# Dyck paths and a bijection for multisets of hook numbers 

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October 23, 2018


#### Abstract

We give a bijective proof of a conjecture of Regev and Vershik [7] on the equality of two multisets of hook numbers of certain skew-Young diagrams. The bijection proves a result that is stronger and more symmetric than the original conjecture, by means of a construction involving Dyck paths, a particular type of lattice path.


## 1 Introduction

Let $n, k$ be positive integers, and $\alpha=\left(\alpha_{1}, \ldots, \alpha_{k}\right)$ be a partition with at most $k$ parts, each part at most $n$, so $n \geq \alpha_{1} \geq \ldots \geq \alpha_{k} \geq 0$. The Young diagram of $\alpha$ is given by

$$
D=\left\{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq \alpha_{k-i+1}\right\}
$$

a collection of unit cells, arranged in rows and columns. Here cell $(i, j)$ appears in row $i$ and column $j$, rows numbered from bottom to top, and columns numbered from left to right. We regard translates of the diagram in the plane as equivalent, and generally place the bottomleft cell at $(1,1)$. (Note, however that this is not the case for $D$ above when $\alpha_{k}=0$.) Also let

$$
\begin{gathered}
R=\{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq n\} \\
T=\left\{(i, j) \mid 1 \leq i \leq k, \alpha_{1}-\alpha_{i}+1 \leq j \leq n+\alpha_{1}-\alpha_{i}\right\} \\
V=\left\{(i, j) \mid k+1 \leq i \leq 2 k, n+\alpha_{1}-\alpha_{i-k}+1 \leq j \leq n+\alpha_{1}\right\}, \\
S Q=T \cup V
\end{gathered}
$$

so $R, T, S Q$ are skew diagrams (in fact, $R$ is also a Young diagram, the $k \times n$ rectangle).
For a skew diagram $G$, let $G^{*}$ be the skew diagram obtained by rotating $G$ through 180 degrees. Thus, for example,

$$
T^{*}=\left\{(i, j) \mid 1 \leq i \leq k, \alpha_{k-i+1}-\alpha_{k}+1 \leq j \leq n+\alpha_{k-i+1}-\alpha_{k}\right\} .
$$

Also, let $G^{\dagger}$ be the collection of cells obtained by reflecting $G$ about a vertical axis.
The arm length $a_{G}(x)$ of a cell $x$ in a skew diagram $G$ is the number of cells of $G$ in the same row of $x$ and to the right of $x$; the leg length $l_{G}(x)$ of a cell $x$ in a skew diagram $G$ is
the number of cells of $G$ in the same column and below. The coleg length of a cell $x$ in a skew diagram is the number of cells in the same column and above. The hook length $h_{G}(x)$ is given by $h_{G}(x)=a_{G}(x)+l_{G}(x)+1$. If $E$ is a subset of the cells of $G$, then $A L_{G}(E)$ is the multiset $\left\{\left(a_{G}(x), l_{G}(x)\right) \mid x \in E\right\}$, and $H_{G}(E)$ is the multiset $\left\{h_{G}(x) \mid x \in E\right\}$. When there is no ambiguity, we write $H_{G}(G)$ as $H(G)$, and $A L_{G}(G)$ as $A L(G)$.

For example, the skew diagrams $D, R, S Q$ are illustrated in Figure 1 for the case $n=$ $6, k=4, \alpha=(6,5,3,1)$. For the three cells labelled $b, c, d$ in Figure 1, we have $a_{D}(b)=$ $1, l_{D}(b)=0, a_{S Q}(c)=4, l_{S Q}(c)=2$ and $a_{R}(d)=0, l_{R}(d)=3$.


R
Figure 1: $D, R, S Q$ for $n=6, k=4, \alpha=(6,5,3,1)$.
Theorem 1.1 below was conjectured by Regev and Vershik [7], and proved by Regev and Zeilberger [6], Janson [2], and Bessenrodt [1] (though only for the case $n=\alpha_{1}$ in [6]).

Theorem 1.1. For all $n, k, \alpha$,

$$
H(S Q)=H(R) \cup H(D)
$$

is a multiset identity.

Regev and Zeilberger note that their proof is not bijective, and ask for a canonical bijection between the multisets. Bessenrodt [1] presents such a bijection, deducing it from a general result about "removable" hooks in Young diagrams. In this paper, we present a different bijection, deducing it from another general result, the main result of the paper. It is convenient to keep arm and leg lengths separately, and thus we prove the following result, which is obviously a generalization of Theorem 1.1.

Theorem 1.2. For all $n, k, \alpha$,

$$
A L(S Q)=A L(R) \cup A L(D)
$$

is a multiset identity.
The next result, our main result, is more symmetric and natural looking than Theorem 1.2, but it implies Theorem 1.2. Independently, this result has also been obtained by Regev [1], and a bijective proof that is different from ours has been given by Krattenthaler [3].

Theorem 1.3. For all $n, k, \alpha$,

$$
A L(T)=A L\left(T^{*}\right)
$$

is a multiset identity.
We delay the proof of Theorem 1.3 until the next section, and proceed now by giving a bijective proof that it implies Theorem [1.2. The proof involves partitioning the cells of $R$ and $T^{*}$ into two regions each, and identifying cells in various regions of skew diagrams whose pairs of arm and leg lengths are immediately equal.
Proof that Theorem 1.3 implies Theorem 1.2: Partition the cells of $R$ into two subsets $R_{1}$ and $R_{2}$, given by

$$
\begin{gathered}
R_{1}=\left\{(i, j) \mid 1 \leq i \leq k, n-\alpha_{k-i+1}+1 \leq j \leq n\right\}, \\
R_{2}=\left\{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq n-\alpha_{k-i+1}\right\}
\end{gathered}
$$

and the cells of $T^{*}$ into two subsets $T_{1}^{*}$ and $T_{2}^{*}$, given by

$$
\begin{gathered}
T_{1}^{*}=\left\{(i, j) \mid 1 \leq i \leq k, \alpha_{k-i+1}-\alpha_{k}+1 \leq j \leq n-\alpha_{k}\right\}, \\
T_{2}^{*}=\left\{(i, j) \mid 1 \leq i \leq k, n-\alpha_{k}+1 \leq j \leq n+\alpha_{k-i+1}-\alpha_{k}\right\} .
\end{gathered}
$$

The significance of these regions in this proof is that $R_{1}^{\dagger}=T_{2}^{*}=V^{*}=D$ and $R_{2}^{\dagger}=T_{1}^{*}$. These equalities (using appropriate translations) are immediate from the definitions of the regions. See Figure 2 for an illustration of these regions in the case $n=6, k=4, \alpha=(6,5,3,1)$, and to check visually the above equalities in this case.
Bijective identification of $A L_{S Q}(V)$ and $A L_{R}\left(R_{1}\right)$ : Now $V^{*}=R_{1}^{\dagger}$, so the $j$ th columns of $V$ and $R$, respectively, have the same lengths, for each $j=1, \ldots, \alpha_{1}$. Furthermore, $V$ appears in $S Q$ with cells added below $V$ to extend all columns of $V$ to length $k$. Similarly, $R_{1}$ appears in $R$ with cells added below $R_{1}$ to extend all columns of $R_{1}$ to length $k$. Thus the arm and leg lengths are equal, for the cells that are $i$ rows from the topmost entry, in the $j$ th column from the left most column, of $V$ in $S Q$ and $R_{1}$ in $R$, respectively. Thus we establish immediately that

$$
\begin{equation*}
A L_{S Q}(V)=A L_{R}\left(R_{1}\right) \tag{1}
\end{equation*}
$$

Bijective identification of $A L_{T^{*}}\left(T_{1}^{*}\right)$ and $A L_{R}\left(R_{2}\right)$ : Now $T_{1}^{*}=R_{2}^{\dagger}$, so the $i$ th rows of $T_{1}^{*}$ and $R_{2}$, respectively, have the same lengths, for each $i=1, \ldots, k$ (some of these lengths are zero


Figure 2: Skew shapes for $n=6, k=4, \alpha=(6,5,3,1)$.
when $\alpha_{1}=n$ ). Furthermore, $T_{1}$ appears in $T$ with cells added to the right of $T_{1}$ to extend all rows of $T_{1}$ to length $n$. Similarly, $R_{2}$ appears in $R$ with cells added to the right of $R_{2}$ to extend all rows of $R_{2}$ to length $n$. Thus, the arm lengths and leg lengths are equal, for the cells that are $j$ columns from the left most entry, in the $i$ th row from the bottom row, of $T_{1}^{*}$ in $T^{*}$ and $R_{2}$ in $R$, respectively. Thus we establish immediately that

$$
\begin{equation*}
A L_{T^{*}}\left(T_{1}^{*}\right)=A L_{R}\left(R_{2}\right) \tag{2}
\end{equation*}
$$

Bijective identification of $A L_{T^{*}}\left(T_{2}^{*}\right)$ and $A L(D)$ : Now $T_{2}^{*}=D$, and $T_{2}^{*}$ appears in $T^{*}$ with no cells added to the right nor below, so we establish immediately that

$$
\begin{equation*}
A L_{T^{*}}\left(T_{2}^{*}\right)=A L(D) \tag{3}
\end{equation*}
$$

The result: Suppose Theorem 1.3 is true. Then, applying (【), we obtain

$$
\begin{equation*}
A L_{S Q}(V) \cup A L(T)=A L_{R}\left(R_{1}\right) \cup A L\left(T^{*}\right) \tag{4}
\end{equation*}
$$

But $A L(T)=A L_{S Q}(T)$, since $T$ appears in $S Q$ with no cells added to the right nor below. Also, $A L\left(T^{*}\right)=A L_{T^{*}}\left(T_{1}^{*}\right) \cup A L_{T^{*}}\left(T_{2}^{*}\right)$, since $T_{1}^{*}$ and $T_{2}^{*}$ partition the cells of $T^{*}$. Making these substitutions into (4) gives

$$
\begin{aligned}
A L_{S Q}(V) \cup A L_{S Q}(T) & =A L_{R}\left(R_{1}\right) \cup A L_{T^{*}}\left(T_{1}^{*}\right) \cup A L_{T^{*}}\left(T_{2}^{*}\right) \\
& =A L_{R}\left(R_{1}\right) \cup A L_{R}\left(R_{2}\right) \cup A L(D),
\end{aligned}
$$

with the second equality from (2) and (3). Now $V$ and $T$ partition the cells of $S Q$, and $R_{1}$ and $R_{2}$ partition the cells of $R$, so the above result becomes $A L(S Q)=A L(R) \cup A L(D)$, and we have established Theorem 1.2.

How is this proof bijective? To prove Theorem 1.3 bijectively, in the next section we determine an explicit bijection $\phi: T \rightarrow T^{*}$, that preserves arm and leg lengths (this means that for each cell $x \in T$ we have $a_{T}(x)=a_{T^{*}}(\phi(x))$ and $\left.l_{T}(x)=l_{T^{*}}(\phi(x))\right)$. Similarly, to give a bijective proof of Theorem 1.2, we must determine an explicit bijection $\psi: S Q \rightarrow R \cup D$, that preserves arm and leg lengths.

In terms of $\phi$, we now describe such a bijection $\psi$ that is implicit in the above proof. First, note that, to establish (11), (2) and (3) above, we have described three simple bijections, and let us call them $\zeta_{1}: V \rightarrow R_{1}, \zeta_{2}: T_{1}^{*} \rightarrow R_{2}$, and $\zeta_{3}: T_{2}^{*} \rightarrow D$.
A bijection $\psi$ that establishes Theorem 1.2; For $x \in S Q$, we obtain $\psi(x) \in R \cup D$ as follows:

For $x \in V$, let $\psi(x)=\zeta_{1}(x)$.
For $x \in T$,

- if $\phi(x) \in T_{1}^{*}$, let $\psi(x)=\zeta_{2}(\phi(x))$,
- if $\phi(x) \in T_{2}^{*}$, let $\psi(x)=\zeta_{3}(\phi(x))$.

This clearly specifies a bijection $\psi$ of the required type, giving a bijective proof of Theorem 1.2.

## 2 Dyck paths and the bijection

In this section, we determine a bijection $\phi: T \rightarrow T^{*}$, that preserves arm and leg lengths, as referred to above at the end of Section 1. This provides a bijective proof of Theorem 1.3.

The bijection is described in terms of a particular type of lattice path that will be associated with $T$ and $T^{*}$, called a Dyck path. A Dyck path of length $2 k, k \geq 0$, is a sequence $\left(i, y_{i}\right), i=0, \ldots, 2 k$, of lattice points in the plane, in which $y_{0}=y_{2 k}=0, y_{i} \geq 0$, for $i=1, \ldots, 2 k-1$, and $y_{i}-y_{i-1}=+1$ or -1 , for $i=1, \ldots, 2 k$. Equivalently, a Dyck path is completely specified by its sequence of steps; if $y_{i}-y_{i-1}=+1$ then the $i$ th step is an up step, and if $y_{i}-y_{i-1}=-1$ then the $i$ th step is a down step. The height of the $i$ th step is $y_{i-1}$, for $i=1, \ldots, 2 k$. Since $y_{2 k}=0$, then the $2 k$ steps consist of $k$ up steps and $k$ down steps. We can visualize a Dyck path as a connected path in the plane by drawing a line segment between the consecutive lattice points in the path.

Let the skew diagrams $T_{[i]}$ and $T_{(i)}$, for $i=1, \ldots, n$, be given by

$$
\begin{aligned}
T_{[i]} & =\left\{x \in T \mid a_{T}(x)=i-1\right\}, \\
T_{(i)} & =\left\{x \in T \mid a_{T}(x) \leq i-1\right\},
\end{aligned}
$$

and define $\left(T^{*}\right)_{[i]}$ and $\left(T^{*}\right)_{(i)}$ in the same way. Consider the skew diagram $T_{(i)}$, for each fixed $i=1, \ldots, n$. Label the $k$ cells of $T_{[i]}$ in $T_{(i)}$, successively, $x_{1}, \ldots, x_{k}$, from bottom to top (there is exactly one cell of $T_{[i]}$ in each of the $k$ rows of $\left.T_{(i)}\right)$. Label the cells of $T_{[0]}$ in $T_{(i)}$, successively, $z_{1}, \ldots, z_{k}$, from top to bottom (similarly, there is exactly one cell of $T_{[0]}$ in each of the $k$ rows of $T_{(i)}$ ). In the case $i=1$, then each cell of $T_{[0]}$ will have two labels, one an $x_{j}$ and the other $z_{k+1-j}$, for some $j=1, \ldots, k$.

Now form a permutation $\sigma_{i}$ of $x_{1}, \ldots, x_{k}, z_{1}, \ldots, z_{k}$ as follows: Place the $x$ 's and $z$ 's from left to right in $\sigma_{i}$ in the order that they appear from left to right as labels in the cells of $T_{(i)}$. For labels in the same column of $T_{(i)}$, order them with the $x$ 's first, followed by the $z$ 's; the $x$ 's are ordered as they appear from bottom to top in the same column, and the $z$ 's from bottom to top also. For example, in the case $n=11, k=9, \alpha=(11,11,9,8,8,6,3,1,0)$, we illustrate $T_{(3)}$ in Figure 3, with the cells labelled as described above. In this case, the permutation $\sigma_{3}$ is given by

$$
\sigma_{3}=x_{1} x_{2} x_{3} z_{9} z_{8} x_{4} x_{5} z_{7} x_{6} z_{6} z_{5} z_{4} x_{7} x_{8} z_{3} x_{9} z_{2} z_{1}
$$



Figure 3: $T_{(3)}$ for $n=11, k=9, \alpha=(11,11,9,8,8,6,3,1,0)$.
Now let $\rho_{i}$ be the lattice path starting at $(0,0)$, whose steps are specified by $\sigma_{i}$ as follows: the $x_{j}$ 's specify the up steps (labelled $x_{j}$ ), and the $z_{j}$ 's specify the down steps (labelled $z_{j}$ ). For example the lattice path $\rho_{3}$ determined from $\sigma_{3}$ in the example above is illustrated in Figure 4.


Figure 4: The Dyck path $\rho_{3}$ determined from $\sigma_{3}$.
It is a straightforward induction to prove that the height of the up step labelled $x_{j}$ in $\rho_{i}$ is equal to the leg length of the cell labelled $x_{j}$ in $T_{(i)}$, and that the height of the down step labelled $z_{j}$ in $\rho_{i}$ is equal to one more than the coleg length of the cell labelled $z_{j}$ in $T_{(i)}$. But since leg and coleg lengths are always nonnegative, the height of every up step in $\rho_{i}$ is
nonnegative, and the height of every down step in $\rho_{i}$ is positive, so $\rho_{i}$ is a Dyck path. For example, the lattice path $\rho_{3}$ illustrated in Figure 4 is clearly a Dyck path.

Now there is a natural bijection between the up steps and down steps in a Dyck path: pair each up step at height $j$ with the first down step at height $j+1$ occurring after that up step (there must be such a down step since the path ends at a vertex with ordinate equal to 0 , and down steps decrease the value of the ordinate by exactly 1 for each step). Suppose that the up step labelled $x_{j}$ is paired with the down step labelled $z_{P_{i}(j)}$ in this way, for $j=1, \ldots, k$. Then $P_{i}$ is a bijection on $\{1, \ldots, k\}$, for each fixed $i$. For example, for the Dyck path illustrated in Figure 4, we have $P_{3}(1)=4, P_{3}(2)=8, P_{3}(3)=9, P_{3}(4)=5$, $P_{3}(5)=7, P_{3}(6)=6, P_{3}(7)=1, P_{3}(8)=3$, and $P_{3}(9)=2$.

Now rotate $T_{(i)}$, with its cells labelled as above, through 180 degrees, to obtain $\delta$. Now $\delta=\left(T_{(i)}\right)^{*}=\left(T^{*}\right)_{(i)}$, and the cells of $T_{[0]}$ in $T_{(i)}$, labelled with $z_{j}$ 's, become the cells of $\left(T^{*}\right)_{[i]}$ in $\delta$. Moreover, the coleg length of a cell labelled $z_{j}$ in $T_{(i)}$ equals the leg length of the corresponding cell in $\delta$, so

$$
l_{T_{(i)}}\left(x_{j}\right)=l_{\left(T^{*}\right)_{(i)}}\left(z_{P_{i}(j)}\right),
$$

where, for example, $l_{T_{(i)}}\left(x_{j}\right)$ means the leg length of the cell labelled $x_{j}$ in $T_{(i)}$. Also,

$$
a_{T_{(i)}}\left(x_{j}\right)=i-1=a_{\left(T^{*}\right)(i)}\left(z_{P_{i}(j)}\right),
$$

since all cells in $T_{[i]}$ and $\left(T^{*}\right)_{[i]}$ have arm length equal to $i-1$, for each fixed $i$. But $T_{(i)}$ appears in $T$ with no cells added to the right nor below, so $l_{T_{(i)}}\left(x_{j}\right)=l_{T}\left(x_{j}\right)$ and $a_{T_{(i)}}\left(x_{j}\right)=a_{T}\left(x_{j}\right)$. Similarly, $l_{\left(T^{*}\right)_{(i)}}\left(z_{P_{i}(j)}\right)=l_{T^{*}}\left(z_{P_{i}(j)}\right)$ and $a_{\left(T^{*}\right)_{(i)}}\left(z_{P_{i}(j)}\right)=a_{T^{*}}\left(z_{P_{i}(j)}\right)$. Thus, putting these equalities together, we have

$$
\begin{equation*}
l_{T}\left(x_{j}\right)=l_{T^{*}}\left(z_{P_{i}(j)}\right), \quad a_{T}\left(x_{j}\right)=a_{T^{*}}\left(z_{P_{i}(j)}\right) . \tag{5}
\end{equation*}
$$

Proof of Theorem 1.3: This follows from Lemma 2.1 immediately, These equations imply that the mapping from the cell labelled $x_{j}$ in $T$ to the cell labelled $z_{P_{i}(j)}$ in $T^{*}$, for each $i=1, \ldots, n$, is arm and leg length preserving, so we have found the bijection $\phi$ that we require, as stated below.

A bijection $\phi$ that establishes Theorem 1.3: For $w \in T$, we obtain $\phi(w) \in T^{*}$ as follows. Each $w$ is contained in $T_{[i]}$ for some unique $i=1, \ldots, n$. If $w$ has label $x_{j}$ in $T_{(i)}$, then $\phi(w)$ is the cell with label $z_{P_{i}(j)}$ in $\left(T^{*}\right)_{(i)}$.

This clearly specifies a bijection, that is arm and leg length preserving from (5), giving a bijective proof of Theorem 1.3 .

## 3 The projective case

A refinement of Theorem 1.2 has been given by Regev [5], in which the partition $\alpha$ has a special form. In order to state this result, we require some adaptations of the notation in Section 1. Let $n=k+1$, and $\alpha$ have the form $\alpha=\left(\lambda_{1}, \ldots, \lambda_{m} \mid \lambda_{1}-1, \ldots, \lambda_{m}-1\right)$, in Frobenius notation, where $k \geq \lambda_{1}>\ldots>\lambda_{m}>0$, so $\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ is a partition with $m$ distinct parts. This means that $D$, the Young diagram of $\alpha$, has exactly $m$ cells on the
(top-left to bottom-right) diagonal, given by the cells $(k+1-j, j)$, for $j=1, \ldots, m$, with $\lambda_{j}$ cells to the right of the $j$ th of these cells in row $k+1-j$, and $\lambda_{j}-1$ cells below this cell in column $j$. Let $\mathcal{B}$ consist of all partitions $\alpha$ of this form, for any $m \geq 0, k \geq 1$ (e.g., $R$ is the Young diagram of a partition in $\mathcal{B}$, with $m=k$ and $\lambda_{j}=k+1-j$, for $j=1, \ldots, k$ ).

For a Young diagram $G$, let $p(G)$ consist of the cells of $G$ on or below the diagonal (as described above), and let $q(G)$ consist of the cells of $G$ strictly above the diagonal. For a skew diagram, extend this notation by describing the diagonal: for $T, T_{[i]}, T_{(i)}, S Q$, where $n=k+1$, and $\alpha \in \mathcal{B}$, the diagonal consists of the cells $\left(k+1-j, \alpha_{1}+j\right)$, for $j=1, \ldots, k-m$; for $T^{*}$, the diagonal consists of the cells $\left(k+1-j, k+1-\alpha_{k}+j\right)$, for $j=1, \ldots, m$. For example, the skew diagrams $D, R, S Q, T$ are illustrated in Figure XXX for the case $k=5, m=2, \alpha=(5,4,2,1)$, corresponding to $\lambda=(4,2)$. In each of these skew diagrams, there is a thick line extending from top left to bottom right, which partitions the diagram $G$ into the cells of $p(G)$, below and to the left of the line, and the cells of $q(G)$, above and to the right of the line.

The following result has been given by Regev [5], whose proof is not bijective. A bijective proof has been given by Krattenthaler [3].

Theorem 3.1. For all $k, m$ and $\alpha \in \mathcal{B}$,

$$
A L(p(S Q))=A L(p(R)) \cup A L(q(D))
$$

is a multiset identity.
In order to prove Theorem 3.1, we first note that

$$
\begin{equation*}
A L(p(S Q))=A L(p(T)) \tag{6}
\end{equation*}
$$

so we shall work with $T$ on the left hand side of the result, instead of $S Q$. For each $i=1, \ldots, k+1$, let $u$ be the smallest row index among the elements of $T_{[i]}$ above the diagonal of $T$. Let $T^{i}$ be the skew diagram obtained from $T$ by shifting rows $u, u+1, \ldots, k$ to the right, where necessary, so that the right most of the $k+1$ cells in each of these rows occurs in column $\alpha_{1}+k+1$. (If no element of $T_{[i]}$ is above the diagonal of $T$, then we define $T^{i}=T$.) The diagonals of $T^{i}$ and $T^{i^{*}}$ are the same as for $T$ and $T^{*}$, respectively. For example, the skew diagrams $D, R, T, T^{i}, T^{i^{*}}$ are illustrated in Figure XXX for the case $k=12, m=6, \alpha=(12,11,11,9,8,8,6,4,3,3,1)$, with $i=5$. In each of these skew diagrams, there is again a thick line partitioning the cells into those given by $p$ and $q$, and there is a dot in every cell with arm length equal to $i-1=4$.

We require the following technical result about the row index $u$, chosen above for each $i$.
Proposition 3.2. Let $\alpha \in \mathcal{B}$, with the diagonal of length $m$, and with $\alpha_{1} \leq k+1$. Let $u$ be the smallest row index among the elements of $T_{[i]}$ above the diagonal of $T$. Then

1. $u-\alpha_{u}>i-1$ and $u-1-\alpha_{u-1} \leq i-1$,
2. $u>m$,
3. $\alpha_{u-i} \geq u \quad$ and $\quad \alpha_{u-i+1} \leq u$,

$$
\text { 4. } \alpha_{u}+i \leq \alpha_{u-i} \quad \text { and } \quad \alpha_{u-1}+i \geq \alpha_{u-i+1},
$$

Proof: In the row of $T$ with index $a$, for $a=1, \ldots, k$, the diagonal cell is in column $\alpha_{1}+k+1-a$, the right most element is in column $\alpha_{1}+k+1-\alpha_{a}$, and the unique element of $T_{[i]}$ is therefore in column $\alpha_{1}+k+1-\alpha_{a}-(i-1)$. This means that the element of $T_{[i]}$ in row $a$ is above the diagonal of $T$ exactly when $\alpha_{1}+k+1-\alpha_{a}-(i-1)>\alpha_{1}+k+1-a$, or $a-\alpha_{a}>i-1$. Part 1 of the result follows immediately.
¿From Part 1, we have $u-\alpha_{u}>i-1 \geq 0$, so $\alpha_{u}<u$. But, since $\alpha \in \mathcal{B}$, then $\alpha_{j} \geq j$ for $j=1, \ldots, m$, where $m$ is the length of the diagonal of $\alpha$, giving Part 2 of the result.

Now let $u-\alpha_{u}=c$ and $u-1-\alpha_{u-1}=d$, where $c>i-1 \geq d$, from Part 1. Thus in the Young diagram $D$ of $\alpha$, the right most cell in row $k+1-u$ is in column $u-c$, and the right most cell in row $k+1-(u-1)$ is in column $u-1-d$. But $\alpha \in \mathcal{B}$, so symmetry of $\mathcal{B}$ implies that the bottom cell in column $u+1$ is in row $k+1-(u-c)$, and the bottom cell in column $u$ is in row $k+1-(u-1-d)$. Thus we have $\alpha_{u-c} \geq u+1, \alpha_{u+1-c}=\ldots=\alpha_{u-1-d}=u, \alpha_{u-d}<u$, and Result 3 follows from $c>i-1 \geq d$.

Part 4 follows immediately from Parts 1 and 3 .
Now we are able to prove Theorem 6, using the bijective proof of Theorem 1.3.
Proof of Theorem 6: Let $M_{1}, M_{2}, M_{3}, M_{4}$ be the multisets of leg lengths of the cells with arm lengths equal to $i-1$, in $T^{i},\left(T^{i}\right)^{*}, p(T), q(D)$, respectively. Now, Theorem 1.3 applied to skew diagram $T^{i}$ gives a bijection between $A L\left(T^{i}\right)$ and $A L\left(\left(T^{i}\right)^{*}\right)$, which contains a bijection between $M_{1}$ and $M_{2}$.

Now, the elements of $M_{1}$ can be partitioned into two subsets: $M_{11}$, corresponding to the cells on or below the diagonal of $T^{i}$; and $M_{12}$, corresponding to the cells above the diagonal. Thus the elements of $M_{11}$ correspond to cells in rows $1, \ldots, u-1$ of $T^{i}$, and the elements of $M_{12}$ correspond to the cells in rows $u, \ldots, k$. But $T$ and $T^{i}$ differ only in rows $u, \ldots, k$, so $M_{11}=M_{3}$. Also, the right most cell of $T^{i}$ is in column $k+1+\alpha_{1}-\alpha_{j}$, for $j=1, \ldots, u-1$. Now let $s$ be chosen so that

$$
\begin{equation*}
\alpha_{s} \leq i-1 \text { and } \alpha_{s-1}>i-1 \tag{7}
\end{equation*}
$$

Then the bottom element of column $k+1+\alpha_{1}-(i-1)$ in $T^{i}$ is in row $s$, so $M_{12}=$ $\{u-s, \ldots, k-s\}$, giving

$$
\begin{equation*}
M_{1}=M_{3} \cup\{u-s, \ldots, k-s\} . \tag{8}
\end{equation*}
$$

Similarly, the elements of $M_{2}$ can be partitioned into three subsets: $M_{21}$, corresponding to the cells in columns $1, \ldots, k+1$ of $T^{i^{*}} ; M_{22}$, corresponding to the cells to the right of column $k+1$ but on or below the diagonal of $T^{i^{*}}$; and $M_{23}$, corresponding to the cells above the diagonal of $T^{i^{*}}$. Now, the right most cell in rows $1, \ldots, k+1-u$ of $T^{i^{*}}$ is in column $k+1$, and the right most cell in row $j$ of $T^{i^{*}}$ is in column $k+1+\alpha_{k+1-j}$, for $j=k+2-u, \ldots, k$. Therefore, from (7), the cells in $M_{21}$ occur in rows $1, \ldots, k+1-s$, and the bottom element in each corresponding column is in row 1 , so $M_{21}=\{0, \ldots, k-s\}$.

Now, let $r$ be the largest row index of the elements of $M_{22}$. Then, since the diagonal elemens of row $j$ is in column $k+1+k+1-j$, for $k=k+2-u, \ldots, k$, we have

$$
\begin{equation*}
k+1-r+i-1 \geq \alpha_{k+1-r} \text { and } k+1-(r+1)+i-1<\alpha_{k-r}, \tag{9}
\end{equation*}
$$

and, from Proposition 3.2(3), we immediately have $k-r=u-i$, or $r=k-u+i$. Also, the bottom element of the columns corresponding to the cells of $M_{22}$ all occur in row $k+2-u$, from the second part of Proposition 3.2(4). Thus, $M_{22}=\{(k+2-s)-(k+2-u), \ldots,(k-$ $u+i)-(k+2-u)\}=\{u-s, \ldots, i-2\}$.

Finally, the leg lengths of the cells of $M_{23}$ are all the same in $T^{i^{*}}$ as in $T^{*}$, from the first part of Proposition 3.2(4). Thus $M_{23}=M_{4}$, and we have

$$
M_{2}=M_{21} \cup M_{22} \cup M_{23}=M_{4} \cup\{0, \ldots, k-s\} \cup\{u-s, \text { ldots }, i-2\} .
$$

The bijection between $M_{1}$ and $M_{2}$ then gives, from (8),

$$
M_{3} \cup\{u-s, \ldots, k-s\}=M_{4} \cup\{0, \ldots, k-s\} \cup\{u-s, l d o t s, i-2\},
$$

and we have

$$
M_{3}=M_{4} \cup\{0, \ldots, i-2\} .
$$

Now, Theorem 3.1 follows from (6), and the fact that the cells in $p(R)$ with arm length equal to $i-1$ in $R$ have leg lengths $0, \ldots, i-2$.

## Acknowledgements

This work was supported by the Natural Sciences and Engineering Research Council of Canada, through a grant to IG, and a PGSA to AY.

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