Generalized domination structure in cubic graphs

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Abstract

In this paper, we consider generalized domination structure in graphs, which stipulates the structure of a minimum dominating set. Two cycles of length 0 mod 3 intersecting with one path are the constituents of the domination structure and by taking every three vertices on the cycles we can obtain a minimum dominating set. For a cubic graph, we construct generalized domination structure by adding edges in a certain way. We prove that the minimum dominating set of a cubic graph is determined in polynomial time.
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1 Notation

In this paper, a graph G is finite, undirected, and simple with the vertex set V and edge set E. We follow [1] for basic notation. For a vertex $v \in V(G)$, the open neighborhood, denoted by $N_G(v)$, is $\{u \in V(G)\colon uv \in E(G)\}$, and the closed neighborhood, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$, also for a set $W \subseteq V(G)$, let $N_G(W) = \bigcup_{v \in W} N_G(v)$ and $N_G[W] = N_G(W) \cup W$. A *dominating set* $X \subseteq V(G)$ is such that $N_G[X] = V(G)$. For a set $S \subseteq V(G)$, as is clear from the context, S denotes $G[S]$. A minimum dominating set, called a *d-set*, is a dominating set of minimum cardinality. Two cycles C_1 and C_2 are said to be *connecting without seams* if $C_1 \cap C_2$ is one path. For a graph G, *structure* H is the union of maximal number of cycles of length 0 mod 3 in G where each cycle of length 0 mod 3 is connecting without seams for some other cycle of length 0 mod 3, or one cycle of length 0 mod 3 in H, in addition, if $V(G - H) = \emptyset$, we call this structure *domination structure*. Let $\mathcal{F}(G)$ be the set of all structures in a graph G.

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We consider a connected graph G, otherwise consider each component one by one. We introduce the construction scheme $\mathbf{K}$ as follows.

$\mathbf{K}$: Input a connected graph G.
(1) Let $G_0 = G$ and $k = 0$.
(2) Let v be a cut vertex of G_k. For every pair of components C_1 and C_2 of $G_k - v$ and for every pair of vertices $v_1 \in C_1 \cap N_{G_k}(v)$ and $v_2 \in C_2 \cap N_{G_k}(v)$, add an edge v_1v_2. Increment k.
(3) Let D_2 be an induced cycle of length 2 mod 3 in G_k. Take a vertex $w \in V(D_2)$, and set $N_{D_2}(w) = \{w_1, w_2\}$. Now, add an edge w_1w_2. Set $w_1ww_2\alpha \subseteq D_2$. Now, add two edges $w_1\alpha$ and $w\alpha$. Increment k.
(4) Let D_1 be an induced cycle of length 1 mod 3 in G_k. Take a vertex $x \in V(D_1)$, and set $N_{D_1}(x) = \{x_1, x_2\}$. Now, add an edge x_1x_2. Set $x_1xx_2\alpha \subseteq D_1$. Now, add an edge $x\alpha$. Increment k.
(5) Repeat (2)-(4).
(6) Return the resulting graph G_k.

The next remark is a basic concept of the following proofs.

Remark 2.1. *Let X be a dominating set of G. Every subset $D \subseteq X$ is a d-set of $N_G[D]$ if and only if X is a d-set of G.*

Let $\mathbf{K}(G)$ be a graph constructed by applying $\mathbf{K}$ to G. Note that $\mathbf{K}(G)$ is not unique and constructed from G arbitrarily.

Proposition 2.1. *$\mathbf{K}(G)$ is domination structure. Moreover, $|\mathcal{F}(\mathbf{K}(G))| = 1$.*

Proof. From the rules, $\mathbf{K}(G)$ is 2-connected. Hence $\mathbf{K}(G)$ has an ear decomposition. Since all induced cycles are of length 0 mod 3, the domination structure H is obtained by finding 0 mod 3 induced cycles connecting without seams one by one, so that $H = \mathbf{K}(G)$. We had the claim. $\square$

Suppose that for all $v \in V(G)$, $d_G(v) \geq 3$.

Fact 2.1. *For domination structure $H = \mathbf{K}(G)$, label every three vertices on the induced cycles that constitute H in order of connecting without seams. Note that a certain induced cycle of H is not counted for the labeling, and may have no labels, where the vertices of the induced cycle are all in other induced cycles of H. There exist at most $|V(G)|$ cases of labeling. (i) For* *every labeling, the set of all labeled vertices is a dominating set of* $\mathbf{K}(G)$*. (ii) For at least one labeling, the set of all labeled vertices is a d-set of* $\mathbf{K}(G)$*.*

Proof. By the first vertex choice for the labeling, all labeled vertices are uniquely determined in $V(G)$ since for all $v \in V(G)$, $d_G(v) \geq 3$, and so there exist at most $|V(G)|$ cases of labeling. The statement (i) is obvious. The statement (ii) follows from Remark 2.1. □

Let Y be a d-set of $\mathbf{K}(G)$ that is obtained by applying Fact 2.1. Let $\mathcal{Y}$ be the set of all Y. Let X be a d-set of G. Let $\mathcal{X}$ be the set of all X.

Proposition 2.2. *For some* $X \in \mathcal{X}$*, and some* $Y \in \mathcal{Y}$*,* $Y \subseteq X$*.*

Proof. For some $Y \in \mathcal{Y}$, if Y is a dominating set of G, then for some $X \in \mathcal{X}$, $X = Y$. Otherwise, for all $Y \in \mathcal{Y}$, Y is not a dominating set of G. Now, for some $v \in Y$, and some $w \in V(G) \setminus Y$, vw is an added edge for $\mathbf{K}(G)$ and $N_G(w) \subseteq V(G) \setminus Y$. Now, we consider such v and w. Let $Z' = \{w \in V(G) \setminus Y\colon v \in Y, vw \in E(\mathbf{K}(G)) \setminus E(G), N_G(w) \subseteq V(G) \setminus Y\}$ and $E' = \{vw \in E(\mathbf{K}(G)) \setminus E(G)\colon w \in V(G) \setminus Y, v \in Y, N_G(w) \subseteq V(G) \setminus Y\}$. Let $E''(vw)$ be an induced cycle of length 0 mod 3 in $\mathbf{K}(G)$ such that for $vw \in E'$, $vw \in E''(vw)$ holds. Let $\mathcal{E}(vw)$ be the set of all $E''(vw)$ for $vw \in E'$ and let $\mathcal{E} = \bigcup_{vw \in E'} \mathcal{E}(vw)$. Let J be the union of all induced cycles of length 0 mod 3 in $\mathbf{K}(G)$ other than the induced cycles in $\mathcal{E}$. By the definition of Y and Remark 2.1, $J \cap Y$ is a d-set of $N_G[J \cap Y]$, and Y is a d-set of $G - Z' = N_G[Y]$. Let W be a subset of $V(G) \setminus Y$ of minimum cardinality such that $Y \cup W$ is a dominating set of G. By the definition of W, W is a d-set of $N_G[W]$. Since for all $x \in V(G)$, $d_G(x) \geq 3$, and by Fact 2.1, $E(G[Y]) = \emptyset$, and $Y \cup W$ is a minimal dominating set of G. Therefore, by Remark 2.1, for some $X \in \mathcal{X}$, $X = Y \cup W$. □

Suppose that Y is a d-set of $\mathbf{K}(G)$ that satisfies Proposition 2.2.

Fact 2.2. *Let* G' *be constructed by deleting* Y*, and for every pair* $w_1, w_2 \in \bigcup_{y \in Y} N_G(y)$*, adding an edge* $w_1 w_2$ *to* G*. Let* Z_1 *be a d-set of* G'*. Let* $G'' = G - N_G[Y]$ *and* Z_2 *be a d-set of* G''*. If* $|Z_1| < |Z_2|$*, then* $Y \cup Z_1$ *is a d-set of* G*, and* $Z_1 \cap \bigcup_{y \in Y} N_G(y) \neq \emptyset$*. If* $|Z_1| \geq |Z_2|$*, then* $Y \cup Z_2$ *is a d-set of* G*.*

Proof. Obviously, $Y \cup Z_2$ is a d-set of G if and only if $|Z_1| \geq |Z_2|$. For a set $A \subseteq V(G)$, suppose that $Y \cup A$ is a d-set of G. Let $A' = A \setminus \bigcup_{y \in Y} N_G(y)$. By Remark 2.1, A' is a d-set of $N_G[A']$. Suppose that $|Z_1| < |Z_2|$, then A' is not a dominating set of G'', and $A \cap \bigcup_{y \in Y} N_G(y) \neq \emptyset$. Now, A is a minimal dominating set of G'. By the definition of G', $Y \cup Z_1$ is a minimal dominating set of G, and so it suffices that $A = Z_1$. Thus $Z_1 \cap \bigcup_{y \in Y} N_G(y) \neq \emptyset$. $\square$

Suppose that G is cubic.

Theorem 2.1. *For some $X \in \mathcal{X}$, X is determined in polynomial time.*

Proof. By Proposition 2.2, $Y \subseteq X$ for some $X \in \mathcal{X}$. Let $G_0 = G$. Let G_1 be constructed by deleting Y, and for every pair $w_1, w_2 \in \bigcup_{y \in Y} N_G(y)$, adding an edge $w_1 w_2$ to G_0. Let $G_2 = G_0 - N_{G_0}[Y]$. Let W_1 be a d-set of G_1 and W_2 be a d-set of G_2. Since G_0 is cubic and by the definition of G_2, each component of G_2 is path or cycle. Thus W_2 is determined in polynomial time. Suppose that $|W_1| < |W_2|$. Let Y_1 be a d-set of $\mathbf{K}(G_1)$ that satisfies Proposition 2.2. By the definition of Y_1, it suffices that $Y_1 \cap \bigcup_{y \in Y} N_{G_0}(y) \neq \emptyset$. Let G_3 be constructed by deleting Y_1, and for every pair $w_1, w_2 \in \bigcup_{y \in Y_1} N_{G_1}(y)$, adding an edge $w_1 w_2$ to G_1. Let $G_4 = G_1 - N_{G_1}[Y_1]$. Let W_3 be a d-set of G_3 and W_4 be a d-set of G_4. By the definition of G_4, each component of G_4 is path. Thus W_4 is determined in polynomial time. Suppose that $|W_3| < |W_4|$. Let Y_2 be a d-set of $\mathbf{K}(G_3)$ that satisfies Proposition 2.2. By the definition of Y_2, it suffices that $Y_2 \cap \bigcup_{y \in Y_1} N_{G_1}(y) \neq \emptyset$. Let G_5 be constructed by deleting Y_2, and for every pair $w_1, w_2 \in \bigcup_{y \in Y_2} N_{G_3}(y)$, adding an edge $w_1 w_2$ to G_3. Let $G_6 = G_3 - N_{G_3}[Y_2]$. Let W_5 be a d-set of G_5 and W_6 be a d-set of G_6. By the definition of G_6, G_6 is independent. Thus $W_6 = V(G_6)$. Suppose that $|W_5| < |W_6|$. Let Y_3 be a d-set of $\mathbf{K}(G_5)$ that satisfies Proposition 2.2. By the definition of Y_3, it suffices that $Y_3 \cap \bigcup_{y \in Y_2} N_{G_3}(y) \neq \emptyset$. Now, Y_3 is a dominating set of G_5 and so it suffices that $W_5 = Y_3$. By Fact 2.2, if $|W_5| < |W_6|$, then it suffices that $W_3 = Y_2 \cup W_5$, otherwise, it suffices that $W_3 = Y_2 \cup W_6$. By Fact 2.2, if $|W_3| < |W_4|$, then it suffices that $W_1 = Y_1 \cup W_3$, otherwise, it suffices that $W_1 = Y_1 \cup W_4$. By Fact 2.2, if $|W_1| < |W_2|$, then it suffices that $X = Y \cup W_1$, otherwise, it suffices that $X = Y \cup W_2$. The proof is complete. $\square$

References

[1] R. Diestel: Graph Theory Fourth Edition. Springer (2010)