ avoids 1^k.

Suppose now that $v(\pi) \in \mathscr{L}^{(k)}_{m-1}$. It means that either $v_i = 0$ or $v_i = 1$, then for any π_i two entries π_p and π_q smaller than π_i, with $p, q > i$, do not exist. Therefore, the permutation π avoid both 321 and 312. We have to prove that π avoids $23\ldots(k+1)1$, too.

Suppose ad absurdum that such a pattern occurs and let $\pi_{j_1}\pi_{j_2}\ldots\pi_{j_k}\pi_{j_{k+1}}$ be an occurrence of it. All the entries π_p between π_{j_i} and $\pi_{j_{i+1}}$ for $i = 1, 2, \ldots, k-1$ are such that $\pi_{j_i} < \pi_p < \pi_{j_{i+1}}$, otherwise either a pattern 312 or 321 occurs in $\pi_{j_i}\pi_p\pi_{j_{k+1}}$ if $\pi_p < \pi_{j_i}$, or a pattern 321 occurs in $\pi_p\pi_{j_{i+1}}\pi_{j_{k+1}}$ if $\pi_p > \pi_{j_{k+1}}$.

This implies that there are at least k consecutive entries $\pi_p\pi_{p+1}\ldots\pi_{p+k-1}$ of π between π_{j_i} and π_{j_k} which are in increasing order ($\pi_p < \pi_{p+1} < \ldots < \pi_{p+k-1}$) and for each of them it is $\pi_{p+r} > \pi_{j_{k+1}}$ for $r = 0, 1, \ldots, k-1$. Therefore in $v(\pi)$ we have $v_p v_{p+1}\ldots v_{p+k-1} = 1^k$, against the hypothesis. Hence, π avoids the pattern $23\ldots(k+1)1$, too.

■

Before giving the Gray code for $S_m(321, 312, 23\ldots(k+1)1)$, let us introduce some notations b.

Given a permutation $\pi \in S_n$, $\pi = \pi_1\pi_2\ldots\pi_n$, and a positive integer p, we denote by $\pi \uparrow p = (\pi_1+p)(\pi_2+p)\ldots(\pi_n+p)$ the permutation of $[p+1, p+2, \ldots, p+n]$ obtained by π by adding p to each entry of π.

If $\rho \in S_p$, $\rho = \rho_1\rho_2\ldots\rho_p$ is a permutation of length p, then we denote by $\rho\cdot(\pi \uparrow p) = \rho_1\ldots\rho_p(\pi_1+p)\ldots(\pi_n+p)$ the permutation of $[1, 2, \ldots, n+p]$ obtained by concatenating ρ with $\pi \uparrow p$.

If $\Pi = \{\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(\ell)}\}$ is a set of permutations, then the set of the permutations $\pi^{(j)} \uparrow p$, for $j = 1, 2, \ldots, \ell$ is denoted by $\Pi \uparrow p$.

We observe that the permutations of $S_m(321, 312, 23\ldots(k+1)1)$ can be recursively defined basing on Definition (14) of $\mathscr{L}^{(k)}_{m-1}$. Indeed, if $v(\pi) \in \mathscr{L}^{(k)}_{m-1}$, then the inversion array $v(\pi') = 0\cdot v(\pi)$ corresponds to the inversion array of the permutation $\pi' = 1\cdot(\pi \uparrow 1)$ since a 0 entry in an inversion array corresponds to an entry of the permutation which is less than all the elements to its rigth. So, the first 0 entry in $v(\pi')$ must correspond to the 1 entry in π'.

More generally, by means of a similar argument, we can prove that if $v(\pi) \in \mathscr{L}^{(k)}_{m-j}$, then $v(\pi') = 1^{j-1}0\cdot v(\pi)$ corresponds to the inversion array of the permutation $\pi' = 23\ldots j1\cdot(\pi \uparrow j)$. Therefore, from (14), by considering B_m defined as in (10), we can define recursively the permutations of the set of $S_m(321, 231, 23\ldots(k+1)1)$ denoted, for short, by $S^{(k)}_m$. We give so the following definition:

$$S_m^{(k)} = \begin{cases} \{\varepsilon\}, & \text{for } m = 0 \\ 1 \cdot (S_{m-1}^{(k)} \uparrow 1) \cup 21 \cdot (S_{m-2}^{(k)} \uparrow 2) \cup \ldots \cup 23 \ldots k1 \cdot (S_{m-k}^{(k)} \uparrow k), & \text{for } m \geq 1 \end{cases}, \tag{16}$$

where $S_i^{(k)} = \emptyset$ if $i < 0$.

We note that, given an inversion array $v(\pi) \in \mathscr{L}_{m-1}^{(k)}$ one can get the corresponding permutation $\pi \in S_m^{(k)}$ in a very simple way. Indeed, it is not difficult to prove that denoting by $i_1, i_2, \ldots, i_r$ the indexes of the positions of the 0's in $v(\pi) \cdot 0$ (i.e. $v_{i_j} = 0$ for $j = 1, 2, \ldots, r$), the entries of π are:

$$\pi_i = \begin{cases} 1 & \text{if } i = i_1 \\ i_j + 1 & \text{if } i = i_{j+1} \quad (j = 1, 2, \ldots, r-1) \\ i + 1 & \text{if } i \neq i_j \quad (j = 1, 2, \ldots, r) \end{cases}.$$

Moreover, if $\pi \in S_m^{(k)}$, the inversion array $v(\pi) = v_1 v_2 \ldots v_{m-1}$ is obtained as follows:

$$v_i = \begin{cases} 0 & \text{if } \pi_i \leq i \\ 1 & \text{if } \pi_i > i \end{cases}.$$

Unlike in the case for strings, two permutations cannot differ by a single position but, at least, by a transposition of two entries. One famous Gray code for S_n (the set of the unrestricted permutations of length n) was given by Johnson [11] and Trotter [14], where each permutation is obtained from the preceding one by a transposition of two consecutive entries.

As a final step of our paper we propose a Gray code $\mathcal{S}_m^{(k)}$ for the set $S_m^{(k)}$ which is induced by the Gray code $\mathcal{L}_m$ defined in (15). We recursively define the list $\mathcal{S}_m^{(k)}$ as

$$\mathcal{S}_m^{(k)} = \begin{cases} \varepsilon, & \text{for } m = 0 \\ 1 \cdot (\bar{\mathcal{S}}_{m-1}^{(k)} \uparrow 1) \circ 21 \cdot (\bar{\mathcal{S}}_{m-2}^{(k)} \uparrow 2) \circ \ldots \circ 23 \ldots k1 \cdot (\bar{\mathcal{S}}_{m-k}^{(k)} \uparrow k), & \text{for } m \geq 1 \end{cases}, \tag{17}$$

where $\mathcal{S}_i^{(k)} = \varepsilon$ if $i < 0$.

Proposition 5.2 *The list of permutation $\mathcal{S}_m^{(k)}$ is a Gray code where two consecutive permutations differ by a transposition of two consecutive entries.*

Proof. First of all we observe that, from (17) for each $m \geq 2$, it is

$$first(\mathcal{S}_m^{(k)}) = first(1 \cdot (\bar{\mathcal{S}}_{m-1}^{(k)} \uparrow 1)) = 1 \cdot first(\bar{\mathcal{S}}_{m-1}^{(k)} \uparrow 1) = 1 \cdot last(\mathcal{S}_{m-1}^{(k)} \uparrow 1)\ . \tag{18}$$

We proceed by induction. Easily, the lists $\mathcal{S}_0^{(k)} = \varepsilon$ and $\mathcal{S}_1^{(k)} = 1$ and $\mathcal{S}_2^{(k)} = 12, 21$ are Gray codes where two consecutive permutations differ for a transposition of two consecutive entries (note that $\mathcal{S}_0^{(k)}$ and $\mathcal{S}_1^{(k)}$ are trivial cases). Let us suppose that $\mathcal{S}_i^{(k)}$ is a Gray code where two consecutive permutations differ by a transposition of two consecutive entries, for $i \leq m-1$.

In order to prove that $\mathcal{S}_m^{(k)}$ is a Gray code we have to show only that the permutation $last(23\ldots(j-1)1\cdot(\bar{\mathcal{S}}_{m-j+1}^{(k)} \uparrow (j-1)))$ and the permutation $first(23\ldots j1\cdot(\bar{\mathcal{S}}_{m-j}^{(k)} \uparrow j))$ differ by a transposition of two consecutive entries, for $j = 2, 3, \ldots, k$. Indeed, by the inductive hypothesis each $23\ldots j1\cdot (\bar{\mathcal{S}}_{m-j}^{(k)} \uparrow j)$ is a Gray code, for $j = 1, 2, \ldots, k$.

We have:

$$last(23\ldots(j-1)1\cdot(\bar{\mathcal{S}}_{m-j+1}^{(k)} \uparrow (j-1))) = 23\ldots(j-1)1\cdot last((\bar{\mathcal{S}}_{m-j+1}^{(k)} \uparrow (j-1))) =$$

$$23\ldots(j-1)1\cdot first((\mathcal{S}_{m-j+1}^{(k)} \uparrow (j-1))) = 23\ldots(j-1)1\cdot(first(\mathcal{S}_{m-j+1}^{(k)}) \uparrow (j-1)) =$$

$$23\ldots(j-1)1\cdot((1\cdot last(\mathcal{S}_{m-j}^{(k)} \uparrow 1)) \uparrow (j-1))) = 23\ldots(j-1)\mathbf{1j}\cdot last((\mathcal{S}_{m-j}^{(k)}) \uparrow j) \quad .$$

On the other hand:

$$first(23\ldots(j-1)j1\cdot(\bar{\mathcal{S}}_{m-j}^{(k)} \uparrow j)) = 23\ldots(j-1)\mathbf{j1}\cdot last(\mathcal{S}_{m-j}^{(k)} \uparrow j)$$

so that the two permutations differ only for the transposition of the two consecutive entries $1j$ in the first one which become $j1$ in the second one. ■

6 Conclusion

As already mentioned in Section 1, this research was started in [4], where interesting properties of the languages arising from particular linear recurrences were given. Later ([1, 2]), the study has been enriched by the possibility of listing the languages in a Gray code order (at least in many cases), depending on the parameters of the linear recurrences. Here, we extend the results by starting from k-generalized Fibonacci recurrences and by involving pattern avoiding permutations, transferring on them the Gray code obtained for the language of strings by means of the inversion arrays.

A further improvement towards this direction could be devoted to different particular recurrences, clearly leading to different languages. An interesting result would be the definition of a Gray code for them, as well as for pattern avoiding permutations possibly associated to.

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