

A Short Proof that “Proper = Unit”

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Abstract. A short proof is given that the graphs with proper interval representations are the same as the graphs with unit interval representations.

An graph is an *interval graph* if its vertices can be assigned intervals on the real line so that vertices are adjacent if and only if the corresponding intervals intersect; such an assignment is an *interval representation*. When the intervals have the same length, we have a *unit interval representation*. When no interval properly contains another, we have a *proper interval representation*. The *unit interval graphs* and *proper interval graphs* are the interval graphs having unit interval or proper interval representations, respectively.

Since no interval contains another of the same length, every unit interval graph is a proper interval graph. Roberts [1] proved that also every proper interval graph is a unit interval graph; the two classes are the same. He proved this as part of a characterization of unit interval graphs as the interval graphs with no induced subgraph isomorphic to the “claw” $K_{1,3}$ (it is immediate that the condition is necessary). Roberts used a version of the Scott-Suppes [2] characterization of semiorders to prove that claw-free interval graphs are unit interval graphs. By eschewing the trivial implication that unit interval graphs are proper interval graphs and instead going from “claw-free” to “proper” to “unit” among the interval graphs, we obtain a short self-contained proof.

In the language of partial orders, our proof also characterizes the semiorders among the interval orders. A partial order is an *interval order* if its elements can be assigned intervals on the real line so that $x < y$ if and only if the interval assigned to x is completely to the left of the interval assigned to y . A partial order is a *semiorder* if its elements can be assigned numbers so that $x < y$ if and only if the number assigned to y exceeds the number assigned to x by more than 1. The poset $\underline{1} + \underline{3}$ is the poset consisting of two disjoint chains of sizes 3 and 1. The semiorders are precisely the interval orders that do not contain $\underline{1} + \underline{3}$.

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THEOREM The following statements are equivalent when G is a simple graph.

- A) G is a unit interval graph.
- B) G is an interval graph with no induced $K_{1,3}$.
- C) G is a proper interval graph.

Proof: In an interval representation of $K_{1,3}$, the intervals for the three leaves must be pairwise disjoint, and then the interval for the central vertex must properly contain the middle of the three intervals for leaves. Thus A and C imply B.

For the converse, let G be a claw-free interval graph, and consider an interval representation that assigns to each $v \in V(G)$ an interval I_v . We first transform this into a proper interval representation. Since G is claw-free, there is no pair $x, y \in V(G)$ such that 1) $I_y \subset I_x$ and 2) I_x intersects intervals to the left and right of I_y that do not intersect I_y . If $I_x = [a, b]$ and $I_y = [c, d]$ with $a < c \leq d < b$, this means that $[a, c]$ or $[d, b]$ is empty of endpoints of intervals that don't intersect I_y . Hence we can extend I_y past the end of I_x on one end without changing the graph obtained from the representation. Repeating this until no more pairs of intervals are related by inclusion yields a proper interval representation.

From a proper interval representation of G , we obtain a unit interval representation. When no interval properly includes another, the right endpoints have the same order as the left endpoints. We process the representation from left to right, adjusting all intervals to length 1. At each step until all have been adjusted, let $I_x = [a, b]$ be the unadjusted interval that has the leftmost left endpoint. Let $\alpha = a$ unless I_x contains the right endpoint of some other interval, in which case let α be the largest such right endpoint. Such an endpoint would belong to an interval that has already been adjusted to have length 1; thus $\alpha < \min\{a + 1, b\}$. Now, adjust the portion of the representation in $[a, \infty)$ by shrinking or expanding $[\alpha, b]$ to $[\alpha, a + 1]$ and translating $[b, \infty)$ to $[a + 1, \infty)$. The order of endpoints does not change, intervals earlier than I_x still have length 1, and I_x also now has length 1. Iterating this operation produces the unit interval representation. ■

This theorem has a standard interpretation for posets. The incomparability graph of an interval order is an interval graph. Existence of the function required in the definition of semiorder is equivalent to having a representation as an interval order using intervals of unit length. The incomparability graph of $\underline{1} + \underline{3}$ is $K_{1,3}$. Thus applying the Theorem above to incomparability graphs yields the following Corollary.

COROLLARY The following statements are equivalent when P is a poset.

- A) P is a unit interval order.
- B) P is a semiorder.
- C) P is a interval order not containing $\underline{1} + \underline{3}$.
- D) P is a proper interval order.

References

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- [2] D. Scott and P. Suppes, Foundational aspects of theories of measurement, *J. Symbolic Logic* 23(1958), 233–247.