# Matching, linear systems, and the ball and beam 

F. Andreev ${ }^{1,3}$, D. Auckly ${ }^{1,4}$, L. Kapitanski ${ }^{1,2,4}$, S. Gosavi ${ }^{1,5}$, W. White ${ }^{1,5}$, A. Kelkar ${ }^{1,6}$<br>${ }^{3}$ Department of Mathematics, Western Illinois University, Macomb, IL 61455, USA<br>${ }^{4}$ Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA<br>${ }^{5}$ Department of Mechanical and Nuclear Engineering, Kansas State University, Manhattan, KS 66506, USA<br>${ }^{6}$ Department of Mechanical Engineering, Iowa State University, Ames, IA 50011, USA


#### Abstract

A recent approach to the control of underactuated systems is to look for control laws which will induce some specified structure on the closed loop system. In this paper, we describe one matching condition and an approach for finding all control laws that fit the condition. After an analysis of the resulting control laws for linear systems, we present the results from an experiment on a nonlinear ball and beam system.


## 1 Underactuated systems and the matching condition

Over the past five years several researchers have proposed nonlinear control laws for which the closed loop system assumes some special form, see the controlled Lagrangian method of [8, 9, 10] the generalized matching conditions of $11,12,13$, the interconnection and damping assignment passivity based control of |7], the $\lambda$-method of [6, 5], and the references therein. In this paper we describe the implementation of the $\lambda$-method of [6] on a ball and beam system. For the readers convenience we start with the statement of the main theorem on $\lambda$-method matching control laws (Theorem 1). We also present an indicial derivation of the main equations. We then prove a new theorem showing that the family

[^0]of matching control laws of any linear time invariant system contains all linear state feedback control laws (Theorem 2). We next present the general solution of the matching equations for the Quanser ball and beam system. (Note, that this system is different from the system analyzed by Hamberg, 11.) As always, the general solution contains several free functional parameters that may be used as tuning parameters. We chose these arbitrary functions in order to have a fair comparison with the manifacturer's linear control law. Our laboratory tests confirm the predicted stabilization. This was our first experimental test of the $\lambda$-method. We later tested this method on an inverted pendulum cart, [3].

Consider a system of the form

$$
\begin{equation*}
g_{r j} \ddot{x}^{j}+[j k, r] \dot{x}^{j} \dot{x}^{k}+C_{r}+\frac{\partial V}{\partial x^{r}}=u_{r} \tag{1}
\end{equation*}
$$

$r=1, \ldots, n$, where $g_{i j}$ denotes the mass-matrix, $C_{r}$ the dissipation, $V$ the potential energy, $[i j, k]$ the Christoffel symbol of the first kind,

$$
\begin{equation*}
[j k, i]=\frac{1}{2}\left(\frac{\partial g_{i j}}{\partial q^{k}}+\frac{\partial g_{k i}}{\partial q^{j}}-\frac{\partial g_{j k}}{\partial q^{i}}\right) \tag{2}
\end{equation*}
$$

and $u_{r}$ is the applied actuation. To encode the fact that some degrees of freedom are unactuated, the applied forces and/or torques are restricted to satisfy $P_{j}^{i} g^{j k} u_{k}=0$, where $P_{j}^{i}$ is a $g$-orthogonal projection. The matching conditions come from this restriction together with the requirement that the closed loop
system takes the form

$$
\left.\widehat{g}_{r j} \ddot{x}^{j}+\widehat{[j k, r}\right] \dot{x}^{j} \dot{x}^{k}+\widehat{C}_{r}+\frac{\partial \widehat{V}}{\partial x^{r}}=0, r=1, \ldots, n
$$

for some choice of $\widehat{g}, \widehat{C}$, and $\widehat{V}$. The matching conditions read

$$
\begin{align*}
& P_{k}^{r}\left(\Gamma_{i j}^{k}-\widehat{\Gamma}_{i j}^{k}\right)=0, P_{k}^{r}\left(g^{k i} C_{i}-\widehat{g}^{k i} \widehat{C}_{i}\right)=0 \\
& P_{k}^{r}\left(g^{k i} \frac{\partial V}{\partial q^{i}}-\widehat{g}^{k i} \frac{\partial \widehat{V}}{\partial q^{i}}\right)=0 \tag{3}
\end{align*}
$$

where $\Gamma_{i j}^{k}$ is the Christoffel symbol of the second kind,

$$
\begin{equation*}
\Gamma_{i j}^{k}=g^{k \ell}[i j, \ell] . \tag{4}
\end{equation*}
$$

If the matching conditions (3) hold, the control law will be given by

$$
\begin{align*}
u_{r} & =g_{r k}\left(\Gamma_{i j}^{k}-\widehat{\Gamma}_{i j}^{k}\right) \dot{q}^{i} \dot{q}^{j}+\left(C_{r}-\widehat{C}_{r}\right) \\
& +g_{r k}\left(g^{k i} \frac{\partial V}{\partial q^{i}}-\widehat{g}^{k i} \frac{\partial \widehat{V}}{\partial q^{i}}\right) . \tag{5}
\end{align*}
$$

The motivation for this method is that $\widehat{H}=$ $\frac{1}{2} \widehat{g}_{i j} \dot{q}^{i} \dot{q}^{j}+\widehat{V}$ is a natural candidate for a Lyapunov function because $\frac{d}{d t} \widehat{H}=-\widehat{g}_{i j} \widehat{c}^{i} \dot{q}^{j}$. Following [6], introduce new variables $\lambda_{i}^{k}=g_{i j} \widehat{g}^{j k}$. We have

Theorem 1 The functions $\widehat{g}_{i j}, \widehat{V}$, and $\widehat{C}$ satisfy (3) in a neighborhood of $x_{0}$ if and only if

$$
\begin{aligned}
P_{k}^{r}\left(g^{k i} C_{i}-\widehat{g}^{k i} \widehat{C}_{i}\right) & =0, \\
P_{k}^{r}\left(g^{k i} \frac{\partial V}{\partial q^{i}}-\widehat{g}^{k i} \frac{\partial \widehat{V}}{\partial q^{i}}\right) & =0,
\end{aligned}
$$

and the following conditions hold. First, there exists a hypersurface containing $x_{0}$ and transverse to each of the vectorfields $\lambda_{i}^{\ell} P_{j}^{i} \partial / \partial x^{\ell}$ on which $\widehat{g}_{i j}$ is invertible and symmetric and satisfies

$$
g_{k i} P_{\ell}^{k}=\lambda_{k}^{j} P_{\ell}^{k} \widehat{g}_{j i} .
$$

Second, $\lambda_{j}^{i} P_{k}^{j}$ and $\widehat{g}_{i j}$ satisfy

$$
\begin{gather*}
\quad P_{k}^{s} P_{t}^{r}\left(g_{\ell s} \frac{\partial \lambda_{r}^{\ell}}{\partial q^{j}}+[\ell j, s] \lambda_{r}^{\ell}-[r j, i] \lambda_{s}^{i}\right. \\
\left.+g_{i r} \frac{\partial \lambda_{s}^{i}}{\partial q^{j}}+[i j, r] \lambda_{s}^{i}-[s j, \ell] \lambda_{r}^{\ell}\right)=0  \tag{6}\\
\lambda_{r}^{\ell} P_{t}^{r} \frac{\partial \widehat{g}_{n m}}{\partial q^{\ell}}+\widehat{g}_{\ell n} \frac{\partial\left(\lambda_{r}^{\ell} P_{t}^{r}\right)}{\partial q^{m}}+\widehat{g}_{\ell m} \frac{\partial\left(\lambda_{r}^{\ell} P_{t}^{r}\right)}{\partial q^{n}} \\
=  \tag{7}\\
P_{t}^{\ell} \frac{p a r t i a l g_{n m}}{\partial q^{\ell}}++g_{\ell n} \frac{\partial P_{t}^{\ell}}{\partial q^{m}} g_{\ell m} \frac{\partial P_{t}^{\ell}}{\partial q^{n}}
\end{gather*}
$$

Although the proof of this proposition may be found in [6], [4], and [5], for convenience, we include an indicial notation derivation of equations (6) and (7). Substitute equations (2), (4) for both $\Gamma_{i j}^{k}$ and $\widehat{\Gamma}_{i j}^{k}$ into the first of equations (3) and multiply the result by the scalar 2 to obtain:

$$
\begin{aligned}
& P_{k}^{r} \widehat{g}^{k \ell} \frac{\partial \widehat{g}_{i j}}{\partial q^{\ell}}-P_{k}^{r} \widehat{g}^{k \ell} \frac{\partial \widehat{g}_{\ell i}}{\partial q^{j}}-P_{k}^{r} \widehat{g}^{k \ell} \frac{\partial \widehat{g}_{j \ell}}{\partial q^{i}} \\
& \quad=P_{k}^{r} g^{k \ell} \frac{\partial g_{i j}}{\partial q^{\ell}}-P_{k}^{r} g^{k \ell} \frac{\partial g_{\ell i}}{\partial q^{j}}-P_{k}^{r} g^{k \ell} \frac{\partial g_{j \ell}}{\partial q^{i}}
\end{aligned}
$$

Multiply by $g_{r t}$ and use that $P$ is self-adjoint i.e., $P_{i}^{k} g_{k j}=g_{i k} P_{j}^{k}$, to get

$$
\begin{align*}
& P_{t}^{r} \lambda_{r}^{\ell} \frac{\partial \widehat{g}_{i j}}{\partial q^{\ell}}-P_{t}^{r} \lambda_{r}^{\ell} \frac{\partial \widehat{g}_{\ell i}}{\partial q^{j}}-P_{t}^{r} \lambda_{r}^{\ell} \frac{\partial \widehat{g}_{j \ell}}{\partial q^{i}} \\
& \quad=P_{t}^{r} \frac{\partial g_{i j}}{\partial q^{r}}-P_{t}^{r} \frac{\partial g_{r i}}{\partial q^{j}}-P_{t}^{r} \frac{\partial g_{j r}}{\partial q^{i}} \tag{8}
\end{align*}
$$

Use $P_{t}^{r} \lambda_{r}^{\ell} \frac{\partial \widehat{g}_{\ell i}}{\partial q^{j}}=\frac{\partial\left(P_{t}^{r} \lambda_{r}^{\ell} \widehat{g}_{e_{i}}\right)}{\partial q^{j}}-\widehat{g}_{\ell i} \frac{\left(\partial P_{t}^{r} \lambda_{r}^{\ell}\right)}{\partial q^{j}}$ and

$$
\begin{equation*}
\lambda_{r}^{\ell} \widehat{g}_{\ell i}=g_{r i} \tag{9}
\end{equation*}
$$

in (8) to obtain (7). To derive (6), first, differentiate (9) with respect to $q^{j}$ to get

$$
\begin{equation*}
\lambda_{r}^{\ell} \frac{\partial \widehat{g}_{\ell i}}{\partial q^{j}}=\frac{\partial g_{r i}}{\partial q^{j}}-\widehat{g}_{\ell i} \frac{\partial \lambda_{r}^{\ell}}{\partial q^{j}} \tag{10}
\end{equation*}
$$

Substitute equation (10) into equation (8) and obtain

$$
\begin{align*}
& P_{t}^{r}\left(\widehat{g}_{\ell i} \frac{\partial \lambda_{r}^{\ell}}{\partial q^{j}}+\widehat{g}_{\ell j} \frac{\partial \lambda_{r}^{\ell}}{\partial q^{i}}+\lambda_{r}^{\ell} \frac{\partial \widehat{g}_{i j}}{\partial q^{\ell}}\right) \\
& =P_{t}^{r}\left(\frac{\partial g_{r i}}{\partial q^{j}}+\frac{\partial g_{r j}}{\partial q^{i}}-\frac{\partial g_{r i}}{\partial q^{j}}-\frac{\partial g_{j r}}{\partial q^{i}}+\frac{\partial g_{i j}}{\partial q^{r}}\right) . \tag{11}
\end{align*}
$$

Multiply by $-P_{k}^{s} \lambda_{s}^{i}$, use (9) and (10) to obtain

$$
\begin{align*}
& P_{k}^{s} P_{t}^{r}\left(g_{\ell s} \frac{\partial \lambda_{r}^{\ell}}{\partial q^{j}}+\lambda_{r}^{\ell} \frac{\partial g_{j s}}{\partial q^{\ell}}-\lambda_{s}^{i} \frac{\partial g_{i j}}{\partial q^{r}}\right) \\
& =P_{k}^{s} P_{t}^{r}\left(\widehat{g}_{i j} \lambda_{r}^{\ell} \frac{\partial \lambda_{s}^{i}}{\partial q^{\ell}}-\lambda_{s}^{i} \widehat{g}_{\ell j} \frac{\partial \lambda_{r}^{\ell}}{\partial q^{i}}\right) \tag{12}
\end{align*}
$$

Finally, to obtain (6), add to equation (12) an equation obtained from (12) by interchanging $k$ and $t, r$ and $s, \ell$ and $i$.

## 2 Matching and constant coefficient linear systems

In this section, we prove that for linear time invariant systems any linear full state feedback control law is a solution to the matching equations.

Theorem 2 When applied to linear, timeindependent systems, the family of matching control laws contains all linear state feedback laws.

Choose coordinates $q^{i}$ so that the desired equilibrium is at the origin, $V=V_{i j} q^{i} q^{j}+v_{k} q^{k}$, and $C_{i}=C_{i j} \dot{q}^{j}$, where $g_{i j}, V_{i j}, v_{k}, C_{i j}$, and $P_{k}^{r}$ are constant, and $P_{k}^{r}$ has rank $n_{u}$. Clearly, $\widehat{g}_{i j}, \widehat{V}=\widehat{V}_{i j} q^{i} q^{j}$, and $\widehat{C}_{i}=\widehat{C}_{i j} \dot{q}^{j}$ is a solution to the matching equations when $\widehat{g}_{i j}, \widehat{V}_{i j}$, and $\widehat{C}_{i j}$ are constant provided $\widehat{g}_{i j}$ and $\widehat{V}_{i j}$ are symmetric, $P_{k}^{r}\left(g^{k i} V_{i j}-\widehat{g}^{k i} \widehat{V}_{i j}\right)=0$, and $P_{k}^{r}\left(g^{k i} C_{i j}-\widehat{g}^{k i} \widehat{C}_{i j}\right)=0$. Let $u_{k}=v_{k}+a_{k i} q^{i}+$ $b_{k i} \dot{q}^{i}$ be an arbitrary linear control law, satisfying $P_{k}^{r} g^{k \ell} u_{\ell}=0$. Comparison with equation (3) gives

$$
g_{r k}\left(g^{k i} V_{i j}-\widehat{g}^{k i} \widehat{V}_{i j}\right)=a_{r j}
$$

and

$$
g_{r k}\left(g^{k i} C_{i j}-\widehat{g}^{k i} \widehat{C}_{i j}\right)=b_{r j}
$$

Thus,

$$
\widehat{V}_{\ell j}=\widehat{g}_{\ell p} g^{p r}\left(V_{r j}-a_{r j}\right)
$$

and

$$
\widehat{C}_{\ell j}=\widehat{g}_{\ell p} g^{p r}\left(C_{r j}-b_{r j}\right)
$$

It remains to check that we can find a symmetric, nondegenerate matrix $\widehat{g}^{k i}$ so that the resulting $\widehat{V}_{\ell j}$ is
also symmetric. The symmetry of $\widehat{V}_{\ell j}$ will follow if we have

$$
\widehat{g}_{\ell p} g^{p r}\left(V_{r j}-a_{r j}\right)-\widehat{g}_{j p} g^{p r}\left(V_{r \ell}-a_{r \ell}\right)=0
$$

and, therefore, we need to find a symmetric, nondegenerate matrix $\widehat{g}_{\ell p}$ satisfying this equation. The existence of such matrix is guaranteed by the following simple observation.

Lemma 1 Given any real $n \times n$ matrix $R$, there is a nondegenerate symmetric matrix $X$ so that

$$
R X-X^{T} R^{T}=0
$$

Indeed, setting $X=Q Y Q^{T}$, results in the following equation for $Y$ :

$$
Q^{-1} R Q Y-Y^{T}\left(Q^{-1} R Q\right)^{T}=0
$$

Hence, without loss of generality we may assume that $Q^{-1} R Q$ is a real Jordan block (see 14),

$$
\left(\begin{array}{cccc}
\lambda & 1 & 0 & \ldots \\
0 & \lambda & 1 & \ldots \\
& & \cdots &
\end{array}\right)
$$

or

$$
\left(\begin{array}{rrrrccc}
a & -b & 1 & 0 & 0 & 0 & \\
b & a & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & a & -b & 1 & 0 & \\
0 & 0 & b & a & 0 & 1 & \cdots \\
\ldots & & \ldots & & \ldots & & \ldots
\end{array}\right)
$$

In each case $Y=\left(\begin{array}{cccc}0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ & & \ldots & \\ 1 & 0 & \ldots & 0\end{array}\right)$ solves the equation.

Note that the result of Lemma 1 is true for matrices with coefficients in any field. This is proved in 15.

## 3 Example: The Ball and Beam

In order to demonstrate the approach described above, we have implemented one of the control laws from the family of control laws described in the first

Table 1: The physical parameters of the system.

| $\ell_{b}=$ length of the beam | $=0.43 \mathrm{~m}$ | $I_{B}=\frac{2}{5} m_{B} r_{B}^{2}=$ ball inertia | $=4.25 \times 10^{-6} \mathrm{Kg} \mathrm{m}$ |
| :--- | :--- | :--- | :--- |
| $\ell_{l}=$ length of the link | $=0.11 \mathrm{~m}$ | $I_{b}=$ inertia of the beam | $=.001 \mathrm{Kg} \mathrm{m}^{2}$ |
| $r_{g}=$ radius of the gear | $=0.03 \mathrm{~m}$ | $I_{s}=$ effective servo inertia | $=0.002 \mathrm{Kg} \mathrm{m}^{2}$ |
| $r_{B}=$ radius of the ball | $=0.01 \mathrm{~m}$ | $g=$ gravitational acceleration | $=9.8 \mathrm{~m} / \mathrm{s}^{2}$ |
| $m_{B}=$ mass of the ball | $=0.07 \mathrm{Kg}$ | $s_{0}=$ desired equilibrium position | $=0.22 \mathrm{~m}$ |
| $m_{b}=$ mass of the beam | $=0.15 \mathrm{Kg}$ | $c_{0}=$ inherent servo dissipation | $=9.33 \times 10^{-10} \mathrm{Kg} \mathrm{m}^{2} / \mathrm{s}$ |
| $m_{l}=$ mass of the link | $=0.01 \mathrm{Kg}$ |  |  |



Figure 1: The ball and beam system
section on a ball and beam system, Figure 1 (this system is commercially available from Quanser Consulting, Ontario, Canada).

The $s$-coordinate is unactuated, the $\theta$-coordinate is actuated by the servo, and the objective is to bring the ball to the center of the beam. The physical parameters of the system are given in Table 1.
One can express $\alpha$ as a concrete function of $\theta$ from the kinematic relation

$$
\begin{aligned}
& \left(\ell_{b}(1-\cos (\alpha))-r_{g}(1-\cos (\theta))\right)^{2} \\
& +\left(\ell_{b} \sin (\alpha)+\ell_{l}-r_{g} \sin (\theta)\right)^{2}=\ell_{l}^{2}
\end{aligned}
$$

The kinetic energy of the system is

$$
T=\frac{1}{2} m_{b} s^{2} \dot{\alpha}^{2}+\frac{1}{2} I_{B}\left(\dot{\alpha}+\frac{1}{r_{B}} \dot{s}\right)^{2}+\frac{1}{2} I_{b} \dot{\alpha}^{2}+\frac{1}{2} I_{s} \dot{\theta}^{2} .
$$

The potential energy is

$$
\begin{aligned}
& V=\frac{1}{2} m_{l} g r_{g} \sin (\theta)+\frac{1}{2}\left(m_{b}+m_{l}\right) g \ell_{b} \sin (\alpha) \\
& +m_{B} g s \sin (\alpha)
\end{aligned}
$$

and the dissipation is $C_{1}=0, C_{2}=c_{0} \dot{\theta}$. After rescaling, we get

$$
\begin{gathered}
\left(1-\cos (\alpha)-a_{2}(1-\cos (\theta))\right)^{2} \\
\quad+\left(\sin (\alpha)+a_{1}-a_{2} \sin (\theta)\right)^{2}=a_{1}^{2} \\
T=\frac{1}{2} \dot{s}^{2}+\frac{1}{2}\left(a_{4}+\left(a_{3}+5 / 2 s^{2}\right)\left(\alpha^{\prime}(\theta)\right)^{2}\right) \dot{\theta}^{2}+\alpha^{\prime}(\theta) \dot{s} \dot{\theta}, C_{1}=0, \\
V=a_{5} \sin (\theta)+\left(s+a_{6}\right) \sin (\alpha(\theta))
\end{gathered}
$$

$C_{1}=0$, and $C_{2}=a_{7} \dot{\theta}$, where the $a_{k}$ are the dimensionless parameters,

$$
\begin{gathered}
a_{1}=\frac{\ell_{l}}{\ell_{b}}, \quad a_{2}=\frac{r_{g}}{\ell_{b}}, \quad a_{3}=\frac{\left(I_{b}+I_{B}\right)}{I_{B}}, \quad a_{4}=\frac{I_{s}}{I_{B}} \\
a_{5}=\frac{m_{l} r_{g}}{2 m_{B} r_{B}}, \quad a_{6}=\frac{\ell_{b}\left(m_{b}+m_{l}\right)}{2 m_{B} r_{B}} \\
a_{7}=\left(\frac{5}{2 r_{B}^{3} g}\right)^{\frac{1}{2}} \frac{c_{0}}{m_{B}}
\end{gathered}
$$

The notation ' is used to denote a derivative of a function of one variable. For general underactuated systems, the use of the powerful $\lambda$-method to solve the matching equations is discussed in [6, 5]. For systems with two degrees of freedom, the $\lambda$-method produces the general solution to the matching equations in an explicit form, [4]. When applied to the ball and beam system, the explicit family of control laws is given by equation (5) with the following expressions for $\widehat{g}, \widehat{V}$, and $\widehat{C}$, where

$$
\begin{gathered}
\widehat{g}_{11}=\psi^{2}(\alpha)\left(h(y(s, \theta))+10 \int_{0}^{\alpha} \frac{d \varphi}{\mu_{1}^{\prime}(\varphi) \psi^{2}(\varphi)}\right), \\
\widehat{g}_{12}=\frac{1}{\mu}\left(g_{11}-\sigma \widehat{g}_{11}\right), \widehat{g}_{22}=\frac{1}{\mu}\left(g_{12}-\sigma \widehat{g}_{12}\right)
\end{gathered}
$$

$$
\begin{aligned}
\widehat{V}(s, \theta) & =w(y)+5\left(y+s_{0}\right) \int_{0}^{\alpha} \frac{\sin (\varphi)}{\mu_{1}^{\prime}(\varphi) \psi(\varphi)} d \varphi \\
& -5 \int_{0}^{\alpha} \frac{\sin (\varphi)}{\mu_{1}^{\prime}(\varphi) \psi(\varphi)} \int_{0}^{\varphi} \psi(\tau) d \tau d \varphi \\
\widehat{C}_{1} & =\left(g_{1 i} \widehat{g}^{i 1}\right)^{-1}\left(C_{1}-g_{1 j} \widehat{g}^{j 2} \widehat{C}_{2}\right)
\end{aligned}
$$

where

$$
\begin{gathered}
\mu(s, \theta)=\frac{\mu_{1}^{\prime}(\alpha(\theta))}{5 s g_{12}} \\
\sigma(s, \theta)=\mu_{1}(\alpha)-\frac{1}{5 s} \mu_{1}^{\prime}(\alpha) \\
\left.y=\psi(\alpha) s-s_{0}+\int_{0}^{\alpha} \psi(\tau) d \tau, \frac{\mu_{1}(\kappa)}{\mu_{1}^{\prime}(\kappa)} d \kappa\right\} .
\end{gathered}
$$

Here $h(y), w(y), \mu_{1}(\alpha)$ are arbitrary functions of one variable, and $\widehat{C}_{2}$ is an arbitrary function which is odd in velocities.

## 4 Experimental Results

Our experiments were conducted on the Quanser ball and beam system. The control signal is a voltage supplied to the servo and the sensed output of the system is $s$ and $\theta$ sampled at 300 Hz . A Quanser MULTIQ ${ }^{\circledR}$ data acquisition card is used for the analog signal input and output. The velocities are computed via numerical differentiation using the forward difference algorithm. The control law produces a voltage signal and is supplied through the $\mathrm{D} / \mathrm{A}$ converter to the DC servomotor via an amplifier. The relation between the control voltage, $v_{i n}$, and the torque, $u\left(=u_{2}\right.$ in equation (5) ) is $K_{m}^{2} N_{g}^{2} \dot{\theta}$, where $R_{m}=$ armature resistance $=2.6 \Omega, N_{g}=$ gear ratio $=70.5, K_{m}=$ motor torque constant $=0.00767$ Volt $\cdot$ sec.

Any stabilizing linear control law for this system is specified by four constants. The nonlinear control laws in our family are specified by the four arbitrary functions: $\mu_{1}(\alpha), h(y), w(y)$, and $\widehat{C}_{2}(s, \theta, \dot{s}, \dot{\theta})$. We
chose

$$
\begin{aligned}
\mu_{1}(\alpha) & =1.0849 \exp (4.7845 \sin (\alpha)) \\
h(y) & =1.1031, w(y)=0.0023 y^{2} \\
\widehat{C}_{2}(s, \theta, \dot{s}, \dot{\theta}) & =-\widehat{g}_{12} \cdot\left(1+\dot{s}^{2}+10 \dot{\theta}^{2}\right)(-\mu \dot{s}+\sigma \dot{\theta})
\end{aligned}
$$

These functions produce the control law, $u$, in rescaled units. The values of the constants $a_{1}$ through $a_{7}$ are as follows

$$
\begin{array}{ll}
a_{1}=0.2547 & a_{5}=0.1889 \\
a_{2}=0.0588 & a_{6}=42 \\
a_{3}=236.294 & a_{7}=5 \times 10^{-6} \\
a_{4}=471.126 &
\end{array}
$$

The final control signal is obtained by converting back into MKS units and using the formula in the preceding paragraph to get the input voltage. These choices were made from the following considerations. The form of the function $\mu_{1}$ was chosen to simplify the integrals in the expressions for $y, \psi$, and $\widehat{g}_{11}$. The form of $\widehat{C}_{2}$ was chosen to ensure that $\widehat{C}_{1} \dot{s}+\widehat{C}_{2} \dot{\theta}$ would be positive (for $\widehat{H}$ to be a Lyapunov function). Finally, the coefficients in these functions were chosen so that the linearization of the nonlinear control law would agree with the linear control law provided by the manufacturer.

Extensive numerical simulations done using Matlab ${ }^{\circledR}$ confirm that the nonlinear control law stabilizes the system. The linear control law appears to stabilize the system for a wider range of initial conditions than the nonlinear control law. This is an empirical observation, not a mathematical fact. Finding an adequate mathematical framework to compare different control laws is a very interesting unresolved problem, see $\sqrt[4]{4}$. Usually, given two locally stabilizing control laws, there exist initial conditions stabilized by one but not by the other. For example, one set of physically unrealistic initial conditions with a large angular velocity $\dot{\theta}=3.6$ (or $158 \mathrm{rad} / \mathrm{sec}$ in physical units) is stabilized by our nonlinear control law but not by the linear one.

We have implemented the nonlinear control law in the laboratory. The laboratory tests confirm the predicted behavior of the nonlinear controller. Figures 2 and 3 show a comparison of the time histories of the ball position $(s)$ and angular displacement $(\theta)$ for the


Figure 2: Ball position response
linear and nonlinear control laws. In both cases the control signal reached the saturation limit for a short duration during the initial rise of the response. The difference in the steady-state values of the responses is attributed to a lack of sensitivity of the resistive strip used to measure the ball position.

## 5 Conclusions

The $\lambda$-method produces explicit infinite-dimensional families of control laws and simultaneously provides a natural candidate for a Lyapunov function. When this method is applied to linear time-invariant systems, the resulting family contains all linear state feedback control laws (Proposition 2). In this paper we also present the results of the first implementation of a $\lambda$-method matching control law on a concrete physical device, the ball and beam system. The experimental results agree with theoretical predictions and numerical simulations. In our experiments we observe that the linear control law performs better than our nonlinear control law for the ball and beam system. However, in a later experiment with an inverted pendulum cart, [3], we found that a properly tuned $\lambda$-method matching control law performed better than the corresponding linear one. At the moment, it is not known for which systems matching


Figure 3: Angular displacement response
control laws will perform better. This is an important problem that must be resolved.

## References

[1] F. Andreev, D. Auckly, S. Gosavi, L. Kapitanski, A. Kelkar, and W. White, Matching, linear systems, and the ball and beam, http://arXiv.org/abs/math.OC/0006121
[2] F. Andreev, D. Auckly, L. Kapitanski, A. Kelkar, W. White, Matching control laws for a ball and beam system, Proc. IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, Princeton (2000) 161-162;
[3] F. Andreev, D. Auckly, L. Kapitanski, A. Kelkar, W. White, Matching and digital control implementation for underactuated systems, Proceedings of the American Control Conference, Chicago, IL, (2000) 3934-3938.
[4] D. Auckly, L. Kapitanski, Mathematical problems in the control of underactuated systems, CRM Proceedings and Lecture Notes 27 (2000) 41-52.
[5] D. Auckly, L. Kapitanski, On the $\lambda$-equations for matching control laws, submitted
[6] D. Auckly, L. Kapitanski and W. White, Control of nonlinear underactuated systems, Communications on Pure Appl. Math. 53 (2000) 354-369.
[7] G. Blankenstein, R. Ortega and A. J. van der Schaft, The matching conditions of controlled Lagrangians and interconnection and damping assignment passivity based control, Preprint (2001)
[8] A. Bloch, N. Leonard and J. Marsden, Stabilization of mechanical systems using controlled Lagrangians, Proc. IEEE Conference on Decision and Control, San Diego, CA (1997) 2356-2361.
[9] A. Bloch, D. Chang, N. Leonard and J. E. Marsden, Controlled Lagrangians and the stabilization of mechanical systems II: Potential shaping, Trans IEEE on Auto. Control 46 (2001) 15561571.
[10] A. Bloch, N. Leonard and J. Marsden, Controlled Lagrangians and a stabilization of mechanical systems I: The first matching theorem, IEEE Trans. Automat. Control 45 (2000) 22532270.
[11] J. Hamberg, General matching conditions in the theory of controlled Lagrangians, in Proc. IEEE Conference on Decision and Control, Phoenix, AZ (1999)
[12] J. Hamberg, Controlled Lagrangians, symmetries and conditions for strong matching, Proc. IFAC Workshop on Lagrangian and Hamiltonian methods for nonlinear control, Princeton, NJ (2000) 62-67.
[13] J. Hamberg, Simplified conditions for matching and for generalized matching in the theory of controlled Lagrangians, Proc. ACC, Chicago, Illinois (2000) 3918-3923.
[14] R. Horn and C. Johnson, Matrix analysis, Cambridge University Press, Cambridge, 1985.
[15] O. Taussky and H. Zassenhaus, On the similarity transformation between a matrix and its transpose, Pacific J. Math., 9 (1959) 893-896.


[^0]:    ${ }^{1}$ Supported in part by NSF grant CMS 9813182
    ${ }^{2}$ Supported in part by NSF grant DMS 9970638

