Pergamon

PII: S0005-1098(97)00055-1

Automatica, Vol. 33, No. 7, pp. 1393–1399, 1997

Printed in Great Britain
0005-1098/97 $17.00 + 0.00

Brief Paper

Tracking Control of Mobile Robots: A Case Study in Backstepping*

ZHONG-PING JIANG† and HENK NIJMEIJER‡

Key Words—Mobile robots; tracking; time-varying feedback; velocity control; backstepping.

Abstract—A tracking control methodology via time-varying state feedback based on the backstepping technique is proposed for both a kinematic and simplified dynamic model of a two-degrees-of-freedom mobile robot. We first address the local tracking problem where initial tracking errors are sufficiently small. Then, under additional conditions on the desired velocities, we treat the global tracking problem where initial tracking errors are arbitrary. Simulation results are provided to validate and analyse our theoretical results. © 1997 Elsevier Science Ltd.

1. *Introduction*

In recent years there has been enormous activity in the study of a class of mechanical control systems called *nonholonomic systems*. In particular, many kinematic models of physical systems (i.e. systems where velocities are treated as input signals) belong to this category, see the survey by Kolmanovsky and McClamroch (1995) and references cited therein. Controlling such nonholonomic systems turns out to be a nontrivial problem for a number of reasons. Even in the simplest case, which we shall study here, the kinematic model of a two-wheel mobile robot, the stabilization (or parking) problem at a given position requires a nontrivial controller (see e.g. Samson, 1991; Pomet, 1992; Murray *et al.*, 1992; Bloch and Drakunov, 1994; Canudas de Wit *et al.*, 1994; McCloskey and Murray, 1994; Oelen *et al.*, 1995). The crucial problem in this stabilization question centers around the fact that the mobile robot model does not meet Brockett's well-known necessary smooth feedback stabilization condition (Brockett 1983), therefore immediately leading to more complex-structured controllers as either time-varying controllers or approximate, practically stabilizing, controllers. The (from an engineering perspective) very interesting tracking problem for mobile robots has been addressed quite rarely (cf. Kanayama *et al.*, 1990; Murray *et al.*, 1992, Oelen and van Amerongen, 1994; Micaelli and Samson, 1993; Fierro and Lewis, 1995). In all these papers, basically a local viewpoint in the stabilizing feedback design has been taken by using the Taylor linearization of the corresponding error model. A dynamic feedback linearization approach was proposed in Canudas de Wit *et al.* (1996, Chapter 8) that allows (local) posture tracking with exponential convergence for restricted mobility robots. Similar results were obtained in Fliess *et al.* (1995a, b) using time-reparametrization and motion-planning properties of differentially flat systems (systems that have the property that they are linearizable using a dynamic state feedback).

* Received 4 April 1996; revised 30 September 1996; received in final form 13 February 1997. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Alberto Isidori under the direction of Editor Tamer Başar.
Corresponding author Dr Zhong-Ping Jiang. Tel. +61 2 9351 4641; Fax +61 2 9351 3847; E-mail zjiang@ee.usyd.edu.au.

† Department of Electrical Engineering, Building J13, University of Sydney, NSW 2006, Australia.

‡ Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands.

The purpose of the present paper is to use Lyapunov's direct method for obtaining semiglobal and global results in the tracking problem for the mobile robot. In particular, under our proposed time-varying controllers, the two-degrees-of-freedom mobile robot can globally follow special paths such as straight lines and circles (see Remark 4 below). We do this for both the kinematic model and an 'integrated' simplified dynamic model of the mobile robot. In both cases, the design technique to obtain a suitable feedback control law is based upon the integrator backstepping procedure. The latter idea was firstly discovered by Koditschek (1987) and then developed in independent work in the context of nonlinear stabilization (see e.g., Byrnes and Isidori, 1989; Tsinias, 1989) and adaptive nonlinear control (see e.g. Krstić *et al.*, 1995). Applications of the backstepping technique to the adaptive control of nonholonomic systems with unknown parameters and the global stabilization of multi-input chained-form nonholonomic systems were recently considered in Jiang and Pomet (1994, 1995) and Jiang (1996).

The theoretical results obtained in this paper are illustrated by means of simulations using the local (semiglobal) controller and the global controller under changing initial conditions.

The organisation of the paper is as follows. We start with basic concepts, stability definitions and preliminary results in Section 2.1. Section 2.2 is devoted to modelling of the tracking configuration for a wheeled mobile robot and the statement of our problems. In Section 3, we first propose time-varying feedback control laws that solve the local tracking problem. Then, under extra (mild) conditions on the desired velocities, we solve the global tracking problem via time-varying state feedback. Along the way, a solution for local exponential stabilization is given. In section 4, we show how to extend our control method to the tracking problem for the mobile robot described by a simplified dynamic model. Several simulation results are presented in Section 5 to demonstrate our theoretical results. We close with some brief concluding remarks in Section 6.

2. *Preliminaries and problem formulation*

2.1. *Preliminaries.* For any bounded function $\psi:(a,b)\to\mathbb{R}$, $\|\psi\|_\infty$ means its L_∞ norm, i.e. $\|\psi\|_\infty=\sup\{\psi(x);\ a<x<b\}$. $L_p(a,b)$ represents the set of measurable functions f from (a,b) to $\mathbb{R}$ such that $\int_a^b|f(x)|^p\,dx<+\infty$. For any differentiable function $\varphi:(a,b)\to\mathbb{R}$, $\varphi'(x)$ is the derivative of φ at x (not to be confused with $\dot\varphi(x(t))$, which is the time derivative of $\varphi(x(t))$). We write $\varphi\in C^\infty$ if φ is a smooth function. For any function $g:\mathbb{R}_+\to\mathbb{R}$, $\liminf_{t\to\infty}g(t)$ denotes the limit inferior of $g(t)$ as $t\to\infty$, i.e. $\liminf_{t\to\infty}g(t)=\sup_{\tau\geq0}[\inf_{t\geq\tau}g(t)]$.

Next, we recall some basic concepts about stability theory (see e.g. Khalil, 1992; Vidyasagar, 1993). A function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ is *of class K* if γ is strictly increasing, continuous and $\gamma(0)=0$. It is *of class* K_∞ if furthermore $\gamma(x)$ goes to ∞ as x goes to ∞. A function $V:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}$ is said to be *positive-definite* if (i) it is continuous, (ii) $V(t,0)=0\ \forall t\geq0$ and (iii) there exists a function γ_1 of class K such that

$$\gamma_1(|x|)\leq V(t,x)\quad\forall(t,x)\in\mathbb{R}_+\times\mathbb{R}^n.\tag{1}$$

V is *decrescent* if there exists a function γ_2 of class K such that

$$V(t,x) \le \gamma_2(|x|) \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{2}$$

V is *radially unbounded* if (1) holds for some continuous function γ_1 (not necessarily of class K) satisfying $\gamma_1(r) \to \infty$ as $r \to \infty$.

Consider a nonautonomous system

$$\dot{x} = f(t,x), \quad x \in \mathbb{R}^n, \tag{3}$$

with f a continuously differentiable function such that $f(t,0) = 0$ for all $t \ge 0$.

Definition

(i) The solutions of the system (3) are *uniformly bounded* if for any $\alpha > 0$ and $t_0 \ge 0$, there exists a $\beta(\alpha) > 0$ such that

$$|x(t_0)| < \alpha, \quad t_0 \ge 0 \Rightarrow |x(t)| < \beta \quad \forall t \ge t_0. \tag{4}$$

(ii) The zero equilibrium (i.e. $x = 0$) of the system (3) is *uniformly stable* if, for each $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$|x(t_0)| < \delta(\varepsilon), \quad t_0 \ge 0 \Rightarrow |x(t)| < \varepsilon \quad \forall t \ge t_0 \tag{5}$$

In the following, we give two technical lemmas that are of frequent use in proving our results. Recall that a function $\phi:(a,b) \to \mathbb{R}$ is uniformly continuous if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $|x_1 - x_2| < \delta$, with $x_1, x_2 \in (a,b)$, then $|\phi(x_1) - \phi(x_2)| < \varepsilon$.

Lemma 1. (Barbălat). If $\varphi:\mathbb{R}_+ \to \mathbb{R}$ is uniformly continuous and if the limit of the integral $\int_0^t \varphi(\tau)\,d\tau$ exists as $t \to \infty$ and is finite then

$$\lim_{t\to\infty} \varphi(t) = 0. \tag{6}$$

Proof. See Popov (1973, p. 211).

In the same vein, the following lemma can be proved.

Lemma 2. Consider a scalar system

$$\dot{x} = -cx + p(t), \tag{7}$$

where $c > 0$ and $p(t)$ is a bounded and uniformly continuous function. If, for any initial time $t_0 \ge 0$ and any initial condition $x(t_0)$, the solution $x(t)$ is bounded and converges to 0 as $t \to \infty$ then

$$\lim_{t\to\infty} p(t) = 0. \tag{8}$$

Proof. See Jiang and Nijmeijer (1996). □

2.2. *Problem formulation.* The problem we study deals with a wheeled mobile robot with two degrees of freedom. The robot's dynamics is described by the following differential equations:

$$\begin{aligned} \dot{x} &= v\cos\theta, \\ \dot{y} &= v\sin\theta, \\ \dot{\theta} &= \omega, \end{aligned} \tag{9}$$

where v is the linear velocity and ω is the angular velocity of the mobile robot; (x, y) are the Cartesian coordinates of the center of mass of the vehicle, and θ is the angle between the heading direction and the x axis (see Fig. 1). Systems like (9), or similar chained systems (see Murray and Sastry, 1993) and further nonholonomic systems have been the subject of much ongoing research; see Kolmanovsky and McClamroch (1995) and references therein.

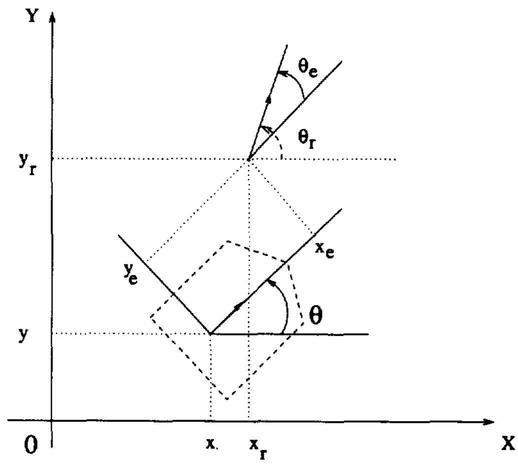

Fig. 1. Tracking configuration square.

The problem we consider here is the tracking problem; that is, we wish to find control laws for v and ω such that the robot follows a *reference robot,* with position $p_r = (x_r, y_r, \theta_r)^T$ and inputs v_r and ω_r (see Fig. 1). Denoting the error coordinates by (see Kanayama *et al.*, 1990)

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \theta_r - \theta \end{bmatrix}, \tag{10}$$

the error dynamics are (see Kanayama *et al.*, 1990)

$$\begin{aligned} \dot{x}_e &= \omega y_e - v + v_r\cos\theta_e, \\ \dot{y}_e &= -\omega x_e + v_r \sin\theta_e, \\ \dot{\theta}_e &= \omega_r - \omega. \end{aligned} \tag{11}$$

In the following sections, we shall examine separately the following two problems.

Local tracking problem. Find appropriate velocity control laws v and ω of the form

$$\begin{aligned} v &= v(x_e, y_e, \theta_e, v_r, \omega_r, \dot{v}_r), \\ \omega &= \omega(x_e, y_e, \theta_e, v_e, \omega_r, \dot{v}_r) \end{aligned} \tag{12}$$

such that, for *small* initial tracking errors $(x_e(0), y_e(0), \theta_e(0))$, the closed-loop trajectories of (11) and (12) are uniformly bounded and converge to zero.

Global tracking problem. Find appropriate velocity control laws v and ω of the form

$$\begin{aligned} v &= v(x_e, y_e, \theta_e, v_r, \omega_r, \dot{v}_r, \dot{\omega}_r), \\ \omega &= \omega(x_e, y_e, \theta_e, v_r, \omega_r, \dot{v}_r, \dot{\omega}_r) \end{aligned} \tag{13}$$

such that, for *arbitrary* initial tracking errors $(x_e(0), y_e(0), \theta_e(0))$, the closed-loop trajectories of (11) and (13) are (globally) uniformly bounded and converge to zero.

2. *Tracking of the kinematic model*

3.1. *The local tracking problem.* Given any fixed $0 < \epsilon < \pi$, let us introduce a set of functions denoted by $\mathcal{S}_\epsilon^\infty$:

$$\begin{aligned} \mathcal{S}_\epsilon^\infty = \{\varphi:\mathbb{R} \to (-\pi+\epsilon, \pi-\epsilon) : \varphi \in C^\infty, \\ \varphi(0) = 0,\ z\varphi(z) > 0\, \forall z \ne 0 \text{ and } \varphi' \text{ is bounded}\}. \end{aligned} \tag{14}$$

Simple examples of functions in $\mathcal{S}_\epsilon^\infty$ include $\varphi(z) = \sigma z/(1+z^2)$ for any $0 < \sigma < 2(\pi - \epsilon)$, and $\varphi(z) = \sigma_1 \arctan(\sigma_2 z)$ for all $0 < \sigma_1 < 2(\pi - \epsilon)/\pi$ and $\sigma_2 > 0$.

In the tracking error model (11), y_e is not directly controlled, and to overcome this difficulty we use the idea of integrator backstepping.

More precisely, for any function φ in $\mathcal{S}_\epsilon^\infty$, when setting $x_e = 0$ and $\theta_e = -\varphi(y_e v_r)$ in the y_e system of (11), the system obtained, $\dot{y}_e = -v_r \sin\varphi(y_e v_r)$ is uniformly stable at $y_e = 0$. With this observation in mind, we introduce a new variable $\bar{\theta}_e$ as follows:

$$\bar{\theta}_e = \theta_e + \varphi(y_e v_r). \tag{15}$$

With (15), the θ_e equation in the system (11) is transformed into

$$\dot{\bar{\theta}}_e = \omega_r - \omega + \varphi'(y_e v_r)(-\omega x_e v_r + v_r^2 \sin\theta_e + y_e \dot{v}_r). \tag{16}$$

Consider the candidate Lyapunov function

$$V_1(t, x_e, y_e, \theta_e) = \frac{1}{2}x_e^2 + \frac{1}{2}y_e^2 + \frac{1}{2\gamma}\bar{\theta}_e^2, \tag{17}$$

with $\gamma > 0$ and $\bar{\theta}_e$ given by (15). As can be directly verified, V_1 is a positive-definite, decrescent and radially unbounded function.

In view of (15) and (16), taking the time derivative of V_1 along solutions of (11) yields

$$\begin{aligned}\dot{V}_1(t, x_e, y_e, \theta_e) &= x_e(\omega y_e - v + v_r \cos\theta_e)\\ &\quad + y_e\{-\omega x_e + v_r \sin[-\varphi(y_e v_r) + \bar{\theta}_e]\}\\ &\quad + \frac{1}{\gamma}\bar{\theta}_e[\omega_r - \omega + \varphi'(y_e v_r)(-\omega x_e v_r + v_r^2 \sin\theta_e + y_e \dot{v}_r)].\end{aligned} \tag{18}$$

Noting that

$$\begin{aligned}\sin[-\varphi(y_e v_r) + \bar{\theta}_e] &= \sin[-\varphi(y_e v_r)]\\ &\quad + \bar{\theta}_e \underbrace{\int_0^1 \cos[-\varphi(y_e v_r) + s\bar{\theta}_e]\, ds}_{\eta(t)},\end{aligned} \tag{19}$$

it follows that (18) implies

$$\begin{aligned}\dot{V}_1(t, x_e, y_e, \theta_e) &= x_e(-v + v_r \cos\theta_e) - y_e v_r \sin[\varphi(y_e v_r)]\\ &\quad + \frac{1}{\gamma}\bar{\theta}_e\{\pi\eta y_e v_r + \omega_r - [1 + \varphi'(y_e v_r) x_e v_r]\omega\\ &\quad + \varphi'(y_e v_r)(v_r^2 \sin\theta_e + y_e \dot{v}_r)].\end{aligned} \tag{20}$$

By choosing the tracking controller v and ω as

$$v = v_r \cos\theta_e + c_1 x_e \tag{21}$$

$$\begin{aligned}\omega &= [1 + \varphi'(y_e v_r) x_e v_r]^{-1}[\gamma\eta y_e v_r + \omega_r\\ &\quad + \varphi'(y_e v_r)(v_r^2 \sin\theta_e + y_e \dot{v}_r) + c_2\gamma\bar{\theta}_e]\end{aligned} \tag{22}$$

with c_1, $c_2 > 0$, we have

$$\dot{V}_1(t, x_e, y_e, \theta_e) = -c_1 x_e^2 - y_e v_r \sin[\varphi(y_e v_r)] - c_2\bar{\theta}_e^2. \tag{23}$$

Note that the control law ω as introduced in (22) may not be defined for every t. However, we shall prove that, for any initial condition $(x_e(0), y_e(0), \theta_e(0))$ in a neighborhood of the origin, $\omega(t)$ does exist for every $t \geq 0$.

Proposition 1. Assume that v_r, $\dot{v}_r$ and ω_r are bounded on $[0, \infty)$. Then there exists a function $\varphi \in \mathcal{S}_\epsilon^\infty$ such that the equilibrium point $(x_e, y_e, \theta_e) = (0, 0, 0)$ of the closed-loop system (11), (21), (22) is uniformly stable. Furthermore, if $v_r(t)$ does not converge to zero then, for small initial conditions $(x_e(0), y_e(0), \theta_e(0))$, the corresponding solution $(x_e(t), y_e(t), \theta_e(t))$ converges to zero, i.e.

$$\lim_{t\to\infty}[|x_e(t)| + |y_e(t)| + |\theta_e(t)|] = 0. \tag{24}$$

Proof. We first prove that there exists a nonempty neighborhood $\Omega \subset \mathbb{R}^3$ of the origin such that for any initial condition $(x_e(0), y_e(0), \theta_e(0)) \in \Omega$, $\omega(t)$ is well defined on $[0, T)$, the maximal interval of definition of the solution $(x_e(t), y_e(t), \theta_e(t))$.

For any r_1, $r_2 \geq 0$, let $B(r_1, r_2)$ stand for the set $\{(x_e, y_e, \theta_e) \in \mathbb{R}^3 : r_1 r_2 |x_e| < 1\}$. Note that $B(0, r_2) = B(r_1, 0) = \mathbb{R}^3$. Also, let Ω be a set given by

$$\Omega = \{(x_e, y_e, \theta_e) \in \mathbb{R}^3 : V_1(t, x_e, y_e, \theta_e) < c^*\ \forall t \geq 0\}, \tag{25}$$

where $c^* > 0$ is the largest constant such that

$$\{(x_e, y_e, \theta_e) \in \mathbb{R}^3 : V_1(t, x_e, y_e, \theta_e) < c^*\} \subset B(\|v_r\|_\infty, \|\varphi'\|_\infty). \tag{26}$$

It follows from (23) that $(x_e(t), y_e(t), \theta_e(t))$ remains in Ω, and therefore $\omega(t)$ is well defined on $[0, T)$. Furthermore, since V_1 is nonincreasing along solutions of the closed-loop system, the boundedness property of the closed-loop trajectory follows readily, and therefore $T = +\infty$. We conclude the uniform stability of the zero equilibrium from (23) and Lyapunov stability theory (see Vidyasagar 1993, Theorem 5.3.14).

To prove the second statement, observe that, with (23), the signals x_e^2, $y_e v_r \sin[\varphi(y_e v_r)]$, $\bar{\theta}_e^2 \in L_1(0, \infty)$. By assumption, the derivatives of these signals are bounded. Hence they are uniformly continuous on $[0, \infty)$. With the help of Barbălat's lemma (Lemma 1), it follows that $x_e(t)$, $y_e(t)v_r(t)$ and $\bar{\theta}_e(t)$ converge to zero as t goes to ∞. With (15), this in turn implies the convergence of $\theta(t)$. Finally, since $V_1(t, x_e, y_e(t), \theta_e(t))$ is decreasing and bounded from below by zero, V_1 tends to a finite nonnegative constant. This implies that the limit of $|y_e(t)|$ exists and is a finite real number l_y. If l_y were not zero, there would exist a sequence of increasing time instants $\{t_i\}_{i=1}^\infty$, with $t_i \to \infty$, such that both the limits of $v_r(t_i)$ and $|y_e(t_i)v_r(t_i)|$ are not zero. This is impossible, because $|y_e(t)v_r(t)|$ was proved to go as zero as $t \to \infty$.

Remark 1. It is of interest to note that we can enlarge the region Ω defined in (25) by choosing an appropriate function φ whose gradient φ' is small enough; see also (26). In this sense, we say that the system (11) is *semiglobally stabilized.*

Remark 2. The local tracking control laws (21) and (22) do not allow us to conclude that the tracking errors converge to zero if $v_r(t)$ tends to zero. The latter case will be addressed in the next subsection via a different controller at the cost of imposing additional asumptions on the desired velocities.

Some related results have been reported on the local tracking problem in the literature (see e.g. Kanayama *et al.*, 1990; Murray *et al.*, 1992, Oelen and van Amerongen, 1994). In particular, asymptotic stabilization was achieved in Oelen and van Amerongen (1994) using input/output linearization, and results on local exponential stabilization were obtained in Kanayama *et al.* (1990) and Murray *et al.* (1992) via linearization (or Lyapunov's indirect method). In the following, we prove that our control laws may also guarantee exponential stability for the system (11). However, our approach is based on Lyapunov's direct method, and does not rely upon the linearization method.

Corollary 1. Under the conditions of Proposition 1, given any refeerence velocity v_r with the property that $\liminf_{t\to\infty} \|v_r(t)| > 0$, it follows that the zero equilibrium of the closed-loop system (11), (21), (22) is exponentially stable if we select a function $\varphi \in \mathcal{S}_\epsilon^\infty$ such that $\varphi'(0) > 0$.

Proof. By choice of φ, $y_e v_r(t) \sin[\varphi(y_e v_r(t)] \geq 0$ for all y_e and all $t \geq 0$. Furthermore,

$$\varphi(y_e v_r(t)) = y_e v_r(t) \int_0^1 \varphi'(\lambda y_e v_r(t))\, d\lambda, \tag{27}$$

$$\begin{aligned}y_e v_r(t) \sin[\varphi(y_e v_r(t))] &= y_e^2 v_r(t)^2 \int_0^1 \varphi'(\lambda y_e v_r(t))\, d\lambda\\ &\quad \times \underbrace{\int_0^1 \cos\left[s y_e v_r(t) \int_0^1 \varphi'(\lambda y_e v_r(t))\, d\lambda\right] ds}_{\chi(t, y_e)}\end{aligned} \tag{28}$$

Note that $\chi_0(y_e) = \sup_{t\geq 0} \chi(t, y_e)$ is a continuous function satisfying $\chi_0(0) = \varphi'(0)$. Letting $l_v = \liminf_{t\to\infty} v_r(t)^2$, it follows from (28), that $y_e v_r(t) \sin[\varphi(y_e v_r(t))] \ge 0.5 l_v \varphi'(0) y_e^2$ for sufficiently small y_e and sufficiently large t. It follows from (23) that

$$\dot{V}_1(t, x_e, y_e, \theta_e) \le -c_1 x_e^2 - 0.5 l_v \varphi'(0) y_e^2 - c_2 \bar{\theta}_e^2 \tag{29}$$

as long as $|y_e| < c_y$ and $t > t^*$ for certain $c_y > 0$ and certain $t^* > 0$.

Let Ω_1 be a subset of Ω defined as follows:

$$\Omega_1 = \{(x_e, y_e, \theta_e) \in \mathbb{R}^3 : V_1(t, x_e, y_e, \theta_e) < c^{**}\}, \tag{30}$$

where $c^{**} > 0$ is the largest constant such that

$$\{(x_e, y_e, \theta_e) \in \mathbb{R}^3 : V_1(t, x_e, y_e, \theta_e) < c^{**}\} \subset \Omega \cap \{(x_e, y_e, \theta_e) \in \mathbb{R}^3 : |y_e| < c_y\}. \tag{31}$$

By (23), $(x_e(t), y_e(t), \theta_e(t))$ remains in Ω_1 as long as $(x_e(0), y_e(0))$ does. In particular, in this case, we have

$$\dot{V}_1(t, x_e, y_e, \theta_e) \le -c_V V_1(t, x_e, y_e, \theta_e) \quad \forall t \ge t^* \tag{32}$$

for some $c_V > 0$. A direct application of Gronwall's inequality (Vidyasagar, 1993), together with (15) and (23), implies the existence of two positive real numbers k_1 and k_2 such that

$$|(x_e(t), y_e(t), \theta_e(t))| \le k_1 e^{-k_2 t} |(x_e(0), y_e(0), \theta_e(0))|. \tag{33}$$

Therefore we conclude the local exponential stability of the closed-loop system (11), (21), (22) at the zero equilibrium for initial conditions $(x_e(0), y_e(0), \theta_e(0))$ belonging to Ω_1. □

3.2. *The global tracking problem.* The tracking control laws proposed in the above section solve the local tracking problem. The purpose of this section is to tackle the global tracking case. In this case, additional conditions are required. As in the previous subsection, the integerator backstepping will also be employed for controller design in the global case. Noticing that $x_e = c_3 \omega y_e$ and $\theta_e = 0$ are stabilizing functions for the y_e system of (11), we introduce a new variable

$$\bar{x}_e = x_e - c_3 \omega y_e, \tag{34}$$

where c_3 is a positive constant.

With (34), the x_e equation in the system (11) is rewritten as

$$\dot{\bar{x}}_e = \omega y_e - v + v_r \cos\theta_e - c_3 \dot{\omega} y_e - c_3 \omega(-\omega x_e + v_r \sin\theta_e) \tag{35}$$

For notational simplicity, denote

$$v_1(t) = \omega(t) y_e(t) + v_r(t)\cos\theta_e(t) - c_3 \dot{\omega}(t) y_e(t) + c_3 \omega(t)[\omega(t) x_e(t) - v_r(t) \sin\theta_e(t)] \tag{36}$$

In this case, instead of (17), consider the function

$$V_2(t, x_e, y_e, \theta_e) = \frac{1}{2}\bar{x}_e^2 + \frac{1}{2} y_e^2 + \frac{1}{2\gamma}\theta_e^2, \tag{37}$$

with $\gamma > 0$ and $\bar{x}_e$ given by (34).

We have, using an identity as in (19),

$$\dot{V}_2(t, x_e, y_e, \theta_e) = -c_3 \omega^2 y_e^2 + \bar{x}_e(-y_e \omega + v_1 - v) + \frac{1}{\gamma}\theta_e\left[\gamma y_e v_r \int_0^1 \cos(s\theta_e)\,ds + \omega_r - \omega\right]. \tag{38}$$

By choosing the tracking controllers v and ω as

$$v = v_1 - y_e \omega + c_4 \bar{x}_e := \alpha_v, \tag{39}$$

$$\omega = \omega_r + \gamma y_e v_r \int_0^1 \cos(s\theta_e)\,ds + c_5 \gamma \theta_e := c_5 \gamma \theta_e := \alpha_\omega, \tag{40}$$

with c_4, $c_5 > 0$, we have

$$\dot{V}_2(t, x_e, y_e, \theta_e) = -c_3 \omega^2 y_e^2 - c_4 \bar{x}_e^2 - c_5 \theta_e^2. \tag{41}$$

We establish the following result.

Proposition 2. Assume that v_r, $\dot{v}_r$, ω_r and $\dot{\omega}_r$ are bounded on $[0, \infty)$. Then all the trajectories of the resulting system composed of (11), (39) and (40) are globally uniformly bounded. Furthermore, if $v_r(t)$ does not converge to zero, or if $v_r(t)$ tends to zero but $\omega_r(t)$ does not converge to zero, then the closed-loop solutions converge to zero, i.e.

$$\lim_{t \to \infty} [|x_e(t)| + |y_e(t)| + |\theta_e(t)|] = 0. \tag{42}$$

Proof. Since V_2 is positive-definite and radially unbounded, as in the proof of Proposition 1, we conclude from (41) that the original trajectories $x_e(t)$, $y_e(t)$ and $\theta_e(t)$ are uniformly bounded and are defined for all $t \ge 0$.

Notice that (41) yields the property that $\omega(t)^2 y_e(t)^2$, $\bar{x}_e(t)^2$, $\theta_e(t)^2 \in L_1(0, \infty)$. By assumption, the derivatives of these signals are bounded. Hence $\omega(t)^2 y_e(t)^2$, $\bar{x}_e(t)^2$ and $\theta_e(t)^2$ are uniformly continuous on $[0, \infty)$. With the help of Barbălat's lemma, it follows that $\omega(t) y_e(t)$, $\bar{x}_e(t)$ and $\theta_e(t)$ converge to zero as t goes to ∞. From the definition of $\bar{x}_e$ in (34), it follows that $x_e(t)$ goes to 0.

It remains to prove that $y_e(t)$ tends to 0. Setting $\eta_1(t) = \int_0^1 \cos[s\theta_e(t)]\,ds$, we have $\eta_1(t)$ going to unity as t goes to ∞. We consider the case where $v_r(t)$ does not converge to zero; the other case proceeds similarly and is therefore omitted. In the closed-loop system, the θ_e equation becomes

$$\dot{\theta}_e = -c_5 \gamma \theta_e - \gamma y_e(t) v_r(t) \eta_1(t). \tag{43}$$

A direct application of Lemma 2 gives that $y_e(t) v_r(t) \eta_1(t)$ tends to 0. By means of the same reasoning as in the proof of Proposition 1, we conclude that $y_e(t)$ must converge to 0. □

Remark 3. Similarly to Corollary 1, we can conclude that, under the additional assumption that $\liminf_{t\to\infty} |\omega_r(t)| > 0$, the zero equilibrium of the closed-loop system (11), (39), (40) is exponentially stable (for small initial errors). In other words, all the closed-loop trajectories go to zero at an exponential rate after a considerable period of time.

Remark 4. (Path following.) It is of interest to mention that the robot under study can *globally* follow two particular types of paths: straight lines and circles. Indeed, putting $\omega_r \equiv 0$ and $v_r \equiv c_v$, with c_v a nonzero constant, the reference trajectories are straight lines of the form $x_r(t) = x_r(0) + tc_v \cos[\theta_r(0)]$ and $y_r(t) = y_r(0) + tc_v \sin[\theta_r(0)]$. In the case where we choose $\omega_r \equiv c_\omega$ and $v_r \equiv c_v$, with c_ω and c_v two nonzero constants, the reference trajectories are circles of radius $|c_v|$ described by $x_r(t) = x_r(0) + c_v \sin(c_\omega t)$ and $y_r(t) = 1 + y_r(0) - c_v \cos(c_\omega t)$.

3.3. *An extension.* In the above sections, we have studied asymptotic posture tracking problems with exponential convergence by Lyapunov's direct method. The main purpose of this subsection is to give a backstepping-based global tracking controller under less restrictive assumptions than Proposition 2. In particular, we relax the conditions of the main Proposition of Samson and Ait-Abderrahim (1991).

Proposition 3. Assume that v_r and ω_r are uniformly continuous and bounded on $[0, \infty)$. Then all the trajectories of the system (11) in closed loop with the controllers

$$v = c_x x_e + v_r \cos\theta_e, \quad c_x > 0, \tag{44}$$

$$\omega = \omega_r + v_r y_e \int_0^1 \cos(\lambda\theta_e)\,d\lambda + c_\theta \theta_e, \qquad c_\theta > 0, \tag{45}$$

are globally uniformly bounded. Furthermore, if either $v_r(t)$ or $\omega_r(t)$ does not converge to zero then the closed-loop solutions converge to zero, i.e.

$$\lim_{t \to \infty} [|x_e(t)| + |y_e(t)| + |\theta_e(t)|] = 0. \tag{46}$$

Proof. Setting $c_3 = 0$ in (34) and $\gamma = 1$ in the definition (37) of V_2, the proof of Proposition 3 follows by mimicking the arguments used in the proof of Proposition 2. □

Note that, unlike in Sections 3.1 and 3.2, the exponential stability of the zero solution of the closed-loop system (11), (44), (45) does not follow from Lyapunov's direct method. Nevertheless, the exponential stability property can be established via Lyapunov's indirect method (Vidyasagar, 1993).

4. *Tracking of a simplified dynamic model*

In this section, we study the augmented system (11) appended with two integrators, i.e.

$$\begin{aligned} \dot{x}_e &= \omega y_e - v + v_r \cos\theta_e, \\ \dot{y}_e &= -\omega x_e + v_r \sin\theta_e, \\ \dot{\theta}_e &= \omega_r - \omega, \\ \dot{v} &= u_1, \\ \dot{\omega} &= u_2, \end{aligned} \tag{47}$$

where u_1 and u_2 may be regarded as torques or generalized force variables of the two-degrees-of-freedom mobile robot. The system (47) is referred to as a simplified *dynamic model* for the mobile robot. It is well known that consideration of models including dynamic effects is interesting from an engineering point of view, although (47) is certainly not a 'complete' dynamic model of the mobile robot, since several other effects acting on the vehicle are not included. However, we wish to demonstrate that the tracking controllers that were developed for the kinematic model can also be obtained for a simple dynamic model as (47), thereby at least making it plausible that a similar controller could be derived for a 'complete' dynamic model. The control objective is to find a control law $u = (u_1, u_2)$ of the form

$$\begin{aligned} u_1 &= u_1(x_e, y_e, \theta_e, v, \omega, v_r, \dot{v}_r, \ddot{v}_r, \omega_r, \dot{\omega}_r, \ddot{\omega}_r), \\ u_2 &= u_2(x_e, y_e, \theta_e, v, \omega, v_r, \dot{v}_r, \ddot{v}_r, \omega_r, \dot{\omega}_r, \ddot{\omega}_r) \end{aligned} \tag{48}$$

in such a way that local or global tracking is achieved. In other words, x_e, y_e and θ_e are forced to converge to zero.

We discuss in this section how the methodology presented in the previous section can be extended to the system (47). For simplicity, we only look at the global tracking case that extends the local tracking result of Fierro and Lewis (1995). The development for the local case is analogous and is omitted.

Introduce the new variables

$$\bar{v} = v - \alpha_v, \quad \bar{\omega} = \omega - \alpha_\omega, \tag{49}$$

where α_v and α_ω are defined as in (39) and (40) respectively.

Following the notation used in Section 3 (see in particular (34)), in the new coordinates $(\bar{x}_e, y_e, \theta_e, \bar{v}, \bar{\omega})$ the system (47) is transformed into

$$\begin{aligned} \dot{\bar{x}}_e &= \omega y_e - c_4 \bar{x}_e - \bar{v}, \\ \dot{y}_e &= -c_3\omega^2 y_e - \omega\bar{x}_e + v_r \sin\theta_e, \\ \dot{\theta}_e &= -c_5\gamma\theta_e - \gamma y_e v_r \int_0^1 \cos(s\theta_e)\,ds - \bar{\omega}, \\ \dot{\bar{v}} &= u_1 - \dot{\alpha}_v, \\ \dot{\bar{\omega}} &= v_2 - \dot{\alpha}_\omega, \end{aligned} \tag{50}$$

where $\dot{\alpha}_v$ and $\dot{\alpha}_\omega$ are given by

$$\begin{aligned} \dot{\alpha}_v = {} & (\cos\theta_e - c_3\omega\sin\theta_e)\dot{v}_r - c_3 y_e \dot{u}_2 \\ & + (c_3\omega^2 + c_4)(\omega y_e - v + v_r\cos\theta_e) \\ & - (c_3 u_2 + c_3 c_4 \omega)(-\omega x_e + v_r \sin\theta_e) \\ & - (v_r \sin\theta_e + c_3\omega v_r \cos\theta_e)(\omega_r - \omega) \\ & + (2c_3\omega x_e - c_3 v_r \sin\theta_e - c_3 c_4 y_e)u_2, \end{aligned} \tag{51}$$

$$\begin{aligned} \dot{\alpha}_\omega = {} & \dot{\omega}_r + \gamma(y_e \dot{v}_r - \omega\dot{x}_e + v_r \sin\theta_e)\int_0^1 \cos(s\theta_e)\,ds \\ & - \gamma y_e v_r(\omega_r - \omega)\int_0^1 \sin(s\theta_e)s\,ds \\ & + c_5\gamma(\omega_r - \omega). \end{aligned} \tag{52}$$

Inspired by the control scheme proposed in the above section, consider the candidate Lyapunov function

$$\begin{aligned} U(t, x_e, y_e, \theta_e, v, \omega) = {} & \frac{1}{2}\bar{x}_e^2 + \frac{1}{2}y_e^2 \\ & + \frac{1}{2\gamma}\theta_e^2 + \frac{1}{2}\bar{v}^2 + \frac{1}{2}\bar{\omega}^2. \end{aligned} \tag{53}$$

It can be directly checked that U is a positive-definite, decrescent and radially unbounded function.

According to the calculation performed in Section 3.2, and in particular (41), the time derivative of U along solutions of (50) satisfies

$$\begin{aligned} \dot{U}(t, x_e, y_e, \theta_e, v, \omega) = {} & -c_3\omega^2 y_e^2 - c_4\bar{x}_e^2 - c_5\theta_e^2 \\ & - \bar{x}_e\bar{v} - \theta_e\bar{\omega} + \bar{v}(u_1 - \dot{\alpha}_v) \\ & + \bar{\omega}(u_2 - \dot{\alpha}_\omega). \end{aligned} \tag{54}$$

Applying the feedback controllers

$$u_1 = \bar{x}_e + \dot{\alpha}_v - c_6\bar{v}, \tag{55}$$

$$u_2 = \theta_e + \dot{\alpha}_\omega - c_7\bar{\omega}, \tag{56}$$

with $c_6, c_7 > 0$, we arrive at

$$\begin{aligned} & \dot{U}(t, x_e, y_e, \theta_e, v, \omega) \\ & \quad = -c_3\omega^2 y_e^2 - c_4\bar{x}_e^2 - c_5\theta_e^2 - c_6\bar{v}^2 - c_7\bar{\omega}^2. \end{aligned} \tag{57}$$

We have the following proposition.

Proposition 4. Under the conditions of Proposition 2, if $\ddot{v}_r$ and $\ddot{\omega}_r$ are bounded then all the trajectories of the resulting system composed of (47), (55) and (56) are globally uniformly bounded. Furthermore, if $v_r(t)$ does not converge to zero, or if $v_r(t)$ converges to zero but $\omega_r(t)$ does not converge to zero, then

$$\begin{aligned} \lim_{t\to\infty} [& |x_e(t)| + |y_e(t)| + |\theta_e(t)| + |v(t) - v_r(t)| \\ & + |\omega(t) - \omega_r(t)|] = 0. \end{aligned} \tag{58}$$

Proof. This follows the same reasoning as the proof of Proposition 2. □

5. *Discussion and simulation results*

With the purpose of illustrating the tracking controllers derived in this paper, a number of simulations have been done. The simulations were carried out using MATLAB, with the following choice for the parameters in the controllers (21), (22), (39) and (40) and the reference velocities

$$\begin{aligned} & c_1 = c_3 = c_5 = c_6 = c_7 = 1, \\ & c_2 = c_4 = 2, \quad v_r \equiv 1, \quad \omega_r \equiv 0. \end{aligned} \tag{59}$$

The simulations not only illustrate the effectiveness of the tracking controllers but are also used for obtaining an insight into the difference between the usefulness of the global versus the local controller under changing initial conditions. Clearly, the local controller (21), (22) assures, by Proposition 1, that the tracking errors converge to zero provided that the initial errors are sufficiently small, but no explicit estimate of how small these errors should be was given. On the other hand, the global controller (39), (40) can be used for arbitrary (large) initial errors (see Proposition 2), but the price will be a relatively slow convergence of the tracking errors. We demonstrate these effects as follows. In Figs 2 and 3, the local controller (21), (22) is applied with initial tracking errors $(x_e(0), y_e(0), \theta_e(0)) = (-0.5, 0.5, 1)$ (respectively $(x_e(0), y_e(0), \theta_e(0)) = (16.6, 1.5, -1)$). Similarly, in Figs 4 and 5, the global controller (39), (40) is applied with the same initial tracking errors as in Fig. 2 (respectively Fig. 3). One can clearly see the difference between the local and global controller under the changing initial tracking errors.

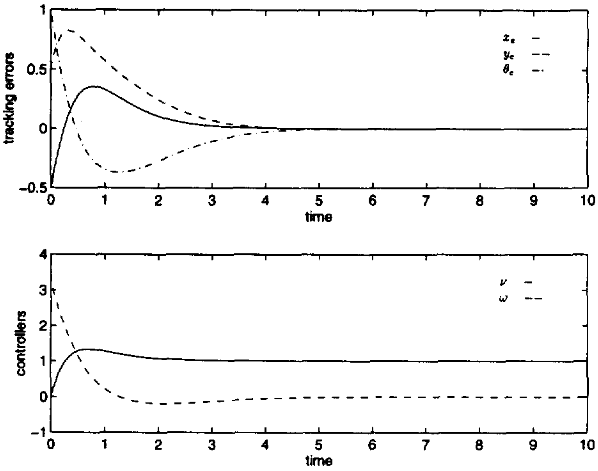

Fig. 2. Local tracking of the kinematic model, with initial errors $(x_e(0), y_e(0), \theta_e(0)) = (-0.5, 0.5, 1)$.

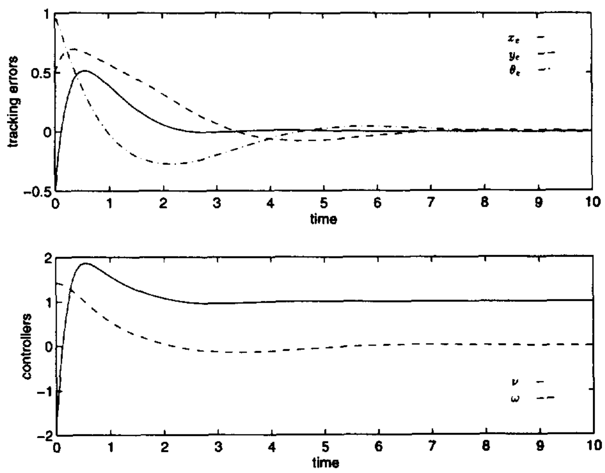

Fig. 4. Global tracking of the kinematic model, with initial errors $(x_e(0), y_e(0), \theta_e(0)) = (-0.5, 0.5, 1)$.

To quantify the difference between the four simulations, one may use the following error measure over the time period $[0, T]$:

$$P = \frac{1}{T}\int_0^T [x_e(t)^2 + y_e(t)^2 + \theta_e(t)^2]\,\mathrm{d}t. \tag{60}$$

corresponding to the simulations described in Figs 2–5, we find the values

$$\begin{aligned} P_2 &= 0.1249, \quad P_3 = 15.6502, \\ P_4 &= 0.1367, \quad P_5 = 7.7944. \end{aligned} \tag{61}$$

Indeed, the above outcomes agree with our expectations in that the local controller performs better for small initial tracking errors, but for large initial tracking errors the global controller (39), (40) is preferable.

6. *Conclusions*

The mobile robot kinematic model, or its simplified dynamic model, serves, as has been shown, as an excellent 'test-bed' for using the backstepping technique in the tracking control problem. Both the local and global tracking problems with exponential convergence have then solved. Our theoretical results have been confirmed by means of a number of simulations together with an analysis of the performance of these controllers. The backstepping tracking control method presented in this paper was recently extended to the more general class of nonholonomic chained systems (Jiang and Nijmeijer 1997).

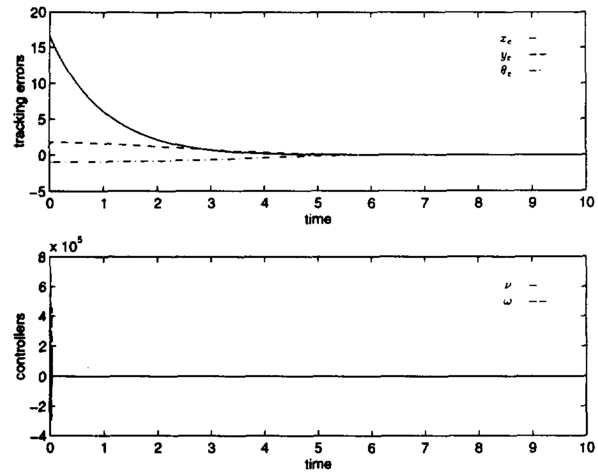

Fig. 3. Local tracking of the kinematic model, with initial errors $(x_e(0), y_e(0), \theta_e(0)) = (16.6, 1.5, -1)$.

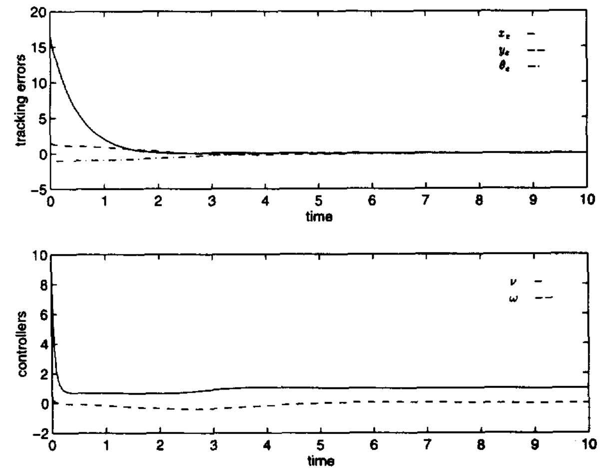

Fig. 5. Global tracking of the kinematic model, with initial errors $(x_e(0), y_e(0), \theta_e(0)) = (16.6, 1.5, -1)$.

As in most previous work on the study of nonholonomic systems, our results are heavily based on a 'nonholonomic' assumption of the form $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$. It should be mentioned that this condition is an idealization of real situations, and is never satisfied by real physical control systems. In d'Andréa-Novel *et al.* (1995), the authors proposed a singular perturbation approach to point-tracking control for nonlinear mechanical systems that do not satisfy ideal velocity constraints. There is still no general answer for the tracking control problem if common velocity constraints are not satisfied by the class of nonholonomic mechanical systems under consideration.

Acknowledgements—Most of this work was done when the first author was with the Department of Systems Engineering, Australian National University, and the second author was visiting the Faculty of Engineering and Information Technology, Australian National University under a research fellowship. The authors acknowledge funding support from the Centre for Robust and Adaptive Systems established under the Cooperative Research Centre Program of the Commonwealth of Australia.

References

d'Andreá-Novel, B., Campion, G. and Bastin, G. (1995) Control of wheeled mobile robots not satisfying ideal velocity constraints: a singular perturbation approach. *Int. J. Robust Nonlinear Control* **5**, 243–267.

Bloch, A. and Drakunov, S. (1994) Stabilization of a nonholonomic system via sliding modes. In *Proc. 33rd IEEE Conf. on Decision and Control,* Lake Buena Vista, FL, pp. 2961–2963.

Brockett, R. W. (1983) Asymptotic stability and feedback stabilization. In Differential Geometric Control Theory, ed. R. W. Brockett, R. S. Milman and H. J. Sussman, pp. 181–191. Birkhäuser, Boston.

Byrnes, C. I. and Isidori, A. (1989) New results and examples in nonlinear feedback stabilization. *Syst. Control Lett.* **12,** 437–442.

Canudas de Wit, C., Berghuis, H. and Nijmeijer, H. (1994) Practical stabilization of nonlinear systems in chained form. In *Proc. 33rd IEEE Conf. on Decision and Control,* Lake Buena Vista, FL, pp. 3475–3480.

Canudas de Wit, C., Siciliano, B. and Bastin, G., eds (1996). *Theory of Robot Control.* Springer-Verlag, London.

Fierro, R. and Lewis, F. L. (1995) Control of a nonholonomic mobile robot: backstepping kinematics into dynamics. In *Proc. 34th IEEE Conf. on Decision and Control,* New Orleans, LA, pp. 3805–3810.

Fliess, M., Levine, J., Martin, P. and Rouchon, P. (1995a) Flatness and defect of non-linear systems: introductory theory and examples. *Int. J. Control* **61,** 1327–1361.

Fliess, M., Levine, J., Martin, P. and Rouchon, P. (1995b) Design of trajectory stabilizing feedback for driftless flat systems. In *Proc. 3rd European Control Conf.*, Rome, pp. 1882–1887.

Jiang, Z. P. (1996) Iterative design of time-varying stabilizers for multi-input systems in chained form. *Syst. Control Lett.* **28,** 255–262.

Jiang, Z. P. and Nijmeijer, H. (1996) Tracking control of mobile robots: a case study in backstepping, memorandum 1321, University of Twente.

Jiang, Z. P. and Nijmeijer, H. (1997) Backstepping-based tracking control of nonholonomic chained systems. Submitted to *European Control Conf.* (ECC'97).

Jiang, Z. P. and Pomet, J.-B. (1994) Combining backstepping and time-varying techniques for a new set of adaptive controllers. In *Proc. 33rd IEEE Conf. on Decision and Control,* Lake Buena Vista, FL, pp. 2207–2212; also *Int. J. Adaptive Control Sig. Process.* **10,** 47–59 (1996).

Jiang, Z. P. and Pomet, J.-B. (1995) Backstepping-based adaptive controllers for uncertain nonholonomic systems. In *Proc. 34th IEEE Conf. on Decision and Control,* New Orleans, LA, pp. 1573–1578.

Kanayama, Y., Kimura, Y., Miyazaki, F. and Noguchi, T. (1990) A stable tracking control scheme for an autonomous mobile robot. In *Proc. IEEE International Conf. on Robotics and Automation,* pp. 384–389.

Khalil, H. K. (1992) *Nonlinear Systems.* Macmillan, New York.

Koditschek, D. E. (1987) Adaptive techniques for mechanical systems. In *Proc. 5th Yale Workshop on Adaptive Systems,* New Haven, CT, pp. 259–265.

Kolmanovsky, H. and McClamroch, N. H. (1995) Developments in nonholonomic control systems. *IEEE Control Syst. Magazine* **15**(6), 20–36.

Krstić, M., Kanellakopoulos, I. and Kokotović, P. V. (1995) *Nonlinear and Adaptive Control Design.* Wiley, New York.

McCloskey, R. T. and Murray, R. M. (1994) Exponential stabilization of driftless nonlinear control systems via time-varying, homogeneous feedback. In *Proc. 33rd IEEE Conf. on Decision and Control,* Lake Buena Vista, pp. 1317–1322.

Micaelli, A. and Samson, C. (1993) Trajectory tracking for unicycle-type and two-steering-wheels mobile robots. INRIA Technical Report No. 2097.

Murray, R. M. and Sastry, S. S. (1993) Nonholonomic motion planning: steering using sinusoids. *IEEE Trans. Autom. Control* **AC-38,** 700–716.

Murray, R. M., Walsh, G. and Sastry, S. S. (1992) Stabilization and tracking for nonholonomic control systems using time-varying state feedback. In *IFAC Nonlinear Control Systems Design,* Bordeaux, ed. M. Fliess, pp. 109–114.

Oelen, W. and van Amerongen, J. (1994) Robust tracking control of two-degrees-of-freedom mobile robots. *Control Engng Practice* **2,** 333–340.

Oelen, W., Berghuis, H., Nijmeijer, H. and Canudas de Wit, C. (1995) Hybrid stabilizing control on a real mobile robot. *IEEE Robotics Automation Magazine* **2,** 16–23.

Pomet, J.-B. (1992) Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *Syst. Control Lett.* **18,** 147–158.

Popov, V. M. (1973) Hyperstability of Control Systems. Springer-Verlag, Berlin.

Samson, C. (1991) Velocity and torque feedback control of a nonholonomic cart. In *Proc. Advanced Robot Control,* ed. C. Canudas de Wit, pp. 125–151. Lecture Notes in Control and Information Sciences, Vol. 162, Springer-Verlag, Berlin.

Samson, C. and Ait-Abderrahim, K. (1991) Feedback control of a nonholonomic wheeled cart in Cartesian space. In *Proc. IEEE International Conf. on Robotics and Automation,* Sacramento, CA, pp. 1136–1141.

Tsinias, J. (1989) Sufficient Lyapunov-like conditions for stabilization. *Math. Control Sig. Syst.* **2,** 343–357.

Vidyasagar, M. (1993) *Nonlinear Systems Analysis,* 2nd ed. Prentice-Hall, Englewood Cliffs, NJ.