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Robust Performance Assessment of Feedback Control Systems

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Abstract. The proper measure of closed-loop performance variation in the presence of model-plant mismatch is discussed in this paper. A *generalized closed-loop error transfer function*, which is a special representation of the dual Youla parameter and has a close relationship with the pointwise ν-gap metric, is proposed as the suitable means of representing closed-loop performance variation in case of plant perturbation, and the closed loop performance variation measure is accordingly defined as its maximum singular value frequency by frequency. It is shown that this measure is essential and informative in characterizing closed-loop performance variation. This measure is also shown to be readily applicable to on-line closed loop performance assessment or monitoring, even without the explicit model of the plant. Its variant, defined as the η-function, which features the relative performance variation as well as generalized stability margin variation with respect to the nominal plant, is also discussed.

Key Words: Performance evaluation; Performance monitoring; Robust performance; Robust stability; Identification; Closed-loop systems.

1. Introduction

In this paper, we deal with robust performance assessment of feedback control systems. There are many methods of assessing robust performance of controllers based on known process models and their uncertainties in the controller design stage. However, there are relatively few methods of assessing achieved robust performance of controllers that have been implemented in the process using plant operating data through some designed closed-loop experiments.

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Since the first systematic study on control loop performance assessment by Harris (1989), it has been now widely recognized that performance assessment is very important in process industry. Research on control loop performance assessment has attracted significant interests from both academia and industry over the last 10 years. Many notable contributions can be found from, for example, Desborough and Harris (1992), Stanfelj et al. (1993), Kozub and Garcia (1993), Lynch and Dumont (1996), Tyler and Morari (1996), Harris et al. (1996), Huang et al. (1997), Kendra and Cinar (1997), Thornhill et al. (1999), Qin (1998), Gustafsson and Graebe (1998), Huang and Shah (1999) and many others.

However, most studies on controller performance assessment have so far focused on evaluating performance such as output variance. One common but also natural question often raised by practicing engineers as well as academic researchers is the robustness or robust performance of control loops. Clearly, the measure of performance itself alone is not sufficient to determine how good a control loop is. A well-designed control loop needs not only to have good performance but also possess certain robustness to tolerate varying process dynamics or model-plant mismatch. Therefore, a measure on robust performance is highly desired in practice.

The robustness issue in control systems has long been an active research area in control community (Zames, 1981). From a pragmatic point of view, robustness problem in control systems can be considered as being consisted of two closely related aspects: stability robustness and performance robustness, with each focusing on a different side of the robustness problem. Roughly speaking, the former mainly concerns with the analysis and design issues on how to stabilize the plant given that the plant to be controlled is subject to certain kind of uncertainty, while the latter mainly focuses on the performance degradation due to the uncertainty. It is of course possible that the two aspects can sometimes be simultaneously considered in the controller design procedure, but this may often lead to a too complex problem that makes it practically intractable. However, the two aspects of robustness issues (stability robustness and performance robustness) are intrinsically related. It is intuitive that inevitably performance will be severely degraded before the closed loop system goes to instability, if the plant is perturbed in a somewhat continuous way. In other words, stability margin deterioration often comes with the degradation of the performance (through a properly defined measure of performance degradation).

One important issue in dealing with the robustness problem is the description of uncertainty. In stability robustness problem, there are several well-known descriptions of plant uncertainty with each leading to different kind of problems and having its own interests in applications. These descriptions of plant uncertainty are, however, not best suited for closed loop performance robustness analysis. The reason is that these uncertainty descriptions are expressed in an “open loop” setting without taking into account the presence of the controller, and thus conservatism is often unavoidable when evaluating the performance variation in closed loop setting. For example, suppose a plant P is perturbed to $P_{\Delta} = P + \Delta$; here we take the additive uncertainty description for the perturbed plant. It is well known that the large open loop uncertainty Δ does not necessarily imply that the difference between the closed loop systems of the two plants with the same controller is also large, due to the presence of the controller. The converse is also true. However, in dealing with the performance robustness problem where performance instead of stability is of major concern, it is reasonable to focus our attention on those perturbations that do not de-stabilize the closed-loop systems. Thus, in describing the uncertainty which is pertain to performance robustness analysis, we can take the advantage of the knowledge of the controller, as will be seen in section 3.

In control engineering practice, one usually designs the controller based on a nominal model. However, plant models are often subject to certain kind of uncertainty. It is thus desirable not only to design a controller which gives a good performance for the nominal plant, but also to guarantee a satisfactory performance for the perturbed plant. In the controller design phase, one can of course take the uncertainty of the plant into account, using some developed robust control theories or techniques. However, existing robust control theories are often either too conservative or too complicated such that they practically prohibit practitioners from using them. In fact, when a controller is claimed to be robust it always (sometimes implicitly) implies that it is against some specific type(s) of uncertainty. Uncertainty descriptions, which are usually chosen by the controller designer based on some *ad hoc* priori knowledge, could seldom be perfect to a particular application. It is thus believed that most often the final justification of the robustness of the designed controller has to be testified through on-line experiments. This is tantamount to answering the following questions, to which the control engineers are often concerned with: does the practical system behave as good as what we design or as good as it is commissioned? How can we evaluate or validate it through a systematic experiment under closed-loop condition? This leads to the requirement for a suitable measure of the performance variation that is easy to be estimated through a properly designed on-line experiment, and is the focus of this paper.

In this paper, we introduce a well defined, informative measure of closed loop performance variation, and show that this measure is not only theoretically elegant, but also has potential to practical applications. In the following sections we restrict our discussion to the performance issue related with feedback properties and all the plants are assumed to be linear time invariant.

2. Practical Performance Consideration

Practical control system performance evaluation may include many aspects. For example, disturbance attenuation, setpoint tracking error, damping ratio and overshoot, etc., are some of the typical performance indices posed in time domain, while some other requirements are expressed in frequency domain, such as bandwidth of the control system, $\mathcal{H}_\infty$ or $\mathcal{H}_2$ performance of a certain augmented closed loop transfer function. Of course, robustness is also one of the major concerns in practical performance consideration, and can also be posed as certain kind of performance index, such as certain type of stability margins. Suppose a controller is designed to satisfy some performance requirements, or to minimize (maximize) certain performance indices for a nominal plant model. Then in assessing the performance variation when the nominal model is perturbed, a question naturally arises: what is the proper measure of the performance variation? One may immediately say that a straightforward way of doing this is to simply measure the difference, caused by the plant perturbations, between those performance indices that are used to design or tune the controller. This measure of difference can give some information about the performance degradation, but is far from sufficient and sometimes inconvenient particularly when one intends to measure it through on-line experiment. Moreover, special caution should be taken when applying this type of methodology. Engineering practice often shows that even if the performance objective degradation due to the plant perturbation is minor, some other practical performances, which are not sufficiently reflected in the design objective, will probably be significant. The closed loop performance for the perturbed system can thus practically be very poor.

What are then the proper features that the measure of performance variation should have? Intuitively, any perceptible and meaningful performance deviations should be able to be observed from the closed-loop input/output properties. This implies that if all of the closed loop transfer functions of a perturbed plant are close to that of the nominal plant, then we can be fairly safe to say that the performance of the controller for the perturbed plant is also satisfactory. We would like to point out that this is true even if the primary controller design objective, such as LQG, eigenvalue/eigenstructure assignment design methodology, is not explicitly expressed as associated with any of the closed loop transfer functions. Refer to Fig.1 of the standard feedback configuration, for any feedback control systems, there are six input/output transfer functions in total, *i.e.*, from external input (u_1 and u_2) to the input/output points of plant/controller respectively, with each having its own interpretation (see *e.g.* [16]). As a matter of fact, all the essential features of feedback systems are contained in the set of all these transfer functions. Obviously, if the plant is perturbed, any perceptible performance variation can be reflected in at least one of the variation of these closed loop transfer functions. In evaluating the difference in any of these transfer functions, a suitable measure is $\bar{\sigma}\left(T_i\left(P_\Delta, C\right) - T_i\left(P, C\right)\right)(\omega)$, *i.e.*, the maximum singular value of the difference transfer function over all the frequency range, where $T_i\left(P, C\right)$ represents any of the closed-loop transfer functions, and $\bar{\sigma}(\bullet)(\omega)$ represents the maximum singular value over frequency. In fact, in measuring the perturbation of a matrix, it seems that we can choose any matrix norm. The reason we prefer the maximum singular value is that it is the induced norm ("gain" of the matrix at each frequency point), and is thus, as is well recognized, best suited for robustness problems among all other matrix norms. Based on this argument, it is quite

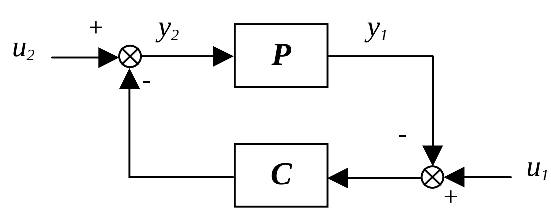

Fig. 1 Standard feedback configuration

safe to say that if all of the variations of closed-loop transfer functions are small, according to the above-defined measure, then the performance degradation is, in any meaningful sense, small. It seems that in evaluating the performance variation, we can (and, to be safe in most cases, have to) check each of these closed-loop transfer function perturbation by the maximum singular value over the frequency range of interest. But it is too cumbersome and sometimes even confusing to check all the six closed-loop transfer function perturbations, particularly when these are to be obtained by on-line experiments. What we are really interested in is: does there exist a single measure that captures all the essential features? In next section, we define a generalized closed-loop error transfer function and show that it has such nice feature.

3. Generalized Closed-Loop Error Transfer Function

In this section, we define a generalized closed-loop error transfer function, and show the nice properties of this notion.

Before proceeding, we need the notion of coprime factorization. It is well known that any transfer function P of a system admits the right and left coprime factorizations as $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ with $M, N \in \mathcal{RH}_\infty$ and $\tilde{M}, \tilde{N} \in \mathcal{RH}_\infty$ are right and left coprime respectively, where $\mathcal{RH}_\infty$ denotes the set of all real rational stable transfer functions. The factorizations are said to be *normalized* if further $M^*M + N^*N = I$ and $\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I$, where $A^*(s) := A^T(-s)$ denotes the conjugate transpose for any $A(s) \in \mathcal{RH}_\infty$. The normalized coprime factorization is known to be unique up to within multiplication by unitary constant matrices. The normalized coprime factorization procedure can be found in, *e.g.*, [5].

We start with a brief introduction of the notion of *dual Youla parameterization*. The following proposition is the well-known dual Youla parameterization of all systems that are stabilized by a given controller.

Proposition 1 (Dual Youla Parameterization) (See *e.g.* [1]).

The set of all plants that are stabilized by controller C is given by

$$\mathcal{S}_C = \left\{ \left(N + V\Delta\right)\left(M + U\Delta\right)^{-1} : \Delta \in \mathcal{RH}_\infty \right\} \tag{1}$$

where $M, N, U, V \in \mathcal{RH}_\infty$, and the two pairs $\left(M, N\right)$ and $\left(U, V\right)$ are both right coprime, and $C = UV^{-1}$ stabilizes $P := NM^{-1}$.

As P in proposition 1 is usually considered as the nominal plant, for a given perturbed plant $P_\Delta \in \mathcal{S}_C$, we can see from (1) that the *Youla parameter* is given by

$$\varDelta = \left(V - P_{\varDelta} U\right)^{-1}\left(P_{\varDelta} - P\right) M \tag{2}$$

This provides a natural uncertainty description of the perturbed plant that is suitable for performance robustness analysis, provided we restrict the perturbed plants to those that do not de-stabilize the original closed-loop system.

However, given any two plants and a stabilizing controller, $\varDelta$ given by (2) is not unique, depending on a somewhat arbitrarily chosen coprime factorizations of the nominal plant and controller. To overcome this, a natural way is to define the coprime factorizations in a manner that is meaningful in the sense to be discussed shortly. We give the following definition:

Definition 1. The *Generalized closed-loop error transfer function* (GCLETF) between two plants, P_1 and P_2, is defined as

$$\varDelta_C\left(P_1, P_2\right) = \left(V - P_1 U\right)^{-1}\left(P_1 - P_2\right)\left(\tilde{V} - \tilde{U} P_2\right)^{-1} \tag{3}$$

where $U,\ V \in \mathcal{RH}_\infty$ and $\tilde{U},\ \tilde{V} \in \mathcal{RH}_\infty$ are right and left normalized coprime factorizations for a given controller C respectively.

Thus, due to the uniqueness of the normalized coprime factorization, given any P_1, P_2 and C, $\varDelta_C\left(P_1, P_2\right)$ defined by (3) is unique. We also note that (3) is the direct consequence of (2) if we require that both the left and right coprime factorizations for the controller are normalized and the right coprime factorization for the nominal plant satisfies the Bezout identity $\tilde{V}M - \tilde{U}N = I$ (this is known to exist as $\left[P, C\right]$ is stable).

When $C = 0$ (this corresponds to open loop situation and is equivalent to $U = 0$, $V = I$, $\tilde{U} = 0$ and $\tilde{V} = I$), (3) reduces to $\varDelta_C\left(P_1, P_2\right) = P_1 - P_2$. This shows that the definition 1 is a natural generalization of the very popular additive uncertainty description for the open loop plant.

A nice feature of definition (3) is that, as an alternative interpretation of proposition 1, if a controller stabilizes a nominal plant, then for a perturbed plant, the control system is still stable if and only if the GCLETF is stable. That is, if $\left[P_1, C\right]$ is stable, then $\left[P_2, C\right]$ is also stable if and only if $\varDelta_C\left(P_1, P_2\right)$ is stable.

One of the main properties of the GCLETF is reflected in the following theorem.

Theorem 1.

$$\bar{\sigma}\left(\Delta_C\left(P_1, P_2\right)\right)(\omega) = \bar{\sigma}\left(T_{P_1,C} - T_{P_2,C}\right)(\omega) \tag{4}$$

where

$$T_{P_i,C} = \begin{bmatrix} I \\ P_i \end{bmatrix} \left(I - CP_i\right)^{-1} \begin{bmatrix} I & -C \end{bmatrix}, \qquad i = 1, 2 \tag{5}$$

is the closed loop transfer function from $\left[u_1^T, u_2^T\right]^T$ to $\left[y_1^T, y_2^T\right]^T$ of Fig.1 with the plant P replaced by P_1 and P_2 respectively.

Proof. By some simple algebraic manipulations we can have

$$\begin{aligned} T_{P_1,C} - T_{P_2,C} &= \begin{bmatrix} I \\ P_1 \end{bmatrix} \left(I - CP_1\right)^{-1} \begin{bmatrix} I & -C \end{bmatrix} - \begin{bmatrix} I \\ P_2 \end{bmatrix} \left(I - CP_2\right)^{-1} \begin{bmatrix} I & -C \end{bmatrix} \\ &= \begin{bmatrix} C \\ I \end{bmatrix} \left(I - P_1 C\right)^{-1} \left(P_1 - P_2\right) \left(I - CP_2\right)^{-1} \begin{bmatrix} I & -C \end{bmatrix} \end{aligned} \tag{6}$$

Substituting $C = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ into (6), we obtain

$$T_{P_1,C} - T_{P_2,C} = \begin{bmatrix} U \\ V \end{bmatrix} \left(V - P_1 U\right)^{-1} \left(P_1 - P_2\right) \left(\tilde{V} - \tilde{U} P_2\right)^{-1} \begin{bmatrix} \tilde{V} & -\tilde{U} \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \Delta_C\left(P_1, P_2\right) \begin{bmatrix} \tilde{V} & -\tilde{U} \end{bmatrix} \tag{7}$$

Note that U, V and $\tilde{U}$, $\tilde{V}$ are right and left normalized coprime factorizations respectively. Therefore

$$\bar{\sigma}\left(T_{P_1,C} - T_{P_2,C}\right)(\omega) = \bar{\sigma}\left(\begin{bmatrix} U \\ V \end{bmatrix} \Delta_C\left(P_1, P_2\right) \begin{bmatrix} \tilde{V} & -\tilde{U} \end{bmatrix}\right)(\omega) = \bar{\sigma}\left(\Delta_C\left(P_1, P_2\right)\right)(\omega)$$

This completes the proof.

This theorem reveals why we need the normalized coprime factorization of the controller in the definition of the GCLETF.

From the proof of theorem 1, the following nice algebraic properties of $\Delta_C\left(P_1, P_2\right)$, which are very useful in on-line model-free closed loop performance assessment as will be shown in section 5, can be readily verified:

(P1) $\Delta_C\left(P_1, P_2\right) = 0$ if and only if $P_1 = P_2$;

(P2) $\Delta_C\left(P_1, P_2\right) = -\Delta_C\left(P_2, P_1\right)$;

(P3) $\Delta_C\left(P_1, P_3\right) = \Delta_C\left(P_1, P_2\right) + \Delta_C\left(P_2, P_3\right)$.

The implication of theorem 1 is reflected in the following proposition, which is a straightforward result from (4) and the properties of the matrix norm.

Proposition 2

$$\bar{\sigma}\left(T_{P_1,C}^{(ij)}-T_{P_2,C}^{(ij)}\right)(\omega)\le\bar{\sigma}\left(\Delta_C\left(P_1,P_2\right)\right)(\omega)\le\sum_{i,\,j=1,\,2}\bar{\sigma}\left(T_{P_1,C}^{(ij)}-T_{P_2,C}^{(ij)}\right)(\omega),\quad i,\ j=1,2 \tag{8}$$

where

$$T_{P_k,C}=\begin{bmatrix}\left(I-CP_k\right)^{-1} & -\left(I-CP_k\right)^{-1}C\\ P_k\left(I-CP_k\right)^{-1} & -P_k\left(I-CP_k\right)^{-1}C\end{bmatrix}:=\left[T_{P_k,C}^{(ij)}\right]_{2\times 2},\quad i,j,k=1,2$$

Thus the important feature of $\Delta_C\left(P_1,P_2\right)$ is that all $\bar{\sigma}\left(T_{P_1,C}^{(ij)}-T_{P_2,C}^{(ij)}\right)(\omega)$ are small if and only if $\bar{\sigma}\left(\Delta_C\left(P_1,P_2\right)\right)(\omega)$ is small. We note that two of the closed loop transfer functions, $(I-PC)^{-1}$ and $(I-CP)^{-1}CP$, are not contained in $T_{P,C}$. But this causes no problem, since $\left(I-P_1C\right)^{-1}-\left(I-P_2C\right)^{-1}=T_{P_1,C}^{(22)}-T_{P_2,C}^{(22)}$ and $\left(I-CP_1\right)^{-1}CP_1-\left(I-CP_2\right)^{-1}CP_2=T_{P_1,C}^{(11)}-T_{P_2,C}^{(11)}$, and hence these two transfer functions are indeed implicitly included in (8). Therefore, proposition 2 implies that all the essential information that reveals the perturbed performance is contained in $\bar{\sigma}\left(\Delta_C\left(P_1,P_2\right)\right)(\omega)$, if we consider P_1 being the nominal plant and P_2 being the perturbed plant. Based on this argument, in assessing the performance robustness, it suffices to check $\bar{\sigma}\left(\Delta_C\left(P_1,P_2\right)\right)(\omega)$. Therefore, $\bar{\sigma}\left(\Delta_C\left(P_1,P_2\right)\right)(\omega)$ provides a safe and informative visualization of the closed loop performance variation over any frequency range of interest.

4. Generalized Stability Margin and the η-Function

One interesting question is, given any two plant models, P_1 and P_2, can we estimate a lower bound of $\bar{\sigma}\left(\Delta_C\left(P_1,P_2\right)\right)(\omega)$ prior to any knowledge of the controller? In this section, we show that this question is naturally related with the pointwise ν-gap metric (for detailed ν-gap metric theory, the reader is referred to [21], [22]), and further show that the GCLETF has more meaning than close loop performance variation in that it is strongly associated with the robustness of the closed loop system. We further introduce the η-function, which can be considered as a relative version of the GCLETF, as a means of evaluating the generalized stability robustness variation.

Let $P_i = N_i M_i^{-1} = \tilde{M}_i^{-1}\tilde{N}_i,\ i = 1,2$ denote the normalized right/left coprime factorizations of P_1 and P_2 respectively. Define the κ-*function* as

$$\kappa\left(P_1, P_2\right)(\omega) = \bar{\sigma}\left(\tilde{N}_2 M_1 - \tilde{M}_2 N_1\right)(\omega) \tag{9}$$

which represents the distance between the two plants P_1 and P_2 frequency by frequency. It is known that $\kappa\left(P_1, P_2\right)(\omega) \le 1\ \forall\omega$. With this κ-function, the ν-*gap metric* is given as follows:

$$\delta_\nu\left(P_1,\ P_2\right) = \begin{cases} sup_\omega\, \kappa\left(P_1, P_2\right)(\omega) & \text{if } det\left(M_1^* M_2 + N_1^* N_2\right)(j\omega) \neq 0 \quad \forall\omega \\ & \text{and } wno\, det\left(M_1^* M_2 + N_1^* N_2\right) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

where $wno(f(s))$ denotes the winding number about the origin of the scalar function $f(s)$ as s follows the standard Nyquist D-contour and $det(A)$ denotes the determinant of a matrix A. Suppose that a controller C stabilizes plant P, then the ρ-*function* is defined as

$$\rho(P, C)(\omega) = \bar{\sigma}^{-1}\left(T_{P,C}\right)(\omega) \tag{10}$$

It is known that $\rho(P, C)(\omega) \le 1\ \forall\omega$. The main feature of the κ-function is given in the following proposition:

Proposition 3. (See [22])

Given a (nominal) plant P, and a function g satisfying $g,\ g^{-1} \in \mathcal{H}_\infty$ and $\left|g(j\omega)\right| < 1$, then $\left[P_\Delta, C\right]$ is stable for all (perturbed) plant P_Δ, satisfying $\delta_\nu\left(P,\ P_\Delta\right) < 1$ and

$$\kappa\left(P, P_\Delta\right)(\omega) \le \left|g(j\omega)\right| \quad \forall\omega \tag{11}$$

if and only if $\left[P, C\right]$ is stable and

$$\rho(P, C)(\omega) > \left|g(j\omega)\right| \quad \forall\omega \tag{12}$$

Thus, based on proposition 3, κ-function can be interpreted as the *pointwise* ν-*gap metric* (provided $\delta_\nu\left(P,\ P_\Delta\right) < 1$), while ρ-function can be interpreted as the *pointwise generalized stability margin*. Usually, a properly designed controller should have a reasonably large ρ-function in order to sustain robust stability against plant uncertainty.

The following proposition reflects the relationship of $\Delta_C(P_1,P_2)$ and $\kappa(P_1,P_2)(\omega)$.

Proposition 4. For any P_1, P_2, C such that $[P_1, C]$ and $[P_2, C]$ are stable, then

$$\kappa(P_1,P_2)(\omega) \le \bar{\sigma}\left(\Delta_C(P_1,P_2)\right)(\omega) \le \frac{\kappa(P_1,P_2)(\omega)}{\rho(P_1,C)(\omega)\rho(P_2,C)(\omega)} \tag{13}$$

Proof. This is a slight generalization of corollary 6.5 in [21] and the proof is omitted here.

One of the implications in Proposition 4 is that, giving any two plants, P_1 and P_2, no matter what controller is used, there exists a performance variation limitation (or performance robustness limitation) bounded below by the (controller independent) pointwise ν-gap metric $\kappa(P_1,P_2)(\omega)$. This also shows that the ν-gap metric is useful not only for stability robustness study, but also for performance robustness evaluation.

For a SISO plant, the ρ-function has an interesting alternative interpretation, as shown in the following corollary:

Corollary 5. For SISO plant we have

$$\lim_{\kappa(P,P_\Delta)(\omega)\to 0} \frac{\bar{\sigma}\left(\Delta_C(P,P_\Delta)\right)(\omega)}{\kappa(P,P_\Delta)(\omega)} = \frac{1}{\rho^2(P,C)(\omega)} \tag{14}$$

Proof. This can easily be obtained from the proof of theorem 6.3 in [21] and is omitted here.

Thus, the inverse square of ρ-function can also be interpreted as the sensitivity function of the closed loop performance with respect to plant perturbation in terms of the pointwise ν-gap metric. This implies that small ρ-function at some frequency indicates that the closed loop performance is more sensitive to those perturbations that affect the plant characteristics at that frequency, and vice versa. Thus, from this performance sensitivity aspect, large ρ-function for closed loop systems is again desired.

It is sometime useful to use the "worst case" version of $\bar{\sigma}\left(\Delta_C(P_1,P_2)\right)(\omega)$, *i.e.*, the $\mathcal{H}_\infty$-norm of $\Delta_C(P_1,P_2)$, defined as

$$\left\|\Delta_C(P_1,P_2)\right\|_\infty := \sup_\omega \bar{\sigma}\left(\Delta_C(P_1,P_2)\right)(\omega) \tag{15}$$

It is easy to show, from the algebraic properties of $\Delta_C(P_1,P_2)$ given in section 3, that $\left\|\Delta_C(P_1,P_2)\right\|_\infty$ is a metric defined on $\mathcal{S}_C$. It is worth noting that if $\Delta_C(P_1,P_2)$ approaches the stability boundary, i.e., some pole(s) of $\varDelta_C\left(P_1, P_2\right)$ approaches $j\omega$ axis, then it is well known from the basic property of the $\mathcal{H}_\infty$-norm that $\left\|\varDelta_C\left(P_1, P_2\right)\right\|_\infty \to \infty$. This property is highly desirable for the measurement of performance degradation since it captures one of the fundamental features of the measure of performance degradation in that significant performance degradation will definitely occur when the closed loop system is becoming unstable. A direct result of Proposition 4 is that (also see [21])

$$\delta_\nu\left(P_1,\ P_2\right) \le \left\|\varDelta_C\left(P_1, P_2\right)\right\|_\infty \le \frac{\delta_\nu\left(P_1,\ P_2\right)}{b_{P_1,C}b_{P_2,C}} \tag{16}$$

This implies that $\delta_\nu\left(P_1,\ P_2\right) \to 0$ if and only if $\left\|\varDelta_C\left(P_1, P_2\right)\right\|_\infty \to 0$. This means that the metric $\left\|\varDelta_C\left(\cdot, \cdot\right)\right\|_\infty$ also induces the same topology, known as the *graph topology* (see [3], [21]), as that by $\delta_\nu\left(\cdot,\cdot\right)$.

In practical applications, it is sometimes more convenient, and perhaps more meaningful, to use the relative measure of performance variation with respect to the nominal plant, defined as the following η-*function*

$$\begin{aligned}\eta_C\left(P, P_\varDelta\right)(\omega) &=: \frac{\bar{\sigma}\left(T_{P_\varDelta,C} - T_{P,C}\right)(\omega)}{\bar{\sigma}\left(T_{P,C}\right)(\omega)} \\ &= \frac{\bar{\sigma}\left(\varDelta_C\left(P, P_\varDelta\right)\right)(\omega)}{\bar{\sigma}\left(T_{P,C}\right)(\omega)} = \rho\left(P, C\right)(\omega)\bar{\sigma}\left(\varDelta_C\left(P, P_\varDelta\right)\right)(\omega)\end{aligned} \tag{17}$$

The rationale of using this measure is seen from the following proposition:

Proposition 6. Suppose that a controller C stabilizes both (nominal) plant P and (perturbed) plant $P_\varDelta$. Then we have the following inequalities

$$1 - \eta_C\left(P, P_\varDelta\right)(\omega) \le \frac{\rho\left(P, C\right)(\omega)}{\rho\left(P_\varDelta, C\right)(\omega)} \le 1 + \eta_C\left(P, P_\varDelta\right)(\omega) \tag{18}$$

and if $\eta_C\left(P, P_\varDelta\right)(\omega) > 1$, then

$$\rho\left(P_\varDelta, C\right)(\omega) \le \frac{\rho\left(P, C\right)(\omega)}{\eta_C\left(P, P_\varDelta\right)(\omega) - 1} \tag{19}$$

Proof. The proof can be readily achieved by using the triangle inequality of the norm and the definition of the η-function, and is omitted here.

Inequality (18) means that if $\eta_C(P, P_\Delta)(\omega) \ll 1$ over some frequency range, the generalized stability margin of the perturbed plant is guaranteed to be close to the unperturbed $\rho(P, C)(\omega)$ in such frequency range. While inequality (19) implies that if $\eta_C(P, P_\Delta)(\omega)$ is significantly larger than 1 around some frequency, the generalized stability margin is degraded significantly around such frequency, *i.e.*, it will decrease at least by a factor of $1/(\eta_C(P, P_\Delta)(\omega) - 1)$ of the original stability margin. Thus, η-function can be interpreted not only as the relative measure of performance variation with respect to nominal performance, but also as an indication of the generalized stability margin degradation.

We also note that, since the ρ-function is always smaller than 1, we have from (17)

$$\eta_C(P, P_\Delta)(\omega) \le \bar{\sigma}(\Delta_C(P, P_\Delta))(\omega) \tag{20}$$

Thus, $\bar{\sigma}(\Delta_C(P, P_\Delta))(\omega)$ gives an upper bound of the η-function. This implies that if $\bar{\sigma}(\Delta_C(P, P_\Delta))(\omega)$ is much smaller than 1, we can be fairly safe to conclude, without referring to the η-function, that the deterioration of the generalized stability margin is small, as this implies that $\eta_C(P, P_\Delta)(\omega)$ is much smaller than 1 by (20). This property is particularly useful since in closed loop performance monitoring the model of the plant may not be required in identifying the GCLETF, while in estimating the η-function the model of plant is necessary, as we will discuss in section 5.

5. **Evaluation of Performance Variation in Practical Systems**

In this section, we show that one of the appealing feature of the generalized closed-loop error transfer function is that it can be estimated through systematic closed loop experiment. This feature is particularly useful in closed loop performance assessment/monitoring for model-based control systems.

Suppose that we have a model of a plant, denoted by P, and a controller designed based on this model. We do not concern about how the model is obtained and how the controller is designed. In other words, we always assume that the controller is designed to the satisfaction of the designer and can thus be taken as a *user-defined benchmark*. The issue we are interested in is: does this controller behave as good as we expect when connected to the actual plant? Or is the control system behaving as good as before in case of suspicious plant changing? In this section, we propose a systematic procedure to handle this issue.

Refer to the practical feedback control system of Fig. 2, where r_u and r_y are known input signals (*e.g.*, excitation or probing signal), v is unknown external (disturbance/measurement noise) input, P_Δ denotes the actual plant, and C is the controller designed based on a model P of the plant.

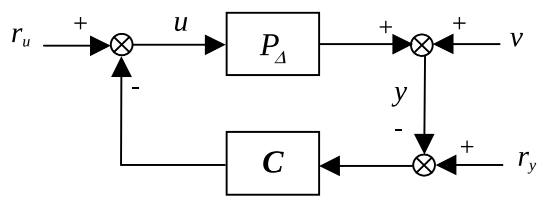

Fig. 2 Practical Closed-Loop System

Define

$$\begin{bmatrix} r \\ z \end{bmatrix} := \begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \tag{21}$$

where $\tilde{U}$, $\tilde{V}$ and U, V are the normalized left and right coprime factorizations of controller C respectively. $P = \tilde{M}^{-1}\tilde{N}$ is a left coprime factorization satisfying the Bezout identity $\tilde{M}V - \tilde{N}U = I$. As

$$\begin{bmatrix} u \\ y \end{bmatrix} = T_{P_\Delta, C} \begin{bmatrix} r_u \\ r_y \end{bmatrix} + \begin{bmatrix} C \\ I \end{bmatrix} \left(I - P_\Delta C\right)^{-1} v \tag{22}$$

left multiply this equation by $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}$ and after some simple algebraic manipulation it yields

$$\begin{bmatrix} r \\ z \end{bmatrix} = \begin{bmatrix} I \\ \Delta_C\left(P,\, P_\Delta\right) \end{bmatrix} \left(\tilde{V} r_u - \tilde{U} r_y\right) + \begin{bmatrix} 0 \\ \left(V - P_\Delta U\right)^{-1} \end{bmatrix} v \tag{23}$$

or,

$$z = \Delta_C\left(P,\, P_\Delta\right) r + \left(V - P_\Delta U\right)^{-1} v \tag{24}$$

where

$$r = \tilde{V}r_u - \tilde{U}r_y \tag{25}$$

z and r can be considered as obtained from passing the measured data u and y through a (stable and inversely stable) known filter $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}$. Also note that r and v are uncorrelated. Thus $\Delta_C\left(P, P_\Delta\right)$ can be identified using standard system identification methods with r and z as input and output data respectively, provided r is properly designed (persistently exciting, for example) (see *e.g.* [13]). Since we use r instead of directly using r_u and/or r_y as input signal, it makes no essential difference whether r_u alone, r_y alone or both exist as the dither signal in estimating $\Delta_C\left(P, P_\Delta\right)$ in the closed loop experiment. The fact that any dual Youla parameter can be obtained through the closed loop experiment plays a key role in a well-known approach for closed-loop system identification (see *e.g.* [1]).

In practice, $\left\|\Delta_C\left(P, P_\Delta\right)\right\|_\infty$ can be used as a primitive index of performance degradation. Usually, small $\left\|\Delta_C\left(P, P_\Delta\right)\right\|_\infty$ implies that the performance degradation is acceptable. However, if $\left\|\Delta_C\left(P, P_\Delta\right)\right\|_\infty$ is large, we can not simply say that the performance degradation is unacceptable. In such case, it is preferable to use the more informative $\bar{\sigma}\left(\Delta_C\left(P, P_\Delta\right)\right)(\omega)$ to further evaluate the model mismatch. This gives not only a distribution of the model mismatch over the frequency range of interest, but may also provide a clue on over what frequency range the model accuracy should be improved, or the controller should be re-designed to meet the robustness performance requirement. Furthermore, it can also provide a guideline in re-designing the controller. It is also worth noting that the η-function, which is preferred for some case, can be readily calculated by (17) once the GLCETF is identified, since $\rho\left(P, C\right)(\omega)$ is available by assumption.

It seems that the perturbed plant can be obtained through some system identification techniques, either in open loop (see *e.g.* [13]) or in closed-loop (see *e.g.* [1]), and then one can directly calculate the GCLETF by (3). However, we would like to point out that this methodology of estimating the GCLETF is risky, because the error of the identified model of the plant may have an unpredictable effect on the accuracy of the calculated GCLETF, thus making it unreliable. This point is basically from the well-known fact, as we mentioned in the first section, that small error of the model from the true plant dynamics dose not necessarily means that their respective closed-loop characters are close, due to the presence of the feedback controller, and vice versa. Moreover, it is well known that closed loop identification is much more involved than open loop identification, mainly due to the presence of the feedback controller that induces the correlation between the unknown external input and the control input. The accuracy of the identified plant model through closed loop identification is usually less predictable than that obtained through open loop identification procedure, and may therefore be less reliable. If we use this "direct method", the error may propagate and accumulate to the GCLETF, thus making it even more unreliable.

In the following we propose a method of identifying the generalized closed loop error transfer function, without the need to have the nominal plant model. This is particularly advantageous in practical applications where the plant model is not available. For example, in some cases where the controller is obtained not based on the model, but through on-line tuning by some experienced control engineer, via physical understanding of the plant. Many PID controllers are tuned in this way, for example.

Suppose P_a is the initial plant for which the controller is tuned. If the plant is changed to P_b, $\Delta_C\left(P_a, P_b\right)$ can be estimated as follows (it is of course always assumed that the controller has not been re-tuned in between):

From the algebraic properties of the generalized closed loop error transfer function we have

$$\Delta_C\left(P_a, P_b\right) = \Delta_C\left(P_M, P_a\right) - \Delta_C\left(P_M, P_b\right) \tag{26}$$

where P_M can be any arbitrarily chosen "plant model" provided it is stabilized by the given controller *C*. Thus, $\Delta_C\left(P_a, P_b\right)$ can be obtained by identifying $\Delta_C\left(P_M, P_a\right)$ and $\Delta_C\left(P_M, P_b\right)$ respectively, using the method developed before, and the estimation of $\Delta_C\left(P_a, P_b\right)$ is accordingly calculated by (26). It is noted that the procedure for identifying $\Delta_C\left(P_a, P_b\right)$ can even be further simplified if we use the same excitation (*i.e.*, the same r) applied to the plant for the two situations, shown as follows:

For the initial plant we have

$$z^a = \Delta_C\left(P_M, P_a\right)r + \left(V - P_a U\right)^{-1} v^a \tag{27}$$

and analogously, for the perturbed plant

$$z^b = \Delta_C\left(P_M, P_b\right)r + \left(V - P_b U\right)^{-1} v^b \tag{28}$$

where r is calculated by (25) (with the same excitation r_u and/or r_y for both situations) and v^a, v^b represent the unknown external disturbances for the two situations respectively. Thus by subtracting (28) from (27), we have

$$\Delta z = \Delta_C\left(P_a, P_b\right)r + \left[\left(V - P_a U\right)^{-1} v^a - \left(V - P_b U\right)^{-1} v^b\right] \tag{29}$$

where

$$\Delta z = z^a - z^b = \tilde{M}_*\left(y^a - y^b\right) - \tilde{N}_*\left(u^a - u^b\right) \tag{30}$$

$\tilde{M}_*$, $\tilde{N}_*$ are the left coprime factorization of the arbitrarily chosen model P_M and satisfy the Bezout identity

$$\tilde{M}_* V - \tilde{N}_* U = I \tag{31}$$

Hence, $\Delta_C\left(P_a, P_b\right)$ can be readily identified by applying any standard identification techniques to the data $\{r, \Delta z\}$. This method may be categorized as *model-free performance assessment/monitoring* as no plant model is needed in this case.

It is worth noting that, as is anticipated, the choice of the arbitrarily model P_M has no effect on identifying $\Delta_C\left(P_a, P_b\right)$. This is seen in the following corollary:

Corollary 7. Suppose P_{m_1} and P_{m_2} are any two models that are stabilized by controller C. Then we have $\Delta z_1 = \Delta z_2$, where $\Delta z_1, \Delta z_2$ are calculated by (30) with respect to P_{m_1} and P_{m_2} respectively with the same excitation r.

Proof. Suppose that the two models have the left coprime factorization $P_{m_1} = \tilde{M}_1^{-1}\tilde{N}_2$ and $P_{m_2} = \tilde{M}_2^{-1}\tilde{N}_2$, and both satisfy the Bezout identity (31). Then from the Youla parameterization theorem, there exists a $Q \in \mathcal{RH}_\infty$ such that

$$\begin{bmatrix}\tilde{M}_2 & -\tilde{N}_2\end{bmatrix} = \begin{bmatrix}\tilde{M}_1 & -\tilde{N}_1\end{bmatrix} + Q\begin{bmatrix}-\tilde{U} & \tilde{V}\end{bmatrix}$$

Also note, from (21) and (25), that $\tilde{V}u^a - \tilde{U}y^a = \tilde{V}u^b - \tilde{U}y^b = r$. Hence,

$$
\begin{aligned}
&\Delta z_2 = \tilde{M}_2\left(y^a - y^b\right) - \tilde{N}_2\left(u^a - u^b\right) \\
&= \left[\tilde{M}_1\left(y^a - y^b\right) - \tilde{N}_1\left(u^a - u^b\right)\right] + Q\left[\left(\tilde{V}u^a - \tilde{U}y^a\right) - \left(\tilde{V}u^b - \tilde{U}y^b\right)\right] \\
&= \Delta z_1
\end{aligned}
$$

This completes the proof.

It is noted that P_M is used only for filtering the measured data to obtain Δz, and r depends only on the controller and the designed excitation signal(s) r_u and/or r_y; therefore the identification algorithm for $\Delta_C\left(P_a, P_b\right)$ does not depend on any particular choice of P_M. We also note that this feature may not hold for the case where the two generalized closed loop error transfer functions on the left side of equation (26) are separately identified.

It is also worthwhile to note that from model validation perspective, $\Delta_C\left(P, P_\Delta\right)$ can be considered as a special *model error models in a closed loop setting*, if P is the model to be evaluated and P_Δ is the true plant (see [4], [14]). However, from the performance monitoring or control-relevant model validation point of view, if the controller is the actual controller, the model quality can be evaluated by $\bar{\sigma}\left(\Delta_C\left(P, P_\Delta\right)\right)(\omega)$ or $\eta_C\left(P, P_\Delta\right)(\omega)$, and is accepted/refused if it is small/large over a frequency range of interest, irrespective of that the model may be *invalidated/validated* from the classical model validation criteria. This method can also be used as a supplementary tool in the iterative identification / controller design procedure.

6. Numerical Examples

In this section, we illustrate by numerical examples how to use the GCLETF and the η_C-function to evaluate the closed loop performance variation. Two cases are considered: the first case is to assess the robust performance of a closed-loop system, given that the controller is designed based on the (identified) model of the plant; while the second case is to monitor the closed loop performance in case of plant change.

6.1 Case (A): Model based robust performance assessment

In this case, we demonstrate through a systematic procedure how to use the GCLETF as well as the η_C-function to assess the robust performance of a closed-loop system. The model of the plant is obtained through open-loop system identification and a controller is consequently designed based on this plant model. The issue we are concerning is: does the closed-loop system behave as good as it is designed?

Let us consider the true plant with an *Output Error* structure

$$y = P_t u + e$$

where *e* is a unit-variance white noise and

$$P_t = \frac{1.28q^{-1} + 1.2q^{-2} + 0.2588q^{-3} + 0.01606q^{-4}}{1 - 0.7932q^{-1} - 0.3698q^{-2} + 0.5184q^{-3} + 0.01298q^{-4}}$$

The sampling time is 1 second. Applying the standard open loop identification procedure we obtain a "reduced order" model of the plant which fits the data well as

$$P_m = \frac{1.258q^{-1} + 0.3066q^{-2}}{1 - 1.53q^{-1} + 0.7397q^{-2}}$$

The input signal used in the identification is a random binary sequence with amplitude of –1 to +1, and the number of collected data is 1000 (the same type of excitation will be used throughout this section.). A controller which is aimed at achieving a good setpoint tracking property is designed based on the model P_m as

$$C = \frac{q^{-1}}{1 - q^{-1}} P_m^{-1} = \frac{q^{-1}\left(1 - 1.53q^{-1} + 0.7397q^{-2}\right)}{\left(1 - q^{-1}\right)\left(1.258q^{-1} + 0.3066q^{-2}\right)}$$

Now we want to verify whether the controller works well for the actual plant. To this end, we connect the designed controller to the true plant and then identify the GCLETF, $\Delta_C\left(P_t, P_m\right)$, via the procedure proposed in section 5. To proceed, we need the normalized coprime factorization of controller C, calculated as,

$$U = \frac{-0.5556 + 0.8499q^{-1} - 0.411q^{-2}}{1 - 0.9926q^{-1} + 0.1092q^{-2}} \qquad V = \frac{-0.6991 + 0.5287q^{-1} + 0.1703q^{-2}}{1 - 0.9926q^{-1} + 0.1092q^{-2}}$$

And the coprime factorization of the plant model P_m that satisfies the Bezout identity, is consequently calculated as

$$N = \frac{-1.8q^{-1} + 1.787q^{-2} - 0.1966q^{-3}}{1 - 1.53q^{-1} + 0.7397q^{-2}} \qquad M = \frac{-1.43 + 1.42q^{-1} - 0.1563q^{-2}}{1 + 0.2437q^{-1}}$$

The identification schematic diagram is shown in Figure 3. Here we apply the dither signal to the feedback loop only at the input node of the plant. We note that as argued in section 5 it makes no essential difference by applying the input signal whether at the input node of the plant or at the input node of the controller, or simultaneously at both nodes. The data u and y are collected when applying the input signal r_u to the feedback control loop, and then post-filter the collected data r_u, u and y through V, N and M respectively to obtain the filtered data r and z, as shown in Figure 3. Using the Box-Jenkins model structure with the data r and z as its input-output data (we also keep these data for later use and will denote z as z_a in the next subsection), a model of $\Delta_C(P_t, P_m)$ that fits the data well is found to be

$$\hat{\Delta}_C(P_t, P_m) = \frac{0.0637q^{-1}}{1 + 1.6198q^{-1} + 0.6726q^{-2}}$$

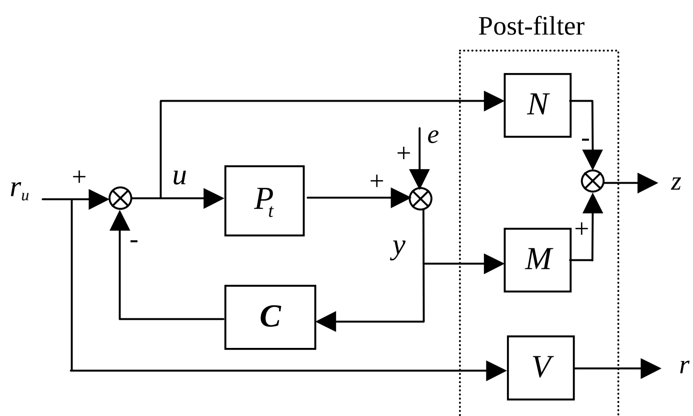

Fig. 3 Schematic diagram for identification of GCLETF

Here, and in the sequel, we use the hat to denote the corresponding estimated quantity.

An estimation of η_C-function $\hat{\eta}_C(P_t, P_m)(\omega)$ can be accordingly computed. Figure 4 shows the plots of $\bar{\sigma}\left(\hat{\Delta}_C(P_t, P_m)\right)(\omega)$ and $\hat{\eta}_C(P_t, P_m)(\omega)$ (where ω is plotted up to the Nyquist frequency), together with the plots of computed $\bar{\sigma}\left(\Delta_C(P_t, P_m)\right)(\omega)$, $\eta_C(P_t, P_m)(\omega)$ and $\rho(P_m, C)(\omega)$ respectively for comparison and analysis. We note that the crossover frequency of the open loop system $P_m C$ is 1.0472 rad/sec. It is shown in Figure 4 that both GCLETF and η_C-function are much smaller than 1 at the frequency range lower than and near the crossover frequency. We note that the maximum value of the identified GCLETF (i.e., $\left\|\hat{\Delta}_C\left(P_t, P_m\right)\right\|_\infty$) is 1.2064 (while the computed one is 1.2109), which is slightly greater than 1. However, we also note that the maximum value of the estimated η_C-function, i.e., 0.2914 (while the computed one is 0.2925), is reasonably smaller than 1. Furthermore, both of the GCLETF and η_C-function achieve their maximum value at the highest frequency of interests, i.e., the Nyquist frequency $\pi\,(rad/\sec)$, which is fairly higher than the crossover frequency. Based on these arguments, we can safely conclude that the closed loop performance is well preserved when the designed controller is applied to the true plant, except for a slight performance deviation at high frequency. Thus, the performance of the designed controller is acceptable when it is applied to the true plant. In fact, we can further verify this by comparing the step responses of the closed loop system when the designed controller is connected to the model and the true plant respectively; the responses are shown in Figure 5 and Figure 6. It is observed that they are very similar.

However, looking at the $\rho\left(P_m, C\right)(\omega)$ from Figure 4 we find that its minimum value is 0.1617 (achieved at the natural frequency of the plant model, another local minimum is at the Nyquist frequency $\pi\,(rad/\sec)$), which is much smaller than 1. This is an indication of poor robustness of the closed loop system as well as high sensitivity, around frequency corresponding to the low values of $\rho\left(P_m, C\right)(\omega)$, of the closed loop performance with respect to plant perturbation. This will be further illustrated in the next subsection.

6.2 Case (B): Model-free robust performance monitoring

In this case, we illustrate the procedure of how to monitor the performance of the closed-loop systems in case of possible plant changes. The problem we are considering here is: is the controller behaving as good as it was commissioned? For this purpose we need two sets of experiment data: the initial experiment data, which was obtained when the controller was commissioned (or tuned), and the new experiment data obtained from the current closed-loop system; and the knowledge of the controller. The GCLETF between the initial plant and the current plant with the given controller can thus be estimated, without knowing the model of either initial plant or the current plant. The performance variation can then be evaluated via the identified GCLETF.

Again we use the same plant structure as that in Case (A), but suppose that the true plant is perturbed to

$$P_t' = \frac{1.272q^{-1} + 1.92q^{-2} + 1.289q^{-3} + 0.2906q^{-4}}{1 - 0.272q^{-1} - 0.4024q^{-2} - 0.266q^{-3} + 0.5789q^{-4}}$$

We want to know if the controller designed in Case (A), which works well for the previous true plant P_t as we demonstrated in previous subsection, still works well for the current (perturbed) plant P_t'. It is noted that the step responses of the open loop system for the two plants P_t and P_t' are hardly distinguishable. However, their closed loop systems when using the designed controller C, differentiates significantly. To show this, we again apply the same input signal as we used in Case (A) to the closed loop system of Figure 3, and obtain the filtered data z (we already have data r in Case (A)). Using the Box-Jenkins model structure with the data r and $(z_a - z)$ as its input-output data, an estimation of $\Delta_C\left(P_t, P_t'\right)$ that fits the data well is found to be

$$\hat{\Delta}_C\left(P_t, P_t'\right) = \frac{0.0169q^{-1} - 0.1275q^{-2} + 0.2641q^{-3} - 0.2407q^{-4} + 0.1122q^{-5} - 0.0248q^{-6}}{1 + 0.5546q^{-1} - 0.5713q^{-2} - 0.7314q^{-3} + 0.4678q^{-4} + 0.4712q^{-5} - 0.0335q^{-6}}$$

The plot of $\bar{\sigma}\left(\hat{\Delta}_C\left(P_t, P_t'\right)\right)(\omega)$ is shown in Figure 7. It is seen that the estimated $\left\|\hat{\Delta}_C\left(P_t, P_t'\right)\right\|_\infty$ is as large as 20.5976, which is significantly greater than 1. This shows that the closed loop performance is dramatically degraded from the originally designed closed loop system with this perturbed plant. In fact, from the step response simulation we can see that the perturbed closed loop system exhibits a highly oscillatory response, as shown in Figure 8.

We note that in identifying $\Delta_C\left(P_t, P_t'\right)$, as is argued in section 5, N and M can be the coprime factorization of any "model" that is stabilized by the controller C, not necessarily being that of the model P_m in Case (A). This is advantageous for the situation where either the controller is designed not based on a model of the plant or the model is not available.

On the other hand, it is worth noting that calculation of GCLETF using open loop identified model can give erroneous results. Suppose we conduct open loop identification for the perturbed plant, and a model of P_t' that fits the data well is obtained as

$$P_m' = \frac{1.2494q^{-1} + 0.3175q^{-2}}{1 - 1.5301q^{-1} + 0.7401q^{-2}}$$

We hence by calculation obtain $\left\| \Delta_C \left(P_m, P_m' \right) \right\|_\infty = 0.0947$, which is much smaller than 1 and implies, contradicting to the reality, that the performance variation is almost negligible! There is no surprise here since, as we argued in the first section, small perturbation evaluated in open-loop plant dose not necessarily implies that the closed-loop performance variation is also small. This means that to evaluate the robustness of the designed controller against plant uncertainty, experiment has to be performed in closed-loop systems, instead of using an open loop identified perturbed plant model to assess it.

6.3 Complementary Remarks

In the previous two subsections, we have demonstrated by numerical examples how to use the GCLETF and/or the η-function to evaluate close-loop performance variations through systematic closed-loop experiments. Though there are some other possible situations on the estimated GCLETF and/or the η-function which do not appear in the discussed numerical cases, the basic principle is clear and simple. The designed closed-loop performance can be claimed, with confidence, to be well-preserved, or the degradation of the generalized stability margin can be claimed minor, only if either the estimated GCLETF or the η-function is fairly small. Otherwise the actual closed-loop performance is different notably from the designed performance and hence re-analysis or re-design of the control systems is most probably needed.

It is worth noting that the proposed method can not tell if the controller is becoming "better" or "worse" when the estimated GCLETF is large. In fact, any benchmark-based evaluation strategies have the same "*limitation*" in the sense that being different from a benchmark does not necessarily mean that it is practically a poor controller. The proposed method in this paper can actually be considered as belonging to this category since we basically use a *user-defined benchmark*. This limitation, however, does not really hinder the effective application of the benchmark-based strategy as discussed next.

First, when the measured variation is large, even if it is resulted from "better" control loop in some sense, most likely we would have reason to question it unless we are able to further identify the root causes and to figure out some reliable evidences to prove that it is indeed “better”. And this requires the designer to go back to check the design process and carry out some further analysis in order to finally claim that the controller indeed becomes better. This is basically the same action that needs to be taken when, in case of large variation, we interpret the controller as becoming “worse”, or at least being not as good as it is expected. In other word, when a significant variation is identified, no matter whether it is resulted from a “better” controller or from a “worse” controller, we are at least being informed that there must be “something wrong” during the design process. And most probably we have no choice but to go back to check the design process. It is in this sense that the proposed method is useful.

On the other hand, it is a common sense in practice that when significant closed loop performance deviation from the nominal one occurs, most probably, but not always, this would practically imply that the controller performance is becoming worse rather than better. The reason is that most often, in order for the control loop to have good performances such as tracking capability and disturbance attenuation, most controllers tend to be fine-tuned on site to make full use of their potentials under some practical constraints. In other words, most controllers are reasonably good controllers. Significant deviation from the nominal performance usually implies that the controller is either becoming over-tuned (excessively large gain) or under-tuned (insufficient gain to achieve its full potentials). It is therefore our opinion that if the closed systems deviate from its nominal performance, most likely the control loop is becoming worse rather than better in a practical sense. In such cases redesign of the controller is often necessary.

7. Conclusion

This paper has proposed a measure of performance variation in frequency domain in terms of the generalized closed loop error transfer function. This measure is shown to be able to capture all the essential information that reflects the performance variation due to plant perturbation. Furthermore, this measure is intimately related with the well-defined generalized stability margin, and is also informative in that it provides a visualization of the performance variation over any frequency range of interest. Also this measure can be used as closed loop performance monitoring through standard “open loop” system identification techniques, even without explicitly knowing the plant model. Such methodology can be considered as using the designed control loop as a user defined benchmark to assess the robust performance of the control system in the frequency domain when the plant is perturbed. The application of the proposed method is illustrated by numerical examples.

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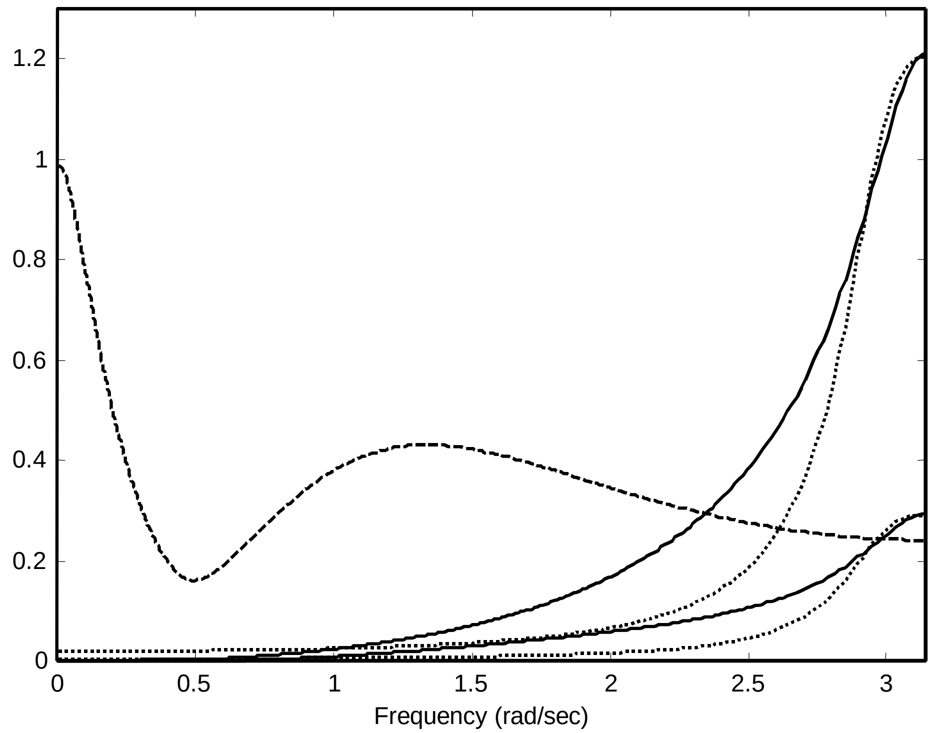

Fig.4 Plots of $\rho\left(P_m, C\right)(\omega)$ **(dashed),** $\bar{\sigma}\left(\hat{\Delta}_C\left(P_t, P_m\right)\right)(\omega)$ **(upper dotted),** $\hat{\eta}_C\left(P_t, P_m\right)(\omega)$ **(lower dotted),** $\bar{\sigma}\left(\Delta\;\left(P, P\;\right)\right)(\omega)$ **(upper solid) and** $\eta\;\left(P, P\;\right)(\omega)$ **(lower solid)**

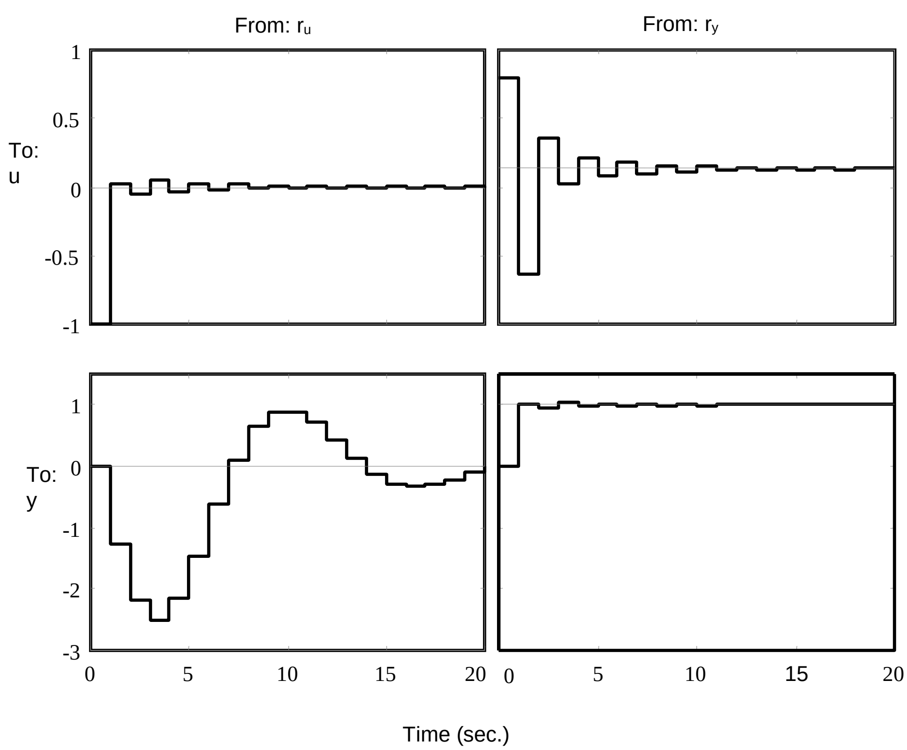

Figure 5: Closed-loop step response when the controller is connected to the true plant

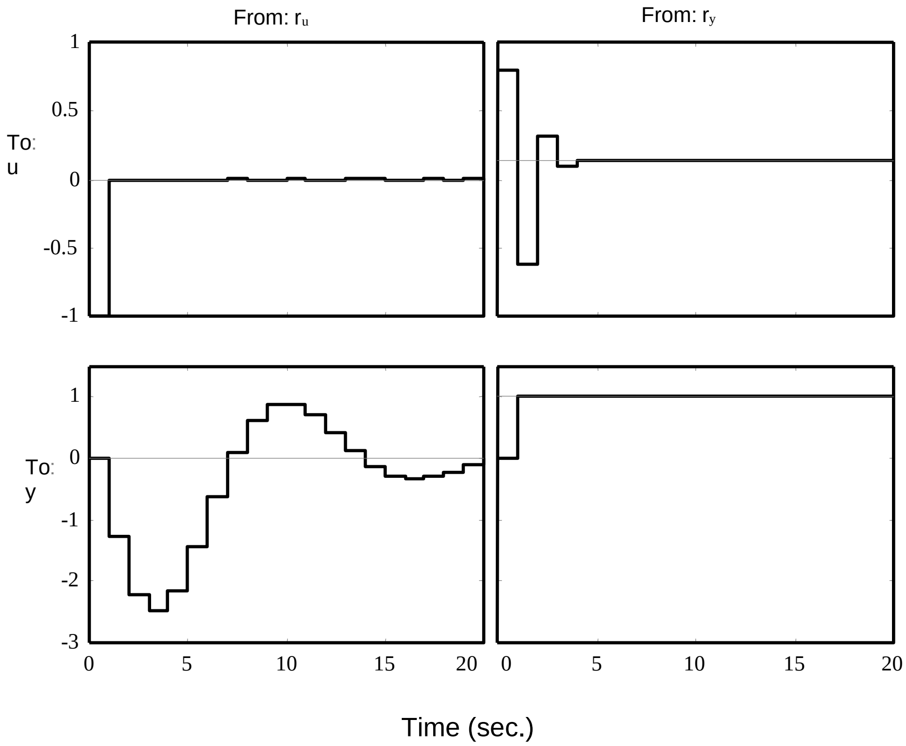

Figure 6: Closed-loop step response when the controller is connected to the nominal model

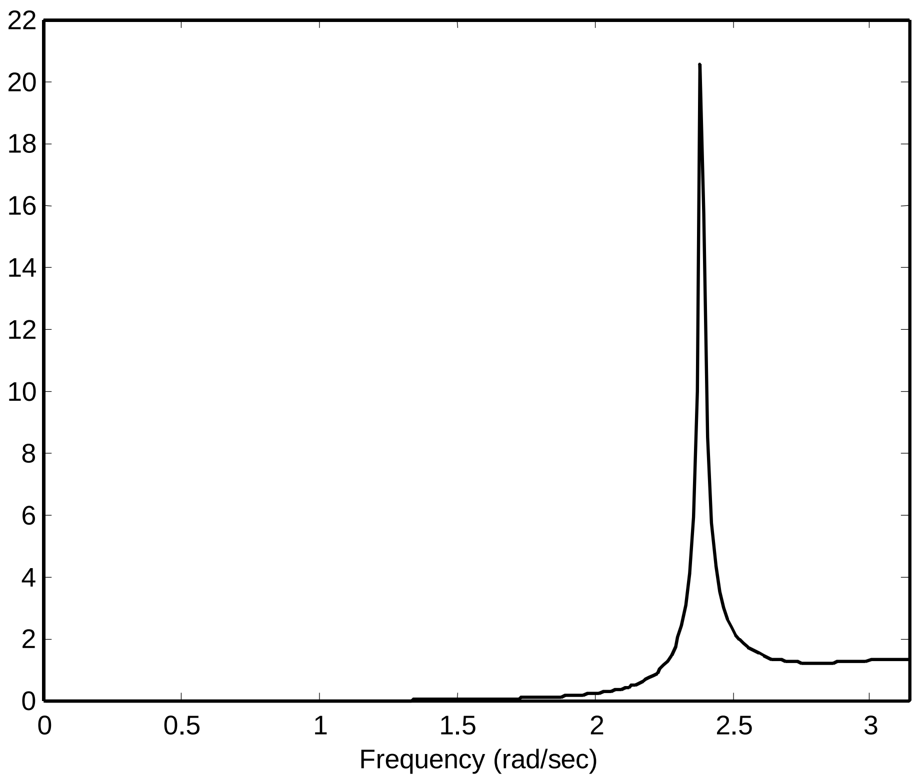

Figure 7: **Plot of** $\bar{\sigma}\left(\hat{\Delta}_C\left(P_t, P_t'\right)\right)(\omega)$

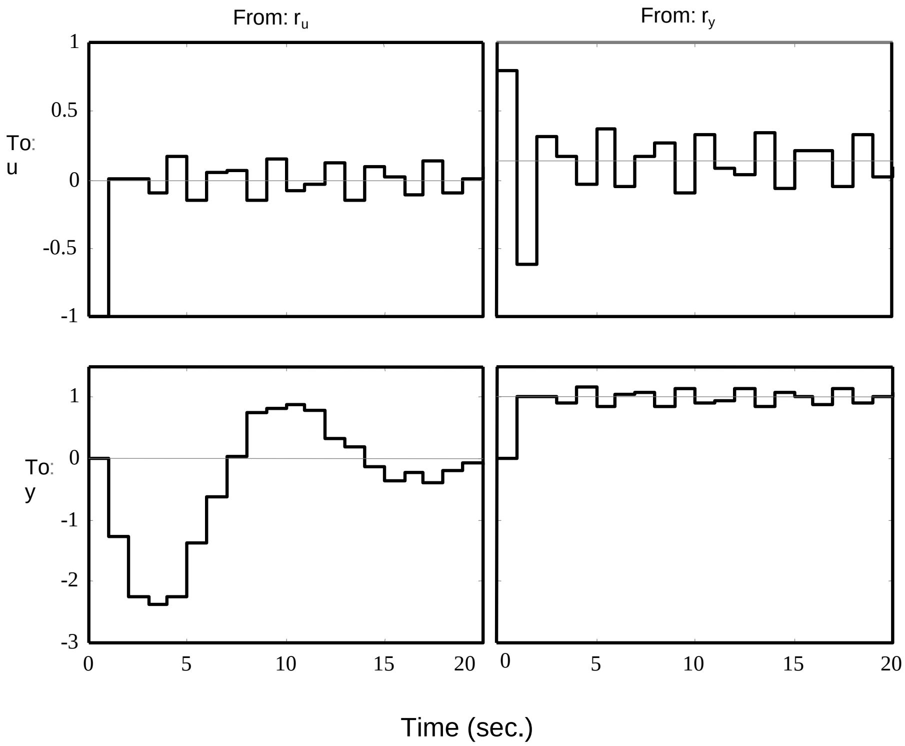

Figure 8: Closed-loop step response when the controller is connected to the perturbed plant