# A Note on $G$-intersecting Families 

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#### Abstract

Consider a graph $G$ and a $k$-uniform hypergraph $\mathcal{H}$ on common vertex set $[n]$. We say that $\mathcal{H}$ is $G$-intersecting if for every pair of edges in $X, Y \in \mathcal{H}$ there are vertices $x \in X$ and $y \in Y$ such that $x=y$ or $x$ and $y$ are joined by an edge in $G$. This notion was introduced by Bohman, Frieze, Ruszinkó and Thoma who proved a natural generalization of the Erdős-Ko-Rado Theorem for $G$-intersecting $k$-uniform hypergraphs for $G$ sparse and $k=O\left(n^{1 / 4}\right)$. In this note, we extend this result to $k=O(\sqrt{n})$.


## 1 Introduction

A hypergraph is said to be intersecting if every pair of edges has a nonempty intersection. The well-known theorem of Erdős, Ko and Rado [2, 3] details the extremal $k$-uniform intersecting hypergraph on $n$ vertices.

Theorem 1 (Erdős-Ko-Rado). Let $k \leq n / 2$ and $\mathcal{H}$ be a $k$-uniform, intersecting hypergraph on vertex set $[n]$. We have $|\mathcal{H}| \leq\binom{ n-1}{k-1}$. Furthermore, $|\mathcal{H}|=\binom{n-1}{k-1}$ if and only if there exists $v \in[n]$ such that $\mathcal{H}=\left\{e \in\binom{n}{k}: v \in e\right\}$.

Of course, for $k>n / 2$ the hypergraph consisting of all $k$-sets is intersecting. So, extremal $k$-intersecting hypergraphs come in one of two forms, depending on the value of $k$.

Bohman, Frieze, Ruszinkó and Thoma [1] introduced a generalization of the notion of an intersecting hypergraph. Let $G$ be a graph on a vertex set $[n]$ and $\mathcal{H}$ be a hypergraph, also on vertex set $[n]$. We say $\mathcal{H}$ is $G$-intersecting if for any $e, f \in \mathcal{H}$, we have $e \cap f \neq \emptyset$ or there are vertices $v, w$ with $v \in e, w \in f$ and $v \sim_{G} w$. We are intersected in the size and structure of maximum $G$-intersecting hypergraphs; in particular, we investigate

$$
N(G, k)=\max \left\{|\mathcal{H}|: \mathcal{H} \subseteq\binom{[n]}{k} \text { and } \mathcal{H} \text { is } G \text {-intersecting }\right\}
$$

[^0]Clearly, Erdős-Ko-Rado gives the value of $N\left(E_{n}, k\right)$ where $E_{n}$ is the empty graph on vertex set [ $n$ ]. For a discusssion of $N(G, k)$ for some other specific graphs see [1].

In this note we restrict our attention to sparse graphs: those graphs for which $n$ is large and the maximum degree of $G, \Delta(G)$, is a constant in $n$. What form can a maximum $G$ intersecting family take? If $K$ is a maximum clique in $G$ then a candidate for a maximum $G$-intersecting family is

$$
\mathcal{H}_{K}:=\left\{X \in\binom{[n]}{k}: X \cap K \neq \emptyset\right\} .
$$

Note that such a hypergraph can be viewed as a natural generalization of the maximum intersecting hypergraphs given by Erdős-Ko-Rado. However, for many graphs and maximum cliques $K$ one can add hyperedges to $\mathcal{H}_{K}$ to obtain a larger $G$-intersecting hypergraph.

Consider, for example, $C_{n}$, the cycle on vertex set $[n]$ (i.e. the graph on $[n]$ in which $u$ and $v$ are adjacent iff $u-v \in\{1, n-1\} \bmod n)$. The set $\{2,3\}$ is a maximum clique in $C_{n}$ and the set

$$
\begin{equation*}
\mathcal{H}_{\{2,3\}} \cup\left\{X \in\binom{[n]}{k}:\{1,4\} \subseteq X\right\} \tag{1}
\end{equation*}
$$

is $G$ intersecting. Bohman, Frieze, Ruszinkó and Thoma showed that

$$
\begin{equation*}
N\left(C_{n}, k\right)=\binom{n}{k}-\binom{n-2}{k}+\binom{n-4}{k-2} \tag{2}
\end{equation*}
$$

(i.e. the hypergraph given in (11) is maximum) for $k$ less than a certain constant times $n^{1 / 4}$. In fact, they showed that for arbitrary sparse graphs and $k$ small, $N(G, k)$ is given by a hypergraph that consists of $\mathcal{H}_{K}$ for some clique $K$ together with a number of 'extra' hyperedges that cover the clique $K$ in $G$ (see Theorem 1 of [1). In this note we extend this result to larger values of $k$.

Theorem 2. Let $G$ be a graph on $n$ vertices with maximum degree $\Delta$ and clique number $\omega$. There exists a constant $C$ (depending only on $\Delta$ and $\omega$ ) such that if $\mathcal{H}$ is a $G$-intersecting $k$-uniform hypergraph and $k<C n^{1 / 2}$ then

$$
|\mathcal{H}| \leq\binom{ n}{k}-\binom{n-\omega}{k}+\binom{\omega(\Delta-\omega+1)}{2}\binom{n-\omega-2}{k-2} .
$$

Furthermore, if $\mathcal{H}$ is a G-intersecting family of maximum cardinality then there exists a maximum clique $K$ in $G$ such that $\mathcal{H}$ contains all $k$-sets that intersect $K$.

An immediate corollary of this Theorem is that (2) holds for $k<C \sqrt{n}$.
Of course, a maximum $G$-intersecting hypergraph will not be of the form ' $\mathcal{H}_{K}$ together with some extra hyperedges' if $k$ is too large. Even for sparse graphs, when $k$ is large enough, there are hypergraphs that consist of nearly all of $\binom{[n]}{k}$ that are $G$-intersecting. In particular, Bohman, Frieze, Ruszinkó and Thoma showed that if $G$ is a sparse graph with
minimum degree $\delta, c$ is a constant such that $c-(1-c)^{\delta+1}>0$ and $k>c n$, then the size of the largest $G$-intersecting, $k$-uniform hypergraph is at least $\left(1-e^{-\Omega(n)}\right)\binom{n}{k}$ (see Theorem 7 of [1]). In some sense, this generalizes the trivial observation that $\binom{[n]}{k}$ is intersecting for $k>n / 2$.

There is a considerable gap between the values of $k$ for which we have established these two types of behavior for maximum $G$-intersecting families. For example, for $C_{n}$ we have (21) for $k<C \sqrt{n}$ while we have $N\left(C_{n}, k\right)>(1-o(1))\binom{n}{k}$ for $k$ greater than roughly $.32 n$. What happens for other values of $k$ ? Are there other forms that a maximum $G$-intersecting family can take? Bohman, Frieze, Ruszinkó and Thoma conjecture that this is not the case, at least for the cycle.
Conjecture 1. There exists a constant $c$ such that for any fixed $\epsilon>0$

$$
\begin{aligned}
& k \leq(c-\epsilon) n \quad \Rightarrow \quad N\left(C_{n}, k\right)=\binom{n}{k}-\binom{n-2}{k}+\binom{n-4}{k-2} \\
& k \geq(c+\epsilon) n \Rightarrow N\left(C_{n}, k\right)=(1-o(1))\binom{n}{k}
\end{aligned}
$$

The remainder of this note consists of the proof of Theorem 2.

## 2 Utilizing $\tau$

Let $\mathcal{H}$ be a hypergraph and $G$ be a graph on vertex set $[n]$. For $X \subseteq[n]$, we define

$$
N(X):=\left\{v \in V(G): v \sim_{G} w \text { for some } w \in X\right\} \cup X
$$

For $x \in[n]$ we write $N(x)$ for $N(\{x\})$. We will define the hypergraph $\mathcal{F}$ by setting $f \in \mathcal{F}$ if and only if $f=N(h)$ for some $h \in \mathcal{H}$. Note that if $\mathcal{H}$ is $G$-intersecting, then

$$
\begin{equation*}
h \in \mathcal{H}, f \in \mathcal{F} \Rightarrow h \cap f \neq \emptyset . \tag{3}
\end{equation*}
$$

The quantity $\tau(\mathcal{F})$ is the cover number of $\mathcal{F}$.
The proof of Theorem 2 follows immediately from Lemma 1 , which deals with the case where $\tau(\mathcal{F}) \geq 2$ and Lemma 2, which deals with the case where $\tau(\mathcal{F})=1$.

Lemma 1. Let $G$ be a graph on $n$ vertices with maximum degree $\Delta$ and clique number $\omega$, both constants. If $k<\sqrt{\frac{\omega n}{2(\Delta+1)^{2}}}$, $\mathcal{H}$ is a $k$-uniform, $G$-intersecting hypergraph on $n$ vertices and $n$ is sufficiently large, then $\tau(\mathcal{F})=1$ or

$$
\begin{equation*}
|\mathcal{H}|<\binom{n}{k}-\binom{n-\omega}{k} . \tag{4}
\end{equation*}
$$

## Proof.

Suppose, by way of contradiction, that $\tau=\tau(\mathcal{F}) \geq 2$ and (4) does not hold. For $v \in[n]$ set $\mathcal{H}_{v}=\{f \in \mathcal{F}: u \in f\}$, and for $Y \subseteq[n]$ set $\mathcal{H}_{Y}=\{f \in \mathcal{F}: Y \subseteq f\}$. Let $\mathcal{F}_{u}$ and $\mathcal{F}_{Y}$ be defined analogously

We first use $\tau>1$ to get an upper bound $\left|\mathcal{H}_{u}\right|$ for an arbitrary $u \in[n]$. First note that, since $\tau>1$, there exists $X_{1} \in \mathcal{F}$ such that $u \notin X_{1}$. It follows from (3) that each $f \in \mathcal{F}_{u}$ must intersect $X_{1}$. In other words, we have

$$
\mathcal{F}_{u}=\bigcup_{u_{1} \in X_{1}} \mathcal{F}_{\left\{u, u_{1}\right\}} .
$$

This observation can be iterated: if $i<\tau$ and $Y=\left\{u=u_{0}, u_{1}, \ldots, u_{i-1}\right\}$ then there exists $X_{i} \in \mathcal{F}$ such that $X_{i} \cap Y=\emptyset$, and we have

$$
\mathcal{F}_{Y}=\bigcup_{u_{i} \in X_{i}} \mathcal{F}_{Y \cup\left\{u_{i}\right\}}
$$

Since $|f| \leq(\Delta+1) k$ for all $f \in \mathcal{F}$, it follows that we have

$$
\begin{equation*}
\left|\mathcal{H}_{u}\right| \leq((\Delta+1) k)^{\tau-1}\binom{n-\tau}{k-\tau} . \tag{5}
\end{equation*}
$$

On the other hand, by the definition of $\tau$, there exists $v \in[n]$ for which

$$
\frac{1}{\tau}\left[\binom{n}{k}-\binom{n-\omega(G)}{k}\right] \leq\left|\mathcal{F}_{v}\right| .
$$

It follows that there exists $u \in[n]$ such that

$$
\frac{1}{\tau(\Delta+1)}\left[\binom{n}{k}-\binom{n-\omega(G)}{k}\right] \leq\left|\mathcal{H}_{u}\right| .
$$

Applying (5) to this vertex we have

$$
\binom{n}{k}-\binom{n-\omega(G)}{k} \leq \tau(\Delta+1)^{\tau} k^{\tau-1}\binom{n-\tau}{k-\tau} .
$$

In order to show that this is a contradiction, we first note that $\tau(\Delta+1)^{\tau} k^{\tau-1}\binom{n-\tau}{k-\tau}$ is a function that is decreasing in $\tau$. Indeed, for $\tau \geq 2$ we have

$$
\frac{n-\tau}{k-\tau} \geq \frac{n-2}{k-2} \geq \frac{3}{2}(\Delta+1) k \geq \frac{\tau+1}{\tau}(\Delta+1) k
$$

(note that the condition $k<\sqrt{\frac{\omega n}{2(\Delta+1)^{2}}}$ is used in the second inequality). It follows that we have

$$
\binom{n}{k}-\binom{n-\omega(G)}{k} \leq 2(\Delta+1)^{2} k\binom{n-2}{k-2},
$$

which is not true if $k<\sqrt{\frac{n \omega(G)}{2(\Delta+1)^{2}}}$ and $n$ is large enough.

Lemma 2. Let $G$ be a graph on $[n]$ with maximum degree $\Delta$, a constant. If $\mathcal{H}$ is a $k$ uniform, $G$-intersecting hypergraph on $[n], k \leq \sqrt{\frac{n}{\Delta(\Delta+1)}}, \tau(\mathcal{F})=1$, $n$ is sufficiently large and $\mathcal{H}$ is of maximum size, then there exists a maximum-sized clique $K$ in $G$ such that $\mathcal{H}$ contains every $k$-set that intersects $K$.

Proof. Let us suppose $\mathcal{H}$ is of maximum size and let $u$ be a cover for $\mathcal{F}$, the hypergraph defined above.

For $v \in[n]$, let $\mathcal{H}_{v}$ denote the members of $\mathcal{H}$ that contain $v$. Since $\mathcal{H}$ is assumed to be extremal, we may assume that $\left|\mathcal{H}_{u}\right|=\binom{n-1}{k-1}$. Let $K$ be the set of $v \in[n]$ such that $\left|\mathcal{H}_{v}\right|=\binom{n-1}{k-1}$. If $n>(\Delta+2) k$ then $K$ must be a clique in $G$; otherwise, we could find two sets that are not $G$-intersecting in $\mathcal{H}$.

We now show that the clique $K$ is maximal. Assume for the sake of contradiction that $v$ is adjacent to every element of $K$ but $v \notin K$ (i.e. $\left|\mathcal{H}_{v}\right|<\binom{n-1}{k-1}$ ). There exists $h \in \mathcal{H}$ that $h$ contains no member of $N(v)$. It follows from (3) that we have

$$
\left|\mathcal{H}_{v}\right|<(\Delta+1) k\binom{n-2}{k-2} .
$$

Since this bounds holds for all vertices in $N(u) \backslash K$, if we have

$$
\begin{equation*}
\Delta(\Delta+1) k\binom{n-2}{k-2}<\binom{n-|K|-1}{k-1} \tag{6}
\end{equation*}
$$

then the number of $k$-sets that contain $v$ but do not intersect $K$ outnumber those edges in $\mathcal{H}$ that contain no member of $K$. In other words, if (6) holds then we get a contradiction to the maximality of $\mathcal{H}$. However, (6) holds for $n$ sufficiently large (here we use $k<\sqrt{\frac{n}{\Delta(\Delta+1)}}$ ).

It remains to show that $K$ is a maximum clique. Since $K$ is maximal, it must be that any member of $\mathcal{H}$ that does not contain a member of $K$ must contain at least 2 members of $N(K) \backslash K$. If

$$
\begin{equation*}
\binom{n}{k}-\binom{n-|K|}{k}+\binom{|K|(\Delta-|K|+1)}{2}\binom{n-|K|-2}{k-2}<\binom{n}{k}-\binom{n-|K|-1}{k} \tag{7}
\end{equation*}
$$

and there is some clique of size $|K|+1$, then $\mathcal{H}$ cannot be maximum-sized. But (77) holds for $k=o(n)$. So the maximum-sized $G$ intersecting family must contain all members of $\bigcup_{v \in K} \mathcal{F}_{v}$ for some $K$ with $|K|=\omega(G)$.

## References

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