Project games

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Abstract

We consider a strategic game, called project game, where each agent has to choose a project among her own list of available projects. The model includes positive weights expressing the capacity of a given agent to contribute to a given project. The realization of a project produces some reward that has to be allocated to the agents. The reward of a realized project is fully allocated to its contributors according to a simple proportional rule. Existence and computational complexity of pure Nash equilibria is addressed and their efficiency is investigated according to both the utilitarian and the egalitarian social function.

Keywords: Pure Nash equilibria, Price of Anarchy, Price of Stability.

1. Introduction

We introduce and study PROJECT GAMES, a model where some agents take part to some projects. Every agent chooses a single project, but several agents can select the same project. This situation happens, for example, when some scientists decide on which problem they work, when some investors choose the business in which they spend their money, when some benefactors select which artistic project they support, etc. Our model includes positive weights which express the capacity of a given agent to contribute to a given project. By assumption, a project is realized if it is selected by at least one agent. The realization of a project produces some reward that has to be allocated to the contributing agents.

We take a game theoretic perspective, i.e., an agent's strategy is to select, within the projects that are available to her, the one inducing the largest piece of reward. Therefore, the way the rewards are allocated is essential to this game. Here, we suppose that the reward of a realized project is fully allocated to its contributors, according to a simple proportional rule based on the aforementioned weights.

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Our motivation is to analyse the impact of this simple and natural allocation rule. Do the players reach a Nash equilibrium, that is, a stable state in which no one wants to deviate from the project she is currently contributing? How bad is a Nash equilibrium compared to the situation where a central authority would, at best, decide by which agent(s) a project is conducted? In other words, does the allocation rule incentivize the players to realize projects that optimize the total rewards?

1.1. The Model

PROJECT GAMES are strategic games with a set of n players $N = \{1, \ldots, n\} := [n]$ and a set of m projects $M = \{1, \ldots, m\} := [m]$. The strategy space of every player i, denoted by S_i, is a subset of M. We assume that $\bigcup_{i \in N} S_i = M$ and a strategy for player i is to select a project $j \in S_i$. Each project $j \in M$ has a positive *reward* r_j. Each player $i \in N$ has a positive weight $w_{i,j}$ when she selects project j. A strategy profile is a vector of strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$, where σ_i denotes the project selected by player i, for each $i \in N$.

The *load* of project j under strategy profile σ, denoted by $L(\sigma, j)$, is the total weight of the players selecting j. Thus, $L(\sigma, j) = \sum_{\{i \in N \,:\, \sigma_i = j\}} w_{i,j}$. A player's utility is defined as a portion of the reward of the realized project that she is contributing to. This portion is proportional to the player's weight. Thus, the utility of player i (that she wants to maximize) under σ is defined as

$$u_i(\sigma) = \frac{w_{i,\sigma_i}}{L(\sigma, \sigma_i)} \, r_{\sigma_i}. \tag{1}$$

We will sometimes consider special cases of PROJECT GAMES. A PROJECT GAME is *symmetric* when $S_i = M$ for every player i. The players' weights are *universal* when, for every player i, $w_{i,j}$ is equal to some positive number w_i for every project j; in particular, they are *identical* when $w_i = 1$ for every player i. The weights are *project-specific* when they are not universal. The projects' rewards are *identical* when the reward is the same for all projects, and this reward is equal to 1 by assumption. When the rewards are not identical, we say that they are *generic* or non-identical.

A strategy profile σ is a *pure Nash equilibrium* if for each $i \in N$ and $j \in S_i$, $u_i(\sigma) \geq u_i(\sigma_{-i}, j)$ where $\sigma' = (\sigma_{-i}, j)$ is defined by $\sigma'_\ell = \sigma_\ell$ for $\ell \in N \setminus \{i\}$ and $\sigma'_i = j$. For a PROJECT GAME G, denote by $\mathsf{NE}(G)$ its set of pure Nash equilibria. An *improving deviation* for player i in strategy profile σ is a deviation to a strategy $j \in S_i$ such that $u_i(\sigma_{-i}, j) > u_i(\sigma)$; a *best response* for i in σ is a strategy $j^* \in S_i$ such that $u_i(\sigma_{-i}, j^*) \geq u_i(\sigma_{-i}, j)$ for each $j \in S_i$. Thus, a pure Nash equilibrium is a strategy profile in which no player has an improving deviation and in which every player plays a best response.

For a strategy profile σ, $P(\sigma) = \{j \in M \;:\; L(\sigma, j) > 0\}$ will denote the set of projects selected by some players in σ. The social utility under strategy profile σ, denoted by $\mathsf{U}(\sigma)$, is defined as the total sum of the rewards of the selected projects (also known as the *utilitarian social welfare*), i.e., $\mathsf{U}(\sigma) = \sum_{j \in P(\sigma)} r_j$. Note that $\mathsf{U}(\sigma) = \sum_{i \in N} u_i(\sigma)$. A social optimum, denoted as σ^*, is a strategy profile maximizing U.

Given a PROJECT GAME G, the *price of anarchy* [26] of G is the worst-case ratio between the social utility of a social optimum and the social utility of a pure Nash equilibrium for G, namely, $\mathsf{PoA}(G) = \sup_{\sigma \in \mathsf{NE}(G)} \frac{\mathsf{U}(\sigma^*)}{\mathsf{U}(\sigma)}$; the *price of stability* [3] of G is the best-case ratio between the social utility of a social optimum and the social utility of a pure Nash equilibrium for G, namely, $\mathsf{PoS}(G) = \inf_{\sigma \in \mathsf{NE}(G)} \frac{\mathsf{U}(\sigma^*)}{\mathsf{U}(\sigma)}$.

For any two integers $n, m > 1$, let $\mathcal{G}_{n,m}$ denote the set of all PROJECT GAMES with n players and m projects. We define $\mathsf{PoA}(n,m) = \sup_{G \in \mathcal{G}_{n,m}} \mathsf{PoA}(G)$ (resp., $\mathsf{PoS}(n,m) = \sup_{G \in \mathcal{G}_{n,m}} \mathsf{PoS}(G)$) as the worst-case price of anarchy (resp., stability) of games with n players and m projects. By definition, we have

$$1 \leq \mathsf{PoS}(n,m) \leq \mathsf{PoA}(n,m). \tag{2}$$

Therefore, in cases where both the price of anarchy and the price of stability are equal to some value x, it suffices to show $x \leq \mathsf{PoS}(n,m)$ and $\mathsf{PoA}(n,m) \leq x$.

1.2. Our Contribution

We focus on existence, computational complexity and efficiency of pure Nash equilibria in PROJECT GAMES. Given the structural simplicity of these games, it will be possible to derive some results from the state of the art of similar classes of games.

For instance, by making use of the notion of strict game isomorphism, we derive that the problem of computing a pure Nash equilibrium in PROJECT GAMES with universal weights belongs to the complexity class PLS (Polynomial Local Search) and can be solved in polynomial time as long as at least one of the following three conditions holds: the game is symmetric, the rewards are identical, the weights are identical. For the more general case of project-specific weights, instead, we show, by means of a potential function argument, that the problem is in PLS as long as the rewards are identical. Without this assumption, the problem gets fairly much more complicated and even the existence of pure Nash equilibria remains an open problem. These results are summarized in Figure 1.

As to the efficiency of pure Nash equilibria, it is easy to see that PROJECT GAMES belong to the class of *valid utility games*. For these games, an upper bound of 2 on the price of anarchy is given in [36]. We show that this bound is tight only for the case of asymmetric games with non-identical rewards and non-identical weights. In all other cases, we give refined bounds parametrized by both the number of players and projects, also with respect to the price of stability. The obtained bounds are summarized in Figures 2, 3 and 4. All these bounds are shown to be tight except for one case involving the price of anarchy of asymmetric games with identical rewards and identical weights. For this particular variant of the game, we also consider an interesting restriction in which all players have at most two available strategies (these games admit a multigraph representation).

Before concluding, we explore the efficiency of equilibria under an alternative notion of social welfare which focuses on the utility of the poorest player (*egalitarian social welfare*).

1.3. Related Work

PROJECT GAMES fall within the class of *monotone valid utility games* introduced in [36] and further considered in [4, 6, 20, 21, 24, 28, 29]. In a monotone valid utility game, there is a ground set of objects V, and a strategy for a player consists in selecting some subset of V. A social function $\gamma : 2^V \mapsto \mathrm{R}$ associates a non-negative value to each strategy profile; γ

strategies \ rewards	generic	identical
symmetric	?	$\in$PLS [*]
asymmetric	?	$\in$PLS [*]

games with project-specific weights

strategies \ rewards	generic	identical
symmetric	$\in$P [2]	$\in$P [2]
asymmetric	$\in$PLS [2]	$\in$P [3]

games with universal weights

strategies \ rewards	generic	identical
symmetric	$\in$P [2,4]	$\in$P [2,3,4]
asymmetric	$\in$P [4]	$\in$P [3,4]

games with identical weights

Figure 1: The complexity of the problem of computing pure Nash equilibria in all possible variants of PROJECT GAMES. Results labelled with [*] are obtained in this paper, all the other ones are derived from previous results in other settings by applying the notion of strict game isomorphism. A question mark means that even the problem of deciding the existence of a pure Nash equilibrium is still open.

$\mathsf{PoA}(n,m)$ and $\mathsf{PoA}(n,m)$ of games with symmetric strategies	Generic rewards	Identical rewards
Any type of project weights	$1+\frac{s-1}{n}$ [Prop. 4, Thm. 8]	1 [Thm. 4]

Figure 2: The price of anarchy $\mathsf{PoA}(n,m)$ and the price of stability $\mathsf{PoS}(n,m)$ of PROJECT GAMES with symmetric strategies expressed as a function of n and m, where $s := \min(n,m)$. All bounds are tight and independent of the project weights.

$\mathsf{PoA}(n,m)$ of games with asymmetric strategies	Generic rewards	Identical rewards
Project-specific & universal weights	2 [Prop. 5, [36]]	$\frac{2t+1}{t+1}$ [Thm. 5, Thm. 6]
Identical weights	$2-1/n$ if $n \leq m$ $\frac{n+1}{n}$ if $n > m = 2$ $2-\frac{1}{m-1}$ if $n > m > 2$ [Prop. 6, Thm. 9]	$[\frac{e}{e-1}, \frac{5}{3}]$ [Thm. 7]

Figure 3: The price of anarchy $\mathsf{PoA}(n,m)$ of PROJECT GAMES with asymmetric strategies expressed as a function of n and m, where $s := \min(n,m)$ and $t := \lfloor \frac{s-1}{2} \rfloor$. The case in which both the rewards and the projects are identical is the only non-tight bound.

$\mathsf{PoS}(n,m)$ of games with asymmetric strategies	Generic rewards	Identical rewards
Project-specific & universal weights	2 [Prop. 5, [36]]	1 [Thm. 3]
Identical weights	$2-1/n$ if $n \leq m$ $\frac{n+1}{n}$ if $n > m = 2$ $2-\frac{1}{m-1}$ if $n > m > 2$ [Prop. 6, Thm. 9]	1 [Thm. 3]

Figure 4: The price of stability $\mathsf{PoS}(n,m)$ of PROJECT GAMES with asymmetric strategies expressed as a function of n and m. All bounds are tight.

is assumed to be monotone and submodular. The utility of player i in a strategy profile σ is at least the value $\gamma(\sigma) - \gamma(\sigma_{-i})$. Moreover, the sum of the players' utilities in σ does not exceed the value $\gamma(\sigma)$. In [36] it is shown that the price of anarchy of these games is at most 2.

Among the special cases of monotone valid utility games considered in the literature, the one that mostly relates to our PROJECT GAMES is the one studied in [24]. They consider a set of projects modelling open problems in scientific research and a set of players/scientists, each of which chooses a single problem to work on. However, there are several differences between the two models which make them incomparable. In fact, in the games studied in [24], players may fail in solving a problem, and so the reward associated with each project is not always guaranteed to be realized; moreover, when a problem is solved, its reward is always shared equally among the solving players. This assumption makes these games instances of *congestion games*, whereas this is not the case in our PROJECT GAMES.

Congestion games [34] is a well known category of strategic games which, by a potential argument [32], always admit a pure Nash equilibrium. In a congestion game, there is a set of resources M and every player's strategy set is a non-empty subset of 2^M. For example, M contains the links of a network from which each player wants to choose a path. Each resource j is endowed with a latency function ℓ_j which depends on the number of players having j in their strategy. A player's cost is the sum of the latencies of the resources that she uses. This model received a lot of attention in the computer science community, see e.g. [1, 2, 13, 14, 17, 22, 23, 27, 30, 31]. Congestion games were generalized to the case where the players have different weights (*weighted congestion games*), or when a resource's latency depends on the identity of the player (*player specific congestion games*) as in [30]. These extensions still admit a pure Nash equilibrium if the players' strategies are singletons. Nevertheless, a pure Nash equilibrium is not guaranteed when we combine weights and player-specific costs, even with singleton strategies [30]. Singleton congestion games with weighted players are also known as *Load Balancing* games [37]: resources and players may represent machines and jobs, respectively. In this context each job goes on the machine that offers her the lowest completion time.

PROJECT GAME is also the name of a very different game studied by [11]. In their setting, there is a single project composed of several activities and the authors study it through the lens of cooperative game theory while our model is a strategic game.

Finally, it is worth mentioning that PROJECT GAMES are remotely connected with *hedonic games* [5, 10] and the *group activity selection problem* [9], as the realized projects induce a partition of the player set.

2. Existence of a Pure Strategy Nash Equilibrium

In this section, we focus on the existence and efficient computation of pure Nash equilibria in PROJECT GAMES. We shall show how several positive results can be obtained from the realm of load balancing games and singleton congestion games by making use of the notion of *strict game isomorphism*. Intuitively, two games are strictly isomorphic when, for every pair of strategy profiles and for every fixed player, they agree when one is better than the other (i.e., they have the same Nash dynamics graph[1]). Formally, we need the following general definitions.

Definition 1. *A game Γ is a tuple $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, where N is a set of $n = |N|$ players, and, for each $i \in N$, S_i is the set of strategies of player i and $u_i : \times_{i\in N} S_i \mapsto \mathbb{R}$ is her utility function.*

In a profit maximization game, every player wants to maximize her utility function (as in our PROJECT GAME), whereas in cost minimization games, utilities need to be minimized. To encompass both types of games under the same umbrella, we introduce the preference relations $\prec$ and $\preceq$ defined as follows. For a game $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, a player $i \in N$, and two strategy profiles σ and σ' of Γ, we write $\sigma \prec_i \sigma'$ (resp., $\sigma \preceq_i \sigma'$) whenever i strictly (resp., weakly) prefers the outcome of σ' to that of σ, that is, when $u_i(\sigma) < u_i(\sigma')$ (resp., $u_i(\sigma) \leq u_i(\sigma')$) if Γ is a profit maximization game and $u_i(\sigma) > u_i(\sigma')$ (resp., $u_i(\sigma) \geq u_i(\sigma')$) if Γ is a cost minimization game. With this notation, we can express the notions of pure Nash equilibrium and improving deviation for any type of game in a unified manner as follows. A pure Nash equilibrium for Γ is a strategy profile σ such that, for each $i \in N$ and $j \in S_i$, $(\sigma_{-i}, j) \preceq_i \sigma$. An improving deviation for player i in σ is a deviation to a strategy $j \in S_i$ such that $\sigma \prec_i (\sigma_{-i}, j)$.

To formally state the definition of strict game isomorphism, we first need to define a game mapping [33, 35].

Definition 2. *Given two games $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ and $\Gamma' = (N, (S'_i)_{i\in N}, (u'_i)_{i\in N})$, a game mapping ψ from Γ to Γ' is a tuple $\psi = (\pi, (\varphi_i)_{i\in N})$, where π is a bijection from N to N and, for any $i \in N$, φ_i is a bijection from S_i to S'_i.*

Observe that a game mapping naturally induces a bijection between the set of strategy profiles of Γ and the set of strategy profiles of Γ', where $\sigma = (\sigma_1, \dots, \sigma_n)$ is mapped to

[1] We recall that the Nash dynamics graph associated with a game is the directed graph whose set of nodes is the set of strategy profiles of the game and there is an arc from profile σ to profile σ' if and only if there exists a player who has an improving deviation in σ leading the game to profile σ'.

$\sigma' = (\sigma'_1, \ldots, \sigma'_n)$ with $\sigma'_{\pi(i)} = \varphi_i(\sigma_i)$ for each $i \in N$. We succinctly define this mapping by $\psi(\sigma)$, thus overloading the use of ψ.

Definition 3. *A strict isomorphism from Γ to Γ' is a game mapping $\psi = (\pi, (\varphi_i)_{i\in N})$ such that, for any $i \in N$, and pair of strategy profiles σ and σ' for Γ, we have $\sigma \prec_i \sigma'$ if and only if $\psi(\sigma) \prec_i \psi(\sigma')$.*

The notion of strict isomorphism is a refinement of that of weak isomorphism given in [15], where $\prec$ is replaced by $\preceq$. By definition, two games which are strictly isomorphic share the same set of pure Nash equilibria. Thus, existential and computational results for one game can be directly applied to the other. By leveraging previously known results, we obtain the following result.

Theorem 1. *Every* PROJECT GAME *with universal weights admits a potential function. Moreover, a pure Nash equilibrium can be computed in polynomial time when at least one of the following conditions is true: the game is symmetric, the rewards are identical, the weights are identical.*

Proof. In a load balancing game with related machines each player owns a task which needs to be processed on one of m available machines (see, for example, [37] for a survey). Every task has a processing time and every machine has a speed. The cost that a player experiences in a strategy profile is equal to the completion time of the chosen machine, where the completion time of a machine is equal to the sum of the weights of all players selecting it, divided by its speed. The game is symmetric if all machines are available to every task. In a load balancing game with identical machines, all speeds are identical and normalized to 1; in a singleton congestion game with linear latency functions, all processing times are identical and normalized to 1.

Fix a PROJECT GAME G with universal weights. By (1), we have that, for each strategy profile σ of G, player $i \in N$ and strategy $j \in S_i$,

$$u_i(\sigma_{-i}, j) > u_i(\sigma) \iff \frac{L(\sigma, j) + w_i}{r_j} < \frac{L(\sigma, \sigma_i)}{r_{\sigma_i}}. \tag{3}$$

If one interprets the set of projects as a set of related machines, where machine j has a speed r_j, and the set of players as a set of tasks, where task i has a processing time w_i, it follows immediately from (3) that G is strictly isomorphic to a load balancing game with related machines. Similarly, a PROJECT GAME with universal weights and identical projects is strictly isomorphic to a load balancing game with identical machines, and a PROJECT GAME with identical weights is strictly isomorphic to a singleton congestion game with linear latency functions.

In [13] it is shown that load balancing games with related machines admit a potential function. A characterization by [32] states that a game admits a potential function if and only if its Nash dynamics graph is acyclic. As the Nash dynamic graphs of two strictly isomorphic games are isomorphic, it follows that also PROJECT GAMES with universal weights admit a potential function. Moreover, since in [13] a polynomial time algorithm for computing a pure Nash equilibrium in load balancing symmetric games with related machines is given, we get an efficient algorithm for computing a pure Nash equilibrium in symmetric PROJECT GAMES with universal weights. For asymmetric PROJECT GAMES with identical rewards, the polynomial time algorithm given in [16] for load balancing games with identical machines can be applied. Finally, for asymmetric PROJECT GAMES with identical weights, the algorithm given in [23] for singleton congestion games with linear latency functions can be applied. □

For the case of project-specific weights, no transformation to other known classes of games are possible (up to our knowledge) and a direct approach needs to be developed. For projects with identical rewards, we show the existence of pure Nash equilibria by providing a potential function argument.

Theorem 2. *For* PROJECT GAMES *with identical rewards, vector* $\langle |P(\sigma)|,\ \Phi(\sigma)\rangle$*, where* $\Phi(\sigma) := \Pi_{j \in P(\sigma)} L(\sigma, j)$*, lexicographically increases after every improving deviation.*

Lexicographic increase means that either $|P(\sigma)|$ increases, or $|P(\sigma)|$ stays the same but $\Phi(\sigma)$ increases.

Proof. Suppose that some player i performs an improving deviation from a given strategy profile σ. The new strategy profile σ' is identical to σ, except that $\sigma_i \neq \sigma'_i$. Let $\sigma_i = j$ and $\sigma'_i = j'$. We have that $u_i(\sigma') > u_i(\sigma)$ is equivalent to

$$\frac{w_{i,j'}}{L(\sigma', j')} > \frac{w_{i,j}}{L(\sigma, j)}. \tag{4}$$

If $L(\sigma', j) = 0$, then player i made a deviation although her utility was maximum (the weights being positive, player i was alone on project j): a contradiction. Thus, $L(\sigma', j) > 0$. If $L(\sigma, j') = 0$, then player i joins an unused project. The combination $L(\sigma', j) > 0$ and $L(\sigma, j') = 0$ indicates that $|P(\sigma')| = |P(\sigma)| + 1$. The vector increased lexicographically.

For the rest of this proof, suppose $L(\sigma, j') > 0$. In this case $|P(\sigma')| = |P(\sigma)|$. The deviation induces a change in Φ:

$$\Phi(\sigma') - \Phi(\sigma) = K\big(L(\sigma', j)L(\sigma', j') - L(\sigma, j)L(\sigma, j')\big),$$

where K is some positive term that is common to both $\Phi(\sigma')$ and $\Phi(\sigma)$. We have $L(\sigma', j) = L(\sigma, j) - w_{i,j}$ and $L(\sigma', j') = L(\sigma, j') + w_{i,j'}$. Thus, $L(\sigma', j)L(\sigma', j') - L(\sigma, j)L(\sigma, j') = (L(\sigma, j) - w_{i,j})(L(\sigma, j') + w_{i,j'}) - L(\sigma, j)L(\sigma, j') = L(\sigma, j)w_{i,j'} - L(\sigma, j')w_{i,j} - w_{i,j}w_{i,j'}$. It follows that:

$$\Phi(\sigma') - \Phi(\sigma) = K\left(L(\sigma, j)w_{i,j'} - L(\sigma, j')w_{i,j} - w_{i,j}w_{i,j'}\right).$$

Inequality (4) gives $L(\sigma, j)w_{i,j'} > w_{i,j}L(\sigma', j') = w_{i,j}L(\sigma, j') + w_{i,j}w_{i,j'}$ which is equivalent to $L(\sigma, j)w_{i,j'} - w_{i,j}L(\sigma, j') - w_{i,j}w_{i,j'} > 0$. We deduce that $\Phi(\sigma') - \Phi(\sigma) > 0$. As $u_i(\sigma') > u_i(\sigma)$

implies $\Phi(\sigma') > \Phi(\sigma)$, Φ must be a potential function of the game. $\square$

It follows from Theorem 2 that any sequence of improving deviations in PROJECT GAMES with identical rewards never cycles: it always converges to a pure Nash equilibrium. As the potential function given in Theorem 2, as well as the one given in [13] for games with universal weights, can be computed in polynomial time, it follows that the problem of computing a pure Nash equilibrium in games with project-specific weights and identical rewards and in games with universal weights belongs to the complexity class PLS; see, for instance, [12].

For the case of generic rewards and project-specific weights, it is easy to see that every PROJECT GAME is strictly isomorphic to a singleton weighted congestion game with player-specific linear latency functions and resource-specific weights. An instance of these games is defined as follows. There is a set of n players $N = \{1, \ldots, n\} = [n]$ and a set of m resources $R = \{1, \ldots, m\} = [m]$. Each player $i \in N$ can choose a resource from a prescribed set $S_i \subseteq R$ and has a weight $w_{i,j} > 0$ on resource $j \in R$. The load (congestion) of resource j in a strategy profile σ is $L(\sigma, j) = \sum_{i \in N: \sigma_i = j} w_{i,j}$. Each resource $j \in R$ has a player-specific linear latency function $\ell_j^i(x) = \alpha_j^i x$, with $\alpha_j^i \geq 0$, for each $i \in N$. The cost of player i in σ is defined as $c_i(\sigma) = \ell_{\sigma_i}^i(L(\sigma, \sigma_i)) = \alpha_{\sigma_i}^i L(\sigma, \sigma_i)$.

To the best of our knowledge, singleton weighted congestion games with player-specific linear latency functions and resource-specific weights have been considered so far in the literature only under the assumption that the players' weights are not resource-specific, i.e., each player $i \in N$ has a weight $w_i > 0$ for each resource $j \in S_i$. These games have been considered in [17] and [19]. In particular, in [17] it is shown that they admit a potential function if and only if $n = 2$, while the existence of a pure Nash equilibrium for the cases of either $n = 3$ or $m = 2$ in proved in [19]; in the latter, a polynomial time algorithm for computing an equilibrium is also provided. However, there is no relationship between these games and our PROJECT GAMES. In fact, if from one perspective PROJECT GAMES are more general than singleton weighted congestion games with player-specific linear latency functions in the definition of the players' weights (which are resource-specific in the former and resource-independent in the latter), on the other hand, singleton weighted congestion games with player-specific linear latency functions are more general than PROJECT GAMES in the definition of the latency functions (which are arbitrary in the former and resource-related in the latter).

We close this section with the most general case of PROJECT GAMES, but for a small number of players.

Proposition 1. *The best response dynamics of a* PROJECT GAME *with two players always converges.*

Proof. Let $N = \{a, b\}$. By contradiction, suppose there is a cycle $\mathcal{C}$ in the best response dynamics. There must be a state s^1 in $\mathcal{C}$, where a and b are on the same project (as long as the players are on distinct projects, every profitable deviation from σ to σ' is such that $\mathsf{U}(\sigma') > \mathsf{U}(\sigma)$). Suppose s^1 is reached from state s^0 by a best response of b, and s^2 immediately follows s^1 in $\mathcal{C}$ by a best response of a. The deviation of a does not decrease b's utility, meaning that b still plays a best response in s^2. Thus s^2 is a pure Nash equilibrium which contradicts the existence of $\mathcal{C}$. $\square$

Proposition 2. *Every* PROJECT GAME *with three players always admits a pure Nash equilibrium.*

Before giving the proof, note that Proposition 1 holds even if each project j has some *residual weight* $\bar{w}_j \geq 0$, and the load of j under strategy profile σ is defined as $\bar{w}_j + \sum_{\{i \in N : \sigma_i = j\}} w_{i,j}$. The residual weight of a project j can be interpreted as the weight of some player whose decision is made regardless of the state (alternatively, some player has strategy space $\{j\}$).

Proof. There are three players a, b, and c. By Proposition 1, a pure Nash equilibrium σ with only players a and b exists. Introduce c in σ and let her play a best response. We shall consider all possible cases.

First, assume that a and b are choosing the same project in σ. If c chooses the same project of both a and b, say 1, then c can play 1 in any case, i.e., playing 1 is a dominant strategy for c. One can consider a 2-player game with residual weight $\bar{w}_1 = w_{c,1}$. As previously mentioned, a pure Nash equilibrium exists and we are done. If c chooses a distinct project, then the state is a pure Nash equilibrium since everyone plays a best response.

From now on, suppose a and b are on projects 1 and 2, respectively. If c is on $j \notin \{1, 2\}$, then the state is a pure Nash equilibrium since everyone plays a best response. Otherwise, c plays 1 or 2. The state is not a Nash equilibrium only if the player who shares the project with c wants to move. That player is the only one who is not playing a best response. Her deviation (best response) preserves the property that at most one player in $\{a, b, c\}$ is not playing a best response. Therefore, the only way for the dynamics to run into a cycle is when the players' strategies are limited to two projects, as described in the next table.

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
a	1	2	2	2	1	1
b	1	1	1	2	2	2
c	2	2	1	1	1	2

State σ_{k+1} is reached from state σ_k by a deviation of one player. State σ_1 is reached from state σ_6 by a deviation of player b, from project 2 to project 1.

The first deviation $\sigma_1 \to \sigma_2$ is due to player a who moves from project 1 to project 2 because:

$$u_a(\sigma_1) = \frac{w_{1a}}{w_{1a} + w_{1b}} r_1 < \frac{w_{2a}}{w_{2a} + w_{2c}} r_2 = u_a(\sigma_2). \tag{5}$$

The other deviations give:

$$\frac{w_{2c}}{w_{2c} + w_{2a}} r_2 \quad < \quad \frac{w_{1c}}{w_{1c} + w_{1b}} r_1 \tag{6}$$

$$\frac{w_{1b}}{w_{1b} + w_{1c}} r_1 \quad < \quad \frac{w_{2b}}{w_{2b} + w_{2a}} r_2 \tag{7}$$

$$\frac{w_{2a}}{w_{2a}+w_{2b}}r_2 < \frac{w_{1a}}{w_{1a}+w_{1c}}r_1 \quad (8)$$
$$\frac{w_{1c}}{w_{1c}+w_{1a}}r_1 < \frac{w_{2c}}{w_{2c}+w_{2b}}r_2 \quad (9)$$
$$\frac{w_{2b}}{w_{2b}+w_{2c}}r_2 < \frac{w_{1b}}{w_{1b}+w_{1a}}r_1 \quad (10)$$

Let us see that this set of inequalities cannot be satisfied. Suppose $w_{2c} > w_{2b}$. We get that $w_{2a} + w_{2c} > w_{2a} + w_{2b} \Leftrightarrow \frac{w_{2a}}{w_{2a}+w_{2c}}r_2 < \frac{w_{2a}}{w_{2a}+w_{2b}}r_2$. Use this inequality with (5) and (8) to get that $\frac{w_{1a}}{w_{1a}+w_{1b}}r_1 < \frac{w_{1a}}{w_{1a}+w_{1c}}r_1 \Leftrightarrow w_{1c} < w_{1b} \Leftrightarrow \frac{w_{1c}}{w_{1b}+w_{1c}}r_1 < \frac{w_{1b}}{w_{1b}+w_{1c}}r_1$. Use this inequality with (6) and (7) to get that $\frac{w_{2c}}{w_{2c}+w_{2a}}r_2 < \frac{w_{2b}}{w_{2b}+w_{2a}}r_2 \Leftrightarrow w_{2c} < w_{2b}$. A contradiction with the hypothesis is reached. □

3. Social Utility and the Price of Anarchy/Stability

In this section, we analyse the quality of pure Nash equilibria in PROJECT GAMES in term of price of anarchy and price of stability. Before presenting our complete characterization of their bounds, we observe that a social optimum can be computed efficiently.

Proposition 3. *Maximizing the utilitarian social welfare of a* PROJECT GAME *can be done in polynomial time.*

Proof. When the game is symmetric, a social optimum can be constructed by selecting the $\min(m, n)$ projects with largest reward. When the game is asymmetric, a social optimum can be constructed in polynomial time if we resort to the following *transversal matroid* (see for example [25] for more details). There are n sets $S_1, \ldots, S_n$, subsets of M, where S_i is the strategy space of player i. A subset of projects is feasible if it is a *partial transversal.* A partial transversal is a set $T \subseteq M$ such that an injective map $\Phi : T \to [n]$ satisfying $j \in S_{\Phi(j)}$ exists. A feasible set of projects which maximizes U can be found by running the classical greedy algorithm over the transversal matroid. □

3.1. Games with Identical Rewards

In this subsection, we give results for games with identical rewards. The first result states that there is always a pure Nash equilibrium that is socially optimal.

Theorem 3. *For any two integers $n, m > 1$, $\mathsf{PoS}(n, m) = 1$.*

Proof. For a fixed pair of integers $n, m > 1$, consider a PROJECT GAME $G \in \mathcal{G}_{n,m}$. Let σ be a Nash equilibrium for G reached after a sequence of improving deviations starting from a social optimum σ^*. As it is never profitable for a player who is the only selector of a project to leave it in favor of another one, we have $P(\sigma) = P(\sigma^*)$ which implies $\mathsf{U}(\sigma) \geq \mathsf{U}(\sigma^*)$. □

Next, we show that, under the assumption of symmetric games, all pure Nash equilibria are socially optimal.

Theorem 4. *For any two integers $n, m > 1$, $\mathsf{PoA}(n, m) = 1$ for symmetric games.*

Proof. For a fixed pair of integers $n, m > 1$, consider a PROJECT GAME $G \in \mathcal{G}_{n,m}$. Fix a Nash equilibrium σ and a social optimum σ^* for G. Assume, by way of contradiction, that $|P(\sigma^*)| > |P(\sigma)|$, then there exists a project selected by at least two players in σ. Any of these two players improves by deviating to a project in $P(\sigma^*) \setminus P(\sigma)$, rising a contradiction. Thus, we derive $|P(\sigma^*)| = |P(\sigma)|$ which implies $\mathsf{U}(\sigma) = \mathsf{U}(\sigma^*)$. $\square$

For asymmetric games, instead, next theorem shows that the price of anarchy rises to almost 2 even when considering universal weights.

Theorem 5. *For any two integers $n, m > 1$ and $s := \min(n, m)$, $\mathsf{PoA}(n, m) \geq \frac{2\left\lfloor \frac{s-1}{2} \right\rfloor + 1}{\left\lfloor \frac{s-1}{2} \right\rfloor + 1}$ for asymmetric games with universal weights.*

Proof. For a fixed pair of integers $n, m > 1$, we are going to consider several cases, whether $s = n$ or $s = m$, and whether s is odd or even. For each of them, we shall define a game with universal weights attaining the claimed lower bound on the price of anarchy.

Suppose $n < m$, i.e., $s = n$, and $n = 2k + 1$ for some positive k. The first k players have weight 1 and the others have a small positive weight ϵ. For each $i \in [k]$, $S_i = \{i, i + k\} \cup \{2k + 2, \cdots, m\}$, and for any other player $i \in \{k + 1, \cdots, 2k + 1\}$, $S_i = \{i, 2k + 1\}$. The maximal number of selected projects is $n = 2k + 1$ in a profile where every player i plays project i. Consider the profile σ in which every player $i \in [k]$ selects project $i + k$ and every other player selects project $2k + 1$. We shall prove that σ is a pure Nash equilibrium. Clearly, none of the first k players improves by deviating, as they are the only selectors of their chosen project. Thus, we only need to establish that no player $i \in \{k + 1, \cdots, 2k + 1\}$ has an incentive in deviating to project i. Since player $i \in \{k + 1, \cdots, 2k + 1\}$ is selecting a project having $k + 1$ participants, and all participants have the same weight ϵ, we get $u_i(\sigma) = (k + 1)^{-1}$ which is a positive constant, as k is fixed (being n fixed). By deviating to project i, player i gets $u_i(\sigma_{-i}, i) = \frac{\epsilon}{1+\epsilon} < (k + 1)^{-1}$ for a suitable choice of ϵ (namely, $\epsilon < 1/k$). Hence, σ is a pure Nash equilibrium in which $k + 1$ projects are selected. Since $s = n = 2k + 1$, we have that $2\left\lfloor \frac{s-1}{2} \right\rfloor + 1 = 2k + 1$ and $\left\lfloor \frac{s-1}{2} \right\rfloor + 1 = k + 1$. The expected ratio is reached.

Suppose $n < m$, i.e., $s = n$, and $n = 2k$ for some positive k. The first $k - 1$ players have weight 1 and the others have a small positive weight ϵ. For each $i \in [k - 1]$, $S_i = \{i, i + k - 1\} \cup \{2k, \cdots, m\}$. For each $i \in \{k, \cdots, 2k - 1\}$, $S_i = \{i, 2k - 1\}$. The last player $2k$ can only play $2k - 1$. The maximal number of selected projects is $2k - 1$ in a profile where every player $i \in [2k - 1]$ plays project i, and player $2k$ plays project $2k - 1$. Consider the profile σ in which player $i \in [k - 1]$ selects project $i + k - 1$, and every other player selects project $2k - 1$. Using the same arguments as in the previous case, it is easy to check that σ is a pure Nash equilibrium for a suitable choice of ϵ (namely, $\epsilon < 1/k$), and k projects are selected. Since $s = n = 2k$, we have that $2\left\lfloor \frac{s-1}{2} \right\rfloor + 1 = 2k - 1$ and $\left\lfloor \frac{s-1}{2} \right\rfloor + 1 = k$. The expected ratio is reached.

Suppose $m \leq n$, i.e., $s = m$, and $m = 2k+1$ for some positive k. The first k players have weight 1 and the others have a small positive weight ϵ. For each $i \in [k]$, $S_i = \{i, i+k\}$. For each $i \in \{k+1, \cdots, 2k\}$, $S_i = \{i, 2k+1\}$. For any other player i, $S_i = \{2k+1\}$. The maximal number of selected projects is $2k+1$ in a profile where every player $i \in [2k]$ plays project i, whereas the other players select project $2k+1$. Consider the profile σ in which player $i \in [k]$ selects project $i+k$ and every other player selects project $2k+1$. Similarly as before, σ can be shown to be a pure Nash equilibrium for a suitable choice of ϵ, and $k+1$ projects are selected. Since $s = m = 2k+1$, we have that $2\left\lfloor \frac{s-1}{2} \right\rfloor + 1 = 2k+1$ and $\left\lfloor \frac{s-1}{2} \right\rfloor + 1 = k+1$. The expected ratio is reached.

Suppose $m \leq n$, i.e., $s = m$, and $m = 2k$ for some positive k. The first k players have weight 1 and the others have a small positive weight ϵ. For each $i \in [k-1]$, $S_i = \{i, i+k-1, 2k\}$. For each $i \in \{k, \cdots, 2k-2\}$, $S_i = \{i, 2k-1\}$. For any other player i, $S_i = \{2k-1\}$. The maximal number of selected projects is $2k-1$ in a profile where every player $i \in [2k-2]$ plays project i, whereas the other players select project $2k-1$. Consider the profile σ in which player $i \in [k-1]$ selects project $i+k-1$ and every other player selects project $2k-1$. Again, σ is a pure Nash equilibrium for a suitable choice of ϵ, and k projects are selected. Since $s = m = 2k$, we have that $2\left\lfloor \frac{s-1}{2} \right\rfloor + 1 = 2k-1$ and $\left\lfloor \frac{s-1}{2} \right\rfloor + 1 = k$. The expected ratio is reached. $\square$

A matching upper bound, which holds for the more general case of project-specific weights is achieved in the following theorem.

Theorem 6. *For any two integers $n, m > 1$, $\mathsf{PoA}(n,m) \leq \frac{2\left\lfloor \frac{s-1}{2} \right\rfloor + 1}{\left\lfloor \frac{s-1}{2} \right\rfloor + 1}$, where $s := \min(n,m)$, holds for asymmetric games with project-specific weights.*

Proof. For a fixed pair of integers $n, m > 1$, consider a PROJECT GAME $G \in \mathcal{G}_{n,m}$. Fix a Nash equilibrium σ and a social optimum σ^* for G and, for the sake of simplicity, denote $P = P(\sigma)$ and $P^* = P(\sigma^*)$.

Given a project $j \in P^* \setminus P$, let $o(j)$ denote any player choosing project j in σ^*. Define $P_1^* = \{j \in P^* \setminus P : \sigma_{o(j)} \in P^*\}$, $P_2^* = (P^* \setminus P) \setminus P_1^*$, $P_1 = \{j' \in P : \exists j \in P_1^* : \sigma_{o(j)} = j'\}$ and $P_2 = \{j' \in P : \exists j \in P_2^* : \sigma_{o(j)} = j'\}$. By definition, we have $P_1 \subseteq P^* \cap P$ and $P_2 \subseteq P \setminus P^*$.

Note that, for each $j \in P^* \setminus P$, player $o(j)$ must be the only selector of project $j' = \sigma_{o(j)} \neq j$ in σ, otherwise she could deviate to j and improve her utility (the rewards are identical by assumption). This property implies that $|P_1^*| = |P_1|$ and $|P_2^*| = |P_2|$.

Assume that $P_1^* \neq \emptyset$, i.e., $|P_1^*| \geq 1$. We claim that $P \setminus P_1 \neq \emptyset$. Indeed, if this does not hold, we have $P \setminus P_1 = \emptyset$ which gives $P = P_1$. Since we know that each project in P_1 has a unique selector in σ, it follows that $|P_1| = n$. As we also know that $P_1 \subseteq P^* \cap P$, we derive $|P^* \cap P| \geq n$. Putting all together, we get $|P^*| \geq |P_1^*| + |P^* \cap P| \geq n+1$, which yields a contradiction. Thus, we have proven the following implication:

$$P_1^* \neq \emptyset \implies P \setminus P_1 \neq \emptyset. \tag{11}$$

We get:

$$\mathsf{PoA}(n,m) = \frac{\mathsf{U}(\sigma^*)}{\mathsf{U}(\sigma)} = \frac{|P_1^*| + |P_2^*| + |P^* \cap P|}{|P_1| + |P_2| + |P \setminus (P_1 \cup P_2)|}.$$

As $|P_2^*| = |P_2|$ and $P_2 \subseteq P \setminus P^*$, $\mathsf{PoA}(n,m)$ is maximized when $P_2^* = P_2 = \emptyset$, by which we get:

$$\mathsf{PoA}(n,m) \le \frac{|P_1^*| + |P^* \cap P|}{|P_1| + |P \setminus P_1|}.$$

Clearly, if $P_1^* = \emptyset$, we derive a price of anarchy of 1, so assume $|P_1^*| \ge 1$. Using (11), we get $|P \setminus P_1| > 0$. By $P_1 \subseteq P^* \cap P$ and $|P_1| = |P_1^*|$, we obtain that $\mathsf{PoA}(n,m)$ is maximized when $|P \setminus P_1| = 1$ and $P \setminus P_1 \subseteq P \cap P^*$, i.e., when $|P^*| = 2|P_1^*| + 1$ and $|P| = |P_1^*| + 1$. By using $|P_1^*| = \left\lfloor \frac{|P^*|-1}{2} \right\rfloor$ and $|P^*| \le \min(n,m) = s$, we derive:

$$\mathsf{PoA}(n,m) \le \frac{2|P_1^*| + 1}{|P_1^*| + 1} \le \frac{2\left\lfloor \frac{s-1}{2} \right\rfloor + 1}{\left\lfloor \frac{s-1}{2} \right\rfloor + 1},$$

which yields the claim. □

By Theorems 5 and 6, we get that $\mathsf{PoA}(n,m) = \frac{2\left\lfloor \frac{s-1}{2} \right\rfloor + 1}{\left\lfloor \frac{s-1}{2} \right\rfloor + 1}$ for games with both project-specific and universal weights.

3.1.1. Identical Weights.

In this subsection, we consider the case of games with identical weights and identical rewards. We shall focus on the basic restriction in which the strategy space of each player is made up of at most two projects, i.e., $|S_i| \le 2$ for each $i \in N$.

Indeed, when bounding the price of anarchy of a game, at most two strategies per player need to be taken into account: the strategy used in the worst-case pure Nash equilibrium and the one adopted in a social optimum. If there are players possessing additional strategies, these strategies can be removed from the game without altering the analysis of the price of anarchy. For such a reason, restricting to games with at most two strategies per player is without loss of generality when characterizing the price of anarchy. Thus, the bounds claimed in the upcoming result (Theorem 7) extend to any PROJECT GAME.

Games with this property admit an interesting representation via multigraphs with possible loops.

Let $G = (V, E)$ be a multigraph where V and E are the set of vertices and edges. The *neighborhood* of vertex $v \in V$, denoted by $N_G(v)$, is the set of vertices adjacent to v, i.e., $N_G(v) = \{u \in V : uv \in E\}$, and the *degree* of v is $d_G(v) = |N_G(v)|$. An *orientation* of G is a digraph $G' = (V, E')$, where each edge $e = uv \in E$ of G has a direction $e' = (u, v)$ from u to v or $e' = (v, u)$ from v to u; such edge e' is called *directed edge* or *arc*. Given a digraph G', the *outgoing* (resp., *incoming*) *neighborhood* of $v \in V$, denoted by $N^+_{G'}(v)$ (resp., $N^-_{G'}(v)$), is the set of vertices connected to v by means of an arc outgoing (resp., incoming) v, i.e., $N^+_{G'}(v) = \{u \in V : (v, u) \in E'\}$ (resp., $N^-_{G'}(v) = \{u \in V : (u, v) \in E'\}$). The *outdegree* (resp., *indegree*) of $v \in V$, denoted by $d^+_{G'}(v)$ (resp., $d^-_{G'}(v)$), is the number of arcs outgoing (resp., incoming) v, i.e., $|N^+_{G'}(v)|$ (resp., $|N^-_{G'}(v)|$); hence, $d_{G'}(v) = d^-_{G'}(v) + d^+_{G'}(v)$. A *source* (resp., a *sink*) of G' is a vertex with $d^-_{G'}(v) = 0$ (resp., $d^+_{G'}(v) = 0$). Given a digraph $G' = (V, E')$, we define the partition $(V_0, \dots, V_d)$ based on indegrees of V in G', where $V_i = \{y \in V : d^-_{G'}(y) = i\}$ and d is the maximum indegree of G'. Note that V_0 is the set of sources of G' and $|E'| = \sum_{i=1}^{d} i \cdot |V_i|$.

We shall be interested in a problem that we call *Nash Orientation Graph Game* (NOGG in short), which is described as follows:

NASH ORIENTATION GRAPH GAME (NOGG)
Input: A multigraph $G = (V, E)$.
Solution: Orientation σ of E, i.e., $G'_\sigma = (V, E')$.
Output: Nash orientation.

A *Nash orientation* σ of $G = (V, E)$, given by digraph $G'_\sigma = (V, E')$, is an orientation satisfying:

$$\forall e' = (u, v) \in E', d^-_{G'_\sigma}(v) \leq d^-_{G'_\sigma}(u) + 1. \tag{12}$$

Any PROJECT GAME can be transformed into an equivalent NOGG and vice versa, by associating projects to vertices and players to edges. In particular, there is an edge $e = uv$ in the graph if and only if player e has two possible projects u and v (it is a loop if there is only one project). A strategy profile σ of the game corresponds to an orientation, where the head of an edge e is the action of player $e \in E$. Moreover, σ is a pure Nash equilibrium if and only if the corresponding orientation σ of G'_σ satisfies (12). Since we study the price of anarchy, we can always assume that the given multigraphs are connected (otherwise, we deal with each connected component separately). The social utility of an orientation G'_σ is given by $\mathsf{U}(G'_\sigma) = |V \setminus V_0|$. The social utility of an optimal orientation σ^* is denoted by $\mathsf{U}(G^*)$. The following result gives a characterization of $\mathsf{U}(G^*)$.

Lemma 1. *Let $G = (V, E)$ be a connected graph with m vertices. If G is a tree, then $\mathsf{U}(G^*) = m - 1$, otherwise, $\mathsf{U}(G^*) = m$.*

Proof. Assume first that G is a tree. Since any orientation of G gives an acyclic graph, this orientation has a source. Thus, $\mathsf{U}(G^*) \leq m - 1$. Now, any orientation from a root to the leaves gives a social utility of $m - 1$. Therefore, $\mathsf{U}(G^*) \geq m - 1$, which yields the claim.

Assume now that G is a connected graph and it contains a cycle. Take a spanning tree T of G and an edge $uv \notin T$ (uv exists because G has a cycle). Let v be the root of T, and orient all the edges from v to the leaves. Edge uv is oriented from u to v, while every remaining edge is arbitrarily oriented. No vertex is a source in the resulting orientation, so its social utility is m. □

The following technical lemma will play a crucial role in the following result of this subsection.

Lemma 2. *For a connected multigraph $G = (V, E)$ and a Nash orientation $G'_\sigma = (V, E')$ of G defining a partition $(V_0, \ldots, V_d)$ with $d \geq 2$, the following properties hold:*

(*i*) *the sub-digraph induced by* $V_0 \cup N_{G'_\sigma}(V_0)$ *is a collection of* $|V_0|$ *stars, each rooted at a vertex of* $|V_0|$, $N_{G'_\sigma}(V_0) \subseteq V_1$ *and* $|N_{G'_\sigma}(V_0) \setminus S_1| \geq |V_0|$, *where* $S_1 = \{x \in V_1 \colon d^+_{G'_\sigma}(x) = 0\}$;

(*ii*) *for every* $1 \leq i < d$, $N^+_{G'_\sigma}(V_i) \setminus (\cup_{j=1}^{i} V_j) \subseteq V_{i+1}$.

Proof. For property (i), observe that the claim holds trivially if $V_0 = \emptyset$. If $V_0 \neq \emptyset$, then consider an arc $(x, y) \in E'$ with $x \in V_0$. If $y \in V_i$ for some $i \geq 2$, which implies $d^-_{G'_\sigma}(y) \geq 2$, it follows that $G'_\sigma = (V, E')$ is not a Nash orientation because Inequality (12) is not satisfied. So, every vertex in V_0 is adjacent to vertices in V_1 only, which yields that V_0 is an independent set and $N_{G'_\sigma}(V_0) \subseteq V_1$. Moreover, $N_{G'_\sigma}(V_0)$ is also an independent set. In fact, if there were an arc between two vertices in $N_{G'_\sigma}(V_0)$, then one of these vertices would have indegree at least two, thus contradicting the fact that $N_{G'_\sigma}(V_0) \subseteq V_1$. Thus, $V_0 \cup N_{G'_\sigma}(V_0)$ induces a collection of $|V_0|$ stars, each rooted at a vertex of $|V_0|$. Finally, since $d^-_{G'_\sigma}(y) = 1$ for each $y \in N_{G'_\sigma}(V_0)$ and G is connected, it follows that, in order to connect the $|V_0|$ stars induced by $V_0 \cup N_{G'_\sigma}(V_0)$, there must be an outgoing arc from at least one leaf of each star, which yields $|N_{G'_\sigma}(V_0) \setminus S_1| \geq |V_0|$.

For property (ii), observe that, by definition, $N^+_{G'_\sigma}(V_1) \cap V_0 = \emptyset$. If $N^+_{G'_\sigma}(V_1) \setminus V_1 \nsubseteq V_2$, then there exists $z \in V_i$ for $i \geq 3$ and $x \in V_1$ such that $(x, z) \in E'$. Thus, Inequality (12) is not satisfied: a contradiction. Inequality $N^+_{G'_\sigma}(V_i) \setminus (\cup_{j=1}^{i} V_j) \subseteq V_{i+1}$ for $i > 1$ is obtained using the same argument. $\square$

We can now prove upper and lower bounds on the price of anarchy.

Theorem 7. *For any connected multigraph* $G = (V, E)$ *with* m *vertices and* n *edges, we have* $1.582 \approx \frac{e}{e-1} \leq \mathsf{PoA}(G) \leq \frac{5}{3} \approx 1.667$. *The lower bound holds even for trees.*

Proof. Fix a Nash orientation G'_σ of G defining a partition $(V_0, \ldots, V_d)$. If $d \leq 1$, then every player selects a different project in σ, yielding $\mathsf{U}(G'_\sigma) = |V| \geq \mathsf{U}(G^*)$, which gives $\mathsf{PoA}(G) = 1$. Thus, we assume $d \geq 2$ which allows us to apply Lemma 2.

Let $V'_1 := N_{G'_\sigma}(V_0) \setminus S_1 \subseteq V_1$ and $V'_2 := N^+_{G'_\sigma}(V_1) \setminus V_1 \subseteq V_2$. From property (i) of Lemma 2, we know that $|V'_1| \geq |V_0|$. Now, observe that $N^-_{G'_\sigma}(V'_2) \subseteq V'_1$. Property (ii) of Lemma 2, and the definition of V_2 give $|V'_2| \geq \frac{1}{2}|V'_1|$. Hence, using these two inequalities, we get:

$$\mathsf{U}(G'_\sigma) = \sum_{i=1}^{d} |V_i| \geq \frac{3}{5} \sum_{i=1}^{d} |V_i| + \frac{2}{5}\left(|V'_1| + |V'_2|\right) \geq \frac{3}{5} \sum_{i=0}^{d} |V_i| = \frac{3}{5}|V| \geq \frac{3}{5}\mathsf{U}(G^*).$$

For the lower bound, consider a tree T_k arranged in $k+1$ levels $0, \ldots, k$, such that each node at level i has $k - i$ children. The number of nodes of T_k is equal to

$$\sum_{i=0}^{k} \frac{k!}{(k-i)!} = k! \sum_{i=0}^{k} \frac{1}{(k-i)!} = k! \sum_{i=0}^{k} \frac{1}{i!}.$$

Moreover, the number of its leaves is equal to $k!$. Consider the orientation T'_k obtained by orienting all edges towards the root. Any node at level i has $k-i$ incoming edges and this immediately implies that T'_k is a Nash orientation. It follows that $\mathsf{PoA}(T_k) \geq \frac{k!\sum_{i=0}^{k}\frac{1}{i!}-1}{k!\sum_{i=0}^{k}\frac{1}{i!}-k!}$. As $\lim_{k\to\infty}\sum_{i=0}^{k}\frac{1}{i!}=e$, we obtain $\lim_{k\to\infty}\mathsf{PoA}(T_k)=\frac{e}{e-1}$. □

3.2. Games with Generic Rewards

In this section, we address the more general case of *generic* rewards. We start by showing a lower bound on the price of stability which holds even for symmetric games with identical weights.

Proposition 4. *For any two integers $n, m > 1$, $\mathsf{PoS}(n,m) \geq 1 + \frac{\min(n,m)-1}{n}$ for symmetric games with identical weights.*

Proof. For any two integers $n, m > 1$, consider a game with n players of weight 1, one project p with reward $n+\epsilon$, where $\epsilon > 0$ is an arbitrary number, and $m-1$ projects with reward 1.

As choosing project p is a dominant strategy for each player, this game has only one pure Nash equilibrium in which all the players select p. Under this strategy profile, the social utility is $n+\epsilon$. In a social optimum, a maximum number of $\min(n,m)$ projects can be selected by some player, so that the social utility is at most $n+\epsilon+\min(n,m)-1$. Thus, by the arbitrariness of ϵ, the price of stability is at least $1+\frac{\min(n,m)-1}{n}$. □

We now show a matching upper bound that holds even for the price of anarchy of symmetric games with project-specific weights.

Theorem 8. *For any two integers $n, m > 1$, $\mathsf{PoA}(n,m) \leq 1 + \frac{\min(n,m)-1}{n}$ for symmetric games with project-specific weights.*

Proof. For a fixed pair of integers $n, m > 1$, consider a PROJECT GAME $G \in \mathcal{G}_{n,m}$. Fix a Nash equilibrium σ and a social optimum σ^* for G and, for the sake of simplicity, denote $P = P(\sigma)$ and $P^* = P(\sigma^*)$. Actually, P^* consists of the $\min(m,n)$ projects with largest reward.

We assume $P^* \setminus P \neq \emptyset$, otherwise σ is a social optimum as well. We can also assume that P^* and P have the project j^* with largest reward in common. Indeed, if the maximum reward for a project of P is strictly less than r_{j^*}, then σ is not a pure Nash equilibrium. Thus, the maximum reward is the same for P^* and P, and we can assume without loss of generality that $j^* \in P^* \cap P$. We deduce that $\min(n,m)-1 \geq |P^* \setminus P| \geq 1$.

As the game is symmetric, for each $j \in P^* \setminus P$, and $i \in N$, we know that $u_i(\sigma) \geq u_i(\sigma_{-i}, j) = r_j$. Thus, we get $|P^* \setminus P| \cdot u_i(\sigma) \geq \sum_{j\in P^*\setminus P} r_j$ which implies that $(\min(n,m)-1)u_i(\sigma) \geq \sum_{j\in P^*\setminus P} r_j$. By summing over all players, we obtain:

$$(\min(n,m)-1)\sum_{i\in N} u_i(\sigma) \geq \sum_{i\in N}\sum_{j\in P^*\setminus P} r_j.$$

Since $\sum_{i\in N} u_i(\sigma) = \mathsf{U}(\sigma)$ and $\sum_{j\in P^*\setminus P} r_j \geq \mathsf{U}(\sigma^*) - \mathsf{U}(\sigma)$, we get $(\min(n,m)-1)\mathsf{U}(\sigma) \geq n\big(\mathsf{U}(\sigma^*) - \mathsf{U}(\sigma)\big)$ which is equivalent to $(n+\min(n,m)-1)\mathsf{U}(\sigma) \geq n\mathsf{U}(\sigma^*)$. That is, the price of anarchy is at most $1 + \frac{\min(n,m)-1}{n}$. ☐

We now move to the case of asymmetric games. We shall prove an upper bound on the price of anarchy for the case of project-specific weights and then provide a matching lower bound on the price of stability which holds for universal weights. As to the upper bound, from [36], we have that for any two integers $n, m > 1$, $\mathsf{PoA}(n,m) \leq 2$ for games with project-specific weights. Now, we show the matching lower bound.

Proposition 5. *For any two integers $n, m > 1$, $\mathsf{PoS}(n,m) \geq 2$ for asymmetric games with universal weights.*

Proof. Given two integers $n, m > 1$, define $s = \min(n,m)$. Consider a game such that each player $i \in [s-1]$ has weight 1 and set of strategies $S_i = \{i, s\}$, while the last $n-s+1$ players have weight $\frac{\epsilon}{n-s+1}$, where $\epsilon > 0$ is an arbitrary value, and set of strategies equal to $M \setminus [s-1]$. Each of the first $s-1$ projects has reward 1, project s has reward $s-1+\epsilon'$ where $\epsilon' > \epsilon$ is also an arbitrary value, and any project in $M \setminus [s]$ has a negligible but positive reward (i.e., choosing a project in $M \setminus [s]$ is always a dominated strategy).

As choosing project s is a dominant strategy for each player, this game has only one pure Nash equilibrium in which all the players select s. Under this strategy profile, the social utility is $s-1+\epsilon'$. In a social optimum, a maximum number of s projects can be selected by some player, so that the social utility is at most $2(s-1)+\epsilon'$. Thus, by the arbitrariness of both ϵ and ϵ', the price of stability is at least 2. ☐

The lower bound for the price of stability given in Proposition 5 does not apply to games with identical weights. This leaves open the possibility to obtain better bounds on both the price of anarchy and the price of stability in this setting. The following two results cover this case. Again, we shall give a lower bound on the price of stability and a matching upper bound on the price of anarchy.

Proposition 6. *For any two integers $n, m > 1$, we have $\mathsf{PoS}(n,m) \geq 2 - \frac{1}{n}$ if $m \geq n$, $\mathsf{PoS}(n,m) \geq \frac{n+1}{n}$ if $n > m = 2$, and $\mathsf{PoS}(n,m) \geq 2 - \frac{1}{m-1}$ if $n > m > 2$ for asymmetric games with identical weights.*

Proof. The lower bounds $\mathsf{PoS}(n,m) \geq 2 - 1/n$ if $m \geq n$ and $\mathsf{PoS}(n,m) \geq \frac{n+1}{n}$ if $n > m = 2$ come from Proposition 4 which holds even for the case of symmetric games. Here, we give a lower bounding instance for the case of $n > m > 2$.

Consider the game with identical weights such that $S_i = \{i, m-1\}$ for each $i \in [m-1]$ and $S_i = \{m\}$ for each $i \geq m$. The first $m-2$ projects have reward 1, project $m-1$ has reward $m-1+\epsilon$ and project m has reward ϵ, where $\epsilon > 0$ is an arbitrary value.

Note that the first $m-2$ players are the only one who can perform a choice and, for each of them, playing project $m-1$ is a dominant strategy. Hence, there exists a unique pure Nash equilibrium σ in which only the last two projects are selected, that is, such that $\mathsf{U}(\sigma) = m - 1 + 2\epsilon$. As in a social optimum all projects can be selected, for an overall social value of $2m-3+2\epsilon$, the claimed lower bound follows by the arbitrariness of ϵ.□

We shall prove the upper bounds by exploiting the primal-dual method developed in [7]. Before doing this, we need some additional notation. Given two strategy profiles σ and σ^*, define $\alpha(\sigma, \sigma^*) := |P(\sigma^*) \setminus P(\sigma)|$; moreover, for each $j \in M$, denote as $C_j(\sigma, \sigma^*) = \{i \in N : \sigma_i = \sigma_i^* = j\}$ the set of players selecting project j in both σ and σ^* and denote as $O_j(\sigma, \sigma^*) = \{i \in N \setminus C_j(\sigma, \sigma^*) : \sigma_i^* = j\}$ the set of players selecting project j in σ^*, but not in σ. In the application of this method, we shall make use of the following technical lemma.

Lemma 3. *Fix an asymmetric game with identical weights. For each strategy profile σ and social optimum σ', there exists a social optimum σ^* such that* (i) *$P(\sigma^*) = P(\sigma')$ and* (ii) *for each $j \in P(\sigma^*) \cap P(\sigma)$, $|C_j(\sigma, \sigma^*)| \geq L(\sigma, j) - \alpha(\sigma, \sigma^*)$.*

Proof. Fix a strategy profile σ and a social optimum σ' and, for the sake of simplicity, set $\alpha = \alpha(\sigma, \sigma')$. Our aim is to slightly modify σ so as to obtain a social optimum σ^* mimicking the assignment of players to projects realized in σ for as much as possible. To do this, consider the following algorithm operating in three steps.

At step 1, for each $j \in P(\sigma') \setminus P(\sigma)$, choose a unique player $o(j)$ such that $j \in S_{o(j)}$ and define $\sigma^*_{o(j)} = j$. Let T_1 be the set of players chosen at this step; clearly, $|T_1| = \alpha$. At step 2, for each $j \in P(\sigma') \cap P(\sigma)$, choose a unique player $o(j)$ in $N \setminus T_1$ such that $j \in S_{o(j)}$ and define $\sigma^*_{o(j)} = j$. Let T_2 be the set of players chosen at this step. At step 3, for each $i \notin T_1 \cup T_2$, set $\sigma_i^* = \sigma_i$ if $\sigma_i \in P(\sigma') \cap P(\sigma)$ and $\sigma_i^* = j$ otherwise, where j is an arbitrary project in $P(\sigma') \cap S_i$.

The existence of σ' implies that there exists a choice for T_1 and T_2 which guarantees that $P(\sigma^*) = P(\sigma')$. To show part *(ii)* of the claim, consider a project $j^* \in P(\sigma) \cap P(\sigma^*)$ such that $L(\sigma, j^*) \geq \alpha$ (if no such project exists, then the claim is trivially true). Let

$$\beta := |\{j \in P(\sigma') \cap P(\sigma) : \{i \in N \setminus T_1 : \sigma_i = j\} = \emptyset\}|$$

be the number of projects in $P(\sigma') \cap P(\sigma)$ that lost all of their users in σ after step 1 of the algorithm. We have that step 1 selects at least β players from β different projects in $P(\sigma) \cap P(\sigma^*)$. This implies that j^* loses at most $\alpha - \beta$ users after step 1. At step 2, j^* can lose at most other β additional users for a total of α users. Hence, at least $L(\sigma, j) - \alpha$ players are assigned to j^* in σ^* at step 3 of the algorithm and this shows claim *(ii)*. □

Theorem 9. *For any two integers $n, m > 1$, we have $\mathsf{PoA}(n,m) \leq 2 - \frac{1}{n}$ if $m \geq n$, $\mathsf{PoA}(n,m) \leq \frac{n+1}{n}$ if $n > m = 2$, and $\mathsf{PoA}(n,m) \leq 2 - \frac{1}{m-1}$ if $n > m > 2$ for asymmetric games with identical weights.*

Proof. For a fixed pair of integers $n, m > 1$, consider a PROJECT GAME $G \in \mathcal{G}_{n,m}$. Fix a pure Nash equilibrium σ and a social optimum σ^* for G and, for the sake of simplicity, set $\alpha = \alpha(\sigma, \sigma^*)$. By Lemma 3, we can assume without loss of generality that, for each $j \in P(\sigma^*) \cap P(\sigma)$, $|C_j(\sigma, \sigma^*)| \geq L(\sigma, j) - \alpha$. We assume $\alpha \geq 1$ as, otherwise, the price of anarchy is trivially equal to 1. By applying the primal-dual method, we get that the inverse of the optimal solution of the following linear program provides an upper bound on $\mathsf{PoA}(n, m)$:

$$\begin{array}{ll}
\min \sum_{j \in P(\sigma)} r_j & \\
s.t. & \\
\frac{r_{\sigma_i}}{L(\sigma, \sigma_i)} - \frac{r_{\sigma_i^*}}{L((\sigma_{-i}, \sigma_i^*), \sigma_i^*)} \geq 0 & \forall i \in N, \\
\sum_{j \in P(\sigma^*)} r_j = 1 & \\
r_j \geq 0 & \forall j \in M
\end{array}$$

For a strategy profile τ and a project j, denote by $\mathbb{1}_j(\tau)$ the indicator function that is equal to 1 if and only if $j \in P(\tau)$. The dual of the above linear program is the following (we associate variable x_i with the first constraint for each $i \in N$ and variable γ with the second one):

$$\begin{array}{ll}
\max \gamma & \\
s.t. & \\
\sum_{i:\sigma_i = j} \frac{x_i}{L(\sigma, j)} - \sum_{i:\sigma_j^*} \frac{x_i}{L((\sigma_{-i}, j), j)} + \gamma \mathbb{1}_j(\sigma^*) \leq \mathbb{1}_j(\sigma) & \forall j \in M, \\
x_i \geq 0 & \forall i \in N
\end{array}$$

By the Weak Duality Theorem, the inverse of the objective value of any feasible solution to this program provides an upper bound on $\mathsf{PoA}(n, m)$.

First of all, we observe that, for any dual solution such that $x_i = x$ for each $i \in N$ and $\gamma = x$, the dual constraint becomes:

$$x \left(\mathbb{1}_j(\sigma) - \frac{|C_j(\sigma, \sigma^*)|}{L(\sigma, j)} - \frac{|O_j(\sigma, \sigma^*)|}{L(\sigma, j) + 1} + \mathbb{1}_j(\sigma^*) \right) \leq \mathbb{1}_j(\sigma). \tag{13}$$

If $\mathbb{1}_j(\sigma^*) = 0$, (13) is satisfied as long as $x \leq 1$. If $\mathbb{1}_j(\sigma^*) = 1$ and $\mathbb{1}_j(\sigma) = 0$, which imply $|C_j(\sigma, \sigma^*)| = 0$, $|O_j(\sigma, \sigma^*)| \geq 1$, and $L(\sigma, j) + 1 = 1$, (13) is satisfied independently of the value of x. The case of $\mathbb{1}_j(\sigma^*) = 1$ and $\mathbb{1}_j(\sigma) = 1$ is then the only one which can cause a price of anarchy higher than 1 and we focus on this case in the remainder on the proof. Note that, in this case, we can always assume $|C_j(\sigma, \sigma^*)| + |O_j(\sigma, \sigma^*)| \geq 1$.

Consider the dual solution such that $x = \frac{n}{2n-1}$. As $L(\sigma, j) + 1 \leq n$, (13) is satisfied. This proves a general upper bound of $2 - 1/n$. However, for the case of $n > m$, better upper bounds can be derived. Note that, in this case, we have $1 \leq \alpha \leq m - 1$.

Assume $\alpha \leq m-2$ and consider the dual solution such that $x = \frac{m-1}{2m-3}$. If $L(\sigma, j) \leq \alpha$, the term within the parenthesis in the left-hand side of (13) is at most $\frac{2m-3}{m-1}$ and the constraint is satisfied. If $L(\sigma, j) > \alpha$, as $|C_j(\sigma, \sigma^*)| \geq L(\sigma, j) - \alpha$, the term within the parenthesis in the left-hand side of (13) is at most $2 - \frac{L(\sigma,j)-\alpha}{L(\sigma,j)} = \frac{L(\sigma,j)+\alpha}{L(\sigma,j)}$ which is maximized for $\alpha = m-2$ and $L(\sigma, j) = \alpha + 1 = m - 1$. Again, (13) is satisfied. This proves an upper bound of $2 - \frac{1}{m-1}$. Note that this bound does not apply to the case of $m = 2$, as α cannot be equal to $m - 2$ in this case.

Assume now $\alpha = m - 1$ and consider the dual solution such that $x = \frac{n}{n+m-1}$. The assumption $\alpha = m - 1$ implies that there exists a unique project $j \in P(\sigma) \cap P(\sigma^*)$ and so $L(\sigma, j) = n$ and $C_j(\sigma, \sigma^*) = n - m + 1$. In this case, the term within the parenthesis in the left-hand side of (13) is exactly $\frac{n+m-1}{n}$ and (13) is satisfied. This proves an upper bound of $\frac{n+m-1}{n}$.

As $2 - \frac{1}{m-1} \geq \frac{n+m-1}{n}$ for $n > m > 2$, the claimed upper bounds follow. □

4. Egalitarian Social Welfare

So far, we have considered the utilitarian social welfare $\mathsf{U}(\sigma) := \sum_{i \in N} u_i(\sigma)$. In this section, we use the *egalitarian social welfare* $\mathsf{E}(\sigma) := \min_{i \in N} u_i(\sigma)$ (to be maximized). For this section, we suppose adapted definitions of the PoA and the PoS which include E instead of U. The motivation for considering E instead of U is fairness among the players.

Proposition 7. *For the egalitarian social welfare, the* PoS *of* PROJECT GAMES *is unbounded even with 4 players, 2 projects, universal weights and identical rewards.*

Proof. Suppose there are 2 players with weight 1 and 2 players with weight $X \gg 1$. If both projects host one player of each kind then the strategy profile is a Nash equilibrium σ where $\mathsf{E}(\sigma) = (X+1)^{-1}$. The egalitarian social welfare is equal to 1/2 for the unstable state when one project gets the 2 players of weight 1, and the other project gets the 2 players of weight X. □

Since PoA $\geq$ PoS, the PoA of PROJECT GAMES is unbounded as well. One can be tempted to try to enforce a social optimum. However, unlike the utilitarian social welfare (see Proposition 3), the problem is intractable.

Proposition 8. *It is **NP**-hard to compute a strategy profile that maximizes the egalitarian social welfare of* PROJECT GAMES *even if there are two projects, identical rewards, and universal weights.*

Proof. Take an instance of PARTITION with n values $a_1, \ldots, a_n$ such that $\sum_{i=1}^{n} a_i = 2B$. The problem is to decide if there exists a subset of indices $I \subset \{1, \ldots, n\}$ such that $\sum_{i \in I} a_i = \sum_{i \notin I} a_i = B$, see [18]. We can suppose without loss of generality that $0 < a_1 \leq a_2 \leq \ldots \leq a_n$. Create a PROJECT GAME with $n+2$ players. There are two projects with identical rewards. Each a_i corresponds to a player i with universal weight a_i, and strategy space $\{1, 2\}$. There are two additional players $n+i$, $i \in \{1, 2\}$, with weight $\epsilon := a_1/2$ and strategy space $\{i\}$. The egalitarian social welfare is equal to the minimum utility between the ones realized by players $n+1$ and $n+2$. It is not difficult to see that this quantity is equal to $\epsilon/(B+\epsilon)$ if and only if the instance of PARTITION is a yes instance. □

Nevertheless, we were able to identify a polynomial case.

Proposition 9. *Maximizing the egalitarian social welfare in* PROJECT GAMES *can be done in polynomial time when the players have identical weights.*

Proof. Since the players have identical weights, there are at most nm possible values for the individual utility of a player. For every possible value u^* of the optimal egalitarian social welfare, construct a bipartite graph $(N \cup M, E)$, where $E := \{(i, j) \in N \times M : j \in S_i\}$. Each node $i \in N$ has capacity $\kappa_i = 1$ and each node $j \in M$ has capacity $\kappa_j = \lfloor \frac{r_j}{u^*} \rfloor$. A b-matching is a set of edges $E' \subseteq E$ such that the number of edges incident to every node v is at most κ_v. A b-matching of maximum cardinality in the bipartite graph can be computed in polynomial time, see e.g. [25]. If a b-matching saturates all the player vertices then a feasible strategy can be derived (edge (i, j) indicates that player i plays project j) where the utility of every player is at least u^*. □

5. Conclusion and Open Problems

We introduced a new class of games sharing similarities with valid utility games, singleton congestion games, and hedonic games. We focused on existence, computational complexity and efficiency of pure Nash equilibria under a natural method for sharing the rewards of the projects that are realized.

Though the existence of a pure Nash equilibrium is showed for many important special cases, proving (or disproving) its existence in general is a challenging task. An interesting special case that is left open is when the number of projects is small (e.g. $m = 2$). Other solution concepts (e.g., strong Nash equilibria) deserve attention.

Our upper bounds on PoA and PoS under the utilitarian social welfare never exceed 2, but it does not prevent to explore other sharing methods. Moreover, closing the gap shown in Theorem 7 is an intriguing open problem.

Regarding the computation of an optimal strategy profile with respect to the egalitarian social welfare, there is a gap between hard and polynomial cases (see Propositions 8 and 9). As a first step, it would be interesting to settle the complexity of the symmetric case. As the PoS is unbounded under the egalitarian social welfare, it is natural to ask if a different reward sharing method can provide better results.

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