# Gathering Problems for Autonomous Mobile Robots with Lights 

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November 30, 2018


#### Abstract

We study the Gathering problem for $n$ autonomous mobile robots in semi-synchronous settings with persistent memory called light. It is well known that Gathering is impossible in a basic model when robots have no lights, if the system is semi-synchronous or even centralized (only one robot is active in each time) [5, 21]. On the other hand, Rendezvous (Gathering when $n=2$ ) is possible if robots have lights of various types with a constant number of colors [10, 22]. If robots can observe not only their own lights but also other robots' lights, their lights are called full-light. If robots can only observe the state of other robots' lights, their lights are called external-light. If robots can only observe their own lights, their lights are called internal-light.

In this paper, we extend the model of robots with lights so that Gathering algorithms can be discussed properly. Then we show Gathering algorithms with three types of lights in the semi-synchronous settings and reveal relationship between the power of lights and other additional assumptions. The most algorithms shown here are optimal in the number of colors they use.


## 1 Introduction

Background and Motivation The computational issues of autonomous mobile robots have been research object in distributed computing fields. In

[^0]particular, a large amount of work has been dedicated to the research of theoretical models of autonomous mobile robots $[1,3,6,6,13,17,20,21$. In the basic common setting, a robot is modeled as a point in a two dimensional plane and its capability is quite weak. We usually assume that robots are oblivious (no memory to record past history), anonymous and uniform (robots have no IDs and run identical algorithms) [9]. Robots operate in Look-ComputeMove (LCM) cycles in the model. In the Look operation, robots obtain a snapshot of the environment and they execute the same algorithm with the snapshot as an input in Compute operation, and move towards the computed destination in Move operation. Repeating these cycles, all robots perform a given task. It is difficult for these too weak robot systems to accomplish the task to be completed. Revealing the weakest capability of robots to attain a given task is one of the most interesting challenges in the theoretical research of autonomous mobile robots.

In this paper, we also explore such weakest capabilities. In particular, we reveal the weakest additional assumptions for the task which cannot be solved in the basic common models. The problem considered in this paper is Gathering, which is one of the most fundamental tasks of autonomous mobile robots. Gathering is the process of $n$ mobile robots, initially located on arbitrary positions, meeting within finite time at a location, not known a priori. When there are two robots in this setting, this task is called Rendezvous. Since Gathering and Rendezvous are simple but essential problems, they have intensively studied and a number of possibility and/or impossibility results have been shown under the different assumptions [1] $3,5,5,6,8,11,14,15,17,20]$. The solvability of Gathering and Rendezvous depends on the activation schedule and the synchronization level. Usually three basic types of schedulers are identified, the fully synchronous (FSYNC), the semi-synchronous (SSYNC) and the asynchronous (ASYNC). In the FSYNC model, there is a common round and in each round all robots are activated simultaneously and Compute and Move are done instantaneously. The SSYNC is the same as FSYNC except that in each round only a subset of robots are activated. In the ASYNC scheduler, there is no restriction about notion of time, Compute and Move in each cycle can take an unpredictable amount of time, and the time interval between successive activations is also unpredictable (but these times must be finite). Gathering and Rendezvous are trivially solvable in FSYNC and the basic model. However, these problems can not be solved in SSYNC without any additional assumptions [9]. In particular, Gathering can not be solvable even in a restricted subclass of SSYNC scheduler, where exactly one robot is activated in each round (called CENT) [5]. If all robots are initially located on different positions (called distinct Gathering), this Gathering can not be solved even in the ROUND-ROBIN scheduler, in
which exactly one robot is activated in each round and always in the same order (5).

In 4, persistent memory called light has been introduced to reveal relationship between the solvability of Gathering and the synchrony of schedulers and they show asynchronous robots with lights equipped with a constant number of colors, are strictly more powerful than semi-synchronous robots without lights. Robots with lights have been also introduced in order to solve Rendezvous without any other additional assumptions. [4, 10, 22]. Table 1 shows results to solve Rendezvous by robots with lights in each scheduler, where the circle $(\bigcirc)$ and the cross $(\times)$ mean Rendezvous is solvable and unsolvable, respectively, the hyphen ( - ) indicates that this part has been solved under weaker conditions or unsolved under the stronger ones, and the question mark (?) means that this part has not been solved. In the table, full-light means that robots can see not only lights of other robots but also their own light, and external-light and internal-ligh ${ }^{11}$ mean that they can see only lights of other robots and only own light, respectively. Full-light is not a weaker assumption than external-light and internal-light, and internal-light seems to be weaker than external-light. Although this relationship is not proved, the results indicate the relationship. For example, Rendezvous is solved by robots with 3 colors of external-light in SSYNC and non-rigid. On the other hand, it is solved by robots with 6 colors of internal-light in SSYNC and rigid, where robots can reach the computed destination in rigid and they may be stopped before reaching it in non-rigid. The number of colors used in the external-light algorithm is less than that in the internal-light one, and the former uses non-rigid assumption but the latter uses the stronger one, rigidness. When robots know the value $\delta$ of a minimum distance movement in non-rigid (denoted by $+\delta$ in the table), the number of colors can be reduced into only 3 in SSYNC and internal-light. Thus, the assumption robots know $\delta$ in non-rigid seems to be stronger than that in rigid. The power of lights to solve other problems are discussed in [7, 16].

Table 1: Rendezvous algorithms by robots with lights.

| scheduler | movement | full-light | external-light | internal-light | no-light |
| :---: | :---: | :---: | :---: | :---: | :---: |
| FSYNC | - | - | - | - | $\bigcirc[9]$ |
| SSYNC | rigid | - | $?$ | $6[10]$ | $\times[9]$ |
|  | non-rigid | - | $3[10]$ | $3(+\delta)[10]$ |  |
| ASYNC | rigid | - | $12[10]$ | $?$ | - |
|  | non-rigid | $2[12]$ | $3(+\delta)[10]$ | $?$ |  |

[^1]Our Contribution In this paper, we discuss Gathering algorithms by robots with lights and reveal relationship about the solvability between kinds of lights and other additional assumptions. First, we extend the model of robots with lights in which Gathering algorithms can be discussed properly. Unlike Rendezvous, the multiplicity detection affects the solvability of Gathering and in fact, Gathering can be solvable by robots without lights if robots detect the exact number of robots on the same points (strong multiplicity detection) [9. When several robots with lights occupy a same location, if we assume that robots can recognize all colors of the robots on the location, the strong multiplicity detection is possible and robots without lights can solve the Gathering problem. Thus, we define the model so that robots can not detect the exact number of robots on the same location provided that robots use only one color. We consider two types of models according to views robots observe. One is called set-view, where robots can recognize sets of colors robots have on the same locations ${ }^{2}$. Another is called arbitraryview, where robots can recognize arbitrary color robots have on the same locations. We usually use the set view assumption in this paper. We also consider the case that several robots occupy a same location and how these robots observe one another. We call local-aware if any robot can recognize other robots located on the same position and local-unaware, otherwise. If we assume the local-awareness, the view of robots obeys the model (set-view or arbitrary-view) we assume. We usually assume the local-unawareness but we show the local-awareness reduces the number of colors for some Gathering algorithm.

In the extended model, we show the following six Gathering algorithms with lights (Table 24).
(a) Algorithm 1:(2, full, SSYNC, non-rigid),
(b) Algorithm 2:(3, external, SSYNC, rigid),
(c) Algorithm 3:(2, external, SSYNC, rigid; local-awareness),
(d) Algorithm 4+5+6:(2, internal, SSYNC, non-rigid; D-distant),
(e) Algorithm 7:(2, external, CENT, non-rigid), and
(f) Algorithm 8:(2, internal, ROUND-ROBIN, rigid),
where Algorithm i: $(\alpha, \beta, \gamma, \delta ; \eta)$ means Algorithm i uses $\beta$-light with $\alpha$ colors and works in $\gamma$ scheduler and movement restriction $\delta$ with additional assumption $\eta$ (if any).

[^2]Table 2 also shows the unsolvability of Gathering and every algorithm using two colors is optimal with respect to the number of colors, since Gathering is not solvable without lights in every case. We show four algorithms in SSYNC. In the full-light model, we construct an algorithm with the weakest assumptions, non-rigid and the least number of colors ( 2 colors). In the external-light model, we construct an algorithm with 3 colors of lights and rigid assumption and we reduce the number of colors used in the algorithm into 2 colors if the local-awareness is assumed. In the internal-light model, with a little bit more assumption we construct an algorithm with lights of 2 colors if robots know the minimum distance $\delta$ of movement in non-rigid and if initial configurations of robots satisfy some condition called $2 \delta$-distant, where arbitrary two robots not occupying the same locations are separated by at least $2 \delta$. We reveal some relationship between view of lights (full, external and internal) and the other assumptions (movement, local-awareness) in our algorithms.

We also construct Gathering algorithms in CENT and ROUND-ROBIN schedulers. Since Gathering and distinct Gathering can not be solved without lights in CENT and ROUND-ROBIN, respectively [5], our algorithms use 2 colors but fairly weak assumptions. We give algorithms in external-light of 2 colors, non-rigid and CENT, and in internal-light of 2 colors, rigid and ROUND-ROBIN, respectively.

Table 2: Our Gathering algorithms by robots with lights.

| scheduler | movement | full-light | external-light | internal-light | no-light |
| :---: | :---: | :---: | :---: | :---: | :---: |
| FSYNC | - | - | - | - | $\bigcirc$ |
| ROUND-ROBIN | rigid | - | - | $2(\mathrm{f})$ | $\times^{*}[5]$ |
| CENT | non-rigid | - | $2(\mathrm{e})$ | $?$ | $\times[5]$ |
| SSYNC | rigid | - | $3(\mathrm{~b}), 2^{* *}(\mathrm{c})$ | $?$ | - |
|  | non-rigid | $2(\mathrm{a})$ | $?$ | $2(+\delta)^{* * *}(\mathrm{~d})$ |  |
| Distinct Gathering, ${ }^{* *}$ local-awareness, ${ }^{* * *} 2 \delta$-distant |  |  |  |  |  |

The remainder of the paper is organized as follows. In Section 2, we define a robot model with lights, gathering problems, and terminologies. Section 3 shows the previous results for Gathering and Rendezvous problems, and Section 4 shows Gathering algorithms of robots with lights in SSYNC on several situations for which Gathering algorithms do not exist by using lights. Section 5 shows Gathering algorithms of robots with lights in CENT. Section 6 concludes the paper.

## 2 Model and Preliminaries

We consider a set of anonymous mobile robots $\mathcal{R}=\left\{r_{1}, \ldots, r_{n}\right\}$ located in $\mathbb{R}^{2}$. Each robot $r_{i}$ has a persistent state $\ell_{i}$ called light which may be taken from a finite set of colors $L$.

We denote by $\ell_{i}(t)$ the color that the light of robot $r_{i}$ has at time $t$ and $p_{i}(t) \in \mathbb{R}^{2}$ the position occupied by $r_{i}$ at time $t$ represented in some global coordinate system. A configuration $\mathcal{C}(t)$ at time $t$ is a multi-set of $n$ pairs $\left(\ell_{i}(t), p_{i}(t)\right)$, each defining the color of light and the position of robot $r_{i}$ at time $t$. When no confusion arises, $\mathcal{C}(t)$ is simply denoted by $\mathcal{C}$.

For a subset $S$ of $L \times \mathbb{R}^{2}, \mathcal{L}(S)$ and $\mathcal{P}(S)$ are denoted as projections to $L$ and $\mathbb{R}^{2}$ from $S$, respectively.

Each robot $r_{i}$ has its own coordinate system where $r_{i}$ is located at its origin at any time. These coordinate systems do not necessarily agree with those of other robots. It means that there is no common knowledge of unit of distance, directions of its coordinates, or clockwise orientation (chirality $\left.{ }^{3}\right)$.

At any point of time, a robot can be active or inactive. When a robot $r_{i}$ is activated, it executes Look, Compute, and Move cycles:

- Look: The robot $r_{i}$ activates its sensors to obtain a snapshot which consists of a pair of light and position for every robot with respect to the coordinate system of $r_{i}$. Let $\mathcal{S S}_{i}(t)$ denote the snapshot of $r_{i}$ at time $t$. We assume robots can observe all other robots(unlimited visibility). Note that $\mathcal{S} \mathcal{S}_{i}(t)$ represents a sub-multi-set of $\mathcal{C}(t)$ according to imposed assumptions by the local coordinate system of $r_{i}$, where $r_{i}$ is at the origin.
- Compute: The robot $r_{i}$ executes its algorithm using the snapshot and the color of its own light (if allowed by the model) and returns a destination point des $_{i}$ expressed in its coordinate system and a light $\ell_{i} \in L$. The robot $r_{i}$ sets its own light $\ell_{i}$ to the color.
- Move: The robot $r_{i}$ moves to the computed destination des $_{i}$. A robot $r$ is said to collide with robot $s$ at time $t$ if $p(r, t)=p(s, t)$ and at time $t r$ is performing Move. The collision is accidental if $r$ 's destination is not $p(r, t)$. Since robots are seen as points, we assume that accidental collisions are immaterial. A moving robot, upon causing an accidental collision, proceeds in its movement without changes, in a "hit-andrun" fashion [9]. If the robot may be stopped by an adversary before

[^3]reaching the computed destination, the movement is said to be nonrigid. Otherwise, it is said to be rigid. If stopped before reaching its destination, we assume that a robot moves at least a minimum distance $\delta>0$. Note that without this assumption an adversary could make it impossible for any robot to ever reach its destination. If the distance to the destination is at most $\delta$, the robot can reach it. If the movement is non-rigid and robots know the value of $\delta$, it is called non-rigid $(+\delta)$.

In Look operation, a snapshot $\mathcal{S S}_{i}$ of $r_{i}$ should contain positions of all robots including $r_{i}$. However, if other robots are located on $p_{i}$ and $r_{i}$ can recognize the other robots, robots have somehow multiplicity detection at this point. Thus, we need some treatment of observation of other robots located on $p_{i}$ for robot $r_{i}$. If any robot $r_{i}$ can observe other robots located on $p_{i}$, it is said to be local-aware. Otherwise, it is said to be local-unaware. Note that if we assume the local-awareness, $r_{i}$ recognizes whether other robots occupy the location $p_{i}$ or not. In the following, we usually use the local-unaware assumption but we will show that we can reduce the number of colors for some Gathering algorithm if we assume the local-awareness.

A scheduler decides which subset of robots is activated for every configuration. The schedulers we consider are asynchronous and semi-synchronous and it is assumed that schedulers are fair, each robot is activated infinitely often.

- ASYNC: The asynchronous scheduler (ASYNC), activates the robots independently, and the duration of each Compute, Move and between successive activities is finite and unpredictable. As a result, robots can be seen while moving and the snapshot obtained with the Look operation and its actual configuration when performing the Compute operation are not the same and so its computation may be done with the old configuration.
- SSYNC: The semi-synchronous scheduler (SSYNC) activates a subset of all robots synchronously and their Look-Compute-Move cycles are performed at the same time. We can assume that activated robots at the same time obtain the same snapshot and their Compute and Move are executed instantaneously. In SSYNC, we can assume that each activation defines discrete time called round and Look-Compute-Move is performed instantaneously in one round. In the following, since we consider SSYNC and its subsets, we use round and time interchangeably.

As a special case of SSYNC, if all robots are activated in each round,
the scheduler is called fully-synchronous (FSYNC). We also consider the following subsets of SSYNC.

- CENT: In CENT scheduler exactly one robot is activated in each round.
- $k$-BOUNDED: In $k$-BOUNDED scheduler there exists some bound $k$ such that between any two consecutive activation of any robot, no other robot is activated more than $k$ times.
- ROUND-ROBIN: ROUND-ROBIN scheduler is 1-BOUNDED and CENT. It means that robots are activated one in each round and always in the same sequence.

Let $\mathcal{C}(t)$ be a configuration at round $t$. When $\mathcal{C}(t)$ reaches $\mathcal{C}(t+1)$ by executing the cycle at $t$, it is denoted as $\mathcal{C}(t) \rightarrow \mathcal{C}(t+1)$. The reflective and transitive closure is denoted as $\rightarrow^{*}$.

Snapshots may be different by using assumptions even if these configurations are the same, and they depend on multiplicity detection or how robots can see lights of other robots when robots equip with lights. The robots are said to be capable of multiplicity detection whether they can distinguish if a point is occupied by at least two robots. The multiplicity detection is strong if the robots can detect the exact number of robots on the same points.

In our settings, robots have persistent lights and can change its color after Compute operation. We consider the following three robot models according to the visibility of lights.

- full-light, the robot can recognize not only colors of lights of other robots but also its own color of light.
- external-light, the robot can recognize only colors of lights of other robots but cannot see its own color of light. Note robot can change its own color.
- internal-light, the robot can recognize only its own color of light but cannot see colors of lights of other robots.

When a robot performs Look operation in the internal-light model, its snapshot is the same as that in the case of robots without lights. On the other hand, in the full-light or external-light model, it obtains a snapshot with locations of other robots and their colors. In this case we consider several types of snapshots according to view robots observe.

Given a snapshot $\mathcal{S} \mathcal{S}_{i}$ of a robot $r_{i}$ and a point $p_{j}(j \neq i)$ included in $\mathcal{P}\left(\mathcal{S S}_{i}\right)$, a view $V_{i}\left[p_{j}\right]$ of $p_{j}$ in $\mathcal{S S}_{i}$ is a subset of $A L_{i}\left[p_{j}\right]=\left\{\ell \mid\left(\ell, p_{j}\right) \in\right.$
$\left.\mathcal{S} \mathcal{S}_{i}, r_{j} \neq r_{i}\right\}$, where $A L_{i}\left[p_{j}\right]$ is a multi-set of colors of other robots that $r_{i}$ can see at point $p_{j}$. For any robot $r_{i}$ and any point $p$ of snapshot of $r_{i}$, if $V_{i}[p]=A L_{i}[p]$, view of robots is called multi-set- view. If $V_{i}[p]$ regards $A L_{i}[p]$ as just a set, it is called set-view. If $V_{i}[p]$ is a set of any single element taken from $A L_{i}[p]$, it is called arbitrary-view. Let $V_{i}$ denote $\cup_{(\ell, p) \in \mathcal{S} \mathcal{S}_{i}} V_{i}[p]$.

Multi-set-view is a strong assumption, because robots without lights (with one color) can have strong multiplicity detection if multi-set-view is assumed. In fact we can solve a Gathering problem by using robots without lights and multi-set-view [9]. On the other hand, set-view and arbitrary-view do not imply multiplicity detection. In the following we assume set-view or arbitrary-view.

An $n$-Gathering problem is defined as follows: given $n(\geq 2)$ robots initially placed at arbitrary positions in $\mathbb{R}^{2}$, they congregate in finite time at a single location which is not predefined. In the following, the case 2-Gathering problem is called Rendezvous and the $n$-Gathering problem $(n \geq 3)$ is simply called Gathering. Gathering is said to be distinct if all robots are initially placed in different positions. An algorithm solving Gathering is said to be self-stabilizing if robots are initially set their lights to arbitrary colors and they start their execution from Look operation.

Let $\mathcal{S}$ be a configuration or a snapshot. Given $\mathcal{S}$, let $\mathcal{H}(\mathcal{S})$ be the convex hull defined by $\mathcal{S}$, let $\partial \mathcal{H}(\mathcal{S})$ denote the set of robots on the border of $\mathcal{H}(\mathcal{S})$, and let $\mathcal{I}(\mathcal{S})$ the set of robots that are interior of $\mathcal{H}(\mathcal{S})$.

Given two points $p, q \in \mathbb{R}^{2}$, we indicate the line segment by $\overline{p q}$ and its length by $|\overline{p q}|$. Given a snapshot $\mathcal{S S}, \operatorname{SEC}(\mathcal{S S})$ denotes the smallest enclosing circle containing $\mathcal{P}(\mathcal{S S})$, and the length of its diameter and its center are denoted by $\operatorname{Diam}(\mathcal{S S})$ and $C T R(\mathcal{S S})$, respectively. A longest distance segment ( $L D S$, for short) in $\mathcal{S S}$ is a line segment $\overline{p q}$ such that $p, q \in \mathcal{P}(\mathcal{S S})$ and $|\overline{p q}|=\max _{x, y \in \mathcal{P}(\mathcal{S S})}|\overline{x y}|$ and a set of the longest distance segments in $\mathcal{S S}$ is denoted by $\operatorname{LDS}(\mathcal{S S})$.

We use the following geometric properties about the smallest enclosing circles and the longest distance segments to obtain a unique longest distance segments.

Let $\mathcal{S}$ be a configuration or a snapshot.
Lemma 1. For any $\overline{p q} \in L D S(\mathcal{S}),|\overline{p q}| \leq \operatorname{Diam}(\mathcal{S})$.
Proof. The two points $p$ and $q$ exist in $S E C(\mathcal{S})$.
Lemma 2. If $\overline{p q} \in L D S(\mathcal{S})$, then $p, q \in \mathcal{H}(S)$.

[^4]Proof. Suppose that $p$ or $q$ is not a vertex of $\mathcal{H}(S)$. Let $\overline{p_{1} p_{2}}$ and $\overline{q_{1} q_{2}}$ be the nearest edges of $\mathcal{H}(S)$ from $p$ and $q$, respectively. Note that $p$ and/or $q$ can be on the edges not vertices. Comparing $\overline{p q}$ and the convex quadrilateral (or the triangle) composed of $\overline{p_{1} p_{2}}$ and $\overline{q_{1} q_{2}}$, the longer diagonal (or the longest edge) is longer than $\overline{p_{1} p_{2}}$. It contradicts that $\overline{p q}$ is the longest distance segment.

Lemma 3. If for any $\overline{p q} \in \operatorname{LDS}(\mathcal{S})$, both $p$ and $q$ are located on $\operatorname{SEC}(\mathcal{S})$, for an endpoint $p$ such that $\overline{p q} \in L D S(\mathcal{S}),|\{q \mid \overline{p q} \in L D S(\mathcal{S})\}| \leq 2$.

Proof. Assume that there are three $L D S$ s with the endpoint $p$ and let the other endpoints be $q_{1}, q_{2}$ and $q_{3}$. Since $q_{1}, q_{2}$ and $q_{3}$ are located on $\operatorname{SEC}(\mathcal{S})$, $\operatorname{SEC}(\mathcal{S})$ must be a circle with the center $p$, which is contradiction that $p$ is also on $\operatorname{SEC}(\mathcal{S})$.

Lemma 4. If both $p$ and $q$ are endpoints of $L D S(\overline{p q} \in L D S(\mathcal{S}))$ and are located on $\operatorname{SEC}(\mathcal{S})$, then $\operatorname{Diam}(\mathcal{S}) / 2<|\overline{p q}|$.

Proof. Let $c=C T R(\mathcal{S})$ and let $a$ and $b$ be distinct points on $\operatorname{SEC}(\mathcal{S})$ such that $\triangle a p c$ and $\triangle b p c$ are equilateral triangles. If $\operatorname{Diam}(\mathcal{S}) / 2 \geq|\overline{p q}|$, all points in $\mathcal{S}$ must exist in the sector consisting of $\overline{c a}$ and $\overline{c b}$ and the shorter arc $\widehat{a b}$. Then $\operatorname{SEC}(\mathcal{S})$ is the circle having $\overline{a b}$ as the diameter. Contradiction.

Lemma 5. If $\overline{p q}, \overline{r s} \in \operatorname{LDS}(\mathcal{S})$ and $p, q, r, s$ are distinct, then $\overline{p q}$ and $\overline{r s}$ have an intersection.

Proof. If $\overline{p q}$ and $\overline{r s}$ do not have any intersection, the longer diagonal of the quadrilateral composed of $\overline{p q}$ and $\overline{r s}$ becomes the longest line segment.

Lemma 6. Let $p$ and $q_{1}\left(\overline{p q_{1}} \in L D S(\mathcal{S})\right)$ be located on $\operatorname{SEC}(\mathcal{S})$ Assume that $\overline{p q_{1}}$ is not a diameter of $\operatorname{SEC}(\mathcal{S})$. Let $q_{2}$ be the intersection of a circle with center $p$ and radius $\left|\overline{p q_{1}}\right|$ and $\operatorname{SEC}(\mathcal{S})$ and not $q_{1}$. Then there are no points on arc $q_{1} q_{2}$ of $S E C(\mathcal{S})$ not containing $p$ except $q_{1}$ and $q_{2}$ (see Figure (1).

Proof. If there is a point $q\left(\neq q_{1}, q_{2}\right)$ on the arc $\widetilde{q}_{1} q_{2},|\overline{p q}|$ is greater than the LDS.

Lemma 7. Assume that for any $\overline{p q} \in L D S(\mathcal{S})$, both $p$ and $q$ are located on $S E C(\mathcal{S})$. If $\mathcal{H}(\mathcal{S})$ is not a regular polygon, there exists a point $p_{0}$ such that $\overline{p_{0} q} \in \operatorname{LDS}(\mathcal{S})$ and $\overline{p_{0} r} \notin \operatorname{LDS}(\mathcal{S})$ for any $r(\neq q) \in \mathcal{P}(\mathcal{S})$. The point $p_{0}$ is called a single-endpoint of $L D S$ (Figure 2(a)).


Figure 1: Proof of Lemma 6.


Figure 2: Proof of Lemma 7.

Proof. If there exists no point $p_{0}$ such that $\overline{p_{0} q} \in L D S(\mathcal{S})$ and $\overline{p_{0} r} \notin L D S(\mathcal{S})$ for any $r(\neq q),|\{q \mid \overline{p q} \in L D S(\mathcal{S})\}| \geq 2$ for any point $p$ on $\operatorname{SEC}(\mathcal{S})$. Since $|\{q \mid \overline{p q} \in \operatorname{LDS}(\mathcal{S})\}| \leq 2$ by Lemma 3 3, $|\{q \mid \overline{p q} \in \operatorname{LDS}(\mathcal{S})\}|=2$ for any point $p$. Let $n$ be the number of points located on $\operatorname{SEC}(\mathcal{S})$. For a point $p$, let $\overline{p q}$ and $\overline{p r}$ be two $L D S$ s having $p$ as the endpoint. Note that $q$ and $r$ are also located on $\operatorname{SEC}(\mathcal{S})$. Let $p p^{\prime}$ be the diameter of $\operatorname{SEC}(\mathcal{S})$. Then $\angle q p p^{\prime}=\angle r p p^{\prime}$ (denoted as $\alpha$ ). Since this property holds for any point on $\operatorname{SEC}(\mathcal{S})$, $\alpha$ must be $(n+2) \pi / n$ (Figure 2(b)). Otherwise, it is easily verified that it contradict that the number of points located on $\operatorname{SEC}(\mathcal{S})$ is $n$. When $\alpha=(n+2) \pi / n$, the points on $\operatorname{SEC}(\mathcal{S})$ constitute a regular polygon and it contradicts the assumption of this lemma.

## 3 Previous Results for Rendezvous and Gathering

Rendezvous is trivially solvable with CENT or FSYNC scheduler. The multiplicity detection does not help to solve Rendezvous and it is generally unsolved with SSYNC, even if chirality is assumed.

Theorem 1. [9] Rendezvous is deterministically unsolvable in SSYNC scheduler even if chirality is assumed.

If robots have a constant number of colors in their lights, Rendezvous can be solved shown in the following theorem (see Table (1).

Theorem 2. [10, 12]
(1) Rendezvous is solved in full-light, non-rigid and ASYNC with 2 colors.
(2) Rendezvous is solved in external-light, non-rigid and SSYNC with 3 colors.
(3) Rendezvous is solved in external-light, rigid and ASYNC with 12 colors.
(4) Rendezvous is solved in external-light, non-rigid and ASYNC with 3 colors and knowledge of a minimum distance $\delta$ robots move.
(5) Rendezvous is solved in internal-light, rigid and SSYNC with 6 colors.
(6) Rendezvous is solved in internal-light, non-rigid and SSYNC with 3 colors and knowledge of a minimum distance $\delta$ robots move.

Impossibility and/or possibility results for Gathering are stated in the following theorems.

Theorem 3. [5, [9] If we do not assume strong multiplicity detection, the followings holds.
(1) Gathering is deterministically unsolvable in SSYNC.
(2) Gathering is deterministically unsolvable in 2-BOUNDED and CENT.
(3) Distinct Gathering is deterministically unsolvable in ROUND-ROBIN.

The impossibility holds even if we assume chirality, and rigid movement or non-rigid movement with knowledge of minimum distance $\delta$.

Theorem 4. [9]
(1) With strong multiplicity detection, $n$-Gathering is solved in SSYNC if and only if $n$ is odd.
(2) In ASYNC, with strong multiplicity detection, distinct $n$-Gathering is solved with $n(\geq 3)$ robots.

Multiplicity detection is a strong assumption to solve Gathering. In the following section, without strong multiplicity detection we consider Gathering algorithms for robots with lights such that the number of colors is the minimum and additional assumptions are the weakest.

## 4 Gathering Algorithms in SSYNC

In this section, we show the following four Gathering algorithms with lights in SSYNC.
(a) Algorithm 1:(2, full, SSYNC, non-rigid),
(b) Algorithm 2:(3, external, SSYNC, rigid),
(c) Algorithm 3:(2, external, SSYNC, rigid; local-awareness), and
(d) Algorithm 4+5+6):(2, internal, SSYNC, non-rigid; $2 \delta$-distant).

The idea of Gathering algorithms shown here is that the algorithm divides into two steps as follows:
(1) We make a configuration where all robots are located on a line segment (called onLDS) from any initial configuration satisfying the corresponding conditions.
(2) We make a gathering algorithm from the configuration onLDS.

Step (1) can be performed with ElectOneLDS [14], that reduces any configuration to one where there is the unique $L D S$ or Gathering is achieved. This algorithm needs chirality assumption but can be performed in SSYNC and without lights. Thus, we will have to use lights to implement Step (2) from the configuration onLDS. It is easily verified that ElectOneLDS can be modified so that it can be performed in the following assumptions, (a) non-rigid moving, SSYNC and chirality, (b) non-rigid moving, CENT and non-chirality.

Lemma 8. With chirality, an algorithm can be constructed to make onLDS from any configuration in SSYNC and non-rigid moving without lights, unless Gathering is achieved. If CENT is assumed, this algorithm can be implemented without chirality.

Proof. The algorithm ElectOneLDS to make onLDS works in rigid movement [9]. We show a modified ElectOneLDS to work in non-rigid movement.

The modified ElectOneLDS works as follows: We call a configuration contractible if (1) its convex hull CH is symmetric and every robot is located at a vertex of CH or at the center of CH , or (2) CH is not symmetric and there are no robots inside CH . The algorithm makes a contractible configuration from the current configuration, unless it already contractible. It can be done by each active robot that is not at a vertex of CH moves to the center of CH if it is symmetric, or each active robot that is neither at a vertex of CH nor on an edge of CH (that is, inside of CH ) moves to a vertex of CH . In making contractible configurations, since the convex hull is not changed, the algorithm can work in non-rigid moving. Note that robots may be located on edges of CH in contractible and non-symmetric configurations but can not be located on edges of $C H$ in contractible and symmetric ones.

If CH becomes contractible, a unique $L D S$ is obtained by decreasing the number of edges of $C H$ or the diameter of $C H$ until the configuration has a unique $L D S$. If CH is contractible and symmetric, any active robot moves to the center of CH . In this case, if CH is not changed the number of robots at vertices of CH decreases via non-contractible configurations. Otherwise, the configuration becomes one of the following ones unless Gathering is attained; (1) it remains contractible and symmetric, (2) the number of edges of the convex hull is decreased and it becomes contractible one via non-contractible configurations. In the both cases, the diameter of CH is eventually reduced.

If CH is contractible and asymmetric, the number of edges is decreased by some robots moving so to contract shortest-length edges in CH . Each robot $r$ on a vertex of CH checks the distance to the clockwise and counterclockwise neighboring robots at vertices of CH . If the edge on the left is a shortest-length edge and the robot is located at the leftmost, this edge becomes contracting. The leftmost robot and the robots on this edge move to the the other vertex of the edge until the edge is contracting. Since all noncontracting robots do not move and the edge remains the shortest-length, this contraction can be done even in non-rigid moving, In this way, the number of edges of the convex hull is decreasing.

In SSYNC, chirality is necessary for determining the leftmost robot of the shortest-length edge. However, since only one robot is activated in CENT, contracting the shortest-length edge can be done without chirality.

### 4.1 Full-light

```
Algorithm 1 Full-Light-Garher \(\left(r_{i}\right)\)
Assumptions: full-light, non-rigid, 2 colors \((A\) and \(B)\), set-view, SSYNC
Input: configuration onLDS, all robots have color \(A\)
    case \(\mathcal{L}\left(\mathcal{S S}_{i}\right)\) of
    \(\{A\}\) :
        if \(\left|\mathcal{P}_{A}\left(\mathcal{S S}_{i}\right)\right|=1\) then des \(_{i} \leftarrow p_{i} / /\) gathered!
        else if \(\left|\mathcal{P}_{A}\left(\mathcal{S S _ { i }}\right)\right|=2\) then \(\ell_{i} \leftarrow B\); des \({ }_{i} \leftarrow\left(p_{n}+p_{f}\right) / 2\)
        else \(/ /\left|\mathcal{P}_{A}\left(\mathcal{S S}_{i}\right)\right| \geq 3\)
            if \(\left(p_{i}=p_{n}\right)\) then \(d e s_{i} \leftarrow p_{i} / / \mathrm{I}\) am at either endpoint
            else des \(_{i} \leftarrow p_{n}\)
    \(\{B\}\) :
        if \(\left(p_{i}=p_{n}\right)\) then \(\ell_{i} \leftarrow A\)
        \(\operatorname{des}_{i} \leftarrow p_{i} / /\) stay
    \(\{A, B\}\) :
        if \(\ell_{i}=A\) then
            if \(\left|\mathcal{P}_{A}\left(\mathcal{S S}_{i}\right)\right|=1\) then des \(_{i} \leftarrow p_{i} / /\) stay
            else if \(\mathcal{P}_{A}\left(\mathcal{S S}_{i}\right)=\left\{p_{n}, p_{f}\right\}\) and \(\mathcal{P}_{B}\left(\mathcal{S S}_{i}\right)=\left\{\frac{p_{n}+p_{f}}{2}\right\}\) then
                    \(\ell_{i} \leftarrow B\)
                    \(\operatorname{des}_{i} \leftarrow\left(p_{n}+p_{f}\right) / 2\)
        else \(/ / \ell_{i}=B\)
            if \(\left|\mathcal{P}_{A}\left(\mathcal{S S}_{i}\right)\right|=1\) then des \(_{i} \leftarrow p\left(\in \mathcal{P}_{A}\left(\mathcal{S S}_{i}\right)\right)\)
            else \(\operatorname{des}_{i} \leftarrow\left(p_{n}+p_{f}\right) / 2 / /\left|\mathcal{P}_{A}\left(\mathcal{S S}_{i}\right)\right|=2\)
    endcase
```

Gathering algorithm with full-light uses two colors of full-light and works in non-rigid movement shown in Algorithm 1. It reduces any configuration of onLDS to a gathering configuration. In the algorithm for robot $r_{i}$, endpoints of the $L D S$ are denoted by $p_{n}$ and $p_{f}$, where $p_{n}$ is the nearest endpoint from $p_{i}$ and $p_{f}$ is the farthest one from $p_{i}$. Note that if $p_{i}$ is either of endpoints, $p_{i}=p_{n}$ and $p_{f}$ is the other endpoint. Since full-light is assumed, robot $r_{i}$ can use colors of other robots in $\mathcal{S S}_{i}$ and its own light $\ell_{i}$ in the algorithm. For snapshot $\mathcal{S S}_{i}$ and color $A$ and $B, \mathcal{P}_{A}\left(\mathcal{S S}_{i}\right)$ and $\mathcal{P}_{B}\left(\mathcal{S S}_{i}\right)$ denote sets of positions on which robots with color $A$ and $B$ are located, respectively.

Initially all robots are located on the unique $L D S$ and have color A . Figure 3 shows the transition diagram of Algorithm 1 , where nodes denote configurations and a directed edge denotes transition from a configuration to a configuration. In Figure 3, labels of directed edges have the following meanings. L. \# means that this transition is shown by Lemma \#. " $-\delta$ " (" $2 \delta$ ") means when the transition occurs, the distance between the endpoints reduces at least $-\delta(2 \delta)$ and "\#A decreases" means the numbers of robots


Figure 3: Transition diagram for Full-Light-Gather.
with color A at the endpoints decrease. Each configuration is denoted by a regular-expression-like sequence of colors robots have from one endpoint to another. Formally we define color-configurations as follows. Let $C(t)$ be a configuration at time $t, p$ and $q$ be the endpoints of the $L D S$. Configuration $C(t)$ has a color-configuration
(1) $\alpha \beta$, if all robots at $p$ have color $\alpha$, all robots at $q$ have color $\beta(\alpha, \beta \in$ $\{A, B\})$ and there are no robots inside the $L D S$,
(2) $\alpha \gamma \beta$, if all robots at $p$ have color $\alpha$, all robots at $q$ have color $\beta(\alpha, \beta \in$ $\{A, B\}$ ), all robots at the mid-point of the $L D S$ have color $\gamma$ and there are no robots except on the three locations, and
(3) $\alpha \gamma^{+} \beta$, if all robots at $p$ have color $\alpha$, all robots at $q$ have color $\beta$ $(\alpha, \beta \in\{A, B\})$ and there is at least one location except the midpoint of the $L D S$ inside the $L D S$ where all robots have color $\gamma$.

Note that $\alpha \gamma^{+} \beta$ contains configurations that the midpoint of the $L D S$ may be occupied by some robot(s) with color $\gamma$, and $\alpha \gamma \beta$ and $\alpha \gamma^{+} \beta$ are exclusive. If either (1) or (3) is satisfied, we denote $\alpha \gamma^{*} \beta$. Let $\operatorname{dis}(C(t))$ denote the length of the $L D S$ in the configuration $C(t)$.

The outline of behaviour of Algorithm $\rceil$ is explained as follows. Suppose that all robots become active (FSYNC). From any initial color-configuration of onLDS $\left(A A^{*} A\right.$ or $\left.A A A\right)$, robots with $A$ located not at endpoints move to endpoints and robots with $A$ located at endpoints stay(lines 5-7) it becomes the color-configuration $A A$. When the color-configuration is $A A$, each robot
changes its color to $B$ and move to the midpoint(line 4). If they can reach to the destination, the gathering is achieved. However, since we assume nonrigid movement, some of them may stop before reaching the midpoint and its configuration becomes $B B^{*} B$ or $B B B$, where the former is $B B$ if the locations all robots stop are only two, otherwise $B B^{+} B$ and the latter is possible when there is a robot moving to the midpoint. In the both cases the length of its $L D S$ is decreased by at least $2 \delta$ (denoted by $-2 \delta$ on the directed edge from $A A$ to $B B^{*} B$ in Fig. (3).

From the color-configuration $B B^{*} B$ or $B B B$, the algorithm changes colors of robots at endpoints to $A$ (lines 8-10) and can change the colorconfiguration directly to $A B^{+} A$ (from $B B^{+} B$ ), $A A$ (from $B B$ ), or $A B A$ (from $B B B$ ). If the color-configuration $A B^{+} A$ occurs, robots with $B$ move to the midpoint (line 19), and robots with $A$ stay until all robots with $B$ move to the midpoint(lines 14-16). In this case the configuration also become $A B A$ and robots with $A$ change the color to $B$ and move to the midpoint (lines 14-16) and Gathering is achieved after repeating the transitions from $A B A$ to $B B^{*} B$, from $B B^{*} B$ to $A B^{+} A$, and from $A B^{+} A$ to $A B A$. If the color-configuration $A A$ occurs, repeating the above transitions Gathering attained.

However, since all robots do not become active at every round in general, the behaviors are complicated and all color-configurations shown in Figure 3 can occur.

Since any cycle in the graph can reduce the length of $L D S$ (denoted by $-\delta$ or $-2 \delta$ in Fig. (3) or the number of robots with $A$ and located at endpoints (denoted by $\# A$ decreases in Figure 3), we can show that Algorithm 1 solves Gathering from any initial configuration of onLDS.

Lemma 9. If $C(t)$ is a configuration at time $t$ with color-configuration $A A^{*} A$ or $A A A$, there is a time $t^{\prime}(\geq t)$ such that $C(t) \rightarrow^{*} C\left(t^{\prime}\right), C\left(t^{\prime}\right)$ has a colorconfiguration $A A$, and $\operatorname{dis}\left(C\left(t^{\prime}\right)\right)=\operatorname{dis}(C(t))$.

Proof. If the color-configuration is $A A$, the lemma holds as $t^{\prime}=t$. Otherwise, since robots located inside the $L D S$ move to the nearest endpoint and robots at the endpoint stay there as long as there is a robot located inside the $L D S$ (lines 5-7), the color-configuration becomes $A A$.

Lemma 10. Let $C(t)$ be a configuration at time $t$ with color-configuration $A A$. If we consider time $t+1$ such that $C(t) \rightarrow C(t+1)$, then
(1) $C(t+1)$ is a gathering configuration,
(2) $C(t+1)$ has a color-configuration $B B^{*} B$ or $B B B$, and $\operatorname{dis}(C(t+1)) \leq$ $\operatorname{dis}(C(t))-2 \delta$,
(3) $C(t+1)$ has a color-configuration $A B^{*} B$ or $A B B$ (or $B B^{*} A$ or $B B A$ ), and $\operatorname{dis}(C(t+1)) \leq \operatorname{dis}(C(t))-\delta$,
(4) $C(t+1)$ has a color-configuration $A B A$, and $\operatorname{dis}(C(t+1))=\operatorname{dis}(C(t))$, or
(5) $C(t+1)$ has a color-configuration $A B^{+} A$, and $\operatorname{dis}(C(t+1))=\operatorname{dis}(C(t))$.

Proof. There are five cases according to the activeness of robots.
(1) If all robots become active and they can reach the destination (the midpoint of the $L D S$ ), the gathering is attained.
(2) If all robots become active but they can not reach the destination, they change their color to $B$ and move at least $\delta$. Thus the colorconfiguration becomes $B B^{+} B, B B$ or $B B B$ as shown in the example above. In any case, since all robots move at least $\delta, \operatorname{dis}(C(t+1)) \leq$ $\operatorname{dis}(C(t))-2 \delta$.
(3) If all robots at one endpoint become active but some robots at the other endpoint are inactive, all active robots change their color to $B$ and all inactive robots do not change their color and stay there. Since the color of all robots at the endpoint inactive robots are located is $A$ and the color of the other robots become $B$ they move at least $\delta$, the colorconfiguration becomes $A B^{*} B A B B$ (or $B B^{*} A$ or $B B A$ ). In this case since all active robots move at least $\delta, \operatorname{dis}(C(t+1)) \leq \operatorname{dis}(C(t))-\delta$.
(4) If there exist inactive robots at the both endpoints and active robots can reach the destination, the color-configuration becomes $A B A$ and $\operatorname{dis}(C(t+1))=\operatorname{dis}(C(t))$.
(5) If there exist inactive robots at the both endpoints and some active robots can not reach the destination, the color-configuration becomes $A B^{+} A$ and $\operatorname{dis}(C(t+1))=\operatorname{dis}(C(t))$.

Lemma 11. If $C(t)$ is a configuration at time $t$ with color-configuration $A B^{+} A$, there is a time $t^{\prime}(\geq t)$ such that $C(t) \rightarrow^{*} C\left(t^{\prime}\right), C\left(t^{\prime}\right)$ has a colorconfiguration $A B A$, and $\operatorname{dis}\left(C\left(t^{\prime}\right)\right)=\operatorname{dis}(C(t))$.

Proof. In the configuration with color-configuration $A B^{+} A$, robots with $A$ do not move until the color-configuration becomes $A B A$ (lines 12-16) robots with $B$ move to the midpoint of the $L D S$. Thus it becomes $A B A$ and $\operatorname{dis}\left(C\left(t^{\prime}\right)\right)=$ $\operatorname{dis}(C(t))$.

Lemma 12. If $C(t)$ is a configuration at time $t$ with color-configuration $B B^{*} B$ or $B B B$, there is a time $t^{\prime}(>t)$ such that $C(t) \rightarrow^{*} C\left(t^{\prime}\right)$ and one of the followings holds,
(1) $C\left(t^{\prime}\right)$ is a gathering configuration,
(2) $C\left(t^{\prime}\right)$ has a color-configuration $A B^{+} A$, and $\operatorname{dis}\left(C\left(t^{\prime}\right)\right)=\operatorname{dis}(C(t))$,
(3) $C\left(t^{\prime}\right)$ has a color-configuration $A B A$, and $\operatorname{dis}\left(C\left(t^{\prime}\right)\right)=\operatorname{dis}(C(t))$, or
(4) $C\left(t^{\prime}\right)$ has a color-configuration $A A$, and $\operatorname{dis}\left(C\left(t^{\prime}\right)\right)=\operatorname{dis}(C(t))$.

Proof. Assume that all robots become active at time $t$. If the color-configuration is $B B$, then the color-configuration becomes $A A$ at time $t^{\prime}=t+1$ (case (4)). If the color-configuration is $B B^{+} B$ or $B B B$, then the color-configuration becomes $A B^{+} A$ or $A B A$ at time $t^{\prime}=t+1$, respectively (case (2) or (3)). These transitions occur from lines 8-10 of the algorithm.

Otherwise, there are two cases, (a) some robots at only one endpoint become active, and (b) robots at the both endpoints become active. Note that active robots not located on the endpoints do nothing in these configurations.
(a) In this case, active robots change their color into $A$ at time $t+1$ and since $\left|\mathcal{P}_{A}(C(t+1))\right|=1$, robots with $A$ stay at the positions (lines 12$13)$ and robots with $B$ move to the position at robots with $A$ (lines 17$18)$ and the gathering is attained (case (1)).
(b) In this case, active robots at the both endpoints change their colors into $A$ at time $t+1$ and since $\left|\mathcal{P}_{A}(C(t+1))\right|=2$ and robots with $A$ do not move and robots with $B$ move to the midpoints of the $L D S$, the color-configuration eventually becomes $A B A$ (case (3)).

Lemma 13. Let $C(t)$ be a configuration at time $t$ with color-configuration $A B A$. If we consider time $t^{\prime}$ such that $C(t) \rightarrow^{*} C\left(t^{\prime}\right)$, then
(1) $C\left(t^{\prime}\right)$ is a gathering configuration,
(2) $C\left(t^{\prime}\right)$ has a color-configuration $A B^{*} A$, and $\operatorname{dis}\left(C\left(t^{\prime}\right)\right)=\operatorname{dis}(C(t))$ and the number of robots at the endpoints and with $A$ is decreased,
(3) $C\left(t^{\prime}\right)$ has a color-configuration $A B^{*} B\left(\right.$ or $\left.B B^{*} A\right)$, and $\operatorname{dis}\left(C\left(t^{\prime}\right)\right) \leq$ $\operatorname{dis}(C(t))-\delta$, or
(4) $C\left(t^{\prime}\right)$ has a color-configuration $B B^{*} B$ or $B B B$, and $\operatorname{dis}\left(C\left(t^{\prime}\right)\right) \leq \operatorname{dis}(C(t))-$ $2 \delta$.

Proof. This lemma can be proved similar to the one of Lemma 9 ,
Lemma 14. If $C(t)$ is a configuration at time $t$ with color-configuration $A B^{*} B$ or $A B B$ (or $B B^{*} A$ or $B B A$ ), there is a time $t^{\prime}(\geq t)$ such that $C(t) \rightarrow^{*}$ $C\left(t^{\prime}\right), C\left(t^{\prime}\right)$ is a gathering configuration.

Proof. Since $\left|\mathcal{P}_{A}(C(t))\right|=1$, robots with $A$ stay at the position (lines 12-13) and robots with $B$ move to the position at robots with $A$ (lines 17-18) and the gathering is attained.

We have the following theorem using Lemmas 8 .14.
Theorem 5. Gathering is solvable in full-light of 2 colors, non-rigid, and SSYNC, with set-view and agreement of chirality.

### 4.2 External-light

```
Algorithm 2 Ext-Light-Gather-with-3colors \(\left(r_{i}\right)\)
Assumptions: external-light, rigid, 3 colors \((A, B, C)\), local-unawareness, set-view, SSYNC
Input: \(\left(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=3\right)\) and \(\left(\left|\overline{p_{f} p_{m}}\right|=\left|\overline{p_{m} p_{n}}\right|\right)\) or \(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=2\), all robots have color \(A\).
    if \(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=2\) then
        if \(C \in \mathcal{L}_{p_{f}}\left(\mathcal{S S}_{i}\right)\) then \(\ell_{i} \leftarrow C\); \(\operatorname{des}_{i} \leftarrow p_{f}\)
            else if \(\mathcal{L}_{p_{f}}\left(\mathcal{S S}_{i}\right)=\{A\}\) then \(\ell_{i} \leftarrow B ;\) des \(_{i} \leftarrow\left(p_{n}+p_{f}\right) / 2\)
            else if \(\mathcal{L}_{p_{f}}\left(\mathcal{S S _ { i }}\right)=\{B\}\) then \(\ell_{i} \leftarrow C\); des \(s_{i} \leftarrow p_{i} / /\) stay
    else if \(\left(\left|\mathcal{P}\left(\mathcal{S} \mathcal{S}_{i}\right)\right|=3\right)\) and \(\left(\left|\overline{p_{f} p_{m}}\right|=\left|\overline{p_{m} p_{n}}\right|\right)\) then
            if \(p_{i}=p_{m}\) then \(/ / \mathrm{I}\) am at the midpoint.
                    \(\ell_{i} \leftarrow B\)
                    \(\operatorname{des}_{i} \leftarrow p_{m} / /\) stay
            else if \(\mathcal{L}_{p_{f}}\left(\mathcal{S} \mathcal{S}_{i}\right)=\{B\}\) and \(\mathcal{L}_{p_{m}}\left(\mathcal{S} \mathcal{S}_{i}\right)=\{B\}\) then
                        \(\ell_{i} \leftarrow C\)
                        \(\operatorname{des}_{i} \leftarrow p_{i} / /\) stay
            else if \(\mathcal{L}_{p_{f}}\left(\mathcal{S} \mathcal{S}_{i}\right) \cap\{A, C\} \neq \emptyset\) and \(\mathcal{L}_{p_{m}}\left(\mathcal{S} \mathcal{S}_{i}\right)=\{B\}\) then
                        \(\ell_{i} \leftarrow B\)
                        \(d e s_{i} \leftarrow p_{m} / /\) go to the midpoint
```

We show two algorithms in external-light. One uses 3 colors in rigid and SSYNC and the other uses only 2 colors in the same assumption as the former with the local-awareness. Algorithm 2 shows the external-light algorithm in rigid and SSYNC with 3 colors and its transition diagram is shown in Figure 4.

Without loss of generality, the algorithm may begin with a configuration of two-location $p_{n}$ and $p_{f}$, where all robots have $A$ or three-location $p_{n}, p_{m}$, and $p_{f}$ such that $\left(\left|\overline{p_{f} p_{m}}\right|=\left|\overline{p_{m} p_{n}}\right|\right)$, where all robots have $A$. Since


Figure 4: Transition diagram for Ext-Light-Gather-with-3colors.
we assume the rigid movement, these configurations can be reduced from onLDS by moving robots to the nearest position among the endpoints and the midpoint. By using notations of the color-configurations, the former has color-configuration $A A$ and the latter has $A A A$ in Fig. (4. Once the configuration of two- or three-location is obtained, this configuration is preserved hereafter since we assume the rigid movement.

Algorithm 2 makes configuration with $B A$ (or $A B$ ) from the initial configuration with $A A A$ or $A A$ via configuration with $A B A$. After the configuration with $B A[5,(1)$ when robots observe color $B$, they change their color to $C$ and stay there, where the location becomes the gathering one, (2) when robots observe color $A$, they change their color to $B$ and go to the midpoint. Then, the color-configuration becomes $B(A+C)$ or $B B(A+C)$, where $A+C$ denotes the endpoint is occupied by robots with color $A$ or $C$. Once the configuration with $B(A+C)$ appears, since robots with $A$ or $C$ observe color $B$ and change their color to $C$, and robots with $B$ observe color $C$ and move to the location having robots with $C$, Gathering is successful. The configuration with $B B(A+C)$ becomes one with $B(A+C)$, because robots with $B$ located at the midpoint stay there, robots with $A$ or $C$ observe two $B$ 's at the endpoint and the midpoint change their color to $C$, robots with $B$ located at the endpoint observe color $C$ at the endpoint and color $B$ at the midpoint move to the midpoint.

Lemma 15. Configuration with $A A$ moves to one with $A B A, A B$ or $B A$. In particular, it moves to the gathering configuration, if all robots become active.

Proof. All active robots perform line 3. If all robots become active, the

[^5]gathering is attained. Otherwise, if all robots at one endpoint only become active, the configuration moves to $A B$ or $B A$. Otherwise, it moves to $A B A$.

Lemma 16. Configuration with $A A A$ moves to one with $A B A$.
Proof. This case is derived from line 3, lines 6-8, and the fact that robots at endpoints cannot move until all robots at the midpoint have color $B$ (line 12).

Lemma 17. Configuration with $A B A$ moves to the two-point configuration with $B A$ (or $A B$ ) unless Gathering is achieved.

Proof. From a configuration with $B A$, robots at the endpoints change their color to $B$ and go to the midpoint (lines 12-14), and robots at the midpoint stay there (lines 6-8). Then if all robots at the both endpoints go to the midpoint, Gathering is achieved. Otherwise, the two-point configuration with $B A$ (or $A B$ ) is obtained.

Lemma 18. Configuration with $B A$ moves to one with $B(A+C)$ or $B B(A+$ $C)$.

Proof. Robots with $A$ observing $B$ change their color to $C$ and stay (line 4) and robots with $B$ observing $A$ change their color to $B$ and move to the midpoint (line 3). Thus, the color-configuration $B(A+C)$ or $B B(A+C)$ is obtained.

Lemma 19. Configuration with $B B(A+C)$ moves to one with $B(A+C)$.
Proof. It can be derived by lines 5-14.
Lemma 20. Configuration with $B(A+C)$ moves to the gathering configuration.

Proof. It can be derived by lines 1-4.
We have the following theorem using Lemma 8 and Lemmas $15+20$.
Theorem 6. Gathering is solvable in external-light of 3 colors, rigid, and SSYNC, if robots have set-view and agreement of chirality.

Although it is unknown whether there exists a 2-color gathering algorithm in external-light, rigid, and SSYNC, we can construct 2-color gathering algorithm in the same condition if we assume the local-awareness (Algorithm 3). In Algorithm 2, we make the location having robots with $C$ the gathering

```
Algorithm 3 Ext-Light-Gather-with-2-Colors ( \(r_{i}\) )
Assumptions: external-light, rigid, 2 colors (A,B), local-awareness, set-view, SSYNC.
Input: \(\left(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=3\right)\) and \(\left(\left|\overline{p_{f} p_{m}}\right|=\left|\overline{p_{m} p_{n}}\right|\right)\) or \(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=2\), all robots have color \(A\).
    if \(\left|\mathcal{P}\left(\mathcal{S S} \mathcal{S}_{i}\right)\right|=2\) then
            if \(B \in \mathcal{L}\left(\mathcal{S S}_{i}\right)\) then \(\ell_{i} \leftarrow B\); des \(i_{i} \leftarrow p\left(\in \mathcal{P}_{B}\left(\mathcal{S} \mathcal{S}_{i}\right)\right)\)
            else if \(\mathcal{L}_{p_{i}}\left(\mathcal{S} \mathcal{S}_{i}\right)=\emptyset\) then no action //I am alone at \(p_{i}\)
            else \(\ell_{i} \leftarrow B ; \operatorname{des}_{i} \leftarrow\left(p_{n}+p_{f}\right) / 2 / / \mathcal{L}_{p_{f}}\left(\mathcal{S S}_{i}\right)=\{A\}\)
        else if \(\left(\left|\mathcal{P}\left(\mathcal{S} \mathcal{S}_{i}\right)\right|=3\right)\) and \(\left(\left|\overline{p_{f} p_{m}}\right|=\left|\overline{p_{m} p_{n}}\right|\right)\) then
            if \(A \in \mathcal{L}_{p_{m}}\left(\mathcal{S S}_{i}\right)\) and \(\left(p_{i}=p_{m}\right)\) then \(\ell_{i} \leftarrow B\)
            else \(\ell_{i} \leftarrow B ;\) des \(_{i} \leftarrow p_{m}\)
```

point. Instead we do not use the third color $C$, we make configurations with $A B$ (or $B A$ ) such that there are at least 2 robots with $B$ at one endpoint utilizing the local-awareness. Thus, since robots with $A$ move to the point having robots with $B$ and robots with $B$ can stay at the point(line 2) using the local-awareness, Gathering is attained. Such configurations are created as follows; When the initial color-configuration is $A A A$, it moves to $A B A$ (line 6) and reaches to $B A$ (or $A B$ ) (lines 7). When one endpoint contains more than one robot in the initial configuration with $A A$, it moves to a configuration with $A B A$ (line 4) and reaches to one with $B A$ (or $A B$ ) (lines 5-7). If one endpoint contains only one robot in configuration with $A A$, since the robot identifies that it is alone due to the local-awareness, the robot makes no action. Thus, all the initial configurations can reach to the desired one with $B A$ (or $A B$ ).

Theorem 7. Assume the local-awareness. Gathering is solvable in externallight of 2 colors, rigid, and SSYNC, if robots have set-view and agreement of chirality.

### 4.3 Internal-light

Views of robots in internal-light are the same as those without lights and so robots must determine their behaviors by these views without colors and their own colors of lights. Thus Gathering algorithms in internal-light can not seem to be constructed without additional knowledge such as distance information. In fact, known Rendezvous algorithms use the minimum distance of moving $\delta$ and/or the unit distance [10].

In our Gathering algorithm in internal-light, we use the minimum distance of moving $\delta$ and assume initial configurations such that if $r_{i}$ and $r_{j}$ do not occupy the same location, $\operatorname{dis}\left(p_{i}, p_{j}\right) \geq D$, where $D$ is a function of $\delta$. This
condition is said to be $D$-distan ${ }^{6}$. If any initial configuration is $D$-distant such that $D \leq 2 \delta$, we can construct a gathering algorithm in internal-light, non-rigid with knowledge of $\delta$, and SSYNC with 2 colors of lights.

Our Gathering algorithm with internal-light is composed of three parts.
(1) From any $D$-distant configuration, we make a 2 -point configuration with the distance is at least $D / 2$ (Algorithm (6).
(2) From any 2-point configuration with distance $d \geq D / 2$, we make a 2-point configuration with $D / 2>d \geq D / 4$ (Algorithm 5).
(3) From any 2-point configuration with $D / 2>d \geq D / 4$, we make a Gathering configuration (Algorithm (4).

We do not use colors to solve (1) and (2) and only use two colors to solve (3). The output of (i) is the input of $(\mathrm{i}+1)(\mathrm{i}=1,2)$ and the explanation of algorithms is stated in reverse order from (3) to (1).

Algorithm 4 solves Gathering in internal-light if initial configurations satisfy $\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=2$ and $\frac{D}{4} \leq\left|\overline{p_{n} p_{f}}\right|<\frac{D}{2}$. Since $D \leq 2 \delta$, every movement in Algorithm 4 is the same as the rigid one. The following lemma is easily verified for Algorithm 4

Lemma 21. Let $\mathcal{C}(t)$ be a configuration at time $t$ appearing in Algorithm4.
(1) If $\mathcal{C}(t)$ satisfies $|\mathcal{P}(\mathcal{C}(t))|=2$ and $\frac{D}{4} \leq \operatorname{Diam}(\mathcal{C}(t))<\frac{D}{2}$ and has a color-configuration $A A$, then (1-I) $\mathcal{C}(t+1)$ is a Gathering configuration, (1-II) $\mathcal{C}(t+1)$ has a color-configuration $A B$ or $B A$ and $\frac{D}{8} \leq \operatorname{Diam}(\mathcal{C}(t+$ 1)) $<\frac{D}{4}$, or (1-III) $C(t+1)$ has a color-configuration $A B A$ and $\frac{D}{4} \leq$ $\operatorname{Diam}(\mathcal{C}(t+1))<\frac{D}{2}$.
(2) If $\mathcal{C}(t)$ has a color-configuration $A B A$ and $\frac{D}{4} \leq \operatorname{Diam}(\mathcal{C}(t))<\frac{D}{2}$, then there is a time $t^{\prime}\left(t^{\prime}>t\right)$ such that $\mathcal{C}(t) \rightarrow^{*} \mathcal{C}\left(t^{\prime}\right)$ and (2-I) $\mathcal{C}\left(t^{\prime}\right)$ is a Gathering configuration, or $\mathcal{C}\left(t^{\prime}\right)$ has a color-configuration $B A$ or $A B$ and $\frac{D}{8} \leq \operatorname{Diam}\left(\mathcal{C}\left(t^{\prime}\right)\right)<\frac{D}{4}$.
(3) If $\mathcal{C}(t)$ has a color-configuration $B A$ or $A B$ and $\frac{D}{8} \leq \operatorname{Diam}(\mathcal{C}(t))<\frac{D}{4}$, then there is a time $t^{\prime}\left(t^{\prime}>t\right)$ such that $\mathcal{C}(t) \rightarrow^{*} \mathcal{C}\left(t^{\prime}\right)$ and $\mathcal{C}\left(t^{\prime}\right)$ is a Gathering configuration.

Next we make 2-point location with distance $d$ satisfying $D / 2>d \geq D / 4$ from any 2-point locations with $d \geq D / 2$. This adjusting task is done by Algorithm 5.

[^6]```
Algorithm 4 Int-Light-Gather \(\left(r_{i}\right)\)
Assumptions: internal-light, 2 colors (A,B), non-rigid with \(\delta, D \leq 2 \delta\), set-view, SSYNC.
Input: \(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=2\) and \(\frac{D}{4} \leq\left|\overline{p_{n} p_{f}}\right|<\frac{D}{2}\), all robots have color \(A\).
    if \(\begin{aligned} \frac{D}{4} \leq\left|\overline{p_{n} p_{f}}\right|<\frac{D}{2} \text { then } \\ \text { if }\left(\left|\mathcal{P}\left(\mathcal{S} \mathcal{S}_{i}\right)\right|=2\right) \text { or }\left(\left(\left|\mathcal{P}\left(\mathcal{S} \mathcal{S}_{i}\right)\right|=3\right) \text { and }\left(\left|\overline{p_{f} p_{m}}\right|=\left|\overline{p_{m} p_{n}}\right|\right)\right) \text { then }\end{aligned}\)
        if \(\ell_{i}=A\) then
                    \(\ell_{i} \leftarrow B\)
                    \(\operatorname{des}_{i} \leftarrow p_{m}\)
    else \(/ / \frac{D}{4}>\left|\overline{p_{n} p_{f}}\right|\left(\geq \frac{D}{8}\right)\)
        if \(\ell_{i}=A\) then
                \(\operatorname{des}_{i} \leftarrow p_{f}\)
    else \(/ / \ell_{i}=B\)
        \(\operatorname{des}_{i} \leftarrow p_{i} / /\) stay
```



Figure 5: Configurations in $A 3 P$ and $A 4 P$.

In Algorithm 5, $A 3 P$ and $A 4 P$ denote the following predicates (Figure (5). If robots on the both endpoints move, its configuration satisfies $A 3 P$ and if robots on only one endpoint move, its configuration satisfies . $A 4 P$

$$
\begin{aligned}
& A 3 P:\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=3, \text { and letting } \mathcal{P}\left(\mathcal{S S}_{i}\right)=\left\{p_{n}, p_{m}, p_{f}\right\},\left|\overline{p_{n} p_{m}}\right| \neq\left|\overline{p_{m} p_{f}}\right|, \\
& \quad \min \left(\left|\overline{p_{n} p_{m}}\right|,\left|\overline{p_{m} p_{f}}\right|\right) \leq \frac{D}{2} \text { and } \max \left(\left|\overline{p_{n} p_{m}}\right|,\left|\left|\overline{p_{m} p_{f}}\right|\right)>\frac{D}{4}\right.
\end{aligned}
$$

$A_{4} P:\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=4$. And letting $\mathcal{P}\left(\mathcal{S S}_{i}\right)=\left\{p_{n}, p_{m 1}, p_{m 2}, p_{f}\right\}$,
$\left|\overline{p_{n} p_{m 1}}\right|=\left|\overline{p_{m 2} p_{f}}\right| \neq\left|\overline{p_{m 1} p_{m 2}}\right|,\left|\overline{p_{n} p_{m 1}}\right|,\left|\overline{p_{m 2} p_{f}}\right| \leq \frac{D}{2}$ and $\left|\overline{p_{m 1} p_{m 2}}\right| \geq \frac{D}{4}$
In Algorithm 5, expression $[p, q]+\alpha-\beta$ means the point from point $p$ with distance $\alpha-\beta$ on the line segment $\overline{p q}$. If $D \leq 2 \delta$, each moving in Algorithm 5 can reach the destination. In Algorithm 5, the final destination is a 2-point location such that the length of the 2-point locations is at least $D / 8$ and less than $D / 4$ from the initial 2-point location via configurations of $A 3 P$ or $A 4 P$. We will show that if any configuration obtained in the algorithm are not 2-point locations, $A 3 P$ or $A 4 P$ holds for the configuration and therefore, any 3-point configuration appearing in Algorithm 4 does not appear. That is, Algorithm 4 cannot be performed during the execution of Algorithm 5. We have the following lemma.

```
Algorithm 5 Reduce-Distance-LDS \(\left(r_{i}\right)\)
Assumptions: non-rigid with \(\delta, D \leq 2 \delta\).
Input: \(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=2\) and \(\frac{D}{2} \leq\left|\overline{p_{n} p_{f}}\right|\)
    if \(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=1\) then no action
    else if \(\left|\overline{p_{f} p_{n}}\right| \geq \frac{D}{2}\) then
        if \(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=2\) then
            if \(\frac{3 D}{2}<\left|\overline{p_{f} p_{n}}\right|\) then \(\operatorname{des}_{i} \leftarrow\left[p_{n}, p_{f}\right]+\frac{D}{2}\)
            else if \(\frac{9 D}{8} \leq\left|\overline{p_{f} p_{n}}\right| \leq \frac{3 D}{2}\) then \(\operatorname{des}_{i} \leftarrow\left[p_{n}, p_{f}\right]+\frac{D}{12}\)
            else \(\operatorname{des}_{i} \leftarrow\left[p_{n}, p_{f}\right]+\frac{\left|\frac{p_{f} p_{n}}{}\right|}{2}-\frac{3 D}{16} / /\left|\overline{p_{f} p_{n}}\right|<\frac{9 D}{8}\)
        else if \(A 3 P\) then
            if \(p_{i}=p_{n}\) and \(\left|\overline{p_{i} p_{m}}\right|<\left|\overline{p_{m} p_{f}}\right|\) then \(\operatorname{des}_{i} \leftarrow p_{m}\) else \(\operatorname{des}_{i} \leftarrow p_{i}\)
        else if \(A 4 P\) then
        if \(p_{i}=p_{n}\) then des \(_{i} \leftarrow p_{m_{1}}\) else \(d e s_{i} \leftarrow p_{i}\)
```

Lemma 22. Let $D \leq 2 \delta$. Without assuming chirality, Algorithm 5 makes a configuration with only two positions on which all robots are located and whose distance $d$ satisfies $\frac{D}{4} \leq d<\frac{D}{2}$ from any configuration with only two positions on which all robots are located and whose distance $d$ is at least $D / 2$. All configurations appearing in Algorithm 5 do not contain any 3 -point location appearing in Algorithm 4 .

Proof. Let $d$ be the distance of the two point-location in the initial configuration. Note that since we assume $D \leq 2 \delta$, each moving in the algorithm always reaches the destination. In the case that $d>3 D / 2$, since robots located on the endpoints move inside by $D / 2$ (line 4 ), the configuration becomes two point-location whose distance reduces by $D / 2$ or $D$, otherwise it satisfies $A 3 P$ or $A 4 P$. The configuration of $A 3 P$ or $A 4 P$ eventually becomes two point-location whose distance reduces by $D / 2$ or $D$ (lines $7-10$ ). This process is called reducing process. Repeating these reducing processes, therefore, a configuration of two point-location whose distance $d^{\prime}$ is greater than $D / 2$ and at most $3 D / 2$ is obtained. Also any 3-point location appearing in Algorithm 4 never appear in these reducing processes.

If $\frac{9 D}{16} \leq d^{\prime} \leq \frac{3 D}{2}$, repeating the reducing processes at most 5 times, the distance becomes less than $\frac{9 D}{8}$ and it is at least $\frac{D}{2}$ at the first time it becomes less than $\frac{9 D}{8}$. Also $A 3 P$ or $A 4 P$ is satisfied every time this process is executed and any 3-point location appearing in Algorithm 4 never appear. If $d<\frac{9 D}{8}$, repeating the redicing processes at most 4 times, the distance becomes at least $\frac{D}{4}$ and less than $\frac{D}{2}$ and $A 3 P$ or $A 4 P$ is satisfied every time line 6 is executed and any 3-point location never appears. Therefore, the lemma holds.

```
Algorithm 6 ElectLDS-Preserving-Distance \(\left(r_{i}\right)\)
Assumptions: no-light, non-rigid with \(\delta\), chirality
Input: \(D\)-distant configuration \((D \leq 2 \delta)\)
Subroutines: Reduce\#LDS, MakeDiameter, MakeEdgeonBorder.
Predicates: AFTER-RP(p),EDGEonBORDER \(\left(\mathcal{S S}_{i}\right)\).
if \(\left|L D S\left(\mathcal{S S}_{i}\right)\right|>1\) then
    if \(\mathcal{H}\left(\mathcal{S} \mathcal{S}_{i}\right)\) is a regular \(f\)-polygon then
        if not \(C L E A N\left(\mathcal{S S}_{i}\right)\) then if \(p_{i} \notin \partial \mathcal{H}\left(\mathcal{S S}_{i}\right)\) then \(\operatorname{des}_{i} \leftarrow C T R\left(\mathbf{S S}_{i}\right)\)
        else if \(\operatorname{Diam}\left(\mathcal{S S}_{i}\right) \leq D\) then \(\operatorname{des}_{i} \leftarrow C T R\left(\mathbf{S S}_{i}\right)\)
            else if \(p_{i} \in \partial \mathcal{H}\left(\mathcal{S S}_{i}\right)\) then des \(_{i} \leftarrow\left[C T R\left(\mathbf{S S}_{i}\right), p_{i}\right]-\frac{D}{2}\)
    else // \(\mathcal{H}\left(\mathcal{S S _ { i }}\right)\) is a non-regular \(f\)-polygon
        if \(\operatorname{Diam}\left(\mathcal{S S} \mathcal{S}_{i}\right) \leq D\) and \(\operatorname{AFTER-RP(p)}\) then \(\operatorname{des}_{i} \leftarrow p\)
        else if \(E D G E O=B O R D E R\left(\mathcal{S} \mathcal{S}_{i}\right)\) then
            if \(\exists \overline{p q} \in L D S\left(\mathcal{S S}_{i}\right)\) s.t. \(|\overline{p q}|=\operatorname{Diam}\left(\mathcal{S S}_{i}\right)\) then
                    if \(\forall p_{j} \exists q\left(\overline{p_{j} q} \in L D S\left(\mathcal{S S}_{i}\right)\right)\) then Reduce\#LDS
                    else if \(\forall q\left(\overline{p_{i} q} \notin L D S\left(\mathcal{S} \mathcal{S}_{i}\right)\right)\) then \(\operatorname{des}_{i} \leftarrow C T R\left(\mathcal{S S}_{i}\right)\)
            else if \(p_{i}\) is a single-endpoint of \(L D S\) then MakeDiameter
        else if \(\overline{p_{i} q} \in L D S\left(\mathcal{S S}_{i}\right)\) and \(q \in S E C\left(\mathcal{S S}_{i}\right)\) then MakeEdgeonBorder
else \(/ /\left|L D S\left(\mathcal{S S}_{i}\right)\right|=1\)
    if \(p_{i} \neq p_{n}\) then \(\operatorname{des}_{i} \leftarrow p_{n}\)
```

Lastly, the problem to be remained is to make the special configuration preserving the length of its $L D S$ is at least $D / 2$ from any $D$-distant initial configuration. Although ElectOneLDS in Lemma 8 makes the unique $L D S$ from any initial configuration, it does not preserve the length of its $L D S$ with $d \geq D / 2$ even if we assume $D$-distant initial configurations. We will prove that Algorithm 6 performs this task from any $D$-distant configuration.

Algorithm 6 is based on ElectOneLDS and produces the unique $L D S$ with its length at least $D / 2$ unless it produces a Gathering configuration. If the unique $L D S$ is obtained, it produces a configuration with only two positions on which all robots are located and its distance $d$ is at least $D / 2$ (lines 14-15).

In Algorithm 6 the following subroutines and predicates are utilized.

## Subroutines

- Reduce\#LDS: Let $q_{0}, \ldots, q_{g-1}$ be $g$ positions on $\operatorname{SEC}\left(\mathcal{S S}_{i}\right)$ in counterclockwise order. Let $C S=\left\{\overline{q_{j} q_{(j+1) \bmod g}}| | \overline{q_{j} q_{(j+1) \bmod g}} \mid=\min \left(\left|\overline{q_{0} q_{1}}\right|, \ldots,\left|\overline{q_{g-1} q_{0}}\right|\right)\right\}$ If $\overline{q_{j} q_{(j+1) \bmod g}} \in C S$, and $p_{i}=q_{j}$, then $r_{i}$ moves from $p_{i}$ to $q=$ $q_{(j+1) \bmod g}$ on $\operatorname{arc} \widetilde{p_{i} \|^{7}}$.

[^7]

Figure 6: Subroutine MakeDiameter.

- MakeDiameter: If $p_{i}$ is the single-endpoint of $L D S$ and $q$ is another endpoint of the $L D S$, let $d^{2} s_{i}$ be the intersection of line passing $q$ and $\operatorname{CTR}\left(\mathcal{S S}_{i}\right)$ and $\operatorname{SEC}\left(\mathcal{S S}_{i}\right)$. Note that the $L D S$ is not a diameter of $\operatorname{SEC}\left(\mathcal{S S}_{i}\right)$. Robot $r_{i}$ moves from $p_{i}$ to des ${ }_{i}$ on arc $p_{i} \overparen{d e s}_{i}$ (Figure (6).
- MakeEdgeonBorder: If $\overline{p_{i} q} \in \operatorname{LDS}\left(\mathcal{S S}_{i}\right)$ and $q \in \operatorname{SEC}\left(\mathcal{S S}_{i}\right)$, let des be the intersection of half-line extending the line segment $\overline{p_{i} q}$ from $p_{i}$ and $\operatorname{SEC}\left(\mathcal{S S}_{i}\right)$.


## Predicates

- $\operatorname{CLEAN}\left(\mathcal{S S}_{i}\right)$ : All robots are located on $\operatorname{CTR}\left(\mathcal{S S}_{i}\right)$ or $\operatorname{SEC}\left(\mathcal{S S}_{i}\right)$. This configuration can be reached from Reg.Polygon (line 3).
- $\operatorname{AFTER}-R P(p)$ : There exists a unique point $p$ such that all robots are located on $p$ and the circle with center $p$. This configuration can be reached after a regular polygon is broken and becomes $\left.C L E A N(\mathcal{S S})_{i}\right)$.
- EDGEonBORDER $\left(\mathcal{S S}_{i}\right)$ : Both endpoints of all $L D S$ s are located on $\operatorname{SEC}\left(\mathcal{S S}_{i}\right)$.

The outline of Algorithm 6 is shown in Figure 7 . There are typical nonfinal configurations, Regular Polygon (Reg.Polygon), Edges on the Border (EDGEonBORDER) and Edges not on the Border (NotEDGEonBORDER), where Reg.Polygon is a configuration the convex hull is a regular polygon, $E D G E o n B O R D E R$ is a configuration both endpoints of all $L D S$ s are located on the smallest enclosing circle and NotEDGEonBORDER is a configuration both endpoints of some $L D S$ are not located on the smallest enclosing circle.

When $\left|L D S\left(\mathcal{S S}_{i}\right)\right|=1$, Algorithm 6 makes 2-point configuration (line 14). Otherwise, if a configuration is Reg.Polygon whose diameter is more


Figure 7: The outline of ElectLDS-Preseving-Distance.
than $D$, Algorithm 6 makes a configuration of a regular polygon with diameter $D$, where robots are located on the corners and the center of the polygon (CLEAN). In the process of making the regular polygon with diameter $D$ or after making the regular polygon, it makes a regular polygon with smaller number of corners, EDGEonBORDER, or NotEDGEonBORDER. If the regular polygon with diameter $D$ is obtained, the algorithm makes each robot located on a corner of the polygon move to the center of the $S E C$ and makes a gathering configuration or a configuration with only one $L D S$ via configurations satisfying $A F T E R-R P(p)$, which is one that some corners are missing from the regular polygon.

If $E D G E o n B O R D E R$ is obtained and its $L D S$ is the diameter of the $S E C$ (lines 8-9), the algorithm reduces the number of $L D S$ s after moving each robot which is located on a vertex of the polygon but is not an endpoint of any $L D S$ to the center of the polygon (lines 10-11). If EDGEonBORDER is obtained but its $L D S$ is not the diameter of the $S E C$, its $L D S$ is changed into the diameter of the $S E C$ by moving the robot which is a single-endpoint of $L D S$ (line 12). Existing such robots is guaranteed by Lemma 7

If NotEDGEonBORDER is obtained, the robot not located on the $S E C$ is moved on the $S E C$ (line 13).

Let $\mathcal{C}(t)$ be a configuration at time $t$ in Algorithm 6. If $\mathcal{H}(\mathcal{C}(t))$ is a


Figure 8: Movements of robots in Lemma 23.
regular $f$-polygon and $\operatorname{CLEAN}(\mathcal{C}(t))$ does not hold, the number of robots located on $C T R(\mathcal{C}(t+1))$ increased (line 3) and therefore, there exists a time $t^{\prime}$ such that $\mathcal{H}\left(\mathcal{C}\left(t^{\prime}\right)\right)$ is a regular $f$-polygon and $\operatorname{CLEAN}\left(\mathcal{C}\left(t^{\prime}\right)\right)$ holds.
Lemma 23. If $\mathcal{H}(\mathcal{C}(t))$ is a regular $f$-polygon, $\operatorname{Diam}(\mathcal{C}(t))>D$ and $\operatorname{CLEAN}(\mathcal{C}(t))$ holds, one of the followings is satisfied.

1. $\mathcal{H}(\mathcal{C}(t))=\mathcal{H}(\mathcal{C}(t+1))$ but the number of robots located on $\operatorname{SEC}(\mathcal{C}(t+$ 1)) decreases.
2. $\mathcal{H}(\mathcal{C}(t+1))$ is the regular $f$-polygon with $\operatorname{Diam}(\mathcal{C}(t+1))=D$.
3. $\mathcal{H}(\mathcal{C}(t+1))$ is a regular $f^{\prime}$-polygon such that $f^{\prime} \leq f$ and $\operatorname{Diam}(\mathcal{C}(t)) \geq$ $\operatorname{Diam}(\mathcal{C}(t+1))>D$.
4. $\mathcal{H}(\mathcal{C}(t+1))$ is a non-regular $f^{\prime}$-polygon such that $f^{\prime} \leq f$ or $\operatorname{Diam}(\mathcal{C}(t))>$ $\operatorname{Diam}(\mathcal{C}(t+1)) \geq D$.

Proof. In this case, the destination of robot $r_{i}$ located on $\partial \mathcal{H}(\mathcal{C}(t))$ is the point of distance $\frac{D}{2}$ from $\operatorname{CTR}(\mathcal{C}(t))$ on the line segment $\overline{p_{i} \operatorname{CTR}(\mathcal{C}(t))}$ (Figure (8). If all vertices on $\partial \mathcal{H}(\mathcal{C}(t))$ remains, 1 . holds. If all robots move to the destinations, 2. holds. Otherwise, 3. or 4. holds according to movements of robots. In the both case, the diameter of the polygon decreases but is at least $D$.

By Lemma 233, there exists a time $t$ such that $\mathcal{H}(\mathcal{C}(t))$ is a regular $f$ polygon $(f \geq 2), \operatorname{Diam}(\mathcal{C}(t))=D$ and $\operatorname{CLEAN}(\mathcal{C}(t))$ holds. If $f=2$, then $\operatorname{LDS}(\mathcal{C}(t))=1$ and 2-point configuration is attained with $\operatorname{Diam}(\mathcal{C}(t))=D$ (line 14-15).

Lemma 24. If $\mathcal{H}(\mathcal{C}(t))$ is a regular $f$-polygon $(f \geq 3), \operatorname{Diam}(\mathcal{C}(t))=D$ and $\operatorname{CLEAN}(\mathcal{C}(t))$ holds, one of the followings is satisfied, and when $f \neq 3 \mathrm{~m}(\mathrm{~m}>$ $1)$, 4. and 5 . never occur.

1. $\mathcal{H}(\mathcal{C}(t))=\mathcal{H}(\mathcal{C}(t+1))$ but the number of robots located on $S E C(\mathcal{C}(t+$ 1)) decreases.
2. $\mathcal{C}(t+1)$ attains Gathering, or $\operatorname{LDS}(C(t+1))=1$ such that $\operatorname{Diam}(\mathcal{C}(t+$ 1) $\geq D / 2$.
3. $\mathcal{C}(t+1)$ satisfies $\operatorname{AFTER}-R P(p)$ and $p=C T R(\mathcal{C}(t))$.
4. $\mathcal{H}(\mathcal{C}(t+1))$ is an isosceles triangle with base of $D / 2$.
5. $\mathcal{H}(\mathcal{C}(t+1))$ is a rhombus with two isosceles triangles with base of $D / 2$.

Proof. In this case, the destination of robot $r_{i}$ located on $\partial \mathcal{H}(\mathcal{C}(t))$ is $\operatorname{CTR}(\mathcal{C}(t))$. Since $D \leq 2 \delta$, active robots always reach the destination. If all points on $\partial \mathcal{H}(\mathcal{C}(t))$ remains, 1. holds. If all robots, robots located on all points except one point, or all points except two points constituting a diameter of $S E C(\mathcal{C}(t))$ move to the destination, 2 . holds.

Otherwise, as long as $f \neq 3 m(m>1)$ or 3 , any 2 points on $\partial \mathcal{H}(\mathcal{C}(t))$ and $C T R(\mathcal{C}(t))$ do not constitute an isosceles triangle with base of $D / 2$. Thus, it is easily verified that 3 . holds and 4 . and 5 . are not satisfied. In the case that $f=3 m(m>1)$, if 4 . and 5 . are not satisfied, it is easily verified that 3. holds.

The following lemmas are easily verified.
Lemma 25. Assume that $\mathcal{H}(\mathcal{C}(t))$ is a non-regular polygon, $\operatorname{Diam}(\mathcal{C}(t))=D$ and $\operatorname{AFTER}-R P(p)$ holds. Then one of the followings holds

1. $\operatorname{Diam}(\mathcal{C}(t+1))=D$ and $\operatorname{AFTER}-R P(p)$ still holds, where $p=\operatorname{CTR}(\mathcal{C}(t))$, but the number of robots located on $S E C(\mathcal{C}(t+1))$ decreases.
2. $\mathcal{C}(t+1)$ is a Gathering configuration.
3. $|\operatorname{LDS}(\mathcal{C}(t+1))|=1$ and $\operatorname{Diam}(\mathcal{C}(t+1)) \geq \frac{D}{2}$.
4. $\mathcal{H}(\mathcal{C}(t+1))$ is an isosceles triangle with base of $D / 2$ or a rhombus with two isosceles triangles with base of $D / 2$.

Lemma 26. If $\mathcal{H}(\mathcal{C}(t))$ is an isosceles triangle with base of $D / 2$ or a rhombus with two isosceles triangles with base of $D / 2$, there is a time $t^{\prime} \geq t$ such that $\mathcal{C}\left(t^{\prime}\right)$ attains Gathering or $\left|L D S\left(\mathcal{C}\left(t^{\prime}\right)\right)\right|=1$ and $\operatorname{Diam}\left(\mathcal{C}\left(t^{\prime}\right)\right) \geq D / 2$.

The remaining case is that $\mathcal{H}(\mathcal{C}(t))$ is a non-regular polygon and $\operatorname{Diam}(\mathcal{C}(t))>$ $D$.

Lemma 27. If $|L D S(\mathcal{C}(t))| \geq 2, \mathcal{H}(\mathcal{C}(t))$ is a non-regular $f$-polygon, $\overline{p q} \in$ $\operatorname{LDS}(\mathcal{C}(t))$ and $|\overline{p q}|=\operatorname{Diam}(\mathcal{C}(t))$, then there exists a time $t^{\prime}(>t)$ such that $\mathcal{H}\left(\mathcal{C}\left(t^{\prime}\right)\right)$ is a non-regular $f^{\prime}$-polygon and $\operatorname{Diam}\left(\mathcal{C}\left(t^{\prime}\right)\right)=\operatorname{Diam}(\mathcal{C}(t))$ but $\operatorname{LDS}\left(\mathcal{C}\left(t^{\prime}\right)\right)<\operatorname{LDS}(\mathcal{C}(t))$.

Proof. In this case, Algorithm 6 performs Reduce\#LDS after any robot located on a point which is not an endpoint of $L D S$ moves to $C T R(\mathcal{C}(t))$ (line 11). Let $C S$ be a set of the minimum segments defined in Reduce $\# L D S$ and let $s_{j}=\overline{q_{j} q_{(j+1) \bmod g}} \in C S$. Robots located on endpoints of $s_{j}$ may be going to move in parallel if they become active in this round. Since the destinations of these robots is uniquely determined due to the chirality, the length of $s_{j}$ decreases by $\delta$. Repeating the process, since $s_{j}$ is a segment that both points are endpoints of a diameter of $S E C(\mathcal{C}(t)),|\operatorname{LDS}(\mathcal{C}(t))|$ decreases. Thus, the lemma holds.

Lemma 28. If $\mathcal{H}(\mathcal{C}(t))$ is a non-regular polygon, $\operatorname{EDGEonBORDER}(\mathcal{C}(t))$ holds and $|\overline{p q}|<\operatorname{Diam}(\mathcal{C}(t))$ for any $\overline{p q} \in \operatorname{LDS}(\mathcal{C}(t))$, there exists a time $t^{\prime}>t$ such that there exist $p^{\prime}$ and $q^{\prime}$ such that $\left|\overline{p^{\prime} q^{\prime}}\right|=\operatorname{Diam}\left(\mathcal{C}\left(t^{\prime}\right)\right), E D$ $\operatorname{GEonBORDER}\left(\mathcal{C}\left(t^{\prime}\right)\right)$ holds and $\operatorname{Diam}\left(\mathcal{C}\left(t^{\prime}\right)\right)=\operatorname{Diam}(\mathcal{C}(t))$.

Proof. In this case, Algorithm 6 performs MakeDiamter. Since there exists a single-endpoint of $L D S$ whose endpoints are located on $\operatorname{SEC}(\mathcal{C}(t))$ and is not a diameter of $S E C(\mathcal{C}(t))$ by Lemma 7, let $p_{0}$ be a single-endpoint of such $L D S$ of $\overline{p_{0} q_{0}}$. The destination of the robot $r_{i}$ is located on $p_{0}$ is the intersection of line passing $q_{0}$ and $C T R(\mathcal{C}(t))$ and $\operatorname{SEC}(\mathcal{C}(t))$. Since there do not exist points on the arc $p_{0} \overparen{\text { des }} s_{i}$ except $p_{0}$ by Lemma 6, the robot $r_{i}$ can move on the $\operatorname{arc} p_{0} \overparen{d e s}_{i}$ and can reach the destination. Then $\overline{\operatorname{des}_{i} q_{0}}$ becomes the diameter of the $S E C$. If there is more than one single-endpoint of $L D S$, these robots on the single-endpoints can move independently by using Lemma 6. Therefore, the lemma holds.

Lemma 29. If $\operatorname{EDGEonBORDER}(\mathcal{C}(t))$ does not hold, then there exists a time $t^{\prime}(>t)$ such that $E D G E$ onBORDER $\left(\mathcal{C}\left(t^{\prime}\right)\right)$ holds and $\operatorname{Diam}\left(\mathcal{C}\left(t^{\prime}\right)\right)=$ $\operatorname{Diam}(\mathcal{C}(t))$.

Proof. In this case, Algorithm 6 performs MakeEdgeonBorder. It can verified that there exists a time $t^{\prime}$ such that both endpoints of all $L D S$ s are located on $\operatorname{SEC}\left(\mathcal{C}\left(t^{\prime}\right)\right)$ and $\operatorname{Diam}\left(\mathcal{C}\left(t^{\prime}\right)\right)=\operatorname{Diam}(\mathcal{C}(t))$

We obtain the following theorem by Lemma $22 \cdot \mid 29$.

Theorem 8. Let $D \leq 2 \delta$. Gathering is solvable in internal-light, non-rigid with $\delta$, and SSYNC, if robots have 2 lights, set-view and agreement of chirality, and the initial configuration is $D$-distant.

## 5 Gathering Algorithms in CENT

In this section, we construct Gathering algorithms in CENT scheduler. Theorem 3 implies that Gathering cannot be solved even in CENT and 2BOUNDED, and distinct Gathering cannot be solved even in ROUND-ROBIN without strong multiplicity. Although it is still open whether distinct Gathering can be solved in ROUND-ROBIN without lights, we show that lights with two colors are enough to solve Gathering in CENT or ROUND-ROBIN with weaker assumptions than those of SSYNC as follows.
(e) Algorithm7:(2, external, CENT, non-rigid), and
(f) Algorithm8:(2, internal, ROUND-ROBIN, rigid).

### 5.1 CENT scehdulers

```
Algorithm 7 Cent-Ext-Light-Gather \(\left(r_{i}\right)\)
Assumption: External-light, 2 colors( \(T\) and \(M\) ), non-rigid, set-view, CENT
Input: Any initial configuration and any color of each robot's light.
    if \(T \notin \mathcal{L}\left(\mathcal{S S}_{i}\right)\) then \(\ell_{i} \leftarrow T\)
    else if \(\left|\mathcal{P}_{T}\left(\mathcal{S S}_{i}\right)\right|=1\) then \(\ell_{i} \leftarrow M ; \operatorname{des}_{i} \leftarrow p\left(\in \mathcal{P}_{T}\left(\mathcal{S S}_{i}\right)\right)\)
    else \(\ell_{i} \leftarrow M\)
```

Algorithm 7 is a Gathering algorithm in external-light, non-rigid and CENT. This algorithm is self-stabilizing and does not use the local-aware assumption. It uses two colors $T$ (Target) and $M$ (Move). It does not use ElectOneLDS and makes a configuration where there is just one position having some robots with $T$.

Since Algorithm 77 is executed in CENT, if each active robot observes at least one position having robots with $T$, the robot changes its color to $M$ (lines 2-3), and the configuration can reach to a configuration of one position (gathering position) having robots with $T$. Once this configuration is obtained, the configuration is unchanged and robots with $M$ not located on the gathering position move to the position, and robots located on the gathering position stay there and change their color to $T$, since these robots do
not observe any position having robots with $T$ due to the local-unawareness. If an initial configuration has no robots with $T$, the algorithm makes the light of a robot being active at this round color $T$ ( line 1). Since we assume external-light, this robot does not know whether its own color is $T$ or $M$. However, the color of the robot becomes $T$ in either case. It can be easily verified this algorithm is performed in non-rigid movement and self-stabilizing. Thus, we have the following theorem.

Theorem 9. Without chirality and local-awareness, Cent-Ext-Light-Gather is a self-stabilizing gathering algorithm in external-light, non-rigid and CENT, with 2 colors and set-view.

If we assume rigid movement and distinct configurations, Algorithm 7 can be transformed into an algorithm performing with arbitrary-view. When the number of positions having robots with $T$ becomes one and robots observe the position, they change their color $T$ and move to the position. It can be easily shown that this modified algorithm solves distinct Gathering with arbitrary-view since robots with different colors do not occupy the same location during the execution.

Corollary 1. Without chirality and local-awareness, distinct Gathering is solvable in external-light, rigid, and CENT with 2 colors and arbitrary-view.

### 5.1.1 ROUND-ROBIN Scheduler

If ROUND-ROBIN scheduler is assumed, Gathering can be done by Algorithm 8 in internal-light and rigid with 2 colors. In this case, we also use ElectOneLDS to obtain configuration onLDS. After onLDS, Algorithm 8 makes a configuration of 2 locations $p_{n}$ and $p_{f}$ (denoted by $A A$ ) or 3 locations $p_{n}, p_{m}, p_{f}$ such that $\left.\left(\left|\overline{p_{f} p_{m}}\right|=\left|\overline{p_{m} p_{n}}\right|\right)\right)($ denoted by $A A A)$, in either case all robots with $A$ are located on these locations. For these cases that the configurations $A A$ or $A A A$, the following lemmas hold by using ROUND-ROBIN property.

Lemma 30. Let $A A$ be a 2-point configuration whose locations are $p$ and $q$ and all robots have color $A$. For the activation order of robots, let last $p_{p}$ and $l_{\text {last }}^{q}$ be the maximum of orders among robots located on $p$ and $q$, respectively, and let $\min =\min \left(\right.$ last $_{p}$, last $\left._{q}\right)$. After $n$ rounds of the executions of Algorithm 8, the configuration becomes $B B$, all robots on the location having the robot with order of $\min$ have the orders less than or equal to $\min$ and all robots on the other location have the orders more than min.

```
Algorithm 8 RR-Int-Light-Gather \(\left(r_{i}\right)\)
Assumption: Internal-light, 2 colors \((A\) and \(B\) ), rigid, set-view, ROUND-ROBIN
Input: on \(L D S\), all robots have color \(A\).
    case \(\ell_{i}\) of
    A:
            if \(\left(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=2\right)\) or \(\left(\left(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=3\right)\right.\) and \(\left.\left(\left|\overline{p_{f} p_{m}}\right|=\left|\overline{p_{m} p_{n}}\right|\right)\right)\) then
                \(\ell_{i} \leftarrow B ; \operatorname{des}_{i} \leftarrow \frac{p_{n}+p_{f}}{2}\)
            else \(d e s_{i} \leftarrow p_{n}\)
    B:
            if \(\left|\mathcal{P}\left(\mathcal{S S}_{i}\right)\right|=2\) then des \(_{i} \leftarrow p_{f}\)
            else des \(_{i} \leftarrow \frac{p_{n}+p_{f}}{2} / /\left|\mathcal{P}\left(\mathcal{S} \mathcal{S}_{i}\right)\right|=3\)
        endcase
```

Proof. Let $p^{\prime}$ and $q^{\prime}$ be the locations of $B B$. Consider the case that last $p_{p}<$ last $_{q}$. In this case $\min =$ last $_{p}$. Since robots with color $A$ move to the midpoint of $p$ and $q$ (denoted as $r$ ) and change their color into $B$ (lines 3-4), the configuration becomes $B A$ whose locations are $r$ and $q$ after last $t_{p}$ rounds and the orders of all robots located on $r(q)$ is less than or equal to last $_{p}$ (is more than last ${ }_{p}$ ). Then after $n-l a s t_{p}$ rounds robots located on $q$ are activated and the configuration becomes $B B$ whose locations are $r$ and the midpoint of $r$ and $q$. Thus the lemma holds. The case that last $p_{p}<$ last $_{q}$ can be proved similarly.

Lemma 31. Let $A A A$ be a 3 -point configuration whose locations are $p, r$ and $q$ from the leftmost and all robots have color $A$. For the activation order of robots, let last ${ }_{p}$, last $r_{r}$ and last ${ }_{q}$ be the maximum of orders among robots located on $p, r$ and $q$, respectively, and let $\min$ and $\min _{2}$ be the minimum and the second minimum of $l a s t_{p}, l a s t_{r}$, and $l_{\text {last }}^{q}$, respectively. After $n$ rounds of the executions of Algorithm 8, the configuration becomes $B B$ or $B B B$ and the followings hold.

1. In the case of $B B$, all robots on the location having the robot with order of $\min$ have the orders less than or equal to $\min$ and all robots on the other location have the orders more than min.
2. In the case of $B B B$, the location having the robot with order of $\min$ is either left side or right side, and all robots on the location having the robot with order of $\min$ have the orders less than or equal to min , all robots on the other side have the orders between $\min +1$ and $\min _{2}$, and all robots on the midpoint have the orders more than $\min _{2}$.

Proof. Consider that case that $\min =$ last $_{p}$. After last $_{p}$ rounds from the
beginning, the configuration becomes $(A+B) A_{8}^{8}$ whose locations are $r$ and $q$ and the orders of all robots with color $B$ located on $r$ are less than or equal to $l a s t_{p}$, and the orders of all robots with color $A$ located on $r$ are between last $_{p}+1$ and $\min _{2}$ (lines 3-4).

If $\min _{2}=l a s t_{r}$, since the next activated robots are ones with color $A$, after the next last $r_{r}-$ last $_{p}$ rounds, the configuration becomes $B B A$ where, these locations are $r$, the midpoint of $r$ and $q$ (denoted as $r^{\prime}$ ) and $q$, all robots located on $r$ have the orders less than or equal to last $_{p}$, all robots located on $r^{\prime}$ have the order between last $_{p}+1$ and last $_{r}$ (lines 3-4). Then the next activated robots are ones located on $q$, and after the next $n$ - last $r_{r}$ rounds, the configuration becomes $B B$ (line 8) whose locations are $r$ and $r^{\prime}$ and the lemma holds. Otherwise $\left(\min _{2}=\right.$ last $\left._{q}\right)$, after the next last ${ }_{q}-$ last $_{p}$ rounds, the configuration becomes $(A+B) B$ whose locations $r$ and $r^{\prime}$ and all robots with color $B$ located on $r$ have the orders less than or equal to last $_{p}$, all robots located on $r^{\prime}$ have the order between last +1 and last $_{q}$ (lines 3-4). Then since the next activated robots are ones with color $A$, after the next $n-$ last $_{q}$ rounds the configuration becomes $B B B$ whose locations $r$, the midpoint of $r$ and $r^{\prime}$ (denoted as $r^{\prime \prime}$ ) and $r^{\prime}$ (lines 3-4) and the lemma holds.

The case that $\min =l a s t_{q}$ can be proved similarly.
Consider the last case that $\min =$ last $_{r}$. After last $_{r}$ rounds from the beginning, the configuration becomes $A B A$ whose locations are $p, r$ and $q$ and all robots with color $B$ have the orders less than or equal to $l_{\text {ast }}^{r}$ (lines 34). If $\min _{2}=l a s t_{p}$, after the next last ${ }_{p}-$ last $_{r}$ rounds, the configuration becomes $B A$ whose locations are $r$ and $q$ and all robots on the location $r$ have the orders less than or equal to last $_{p}$ (lines 3-4). Then after the next $n-l_{\text {last }}^{p}$ rounds, the configuration becomes $B B$ whose locations $r$ and $r^{\prime}$ (lines 3-4) and the lemma holds. The case that $\min _{2}=l_{\text {ast }}^{q}$ is the symmetrical one and so can be proved similarly.

We have the following theorem by using Lemmas 30 and 31 .
Theorem 10. Without chirality, Gathering is solvable in internal-light, rigid, and ROUND-ROBIN, if robots have 2 lights and set-view.

Proof. By using Lemmas 30 and 31, the configuration becomes $B B$ or $B B B$ satisfying the conditions of Lemmas 30 and 31 .

In the case of $B B$, since all robots on the location having the robot with order of $\min$ have the orders at most $\min$ and all robots on the other location

[^8]have the orders more than min, the configuration becomes a Gathering one after min rounds from the configuration $B B$ (line 7).

In the case of $B B B$, let $p, q$ and $r$ be the location having the robot with order of $\min$, the other endpoint and the midpoint, respectively. since all robots located at $p, q$ and $r$ have the orders at most min, between min and $\min _{2}$, and more than $\min _{2}$, respectively, the configuration becomes $B B$ after $\min$ rounds (line 8), from which the Gathering attains after more $\mathrm{min}_{2}$ rounds (line 7).

## 6 Concluding Remarks

We have shown Gathering algorithms by mobile robots with lights in SSYNC and CENT schedulers and we have obtained some relationship between the power of lights (full, external and internal) and assumptions of robots' moving (rigid and non-rigid). Interesting open questions are determining the relationship of precise computational power between external-light and internallight and constructing Gathering algorithms in ASYNC. For full-light, since it is known that any algorithm of robots in full-light of $k$ colors and SSYNC can be simulated by robots in full-light of $5 k$ colors and ASYNC [4, Gathering is solvable in full-light, non-rigid, and ASYNC, if robots have 10 colors, set-view and agreement of chirality by using the result in this paper. Reducing the number of colors in full-light and Gathering algorithms with external-light or internal-light in ASYNC remains as interesting open problems.

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[^1]:    ${ }^{1}$ In [10], external-light and internal-light are called FCOMM and FSTATE, respectively.

[^2]:    ${ }^{2}$ Recognition of multi-set-view enables us the strong multiplicity detection.

[^3]:    ${ }^{3}$ In some cases, we may assume chirality.

[^4]:    ${ }^{4}$ Note that $\ell_{i}$ is not in $A L_{i}\left[p_{j}\right]$, and $A L_{i}\left[p_{j}\right]$ is not defined for $p_{j}$ such that $r_{j}$ is located at $p_{i}$ under the local-unawareness.

[^5]:    ${ }^{5} A B$ can be treated similarly

[^6]:    ${ }^{6}$ This assumption does not imply distinct Gathering.

[^7]:    ${ }^{7}$ Since robots cannot move on arc, they move on the line segment of length $\delta$ repeatedly. That is, if $\left|\overline{p_{i} d e s_{i}}\right| \leq \delta$, then $r_{i}$ moves to the destination. Otherwise, the destination is the intersection of the circle with center $p_{i}$ and radius $\delta$, and $\operatorname{SEC}\left(\mathcal{S S}_{i}\right)$.

[^8]:    ${ }^{8} A+B$ means that robots with color $A$ and robots with color $B$ are located on the same location.

