Possibilistic reasoning—a mini-survey and uniform semantics [*]

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Abstract

In this paper, we survey some quantitative and qualitative approaches to uncertainty management based on possibility theory and present a logical framework to integrate them. The semantics of the logic is based on the Dempster's rule of conditioning for possibility theory. It is then shown that classical modal logic, conditional logic, possibilistic logic, quantitative modal logic and qualitative possibilistic logic are all sublogics of the presented logical framework. In this way, we can formalize and generalize some well-known results about possibilistic reasoning in a uniform semantics. Moreover, our uniform framework is applicable to nonmonotonic reasoning, approximate consequence relation formulation, and partial consistency handling.

Keywords: Nonclassical logics; Possibility theory; Conditional possibility; Modal logic; Conditional logic

1. Introduction

There are essentially two kinds of logical formalisms for reasoning about possibility and necessity. On the one hand, the quantitative approach represents numerical possibility and necessity of logical formulas in the language directly. The most remarkable cases of this approach are possibilistic logic (PL, [16]) and quantitative modal logic (QML, [34]). For example, in PL, the well-formed formulas $(f \ N \ c)$ and $(f \ \Pi \ c)$ denote that the necessity and the possibility of the sentence f are greater than or equal to c respectively. On the other hand, the qualitative approach represents the relative magnitude

of possibility degrees between two formulas. For example, in the representative case qualitative possibility logic (QPL, [19]), the wff $f \geqslant g$ means the possibility degree of f is greater than or equal to that of g. It is shown that QPL is equivalent to a system of conditional logic. Obviously, in the quantitative approach, we cannot represent the comparative possibility or necessity between different formulas, whereas in the qualitative approach, we cannot reason about the numerical uncertainty directly. Thus, it is intended to have a uniform formalism that can integrate both kinds of logics. In this paper, we will propose a logic for conditional possibility (LCP) that can serve the purpose.

2. A mini-survey

In this section, we will review some logics for reasoning about possibility and necessity. These include modal logic, conditional logic, possibilistic logic, quantitative modal logic and qualitative possibility logic. However, the first step is to fix a propositional language. The syntax of the propositional language is as follows:
(1) The alphabet:
- Logical constants: $\top$ (*verum* or *truth* constant) and $\perp$ (*falsum* or *false* constant).
- Propositional variables: $PV = \{p, q, r, \ldots\}$.
- Classical connectives: $\neg$ (negation), $\vee$ (or).
(2) The well-formed formulas (wffs):
- All propositional variables and propositional constants are wffs, also called atomic formulas.
- If f and g are wffs, so are $\neg f$ and $f \vee g$.
- Nothing except those determined by the above are wffs.
(3) Some abbreviations:
- $f \wedge g = \neg(\neg f \vee \neg g)$.
- $f \supset g = \neg f \vee g$.
- $f \equiv g = (f \supset g) \wedge (g \supset f)$.

Let $\mathcal{L}$ denote the set of all wffs of the propositional language.

2.1. Modal logic

Just like many branches of logics, the origin of modal reasoning can be traced back to the Aristotelian age. However, the first modern logic system for reasoning about possibility and necessity appeared in 1912 [32]. Since then, modal logic has received much attention from philosophical logicians. After the publication of Kripke's influential paper on the semantics of modal logics [30], the notion of possible worlds has been associated with modal logic closely. In addition to its philosophical interest, modal logic has also been applied to program verification (e.g. temporal logic and dynamic logic), AI (epistemic logic) and other fields of computer science [26,27,43].

We will introduce the syntax and possible world semantics for classical modal logic in what follows. Only the essential of modal logic will be touched upon. For more detail, the readers can see the introductory textbook [9].

To form the wffs of modal logic, a new unary connective $\Box$ is added to propositional logic and an additional rule for formation of modal formulas is needed:

- if f is a wff, then $\Box f$ is, too.

Furthermore, $\neg \Box \neg f$ is abbreviated as $\Diamond f$.

A possible world model for modal logic is a triplet $M = \langle W, R, V \rangle$, where W is a set of possible worlds, $R \subseteq W \times W$ is a binary relation called *accessibility relation* on W, and $V : PV \rightarrow 2^W$ assigns to each propositional symbol in PV a subset of W. If R satisfies the condition that for all $w \in W$ there exists $u \in W$ such that $(w, u) \in R$, then the model is called *serial*. Given a model M and a wff f, we can define the truth set $|f|_M$ as

$$
|f|_M = \begin{cases}
V(f), & \text{if } f \in PV, \\
W \backslash |g|_M, & \text{if } f = \neg g, \\
|g|_M \cup |h|_M, & \text{if } f = g \vee h, \\
\{w \mid \forall u((w, u) \in R \Rightarrow u \in |g|_M)\}, & \text{if } f = \Box g.
\end{cases}
$$

A formula f is satisfiable iff there exists M such that $|f|_M \neq \emptyset$, and is valid in M iff $|f|_M = W$. The subscript M will be dropped when it is clear from the context. We use $M \models f$ to denote f is valid in M. Let S be a set of wffs, then $S \models_D f$ iff for all serial models M, if for all $g \in S$, $M \models g$, then $M \models f$. Also let D denote the set of wffs f such that $\models_D f$ holds.

2.2. Conditional logic

The original purpose of conditional logic is to provide a formal tool for the analysis of subjunctive conditional in natural language [39]. Recently, there have been a number of applications of conditional logic to nonmonotonic reasoning and belief revision in AI research [22, 29].

The syntax of conditional logic is an extension of the propositional language with a binary connective $\rightarrow$ and the following formation rule:

- if f and g are wffs, then $f \rightarrow g$ is also a wff.

As for the semantics, there are some competitive paradigms which lead to different systems of conditional logics [39]. The one most closely related to possibilistic reasoning is the system-of-spheres semantics proposed by Lewis [33]. According to the reformulation in [7], we describe a system-of-spheres model (s-model) of conditional logic as a triple $M = \langle W, (\leqslant_w)_{w \in W}, V \rangle$, where W and V are the same as in the possible world models for modal logic, and for each $w \in W$, $\leqslant_w \subseteq W \times W$ is a binary relation (called preference relation) on W satisfying almost reflexivity, transitivity, almost connectedness, and nonvacuity.[2] Intuitively, a world u is more possible (preferred, closer) than v from the viewpoint of an agent in w if $v \leqslant_w u$. The set $W_w =_{\text{def}} \{u \mid \exists v, v \leqslant_w u\}$ is called the *accessible worlds* from w. The worlds not in W_w are considered impossible from w. Then, according to *Ramsey test*, $f \rightarrow g$ is true in w iff an agent in w comes

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[2] $\forall u \in W_w$, $u \leqslant_w u$ (almost reflexivity), $\forall t, u, v \in W$, $(t \leqslant_w u \wedge u \leqslant_w v) \supset t \leqslant_w v$ (transitivity), $\forall u, v \in W_w$, $u \leqslant_w v \vee v \leqslant_w u$ (almost connectedness), and $\forall w$, $W_w \neq \emptyset$ (nonvacuity).

to accept g when it revises its belief to accommodate f. Therefore, in addition to the definition of truth set for classical connectives, we can define

$$|f \to g| = \{w \mid W_w \cap |f| = \emptyset \text{ or}$$

$$\exists u \in W_w \cap |f| \, \forall v (u \leqslant_w v \Rightarrow v \in |f \supset g|) \}.$$

The definition of satisfiability and validity is the same as above. For any set of wffs S and wff f, $S \models_{VN} f$ is defined as $S \models_D f$ except that serial models are replaced by s-models. Let VN also denote the set of wffs such that $\models_{VN} f$ holds.

2.3. Possibility theory and related logics

2.3.1. Possibility theory

Possibility theory is developed by Zadeh from fuzzy set theory [44]. Given a universe W, a *possibility distribution* on W is a function $\pi : W \to [0, 1]$. In general, we require the normalized condition is satisfied. That is, $\sup_{w \in W} \pi(w) = 1$ must hold. Obviously, π is a characteristic function of a fuzzy subset of W. Two measures on W can be derived from π. They are called possibility and necessity measures and denoted by Π and N respectively. Formally, $\Pi, N : 2^W \to [0, 1]$ are defined as

$$\Pi(A) = \sup_{w \in A} \pi(w),$$

$$N(A) = 1 - \Pi(\overline{A}),$$

where $\overline{A}$ is the complement of A with respect to W.

It can easily be shown that the possibility measure satisfies the following conditions:
 (i) $\Pi(W) = 1$ and $\Pi(\emptyset) = 0$,
 (ii) $\Pi(\bigcup_{i \in I} A_i) = \sup_{i \in I} \Pi(A_i)$, where I is a (possibly infinite) index set. [3]
Conversely, if a possibility measure Π is given that satisfies the two properties, then the possibility distribution corresponding to Π can be defined as $\pi(w) = \Pi(\{w\})$ for all $w \in W$.

Just like in probability theory, we can consider the definition of conditional possibility distribution. However, how to define it is still controversial. In [2], the following definition from [15] is adopted. Given a subset A of W, the conditional possibility distribution is defined as

$$\pi(x|A) = \begin{cases} 1, & \text{if } \pi(x) = \Pi(A), \ x \in A, \\ \pi(x), & \text{if } \pi(x) < \Pi(A), \ x \in A, \\ 0, & \text{if } x \notin A. \end{cases} \tag{1}$$

An alternative definition of conditional possibility is by the Dempster's conditioning rule introduced in [41]. According to this rule, if $\Pi(A) \neq 0$, we define

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[3] When W is finite, we can restrict $I = \{1, 2\}$ and reduce the equation to $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$.

$$\pi(x|A) = \begin{cases} \dfrac{\pi(x)}{\Pi(A)}, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases} \tag{2}$$

and if $\Pi(A) = 0$, we will define $\pi(x|A) = 1$ for all $x \in W$.

Since in our logic the necessity measure of a proposition will be interpreted as epistemic uncertainty of some agent, we prefer (2) that is the conditioning rule for consonant beliefs as indicated in [41]. Furthermore, by (2), the logic LCP we will introduce can be easily extended to accommodate the belief function model. We need only replace possibility distributions by basic probability assignments [41] in the semantics and modify the satisfaction clauses appropriately, then the probabilistic logic formulas introduced in [18] can be represented in our logic. In Section 5.1, we will also see that the definition facilitates the representation of Goldszmidt and Pearl's system Z^+ [24] in our logic.

Another reason that we adopt (2) is that it is technically more appropriate than (1). According to (1), $\pi(\cdot|A)$ is still a normalized possibility distribution on A when A is a finite and nonempty subset of W. However, if W is infinite, it is possible that A is an infinite subset of W such that $\Pi(A) < 1$ and for all $x \in A$, $\pi(x) < \Pi(A)$. In this case, we will have $\Pi(W|A) = \Pi(A|A) =_{\text{def}} \sup_{x \in A} \pi(x|A) = \Pi(A) < 1$, so $\pi(\cdot|A)$ is no longer normalized. Since in the following logics the universe will be the set of possible worlds that may be infinite, we consider (1) inappropriate. From now on, when we mention conditional possibility, we will refer to the definition (2).

2.3.2. Possibilistic logic

Based on possibility theory, Dubois and Prade propose the possibilistic logic (PL) [13, 14, 16]. Recall our propositional language $\mathcal{L}$. The wffs of PL based on $\mathcal{L}$ are one of the forms $(f \, N \, c)$ or $(f \, \Pi \, c)$, where $f \in \mathcal{L}$ and $c \in (0, 1]$.

A model for PL is also a triplet $\langle W, \pi, V \rangle$, where W and V are still as above and π is a possibility distribution on W. For any $f \in \mathcal{L}$, we can define $|f|$ as above. In what follows, we will identify the truth sets $|f|$ and f in the context of possibility or necessity measures. For example, $N(|f|) = N(f)$, $\Pi(|f|) = \Pi(f)$, etc. Then the truth sets are

$$|(f \, N \, c)| = \begin{cases} W, & \text{if } N(f) \geqslant c, \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$|(f \, \Pi \, c)| = \begin{cases} W, & \text{if } \Pi(f) \geqslant c, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let $S \cup \{f\}$ be a set of wffs in PL, then $S \models_{\text{PL}} f$ iff for any PL model M, $|S|_M = W$ implies $|f|_M = W$. We can also define the valuation functions Val_N and $Val_\Pi : \mathcal{L} \times 2^{\text{PL}} \to [0, 1]$ as

$$Val_N(f, S) = \sup\{c \mid S \models_{\text{PL}} (f \, N \, c)\},$$

$$Val_\Pi(f, S) = \sup\{c \mid S \models_{\text{PL}} (f \, \Pi \, c)\}.$$

Note that our presentation of PL is different with the original one proposed by Dubois and Prade. We will analyze the difference and prove the two presentations are technically equivalent in the following.

First, from the syntactic aspect, PL is just the propositional part of PL2 defined in [14, p. 474]. However, since the extension of the base logic $\mathcal{L}$ to first order logic is straightforward, the difference is not essential. The main difference is thus the semantic one. Let Ω denote the set of all interpretations for $\mathcal{L}$. Then a model for PL2 is just a possibility distribution π on Ω. For the possibility and necessity measures Π and N induced by π, the satisfaction relation is defined by

$$\pi \models (f \ N \ c) \Leftrightarrow N(|f|) \geqslant c$$

$$\pi \models (f \ \Pi \ c) \Leftrightarrow \Pi(|f|) \geqslant c,$$

where $|f|$ is now the set of all propositional models of f.

If π is a PL2 model, then $M_\pi = \langle \Omega, \pi, V \rangle$ is a PL model, where $V(p) = \{\omega \in \Omega \mid \omega \models p\}$ for all $p \in PV$. Obviously, we have $\pi \models f$ iff $|f|_{M_\pi} = \Omega$ for any PL2 wff f. On the other hand, given a PL model $M = \langle W, \pi, V \rangle$, we can define V_u as the propositional interpretation associated with $u \in W$ from the truth assignment V. Then let π_M be a possibility distribution on Ω defined as

$$\pi_M(\omega) = \sup\{\pi(u) \mid V_u = \omega, \ u \in W\},$$

where we assume $\sup \emptyset = 0$ by convention. We can show that $|f|_M = W$ iff $\pi_M \models f$ for any PL wff f. Therefore, the semantics for PL and PL2 are equivalent in a technical sense.

Nevertheless, the philosophical meaning of a world and an interpretation is quite different. In general, the wffs in $\mathcal{L}$ describe the objective facts of a world, so two worlds may coincide in the objective aspect and be different in the others. From an epistemic or doxastic perspective, it is natural to consider a possibility distribution as an agent's view (with uncertainty) on the worlds. Since an agent may have different views in two worlds with the same objective facts, the semantics for PL is more general to reflect the situation. Of course, the difference is irrelevance when we are reasoning only about the agent's beliefs on the objective world instead of nested beliefs. This is why the semantics for PL and PL2 are equivalent when we only consider the possibilistic logic formulas.

Recently, the semantics for possibilistic logic has been generalized to handling partial inconsistency [13, 14]. In the inconsistency-tolerant semantics, an absurd interpretation $\omega_\perp$ is added to Ω, then $\Omega_\perp = \Omega \cup \{\omega_\perp\}$, and a PL2 model is just a possibility distribution $\hat{\pi}$ on $\Omega_\perp$. The absurd interpretation is defined such that $\omega_\perp \models f$ for all $f \in \mathcal{L}$. Define

$$\Pi(f) = \sup_{\omega \in \Omega}\{\hat{\pi}(\omega) \mid \omega \models f\},$$

$$\hat{\Pi}(f) = \sup_{\omega \in \Omega_\perp}\{\hat{\pi}(\omega) \mid \omega \models f\},$$

$$N(f) = \inf_{\omega \in \Omega}\{1 - \hat{\pi}(\omega) \mid \omega \models \neg f\},$$

$$\hat{N}(f) = \inf_{\omega \in \Omega_\perp} \{1 - \hat{\pi}(\omega) \mid \omega \not\models f\}.$$

Then, $\hat{\Pi}(f) = \max(\Pi(f), \hat{\pi}(\omega_\perp))$ and $\hat{N}(f) = N(f)$, and the truth condition is defined as $\hat{\pi} \models (f \; \Pi \; c)$ iff $\hat{\Pi}(f) \geqslant c$ and $\hat{\pi} \models (f \; N \; c)$ iff $\hat{N}(f) \geqslant c$.

Our main results may be generalized to the inconsistency-tolerant setting. However, for simplicity, we exhibit only the basic semantics now, and the general consideration will be deferred to Section 5.3.

2.3.3. Quantitative modal logic

Though PL is useful in reasoning about an agent's uncertain beliefs, it is not suitable for introspective agents, i.e., the agents reasoning about the beliefs of itself. For example, we may want to represent the following sentence about a reflective agent:

> *The agent considers it completely possible that what he believes with half certainty is wrong.*

The sentence can be represented easily in modal logics as $\Diamond(\Box p \land \neg p)$ if the term "half certainty" is replaced by "complete certainty".

By the analogy between necessity (respectively possibility) measure and the modal operator $\Box$ (respectively $\Diamond$) indicated in [16], it is reasonable to express the sentence as

$$\left(\left(\left(p \; N \; \tfrac{1}{2}\right) \land \neg p\right) \; \Pi \; 1\right).$$

However, since PL does not allow the nested use of necessity and possibility operators, it is illegal in PL. Although there is no essential difficulty in generalizing the syntax of PL to cover these cases, the semantics of PL seems somewhat restrictive. Since only a single possibility distribution is associated with a PL model, the semantics corresponds to the so-called absolute semantics in [38]. It can be easily checked that the wff so represented is unsatisfiable in the absolute semantics, though the situation described by the sentence is intuitively possible.

However, since possibility distributions reflect the epistemic states of some agent, just as the accessibility relations in modal logic do, it is very likely that in different possible worlds, the agent has different epistemic states. Thus, we can associate with each possible world a possibility distribution independently. Then we can get a kind of variable semantics [38]. This motivates the proposal of quantitative modal logic (QML).

In QML, to represent nested necessity and possibility measures, we adopt a less cumbersome notation that is compatible with modal operators. In fact, QML can be viewed as a logic with multimodal operators. We add to propositional logic two classes of modal operators $[c]$ and $[c]^+$ for all $c \in [0, 1]$ and the formation rule:

- if f is a wff, then $[c]f$ and $[c]^+f$ are, too.

We also abbreviate $\neg[1 - c]\neg f$ and $\neg[1 - c]^+\neg f$ as $\langle c \rangle^+ f$ and $\langle c \rangle f$ respectively. The intuitive interpretation of $[c]f$ (respectively $[c]^+f$) is that an agent believes f with certainty at least (more than) c. Thus, the above-mentioned sentence can be translated into the following QML wff: $\langle 1 \rangle([\tfrac{1}{2}]p \land \neg p)$.

A model for QML is a triplet $\langle W, R, V \rangle$ just as in modal logic, but $R : W \times W \rightarrow [0, 1]$ is now a serial fuzzy relation on W. For each $w \in W$, a possibility distribution π_w can be defined as $\pi_w(u) = R(w, u)$ for all $u \in W$. Let N_w denote the necessity measure corresponding to π_w for each $w \in W$. Then the truth sets are

$$|[c]f| = \{w \mid N_w(f) \geqslant c\},$$

$$|[c]^+ f| = \{w \mid N_w(f) > c\}.$$

Let us call the resultant system QD, then the definitions of satisfiability, validity, and $S \models_{QD} f$ are analogous to those for D.

Though QML is motivated by an epistemic or doxastic interpretation of possibilistic reasoning, the system QD may be not sufficiently strong for reasoning about uncertain beliefs. Some further constraints on R, such as transitivity and symmetry may be imposed on R to reflect the properties of uncertain beliefs. However, the technical details are beyond the scope of the paper, so we will concentrate on the most basic QML system QD and refer the interested readers to [35].

2.3.4. Qualitative possibility logic

While PL and QML reason about the possibility and necessity degrees of the wffs, qualitative possibility logic (QPL) [19] concerns mainly the relative comparison of possibility measures between two wffs.

The syntax of QPL is an extension of a propositional language with a binary connective "$\geqslant$" and the following formation rule:

- if f and g are wffs, then $f \geqslant g$ is also a wff.

The wff "$f \geqslant g \wedge \neg(g \geqslant f)$" is abbreviated as "$f > g$".

Although QPL is proposed without accompanied formal semantics, the semantics for QML can be used here, too. Given such a model as above, let Π_w denote the possibility measure corresponding to π_w. Then

$$|f \geqslant g| = \{w \mid \Pi_w(f) \geqslant \Pi_w(g)\}.$$

The definition of satisfiability and validity follows directly, and we denote the consequence relation by $\models_{QPL}$.

3. The logic LCP

In the above-mentioned logics, QML can represent the possibility and necessity degrees of wffs directly, however the relative magnitude of possibility measures between two wffs cannot be expressed. Conversely, we cannot express the quantitative aspect of possibility theory in QPL. Sometimes, it may be useful to express the quantitative and qualitative information in a sentence. For example, $\langle 0.8 \rangle f \wedge \neg \langle 0.5 \rangle^+ g \supset f \geqslant g$ may be meaningful from a semantic viewpoint. However, neither QML nor QPL have the expressive power.

It is suggested that each wff in QPL can be translated into one in QML in [19]. In that paper, a multimodal logic PL_P is proposed based on a parameter set $P \subseteq [0, 1]$.

The syntax and semantics of PL_P is the same as those for QML except that the modal operators $[c]$ and $\langle c \rangle$ are restricted to $c \in P$. When P is finite, a translation scheme is suggested. Assume $P = \{c_1, c_2, \ldots, c_n\}$ such that $0 \leqslant c_1 < c_2 < \cdots < c_n \leqslant 1$, then Tr is a mapping from the wffs of QPL to those of PL_P such that $Tr(p) = p$ for all $p \in PV$,

$$Tr(f \geqslant g) = \bigwedge_{i=1}^{n} \langle c_i \rangle Tr(g) \supset \langle c_i \rangle Tr(f),$$

and Tr is homomorphic with respect to classical connectives. However, this translation scheme is not completely faithful from the viewpoint of our semantics. In fact, we can easily imagine a QPL model (or QML model since these two are the same) and a possible world such that $f \geqslant g$ is false in w, while $Tr(f \geqslant g)$ is true there. This can be achieved when there exists an i such that $c_{i+1} > \Pi_w(g) > \Pi_w(f) > c_i$. Thus, if we want to express the quantitative and qualitative information in a sentence, the two logics must be combined.

It is not too hard to put QPL and QML together since the two languages have the same semantics. We need only allow both $\geqslant$ and quantitative modal operators in our combined language and add both formation rules to those for propositional language, then the definition of QML (or QPL) models can be used for the semantics of the combined language. Though this is an easy solution to enhance the expressive powers of both languages, we can find an even more general language. To completely exploit the power of the QML semantics and conditional possibility theory, we can define a logic for conditional possibility. In this language, we cannot only express wffs of both QPL and QML, but also reason about the conditional possibility of some wff given another wff. The logic is first proposed in [36] and some preliminary results appear there.

3.1. Syntax and semantics

We need two types of conditional connectives: $\xrightarrow{|c|}$ and $\xrightarrow{|c|^+}$ for all $c \in [0,1]$. The additional formation rule is:
- if f and g are wffs and $c \in [0,1]$, then $f \xrightarrow{|c|} g$ and $f \xrightarrow{|c|^+} g$ are, too.
We also use the abbreviations:

$$f \xrightarrow{\langle c \rangle^+} g = \neg(f \xrightarrow{|1-c|} \neg g),$$

$$f \xrightarrow{\langle c \rangle} g = \neg(f \xrightarrow{|1-c|^+} \neg g).$$

The semantics for QML and QPL is also applicable to LCP. Let $M = \langle W, R, V \rangle$ be a QML model and π_w, Π_w and N_w denote the possibility distribution, possibility measure, and necessity measure associated with each world w respectively, then according to conditional possibility defined by Dempster's conditioning rule, we have

$$\Pi_w(B|A) = \sup_{u \in B} \pi_w(u|A),$$

$$N_w(B|A) = 1 - \Pi_w(\overline{B}|A),$$

where $A, B \subseteq W$. Then the truth sets are

$$|g \xrightarrow{|c|} h| = \{w \mid N_w(h|g) \geq c\},$$

$$|g \xrightarrow{|c|'} h| = \{w \mid N_w(h|g) > c\}.$$

In the next section, we will explore the relationship between LCP and the logics described above.

4. LCP as a uniform framework

To show that LCP is expressive enough for possibilistic reasoning, let us present some translation lemmas in this section. Let L_1 and L_2 be two logics with sets of wffs $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are constructed from the same set of propositional variables, then a translation mapping from L_1 to L_2 is $\tau : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ satisfying the following classical morphism conditions:

  (i) $\tau(p) = p$ if $p \in PV$,
  (ii) $\tau(\neg f) = \neg \tau(f)$, and
  (iii) $\tau(f \vee g) = \tau(f) \vee \tau(g)$.

Let $\tau(S)$ denote $\{\tau(f) \mid f \in S\}$ for all $S \subseteq \mathcal{L}_1$.

Recall that a model for conditional logic is called an s-model. In what follows, we will call the models for QML, QPL, or LCP p-models (for possibility models) and those for modal logic d-models (for the system D). Furthermore, a model for PL is called an a-model (for absolute models).

4.1. QML and LCP

The logic most closely related to LCP is QML, since the former is just a direct generalization of the latter to the conditional version. Let us consider the translation mapping τ_1 from QML to LCP. The mapping τ_1 satisfies, besides (i)-(iii), the following two requirements:

  (iv$_1$) $\tau_1([c]f) = \top \xrightarrow{|c|} \tau_1(f)$,
  (v$_1$) $\tau_1([c]^+ f) = \top \xrightarrow{|c|'} \tau_1(f)$.

Lemma 1. *Let M be any p-model, then for any wff f of QML, we have $|f|_M = |\tau_1(f)|_M$.*

Proof. Since the fuzzy accessibility relation for the p-model is serial, we have $\Pi_w(f) = \Pi_w(f|\top)$ for each possible world w. The result then follows from an induction on the structure of wffs. $\square$

Proposition 2. *Let S be a set of wffs in QML and f be a QML wff, then $S \models_{QD} f$ iff $\tau_1(S) \models_{LCP} \tau_1(f)$.*

4.2. QPL and LCP

Another logic having the same class of models with LCP is QPL. The translation mapping τ_2 from QPL to LCP satisfies the following requirement:

$\text{(iv}_2)\quad \tau_2(f \geqslant g) = (\tau_2(f) \vee \tau_2(g) \xrightarrow{(1)} \tau_2(f))$.

Lemma 3. *Let M be any p-model, then for any wff f of QPL, we have $|f|_M = |\tau_2(f)|_M$.*

Proof. By induction on the complexity of wffs. The only nonclassical case is $|f \geqslant g|$. By the induction hypothesis, we let $|f|_M = |\tau_2(f)|_M = A$ and $|g|_M = |\tau_2(g)|_M = B$. Then $w \in |f \geqslant g|$ iff $\Pi_w(A) \geqslant \Pi_w(B)$ iff $\Pi_w(A|A \cup B) = 1$ iff $w \in |\tau_2(f \geqslant g)|$. $\quad\square$

Proposition 4. *Let S be a set of wffs in QPL and f be a QPL wff, then $S \models_{QPL} f$ iff $\tau_2(S) \models_{LCP} \tau_1(f)$.*

4.3. QPL and conditional logic

Unlike QML and QPL, conditional logic has different semantic models with LCP. However, by a transformation between numerical and ordinal scales, we can transform these two kinds of models into each other.[4] Consequently, we can prove the translation results between QPL and conditional logic.

First, let us consider the translation mapping τ_3 from QPL to VN that satisfies the following requirement:

$\text{(iv}_3)\quad \tau_3(f \geqslant g) = (\tau_3(f) \vee \tau_3(g) \rightarrow \bot) \vee \neg(\tau_3(f) \vee \tau_3(g) \rightarrow \neg\tau_3(f))$.

Lemma 5.
  (1) *Let M be any finite p-model, then we can find an s-model M' such that for any wff f of QPL, we have $|f|_M = |\tau_3(f)|_{M'}$.*
  (2) *Let M be any finite s-model, then we can find a p-model M' such that for any wff f of QPL, we have $|f|_{M'} = |\tau_3(f)|_M$.*

Proof. (1) Assume $M = \langle W, R, V \rangle$, then $M' = \langle W, (\leqslant_w)_{w \in W}, V \rangle$, where for each $w \in W$, we define

$$W_w = \{u \mid R(w, u) > 0\}$$

and

$$x \leqslant_w y \quad \text{iff} \quad (x \notin W_w \wedge y \in W_w) \vee (x, y \in W_w \wedge R(w, x) \leqslant R(w, y)).$$

Then by the induction hypothesis, let $|f|_M = |\tau_3(f)|_{M'} = A$ and $|g|_M = |\tau_3(g)|_{M'} = B$. It can be verified that $\Pi_w(A) \geqslant \Pi_w(B)$ iff $\Pi_w(A \cup B) = 0$ or for all $u \in A \cup B$ and $R(w, u) > 0$, there exists v such that $R(w, v) \geqslant R(w, u)$ and $v \in A$ since M is *finite*. The first disjunct corresponds to the truth of $w \in |(\tau_3(f) \vee \tau_3(g) \rightarrow \bot)|_{M'}$ and the second to $w \in |\neg(\tau_3(g) \vee \tau_3(f) \rightarrow \neg\tau_3(f))|_{M'}$.

---

[4] More precisely, only finite models can be transformed from s-models into p-models.

(2) Assume $M = \langle W, (\leqslant_w)_{w\in W}, V\rangle$, we can define $M' = \langle W, R, V\rangle$ such that R satisfies the following three requirements:

(a) $R(w, u) = 0$ iff $u \notin W_w$,

(b) $R(w, u) = 1$ for all $\leqslant_w$-maximal elements u, and

(c) $R(w, u) \leqslant R(w, v)$ iff $u \leqslant_w v$.

Since W is finite, the required R exists, so the result follows from an analogous induction as above. $\square$

Proposition 6. *Let S be a finite set of QPL wffs and f be a QPL wff, then $S \models_{QPL} f$ iff $\tau_3(S) \models_{VN} \tau_3(f)$.*

Proof. By using the finite model properties of QPL and VN and the preceding lemma. Note the finite model property of VN can be found in [33, pp. 134–135], while that for QPL can be obtained from [37, Theorem 2] directly. $\square$

In [19], an axiomatic system for QPL is given and it is shown that the system is equivalent to an axiomatization of VN. More precisely, let $\vdash_{QPL}$ and $\vdash_{VN}$ denote the binary provability relations in the respective systems, then for any set S of QPL wffs and QPL wff f, $S \vdash_{QPL} f$ iff $\tau_3(S) \vdash_{VN} \tau_3(f)$. Since the completeness of VN axiomatization has been established [7,33], as a byproduct of Proposition 6, we have the completeness theorem of the QPL system with respect to the current semantics.

Proposition 7. *Let S be a finite set of QPL wffs and f be a QPL wff, then $S \models_{QPL} f$ iff $S \vdash_{QPL} f$.*

On the other hand, we can translate conditional wffs into QPL ones. The translation mapping τ_4 satisfies

(iv_4) $\tau_4(f \to g) = (\bot \geqslant \tau_4(f)) \vee (\tau_4(f) \wedge \tau_4(g) > \tau_4(f) \wedge \neg\tau_4(g))$.

By using the same model transformation as above, we can prove the following results.

Lemma 8.

(1) *Let M be any finite s-model, then we can find a p-model M' such that for any wff f of conditional logic, we have $|f|_M = |\tau_4(f)|_{M'}$.*

(2) *Let M be any finite p-model, then we can find an s-model M' such that for any wff f of conditional logic, we have $|f|_{M'} = |\tau_4(f)|_M$.*

Proposition 9. *Let S be a finite set of conditional logic formulas and f be a conditional logic formula, then $S \models_{VN} f$ iff $\tau_4(S) \models_{QPL} \tau_4(f)$.*

4.4. LCP and conditional logic

We can provide a translation mapping from conditional logic to LCP by the composition of τ_4 and τ_2. However, it is also possible to give a more concise translation mapping directly. Let τ_5 satisfy the following requirement:

(iv$_5$) $\tau_5(f \rightarrow g) = \neg(\top \xrightarrow{\langle 0 \rangle^i} \tau_5(f)) \vee (\tau_5(f) \xrightarrow{[0]^i} \tau_5(g))$.

Lemma 10.

(1) *Let M be any finite s-model, then we can find a p-model M' such that for any conditional logic wff f, we have $|f|_M = |\tau_5(f)|_{M'}$.*

(2) *Let M be any finite p-model, then we can find an s-model M' such that for any conditional logic wff f, we have $|f|_{M'} = |\tau_4(f)|_M$.*

Proof. The model transformations are the same as those used in the preceding subsection. We just note that for any sets $A, B \subseteq W$, $N_w(B|A) > 0$ iff $\Pi_w(B^c|A) < 1$ iff $\exists u \in A \cap W_w \forall v(u \leqslant_w v \Rightarrow v \in \overline{A} \cup B)$, and $\Pi_w(A) = 0$ iff $A \cap W_w = \emptyset$, where Π_w, W_w and $\leqslant_w$ are defined in the respective models. □

Proposition 11. *Let S be a finite set of wffs and f be a wff in conditional logic, then $S \models_{VN} f$ iff $\tau_5(S) \models_{LCP} \tau_5(f)$.*

Proof. The proof also depends on the preceding lemma and the finite model property for LCP. The latter can be obtained directly by the filtration technique introduced in [37]. □

4.5. Modal and conditional logics

The relationship between modal and conditional logic has been explored by Lewis [33]. He shows that the modal operators of the system D can be translated into the inner or outer modalities of VN.

For the outer translation mapping, let τ_6^o satisfy the following condition:

(iv$_6^o$) $\tau_6^o(\Box f) = \neg\tau_6^o(f) \rightarrow \bot$.

For the inner translation mapping, let τ_6^i satisfy the following condition:

(iv$_6^i$) $\tau_6^i(\Box f) = \top \rightarrow \tau_6^i(f)$.

Let $M = \langle W, R, V \rangle$ be a d-model, then a corresponding s-model can be easily constructed as $M' = \langle W, (\leqslant_w)_{w \in W}, V \rangle$, where for all $x, y, w \in W$, $x \leqslant_w y$ iff $(w, y) \in R$. By induction, we have $|f|_M = |\tau_6^o(f)|_{M'} = |\tau_6^i(f)|_{M'}$ for any modal logic wffs.

On the other hand, if $M = \langle W, (\leqslant_w)_{w \in W}, V \rangle$ is an s-model, then we can construct the outer d-model $M^o = \langle W, R, V \rangle$ such that $(w, u) \in R$ iff $u \in W_w$ since $W_w \neq \emptyset$. Moreover, if W is finite, we can have the inner d-model $M^i = \langle W, R, V \rangle$ such that $(w, u) \in R$ iff u is a $\leqslant_w$-maximal element of W. Then, for every modal logic wff f, we have $|f|_{M^o} = |\tau_6^o(f)|_M$ and $|f|_{M^i} = |\tau_6^i(f)|_M$.

Since the system D has also the finite model property, the following result holds.

Proposition 12. *Let S be a set of modal logic wffs and f be a modal logic wff, then $S \models_D f$ iff $\tau_6^o(S) \models_{VN} \tau_6^o(f)$. Furthermore, if S is finite, then $S \models_D f$ iff $\tau_6^i(S) \models_{VN} \tau_6^i(f)$.*

4.6. Modal logic and QML

There are two classes of modal operators in QML, so the necessity modality of modal logic can be represented in either way. First, for each $c > 0$, we can have a translation mapping τ_7^c satisfying the following requirement:

(iv_7^c) $\quad \tau_7^c(\Box f) = [c]\tau_7^c(f)$.

Since a d-model can be considered as a p-model by viewing a crisp relation as a fuzzy one, we have $|f|_M = |\tau_7^c(f)|_M$ for any modal wff f and d-model M. On the other hand, we have

Lemma 13. *For each p-model $M = \langle W, R, V \rangle$, we can find a d-model $M' = \langle W, R', V \rangle$ such that $|f|_{M'} = |\tau_7^c(f)|_M$ for all modal wffs f.*

Proof. Let R' be such that $(u, v) \in R'$ iff $R(u, v) > 1 - c$. Then, by induction, we have $w \in |\Box f|_{M'}$ iff $\forall u(R(w, u) > 1 - c \Rightarrow u \in |\tau_7^c(f)|_M)$ iff $\Pi_w(|\neg\tau_7^c(f)|_M) \leqslant 1 - c$ iff $w \in |[c]\tau_7^c(f)|_M$. $\quad \Box$

Second, for each $c \geqslant 0$, the translation mapping τ_7^{c+} satisfies the following requirement:

(iv_7^{c+}) $\quad \tau_7^{c+}(\Box f) = \lfloor c \rfloor^+\tau_7^{c+}(f)$.

Lemma 14. *For each finite p-model $M = \langle W, R, V \rangle$, we can find a d-model $M' = \langle W, R', V \rangle$ such that $|f|_{M'} = |\tau_7^{c+}(f)|_M$ for all modal wffs f.*

Proof. Let R' be such that $(u, v) \in R'$ iff $R(u, v) \geqslant 1 - c$. Then, by induction, we have $w \in |\Box f|_{M'}$ iff $\forall u(R(w, u) \geqslant 1 - c \Rightarrow u \in |\tau_7^{c+}(f)|_M)$ iff (using the finitary assumption here) $\Pi_w(|\neg\tau_7^{c+}(f)|_M) < 1 - c$ iff $w \in |[c]^+\tau_7^{c+}(f)|_M$. $\quad \Box$

Finally, we have

Proposition 15. *Let S be a set of modal logic wffs and f be a modal logic wff, then $S \models_D f$ iff $\tau_7^c(S) \models_{QD} \tau_7^c(f)$ and if S is finite, then $S \models_D f$ iff $\tau_7^{c+}(S) \models_{QD} \tau_7^{c+}(f)$.*

4.7. PL and QML

It is pointed in [34] that QML is a common generalization of PL and modal logic. The preceding subsection shows the translation mappings from modal logic to QML. In this subsection, a translation mapping from PL to QML will be given. However, because the syntax of PL is rather restrictive,[5] we can define the translation mapping directly without the need of conditions (i)–(iii).
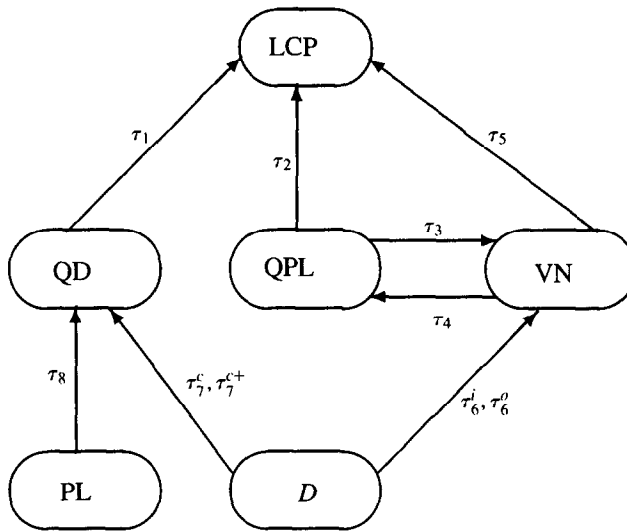
---

[5] In fact, it contains only wffs of modal degree 1.

Fig. 1. Relationships between different logics.

Define τ_8 such that $\tau_8((f \, N \, c)) = [c]f$ and $\tau_8((f \, \Pi \, c)) = \langle c \rangle f$. Then it is easy to prove the following result.

Proposition 16. *Let S be a set of PL wffs and f be a PL wff, then $S \models_{PL} f$ iff $\tau_8(S) \models_{QD} \tau_8(f)$.*

4.8. Summary

The results presented in this section may be summarized in Fig. 1.

Each node of the graph is labeled by a kind of logic, and each arrow between two nodes represents a (class of) translation mapping(s) between them. Since the node labeled LCP is reachable from all nodes of the graph, many important logics for reasoning about possibility and necessity can be represented in the LCP framework. In other words, LCP is the most expressive one among all logics described in this paper.

Now, we will consider some examples that utilize the expressive power of LCP. Since we have shown that all the above-mentioned logics can be expressed in LCP, we will freely use the wffs of all these logics in our examples.

Example 17. Let us consider the following knowledge base:

> *If John's height is beyond 180cm, then it is very possible that he is in the basketball team.*
> *It is quite possible that John's height is beyond 180cm.*

Let p and q denote "John's height is beyond 180cm" and "John is in the basketball team", respectively and assume that c and d are such that $1 \geqslant c > d > 0$. Then we can have at least three different representations of the knowledge base in LCP.

First, we can represent it as $\{\langle c\rangle(p \supset q), \langle d\rangle p\}$. In fact, the representation uses only the expressive power of PL. Second, we can use a slight extension of PL to encode it as $\{p \supset \langle c\rangle q, \langle d\rangle p\}$. From the two representations, we can derive only the trivial information about the possibility of q, i.e., $\langle 0\rangle q$. However, if we encode it as $\{p \xrightarrow{\langle c\rangle} q, \langle d\rangle p\}$ by utilizing the full expressive power of LCP, then we can derive the quite reasonable result, $\langle c \cdot d\rangle q$, i.e., it is fairly possible that John is in the basketball team, since $\Pi(q) \geqslant \Pi(p \wedge q) \geqslant c \cdot \Pi(p) \geqslant c \cdot d$.

Example 18. Continuing the last example, if we have now the additional information

It is more possible that Peter is in the basketball team than that John is,

then we can add a QPL wff $r \geqslant q$ to our knowledge base where r means that Peter is in the basketball team, and the result $\langle c \cdot d\rangle r$ is also derivable.

Example 19. Let p and q denote "Smoking causes lung cancer" and "John will give up smoking" respectively. We will give our modal operators $[c]$ a doxastic interpretation, so the fuzzy relation on the semantics of LCP will be imposed at least the transitivity constraint. This induces that the following positive introspection schema is valid,

$$[c]p \supset [c][c]p.$$

Thus, when $c \approx 1$, $[c]p$ means that John believes p very certainly. Assume normally, John will take care of himself. This fact can be reflected as $[c]p \xrightarrow{[0]^+} q$ or simply $[c]p \to q$. Assume further John is indeed very certain about p, so we have the knowledge base $\{[c]p \xrightarrow{[0]^+} q, [c]p\}$. From this we can derive $[0]^+q$, i.e., John is somewhat certain he will give up smoking. The derivation process is roughly as follows.

$$
\left\{
\begin{array}{lll}
1. & [c]p \xrightarrow{[0]^+} q & \text{assumption,} \\
2. & [c]p & \text{assumption,} \\
3. & [c]p \supset [c][c]p & \text{axiom,} \\
4. & [c]p \supset [0]^+[c]p & 3, \text{QD,} \\
5. & [c]p \supset \langle 1\rangle[c]p & 4, \text{QD,} \\
6. & \langle 1\rangle[c]p & 2, 5, \text{MP,} \\
7. & [0]^+([c]p \supset q) & 1, 6, \text{LCP,} \\
8. & [0]^+[c]p \supset [0]^+q & 7, \text{QD,} \\
9. & [0]^+q & 2, 4, 8, \text{MP,}
\end{array}
\right.
$$

where MP is modus ponens and QD and LCP represent the deductions in QD and LCP respectively.

These examples show the general applications of LCP. In addition to these, we will investigate two particular applications of LCP in the next section.

5. Applications and generalization

In this section, we consider two applications of our framework and its generalization to handing of inconsistent information.

5.1. Formulation of nonmonotonic inference relations

Since the pioneering works of Gabbay [21] was published, there have been vast amounts of literatures on the topic about the properties of general nonmonotonic inference relations. One of the most important among them is the work by Kraus et al. [29,31]. They introduce the following properties for a nonmonotonic inference relation $\mid\!\sim$. Let f, g, h be wffs of the propositional language $\mathcal{L}$ throughout this and the next subsections.

$$
\begin{cases}
\text{(R)} & f \mid\!\sim f, \\[2mm]
\text{(RW)} & \dfrac{\models f \supset g, h \mid\!\sim f}{h \mid\!\sim g}, \\[4mm]
\text{(AND)} & \dfrac{f \mid\!\sim g, f \mid\!\sim h}{f \mid\!\sim g \wedge h}, \\[4mm]
\text{(RM)} & \dfrac{f \mid\!\sim h, f \not\mid\!\sim \neg g}{f \wedge g \mid\!\sim h}, \\[4mm]
\text{(LLE)} & \dfrac{\models f \equiv g, f \mid\!\sim h}{g \mid\!\sim h}, \\[4mm]
\text{(CM)} & \dfrac{f \mid\!\sim g, f \mid\!\sim h}{f \wedge g \mid\!\sim h}, \\[4mm]
\text{(OR)} & \dfrac{f \mid\!\sim h, g \mid\!\sim h}{f \vee g \mid\!\sim h}.
\end{cases}
$$

The system is called $\mathcal{R}$. Let us call $f \mid\!\sim g$ a positive sequent (p-sequent) and $f \not\mid\!\sim g$ a negative sequent (n-sequent), and assume that S is a set of sequents and A is a p-sequent, then we write $S \vdash_{\mathcal{R}} A$ iff there is a derivation[6] of A from S by the axiom and inference rules in $\mathcal{R}$.

In [11], a general translation scheme from nonmonotonic inference relations to conditional logics is provided. According to the scheme, denoted by τ here, $\tau(f \mid\!\sim g) = f \rightarrow g$ and $\tau(f \not\mid\!\sim g) = \neg(f \rightarrow g)$. The wffs of the form $f \rightarrow g$ are called conditional atoms and conditional literals are conditional atoms or their negations. Then $\tau(S)$ is defined as the set of conditional literals translated from the sequents in S by τ. According to the results in [1], if S is a set of sequents and A is a p-sequent, then $S \vdash_{\mathcal{R}} A$ iff $\tau(S) \models_V \tau(A)$ iff $\tau(S) \models_{VW} \tau(A)$ iff $\tau(S) \models_{VTA} \tau(A)$, where V, VW, and VTA are all conditional systems introduced in [33]. Since VN is an intermediary logic between V and VW [33, p. 131], we have the following result.

---

[6] The notion of derivation is the same as that for classical Gentzen sequent calculus.

Proposition 20. *If S is a set of sequents and A is a p-sequent, then $S \vdash_{\mathcal{R}} A$ iff $\tau(S) \models_{VN} \tau(A)$ iff $\tau_5 \circ \tau(S) \models_{LCP} \tau_5 \circ \tau(A)$.*

In the above formulation of nonmonotonic reasoning, a conditional $f \rightarrow g$ is considered as a default, read as "Typically, f is g", and all defaults are supposed to have the same degree of strength. Thus the result in the last proposition is also applied to the 0-entailment of Pearl's system Z [2]. However, the quantitative aspect of LCP allows us to distinguish the different degrees of strength for defaults. This corresponds exactly to Goldszmidt and Pearl's system Z^+ [24].

In [24], a default is of the form $(f \rightsquigarrow g, n)$, where $f, g \in \mathcal{L}$ and n is a *positive* integer.[7] Note that we restrict n to be a positive integer instead of just a nonnegative one as in [24] because this will induce the existence of the least specific a-model for a set of defaults in the following presentation. Let $\Delta = \{(f_i \rightsquigarrow g_i, n_i)\}$ be a set of defaults and assume the base language $\mathcal{L}$ is finite. An interpretation ω is said to verify $f_i \rightsquigarrow g_i$ if $\omega \models f_i \wedge g_i$, to falsify it if $\omega \models f_i \wedge \neg g_i$, and to satisfy it if $\omega \models f_i \supset g_i$. Now, a ranking function is an assignment of nonnegative integers to the interpretations of $\mathcal{L}$. A ranking function κ is said to be admissible relative to Δ if it satisfies

$$\min\{\kappa(\omega) \mid \omega \models f_i \wedge g_i\} + n_i \leqslant \min\{\kappa(\omega) \mid \omega \models f_i \wedge \neg g_i\},$$

for every $(f_i \rightsquigarrow g_i, n_i) \in \Delta$. Note that since n_i is positive, we can use $\leqslant$ instead of $<$ in the above inequality. A set Δ is consistent if there exists an admissible κ for Δ. It is shown that if Δ is consistent, then the following mutual recursive equations give the minimum admissible ranking κ^+ for Δ.

Definition 21 (*see* [24]). Define $\kappa^+(\omega) = 0$ if ω does not falsify any rule in Δ, and otherwise,

$$\kappa^+(\omega) = \max\{Z^+(r_i) \mid \omega \models f_i \wedge \neg g_i\},$$

$$Z^+(r_i) = \min\{\kappa^+(\omega) \mid \omega \models f_i \wedge g_i\} + n_i$$

where $r_i = (f_i \rightsquigarrow g_i, n_i) \in \Delta$.

There is a slight difference between the present definition and the original one, where $\kappa^+(\omega) = \max\{Z^+(r_i) \mid \omega \models f_i \wedge \neg g_i\} + 1$. We can use the present definition because all n_i are positive. We can now define the 0- and 1-entailment for the Z^+ system.

Definition 22. Let Δ be a set of defaults and $(f \rightsquigarrow g, n)$ be a default.
  (1) Δ 0-entails $(f \rightsquigarrow g, n)$, denoted by $\Delta \models_0 (f \rightsquigarrow g, n)$, iff all ranking functions admissible to Δ are also admissible to $\{(f \rightsquigarrow g, n)\}$.
  (2) Δ 1-entails $(f \rightsquigarrow g, n)$, denoted by $\Delta \models_1 (f \rightsquigarrow g, n)$, iff the minimum admissible ranking function of Δ is admissible to $\{(f \rightsquigarrow g, n)\}$.

---

[7] The original notation of a default used in [24] is $f \xrightarrow{n} g$. We change it to avoid the confusion with our LCP operators.

A ranking function κ is in fact the ordinal conditional function (OCF) defined in [42]. Based on the known connection between OCF and possibility theory established in [17], we can obtain the correspondence between the Z^+ system and LCP. First, each default $r = (f \rightsquigarrow g, n)$ is translated into $\tau(r) = f \xrightarrow{[c]} g$ where $c = 1 - 2^{-n}$. Note $c > 0$ because n is positive. Second, for each ranking function κ, we can find a possibility distribution π_κ on the set of interpretations Ω such that $\pi_\kappa(\omega) = 2^{-\kappa(\omega)}$. Third, the admissibility condition for κ and r corresponds exactly to

$$\Pi_\kappa(f \wedge \neg g) \leqslant (1 - c) \cdot \Pi_\kappa(f \wedge g).$$

This means that κ is admissible for Δ iff π_κ is a PL2 model for $\tau(\Delta) =_{\text{def}} \{\tau(r) \mid r \in \Delta\}$. Forth, if π_1 and π_2 are two possibility distributions on Ω, we say that π_1 is more specific than π_2 if $\pi_1(\omega) \leqslant \pi_2(\omega)$ for all $\omega \in \Omega$. Let $S = \{f_i \xrightarrow{[c_i]} g_i \mid c_i \in (0, 1], f_i, g_i \in \mathcal{L}\}$, then it can readily be shown that the following π^+ is the least specific possibility distribution satisfying S, if S is satisfiable, by using the same technique as in [24] and our translation.

Definition 23. Define $\pi^+(\omega) = 1$ if for each i, $\omega \not\models f_i \wedge \neg g_i$ and otherwise,

$$\pi^+(\omega) = \min\{C^+(r_i) \mid \omega \models f_i \wedge \neg g_i\},$$

$$C^+(r_i) = \max\{\pi^+(\omega) \mid \omega \models f_i \wedge g_i\} \cdot (1 - c_i)$$

where $r_i = f_i \xrightarrow{[c_i]} g_i$.

Lemma 24. *Let Δ be a set of defaults, then κ is the minimal admissible ranking for Δ iff π_κ is the least specific possibility distribution satisfying $\tau(\Delta)$.*

Proof. This follows from the facts that
(1) κ^+ is the minimum admissible ranking for Δ [24],
(2) π^+ is the least specific possibility distribution satisfying $\tau(\Delta)$, and
(3) π_{κ^+} is the solution of Definition 23, so is equal to π^+. $\square$

Finally, we note that for flat conditionals, the LCP semantics are equivalent to PL semantics. For two PL models $M_1 = \langle W_1, \pi_1, V_1 \rangle$ and $M_2 = \langle W_2, \pi_2, V_2 \rangle$, M_1 is said to be more specific than M_2, written as $M_1 \sqsubseteq M_2$, if for all $w_1 \in W_1$ there exists $w_2 \in W_2$ such that $V_1(w_1) = V_2(w_2)$ and $\pi_1(w_1) \leqslant \pi_2(w_2)$. Let

$$S = \{f_i \xrightarrow{[c_i]} g_i \mid c_i \in (0, 1], f_i, g_i \in \mathcal{L}\},$$

then we write $S \models_\sqsubseteq f \xrightarrow{[c]} g$ if all $\sqsubseteq$-maximal PL models satisfying S are also models of $f \xrightarrow{[c]} g$, where $f, g \in \mathcal{L}$. Note the $\sqsubseteq$-maximal PL models satisfying S may not be unique. However, this does not matter because $M^+ = \langle \Omega, \pi^+, V \rangle$ is a $\sqsubseteq$-maximal PL model such that $M^+ \models f \xrightarrow{[c]} g$ iff $M \models f \xrightarrow{[c]} g$ for any $\sqsubseteq$-maximal PL model M, $f, g \in \mathcal{L}$ and $c > 0$. Then we have the following result.

Proposition 25. *Let Δ be a set of defaults, $(f \rightsquigarrow g, n)$ be a default, and $c = 1 - 2^{-n}$, then*

(1) $\Delta \models_0 (f \rightsquigarrow g, n)$ *iff* $\tau(\Delta) \models_{1,CP} f \xrightarrow{|c|} g$.

(2) $\Delta \models_1 (f \rightsquigarrow g, n)$ *iff* $\tau(\Delta) \models_C f \xrightarrow{|c|} g$ *iff* $M^+ \models f \xrightarrow{|c|} g$, *where M^+ arises from Definition 23 w.r.t. $\tau(\Delta)$.*

We employ only the flat conditionals in the above formulation. Since for flat conditionals, the absolute semantics and the variable one coincide, we do not fully utilize the expressive power of LCP. However, LCP indeed facilitates a more complex nonmonotonic reasoning scheme. In particular, we can represent defaults about uncertain beliefs. For example, the following sentence:

> *Typically, an agent very certainly believing the existence of God will quite certainly believe the Biblical words are true.*

may be encoded in LCP as a wff of the form $|c|p \rightarrow |d|q$ with $c > d$. In addition, when the fuzzy relation R in the semantics of LCP is a similarity relation, the quantitative aspect of LCP makes it easy to do similarity-based reasoning; we will discuss the topic in the next subsection.

5.2. Formulation of similarity-based consequence relations

A fuzzy relation R on X is called a similarity relation iff it satisfies the following properties:

(1) Reflexivity: $R(x, x) = 1$ for all $x \in X$.

(2) Symmetry: $R(x, y) = R(y, x)$ for all $x, y \in X$.

(3) $\otimes$-transitivity: $R(x, y) \otimes R(y, z) \leqslant R(x, z)$ for all $x, y, x \in X$, where $\otimes$ is a t-norm[8] in $[0, 1]$.

The reasoning based on the similarity relation is first proposed by Ruspini [40]. In a recent article, Dubois et al. [12] define three types of similarity-based consequence relations. Their work is based on the propositional logic $\mathcal{L}$. First, let Ω denote the set of all propositional interpretations of $\mathcal{L}$ and define a similarity relation R on Ω. Then any wff $f \in \mathcal{L}$ is blurred into a fuzzy proposition f^* such that the characteristic function $\mu_{f^*} : \Omega \rightarrow [0, 1]$ is defined as

$$\mu_{f^*}(\omega) = \sup\{R(\omega, \omega') \mid \omega' \models f\}.$$

Furthermore, if $f, g \in \mathcal{L}$, then another fuzzy proposition $f^* \Rightarrow g^*$ is characterized by

$$\mu_{f^* \Rightarrow g^*}(\omega) = \mu_{f^*}(\omega) \otimes\rightarrow \mu_{g^*}(\omega),$$

where $\otimes\rightarrow$ is the residuated implication w.r.t. $\otimes$ defined as $a \otimes\rightarrow b = \sup\{x \mid a \otimes x \leqslant b\}$. If A is a fuzzy proposition, then $[A]$ denotes the fuzzy subset of Ω-characterized by μ_A and $[A]_c$ is the c-cut of $[A]$ for $c \in [0, 1]$.

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[8] $\otimes : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a t-norm iff it is associative, commutative, and increasing in both places, and $1 \otimes a = a$ and $0 \otimes a = 0$ for all $a \in [0, 1]$.

Let $K \subseteq \mathcal{L}$ denote the background knowledge. We will identify K with $\bigwedge K$, i.e., the conjunction of all wffs in K. Then three types of graded entailment relations are defined as

(1) type I:

$$f \models_1^{K,c} g \quad \text{iff} \quad |K| \cap |f| \subseteq [g^*]_c,$$

(2) type II:

$$f \models_2^{K,c} g \quad \text{iff} \quad |K| \subseteq [f^* \Rightarrow g^*]_c,$$

(3) type III:

$$f \models_3^{K,c} g \quad \text{iff} \quad |K| \subseteq [f^* \Rightarrow (f \wedge g)^*]_c,$$

for all $f, g \in \mathcal{L}$ and $c \in [0,1]$. Note that all types of graded consequence relations rely implicitly on a given similarity relation R. Apparently, a similarity relation R corresponds to a special p-model $M_R = \langle \Omega, R, V \rangle$, where the set of possible worlds are identified with Ω and V assigns truth values in an obvious way. In our previous terms,

$$\mu_{f^*}(\omega) = \Pi_\omega(f),$$

$$\mu_{f^* \Rightarrow g^*}(\omega) = \Pi_\omega(f) \otimes \rightarrow \Pi_\omega(g),$$

so we have the following results when $\otimes = \min$:

(1) $f \models_1^{K,c} g$ iff $M_R \models K \wedge f \supset \langle c \rangle g$,
(2) $f \models_2^{K,c} g$ iff $M_R \models K \supset (f \leqslant g \vee \langle c \rangle g)$, and
(3) $f \models_3^{K,c} g$ iff $M_R \models K \supset ((f \leqslant f \wedge g) \vee \langle c \rangle (f \wedge g))$.

Moreover, if $\otimes = \cdot$ (i.e., the numerical product), then

$$f \models_3^{K,c} g \quad \text{iff} \quad M_R \models K \supset \left(f \leqslant \perp \vee f \xrightarrow{\langle c \rangle} g \right).$$

Thus all three types of similarity-based consequence relations can be formulated in LCP as wffs valid in a special model. However, the main advantage of logics is that we can do reasoning without involving a particular model. For example, in [12, Theorem 1], it is claimed that $\models_1^{K,1}$ is just the classical logic consequence relation. However, the claim is just wrong when R is the universal relation (i.e., $R(\omega, \omega) = 1$ for all $\omega \in \Omega$). Therefore, by using object level reasoning in LCP, we can reformulate the generic graded entailment relations. Let $Q5(\otimes)$ denote the class of all p-models having $\otimes$-similarity accessibility relations, S be a set of wffs in LCP and A be a wff in LCP, then $S \models_{Q5(\otimes)} A$ iff for all p-models $M \in Q5(\otimes)$, $M \models S$ implies $M \models A$. The system $Q5(\min)$ is just abbreviated as Q5.

Definition 26. Let $K \subseteq \mathcal{L}$, $f, g \in \mathcal{L}$, and $c \in [0,1]$, then

(1) $f \mathrel{\vfrac{\approx}{}}_1^{K,c} g$ iff $K \models_{Q5} f \supset \langle c \rangle g$,
(2) $f \mathrel{\vfrac{\approx}{}}_2^{K,c} g$ iff $K \models_{Q5} (f \leqslant g \vee \langle c \rangle g)$, and

(3) $f \approx_3^{K,c} g$ iff $K \models_{Q5} (((f \leqslant f \wedge g) \vee \langle c \rangle (f \wedge g)))$,

are respectively the type I, II, and III generic graded entailment relations.

Though the definition is only for the similarity relations based on the t-norm min, the semantics of LCP allows us to generalize the definition to any t-norm. However, the development of the general logics will be left as further research.

We observe that the graded entailment relations defined above do not utilize the expressive power of nested modalities. However, some axioms of the Q5($\otimes$) system involving nested use of modalities indeed reflect the properties of similarity relations. Intuitively, $w \models \langle c \rangle f$ means that the world w is similar to the f-worlds at least to the extent c (or in short, w is *c-similar* to f-worlds). Dually, $w \models [c] f$ means w is discernible with $\neg f$-worlds to the extent c or w is *c-characterized* by f.

Now, three characteristic axioms of Q5($\otimes$) correspond to the three properties of similarity relations. First, the schema T

$$f \supset \langle 1 \rangle f$$

says that a world satisfying f is completely similar to f-worlds. This reflects the reflexivity. Second, the schema 4

$$\langle c \rangle \langle d \rangle f \supset \langle c \otimes d \rangle f$$

corresponds to transitivity, that says a world c-similar to worlds d-similar to f-worlds is itself $c \otimes d$-similar to f-worlds. Finally, the schema B for symmetry

$$f \supset [c] \langle 1 - c \rangle^+ f$$

means that if a world satisfies f, then we can c-discernible it from those worlds that are not $(1 - c)$-similar to f-worlds. Putting it in more qualitative terms, this means that if f is true in a world, then it is strongly discernible from those worlds only little similar to f-worlds.

This shows that the epistemic and similarity interpretations of LCP wffs are just two sides of a coin. In fact, when R is viewed as an ordinary fuzzy relation, it is the similarity relation between worlds, whereas when it is viewed as a collection of possibility distributions $\{\pi_w \mid w \in W\}$, it indeed reflects the epistemic possibilities of an agent. Thus, under the semantics, the more similar to f-worlds a world is, the more possible an agent consider f is true. Consequently, the semantics of LCP allows us to do epistemic uncertain reasoning and the similarity-based one in the same framework.

Another point we would like to consider is the difference between possible worlds and interpretations. We have mentioned the significance of the difference in Section 2.3.2, however, the similarity interpretation provide a concrete example to illustrate it.

Example 27. Consider a statement $p = $ "X is A" in the interpolative reasoning example of [12], where X is a variable; take its value in a domain U. Assume that U is infinite, e.g., the positive real number and that A is a subset of U. Then we have only two propositional interpretations, $\{p\}$ or $\{\neg p\}$. However, we may have infinitely many possible worlds, each corresponding to a point in U, and the similarity between worlds

is decided not only by the truth value of p, but also by the distance of the two points. In other words, we may have two worlds in both of which p is false, but one is very close to p-worlds, while the other is very far from p-worlds. Henceforth, the similarity relation on Ω may be not sufficient for some real applications.

5.3. Handling partial consistency

We have mentioned previously that a recent development of possibilistic logic is the handling of partial inconsistency. Here, we will see how the general semantics can be assimilated into our framework. For the purely qualitative logics D and VN, we only need to drop the seriality and nonvacuity constraints on their models respectively. Consequently, we will use the modal logic K [9] and the conditional logic V [33]. For the other logics, let us first modify the definition of conditional possibility measures. Assume π is a (not necessarily normalized) possibility distribution on W, N and Π are the associated measures, and $l \in [0,1]$, then we can define

$$\Pi^l(A) = \max(\Pi(A), l),$$

$$\Pi^l(B|A) = \begin{cases} \dfrac{\Pi(A \cap B)}{\Pi^l(A)}, & \text{if } \Pi^l(A) \neq 0, \\ 1, & \text{otherwise,} \end{cases}$$

$$\hat{\Pi}^l(B|A) = \begin{cases} \dfrac{\Pi^l(A \cap B)}{\Pi^l(A)}, & \text{if } \Pi^l(A) \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

Let $N^l(B|A)$ denote $1 - \Pi^l(\overline{B}|A)$.

For possibilistic logic, we define an inconsistency-tolerant a-model (ita-model) as a tuple $M = \langle W, \pi, V, l \rangle$, where $\langle W, \pi, V \rangle$ is a (possibly subnormal) a-model and $l \in [0,1]$ is such that

$$\max \left(\sup_{w \in W} \pi(w), l \right) = 1,$$

and define

$$|(f \Pi c)| = \begin{cases} W, & \text{if } \Pi^l(f) \geq c, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Here, the number l plays the role of $\hat{\pi}(\omega_\perp)$ in the semantics with absurd interpretation.

For QML, QPL, and LCP, we define an inconsistency-tolerant p-model (itp-model) as a tuple $M = \langle W, R, V, L \rangle$, where $\langle W, R, V \rangle$ is a p-model but R may not be serial and $L : W \rightarrow [0,1]$ is a threshold level function such that $\max(\sup_{u \in W} R(w,u), L(w)) = 1$ for all $w \in W$. Then the syntax and semantics of the three logics are modified as follows.

(1) QML: two classes of new modal operators (c) and $(c)^+$ are introduced.

    (a) Syntax: the following formation rule is added
        • if f is a wff, then $(c)f$ and $(c)^+f$ are, too.

(b) Semantics: for the new wffs,

$$|(c)f| = \{w \mid \Pi^l_w(f) \geq c, l = L(w)\},$$

$$|(c)^+ f| = \{w \mid \Pi^l_w(f) > c, l = L(w)\}.$$

(2) QPL: the syntax remains unchanged and the truth set for $f \geq g$ is now defined as

$$|f \geq g| = \{w \mid \Pi^l_w(f) \geq \Pi^l_w(g), l = L(w)\}.$$

(3) LCP: two classes of new connectives $\xrightarrow{(c)}$ and $\xrightarrow{(c)^+}$ are introduced.
   (a) Syntax: we add the following formation rule
      • if f and g are wffs, then $f \xrightarrow{(c)} g$ and $f \xrightarrow{(c)^+} g$ are, too.
   (b) Semantics:

$$|f \xrightarrow{|c|} g| = \{w \mid N^l_w(g|f) \geq c, l = L(w)\},$$

$$|f \xrightarrow{|c|^+} g| = \{w \mid N^l_w(g|f) > c, l = L(w)\},$$

$$|f \xrightarrow{(c)} g| = \{w \mid \hat{\Pi}^l_w(g|f) \geq c, l = L(w)\},$$

$$|f \xrightarrow{(c)^+} g| = \{w \mid \hat{\Pi}^l_w(g|f) > c, l = L(w)\}.$$

As for the translation mappings, the respective changes are as follows. For τ_1, the following two additional conditions are imposed.

(vi$_1$) $\tau_1((c)f) = \top \xrightarrow{(c)} \tau_1(f)$.

(vii$_1$) $\tau_1((c)^+f) = \top \xrightarrow{(c)^+} \tau_1(f)$.

For τ_2, the condition (iv$_2$) is modified to

(iv$_2'$) $\tau_2(f \geq g) = \tau_2(f) \vee \tau_2(g) \xrightarrow{(c)} \tau_2(f)$.

For τ_5, the condition (iv$_5$) is modified to

(iv$_5'$) $\tau_5(f \rightarrow g) = \tau_5(f) \xrightarrow{(1)} \bot \vee \neg(\tau_5(f) \xrightarrow{(1)} \neg\tau_5(g))$.

The mapping τ_8 is modified such that $\tau_8((f \text{ } // \text{ } c)) = (c)f$. The other mappings all remain unchanged. Then the results in Section 4 still hold.

To understand how the extended systems are applied to reasoning with partially consistent information, let LCP* denote the extended LCP system and S be a set of wffs in LCP*. Then we can define the nontrivial deduction relation $\widehat{\models}$ as follows:

$$S \widehat{\models} f \xrightarrow{|c|} g \quad \text{iff} \quad S \models_{\text{LCP*}} f \xrightarrow{|c|} g \text{ and } S \not\models_{\text{LCP*}} f \xrightarrow{|c|} \bot.$$

The definitions of $S \widehat{\models} f \xrightarrow{|c|^+} g$, $S \widehat{\models} f \xrightarrow{(c)} g$, and $S \widehat{\models} f \xrightarrow{(c)^+} g$ are given analogously. In particular, when $f = \top$, the definition coincides with that proposed by Dubois et al. for PL [14, p. 466]. In other words, we can deduce $[c]g$ nontrivially only when $c > \sup\{d \mid S \models_{\text{LCP*}} [d]\bot\}$.

6. Related work and further research

There have been quite many works on the qualitative and quantitative aspects of possibilistic reasoning in the literature. Here, we can only touch upon very small parts of the works most closely related to the present one. [9]

In [2], instead of the conditional connective $\rightarrow$, a meta level nonmonotonic reasoning consequence $\models_\pi$ is defined with respect to a possibility distribution π. It is shown that $f \models_\pi g$ iff $\Pi(f \wedge g) > \Pi(f \wedge \neg g)$ iff $N(g|f) > 0$. The property is heavily used in the proof of our main lemma. Although their conditional necessity is defined by using the conditioning rule (1), the property holds as well for Dempster's rule. However, since $\models_\pi$ is a meta level construct, the nested conditional is not allowed. Thus, in $f \models_\pi g$, f and g are restricted to the classical wffs and the semantics for PL is sufficient for the interpretation of $f \models_\pi g$. Furthermore, the underlying propositional language is assumed to be finitary.

In [20], the finitary assumption of the underlying language is lifted, and a binary connective $\preceq$ is introduced, so the logic VA defined there is syntactically equivalent to QPL. However, their semantics is the absolute sphere model in [33]. That is, a model is a triplet $\langle W, \leqslant, V \rangle$, where $\leqslant$ is a preference relation on W. Since there is just one preference relation for the whole model instead of each world, the semantics is different with that presented here.

Both the papers cited in the preceding paragraphs are restricted to the discussion of the qualitative aspect of possibilistic reasoning, so the mechanisms described there cannot represent quantitative measures directly. On the other hand, another logic similar to QML is proposed in [8]. The logic is called lattice-based graded logic. In that logic, the modal operators $[c]$ are formed for all c from a lattice structure instead of the interval $[0, 1]$, so the possibility distribution π_w in the semantics for QML is generalized to L-fuzzy sets [23] in graded logic. The graded logic is more general than QML in some rough sense. [10] Though the graded logic cannot be fitted into the present framework completely since the semantics used here is restricted by using $[0, 1]$-valued possibility distributions, we can easily imagine how to generalize QPL to the lattice-based case. However, it is not clear yet how the Dempster rule can be generalized for the definition of lattice-based conditional possibility distributions because we lack the division operation in a lattice. Thus, how to incorporate the lattice-based multimodal logic into the present framework remains an interesting theoretical problem.

It is also interesting to compare the present framework with that proposed by Boutilier [3–6]. He introduces the logic CO whose syntax is the extension of propositional language by two modal operators $\Box$ and $\overleftarrow{\Box}$ and a CO model is a triplet $\langle W, R, V \rangle$, where R is a transitive and connected binary relation on W. The truth set of $\overleftarrow{\Box} f$ is defined by

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[9] It is suggested by an anonymous referee that our work is much connected to a paper by Dubois and Prade in [10]. However, unfortunately, we could not get the cited book in time when the final version of this paper is finished, so the comparison of their results with ours will not be included here.

[10] This is not exactly true since the lattice structure in graded logic is required to be finitely generated while the interval $[0, 1]$ is not.

$$\{w \mid \forall u((w,u) \notin R \Rightarrow u \in |f|)\}.$$

Since R is a ranked relation, it is apparent that a PL model $\langle W, R, V \rangle$ can be changed into a CO model by letting xRy iff $\pi(x) \leqslant \pi(y)$ for all $x, y \in W$, and vice versa for finite models. Thus the CO semantics is an absolute one. Furthermore, CO logic only allows qualitative wffs. In particular, $f \geqslant g$ in QPL can be written as $\overset{\leftrightarrow}{\Box} (g \supset \Diamond f)$, where $\overset{\leftrightarrow}{\Box} f$ denotes $\overset{\leftarrow}{\Box} f \wedge \Box f$.

An interesting development is that probability can be introduced into a CO model so that an alternative way to combining qualitative and quantitative approaches for uncertainty reasoning is provided [6]. The resultant model is called counterfactual probability model (CPM). A CPM model is a tuple $\langle W, \pi, V, \mu \rangle$ where $\langle W, \pi, V \rangle$ is just a PL model and μ is a probability measure on W. For any wff $f \in \mathcal{L}$, define $Pl(f)$ as the set of most possible f-worlds. That is, $w \in Pl_M(f)$ iff $w \models f$ and for all $u \in W$, if $u \models f$, then $\pi(w) \geqslant \pi(u)$. Then the counterfactual probability of g given f is defined as

$$P(g \uparrow f) = \frac{\mu(Pl(f) \cap |g|)}{\mu(Pl(f))}.$$

The difference between CPM logic and LCP is that possibility distributions serve only qualitative purpose in CPM while probability is not considered in LCP at all. Therefore, how to add probability to LCP is still an open question. This may be a combination of LCP models and those proposed by Fagin and Halpern for probabilistic reasoning [18].

In [25], a temporal logic essentially equivalent to CO, called MTL* is proposed, where the modal operators $\Box$ and $\overset{\leftrightarrow}{\Box}$ are replaced by H and $\hat{G}$ respectively. Thus, in the logic, $\hat{G}f$ means that in the present or future, f always holds, while Hf means that in the past, f always holds. A comparative structure, as they call it, is just a many-valued CO model written as $\langle W, \leqslant, V \rangle$, where $\leqslant$ is a ranked relation and V is a truth assignment $V : W \times PV \to \mathcal{T}$ with the truth value set $\mathcal{T} = \{0, 1/n, 2/n, \ldots, 1\}$. The truth valuation V is extended to the whole temporal language by the following equations:

$$V(w, Hf) = \inf_{u < w} V(u, f),$$

$$V(w, \hat{G}f) = \inf_{w \leqslant u} V(u, f).$$

The definition of QPL wff $f \geqslant g$ in terms of temporal operators is the same as above by using CO modal operators. Therefore, MTL* may be seen as a many-valued version of CO. Its semantics is thus absolute and qualitative in the reasoning about uncertainty. However, it possesses the capability of quantitative reasoning about vagueness or partial truth by its many-valued semantics. The addition of many-valued aspects to LCP will be a worthwhile research topic.

In [28], finite CO models are used to construct a modal interpretation for possibility theory. They consider a finite CO model $\langle M, R, V \rangle$ with the cardinality of W being n. Then for any $f \in \mathcal{L}$, the function $\Pi : \mathcal{L} \to [0, 1]$ defined as

$$\Pi_M(f) = \frac{\sharp(|\Diamond f|)}{n}$$

is shown to be a possibility measure and dually the function

$$N_M(f) = \frac{\sharp(|\,\square\,f|)}{n}$$

is a necessity measure, where $\sharp(A)$ denotes the cardinality of A. Indeed, since R is a ranked ordering, we can easily get a possibility distribution $\pi : W \rightarrow [0,1]$ by

$$\pi(w) = \frac{\sharp(\{u \mid R(u,w)\})}{n},$$

and it is clear that Π_M and N_M are the possibility and necessity measures associated with π. In fact, we use the result implicitly in proving the equivalence of QPL and VN, where we claim that a finite p-model can be constructed from a finite s-model (see Lemma 5(2)). The result here provides a concrete technique to such construction.

7. Conclusion

We have proposed a uniform logic that can reason about quantitative and qualitative uncertainty based on possibility theory. Instead of combining QPL and QD in a modular way, we use Dempster's conditioning rule to provide the semantics for the conditional necessity formulas so that we can also reason about the conditional possibility and necessity measures quantitatively. We show that sublogic relations hold between the different qualitative and quantitative logics appeared previously in the literatures. The general framework is then shown to be useful in formulating nonmonotonic and similarity-based consequence relations.

In this concluding section, we emphasize again that the shift of absolute semantics from possibilistic reasoning to LCP plays the key role in our work. The semantic shift facilitates the epistemic or doxastic interpretation of necessity measures and justifies the use of nested modalities. The use of nested modalities improves the expressive power of the original PL, as our examples show. This also makes it easy to represent defaults about uncertain beliefs when applied to nonmonotonic reasoning and to express the properties of similarity relations when applied to graded reasoning.

Because the main concern of this paper is a uniform semantic framework, we do not develop a proof theory for LCP. However, we exhibit an axiomatic system for LCP in the appendix. The soundness of the system can be readily constructed. However, the completeness is not established yet. The main difficulty lies in the infiniteness of the language. First, the compactness theorem has failed in QML [34]. We can easily find an infinite set of wffs that is unsatisfiable and each of its finite subsets is satisfiable, so we will only try to prove that $S \models f$ iff $S \vdash f$ when S is finite. Then the techniques for proving the completeness of conditional logic [7] can be adopted. However, the numerical characteristic of possibility distribution adds further complexity, so it may be necessary to combine the method provided in [18]. The details of the proof will be left for further research.

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Appendix A. An axiomatic system for LCP

This is a modification of Burgess' system [7]. In the following presentation, we will write $c \wedge d$ instead of $\min(c, d)$ for $c, d \in [0, 1]$.

(1) Axiom schemata:

    (a) All instances of propositional tautologies.

    (b) Inequality constraints:

        (i) Monotonicity:

$$f \xrightarrow{\ [c]\ } g \supset f \xrightarrow{\ [d]\ } g \text{ if } c > d.$$

        (ii) Dichotomy:

$$f \xrightarrow{\ [c]^-\ } g \supset f \xrightarrow{\ [c]\ } g.$$

        (iii) Boundary:

$$f \xrightarrow{\ [0]\ } g \wedge \neg \left(f \xrightarrow{\ [1]^-\ } g \right).$$

    (c) Reflexivity:

$$\langle 0 \rangle^+ f \supset \left(f \xrightarrow{\ [1]\ } f \right).$$

    (d) Right and:

$$\left(f \xrightarrow{\ [c]\ } g \right) \wedge \left(f \xrightarrow{\ [d]\ } h \right) \supset \left(f \xrightarrow{\ [c \wedge d]\ } g \wedge h \right),$$

$$\left(f \xrightarrow{\ [c]^-\ } g \right) \wedge \left(f \xrightarrow{\ [d]^-\ } h \right) \supset \left(f \xrightarrow{\ [c \wedge d]^-\ } g \wedge h \right).$$

    (e) Right weakening:

$$\left(f \xrightarrow{\ [c]\ } g \wedge h \right) \supset \left(f \xrightarrow{\ [c]\ } g \right),$$

$$\left(f \xrightarrow{\ [c]^-\ } g \wedge h \right) \supset \left(f \xrightarrow{\ [c]^-\ } g \right).$$

    (f) Rational monotony:

$$\left(f \xrightarrow{\ [0]^-\ } g \right) \wedge \left(f \xrightarrow{\ [c]\ } h \right) \supset \left(f \wedge g \xrightarrow{\ [c]\ } h \right).$$

$$\left(f \xrightarrow{\ |0|'\ } g\right) \wedge \left(f \xrightarrow{\ |c|'\ } h\right) \supset \left(f \wedge g \xrightarrow{\ |c|^+\ } h\right).$$

(g) Left or:

$$\left(f \xrightarrow{\ |c|\ } h\right) \wedge \left(g \xrightarrow{\ |d|\ } h\right) \supset \left(f \vee g \xrightarrow{\ |c \wedge d|\ } h\right),$$

$$\left(f \xrightarrow{\ |c|'\ } h\right) \wedge \left(g \xrightarrow{\ |d|'\ } h\right) \supset (f \vee g \xrightarrow{\ |c \wedge d|'\ } h).$$

(h) Dempster's conditioning:

$$\left(f \xrightarrow{\langle c \rangle} g\right) \wedge \left(f \wedge g \xrightarrow{\langle d \rangle} h\right) \supset \left(f \xrightarrow{\langle c \cdot d \rangle} g \wedge h\right),$$

$$\left(f \xrightarrow{\langle c \rangle'} g\right) \wedge \left(f \wedge g \xrightarrow{\langle d \rangle} h\right) \supset \left(f \xrightarrow{\langle c \cdot d \rangle^+} g \wedge h\right),$$

$$\left(f \xrightarrow{\langle c \rangle} g\right) \wedge \left(f \wedge g \xrightarrow{\langle d \rangle'} h\right) \supset \left(f \xrightarrow{\langle c \cdot d \rangle^+} g \wedge h\right).$$

(i) Nonvacuity:

$$\langle 1 \rangle \top.$$

(2) Inference rules:

(a) MP:

$$\frac{f \quad f \supset g}{g}.$$

(b) RPE:

$$\frac{f \quad g \equiv h}{f(h/g)},$$

where $f(h/g)$ is a result of replacing some subformulas of f of form g by h.

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