Technical Report

Department of Computer Science
and Engineering
University of Minnesota
4-192 EECS Building
200 Union Street SE
Minneapolis, MN 55455-0159 USA

TR 99-010

New Bounds on a Hypercube Coloring Problem

Hung Q. Ngo and Ding-zhu Du

March 03, 1999

New Bounds on a Hypercube Coloring Problem

Hung Quang Ngo* Ding Zhu Du*

Abstract

On studying the scalability of optical networks, one problem arising is to color the vertices of the n-cube with as few colors as possible such that any two vertices whose Hamming distance is at most k are colored differently. Determining the exact value of $\chi_{\bar{k}}(n)$, the minimum number of colors needed, is a difficult problem. In this paper, we improve the lower and upper bounds of $\chi_{\bar{k}}(n)$ and indicate the connection of this coloring problem to linear codes.

Keywords: hypercube, coloring, codes

*Department of Computer Science and Engineering, University of Minnesota, 200 Union street, EE/CS Building, room 4-192, Minneapolis, MN 55455, USA. e-mail: {hngo, dzd}@cs.umn.edu. Support in part by by the National Science Foundation under grant CCR-9530306.

1 Introduction

An n-cube (or n dimensional hypercube) is a graph whose vertices are vectors of the n dimensional vector space over the field $\{0,1\}$. There is an edge between two vertices of the n-cube whenever their Hamming distance is exactly 1, where the Hamming distance of two vectors is the number of coordinates they differ. Given n and k, our problem is to find $\chi_{\bar{k}}(n)$, the minimum number of colors needed to color the vertices of an n-cube so that any two vertices of (Hamming) distance at most k have different colors.

Wan [1] proved that

$$n+1 \leq \chi_{\bar{2}}(n) \leq 2^{\lceil \log_2(n+1) \rceil} \tag{1}$$

and conjectured that $\chi_{\bar{2}}(n) = 2^{\lceil \log_2(n+1) \rceil}$

Kim et al. [2] showed that

$$2n \leq \chi_{\bar{3}}(n) \leq 2^{\lceil \log_2 n \rceil + 1} \tag{2}$$

$$\left(\binom{n}{k/2} \right) \leq \chi_{\bar{k}}(n) \leq (k+1)\left(\frac{k+2}{2}\right)^{\frac{k(k+2)}{8} \lceil \log_2 n \rceil} \tag{3}$$

$$2\left(\binom{n-1}{\frac{k-1}{2}} \right) \leq \chi_{\bar{k}}(n) \leq (k+1)\left(\frac{k+2}{2}\right)^{\frac{k(k+2)}{8} \lceil \log_2 n \rceil} \tag{4}$$

where $\left(\binom{n}{m}\right) = \sum_{i=0}^{m} \binom{n}{i}$

The upper bounds in (1) and (2) are fairly tight. In (1), the upper and lower bounds coincide when $n+1$ is an exact power of 2, while if n is an exact power of 2 upper and lower bounds of (2) meet. However, upper bounds in (3) and (4) are not tight. In fact, when $k=2$ and $k=3$ they are different from those of (1) and (2). A natural approach to get an upper bound of $\chi_{\bar{k}}(n)$ is to find a coloring of the n-cube with as few colors as possible. We shall use this idea and properties of linear codes (to be introduced in the next section) to give tighter bounds for general k which imply (1) when $k=2$ and (2) when $k=3$. In fact, the upper bounds in (1) and (2) are straight application of the Hamming code [3].

All existing lower bounds can be improved slightly by applying existing results on the main coding theory problem [3].

The remaining of the paper is organized as follows. Section 2 introduces concepts and results from coding theory needed for the rest of the paper. Section 3 discusses our results and section 4 gives general discussions about the problem.

2 Preliminaries

The following concepts and results can be found in many standard texts on coding theory such as [3].

Let $A = \{0, 1, \ldots q-1\}$ where $q \geq 2$ is an integer, and A^n be the set of all n-dimensional vectors (or strings of length n) over A. Any non-empty subset C of A^n is called a *q-ary block code*. Our main concern is when $A = \{0, 1\}$, in which case C is called a *binary code*. From now on, the term codes refers to binary codes unless specified otherwise. Each element of C is called a *codeword*. Let $M = |C|$ then C is called an *(n, M)-code*. The *Hamming distance* between any two codewords $c = c_1 c_2 \ldots c_n$ and $d = d_1 d_2 \ldots d_n$ are defined to be $d(c, d) = |\{i : c_i \neq d_i\}|$. For $c \in C$, the *weight* of c denoted by $w(c)$ is the number of 1's in c. The *minimum distance* $d(C)$ of a code C is the least Hamming distance between two different codewords in C. If $C \subset A^n$, $|C| = M$, and $d(C) = d$ then C is called an *(n, M, d)-code*.

One of the most important problem in coding theory is to find $A_q(n, d)$, the largest size M such that a q-ary (n, M, d)-code exists. This problem is so important that it is refered to as *the main coding theory problem*. In case $q = 2$, we will write $A(n, d)$ instead of $A_2(n, d)$. The following theorems are standard results in coding theory and the reader is refered to [3] for proofs.

Theorem 1 $A(n, 2t+1) = A(n+1, 2t+2)$

Theorem 2

$$A(n, 2t+1) \leq \frac{2^n}{\sum_{i=0}^{t} \binom{n}{i} + \frac{1}{\left\lfloor \frac{n}{t+1} \right\rfloor} \binom{n}{t} \left(\frac{n-t}{t+1} - \left\lfloor \frac{n-t}{t+1} \right\rfloor \right)}$$

Theorem 2 is a special case of the Johnson bound [3].

It is clear that all n-dimensional vectors over $\{0,1\}$ form an n-dimensional vector space, which we denote by $V_n(2)$. A code $C \subset V_n(2)$ is called a *linear code* if C is a linear subspace of $V_n(2)$. Moreover, C is called a $[n,m]$-*code* if it has dimension m. As expected, if C also has minimum distance d then it is called an $[n,m,d]$-*code*. Notice that the square brackets automatically refer to linear codes. A $m \times n$ matrix G is called the *generator matrix* of an $[n,m]$-code C if its rows form a basis of C. In other words, every codewords in C is a linear combination of some rows of G. Given an $[n,m]$-code C, an $(n-m) \times n$ matrix H is called the *parity check matrix* of C if $c \in C \Leftrightarrow cH^T = 0$. From coding theory, we know that specifying a linear code using generator matrix and using parity check matrix are equivalent. In fact, there are ways to construct the parity check matrix from the generator matrix of a code and vice versa. For a vector $x \in V_2(n)$, the *syndrom* of x associated with a parity check matrix H is defined to be $synd(x) = xH^T$.

Given an $[n,m,d]$-code C, the *standard array* of C is a $2^{n-m} \times 2^m$ table where each row is a (left) coset of C. This table is well defined since elements of C form an Abelian subgroup of $V_2(n)$ under addition, and from basic algebra we know that the cosets of a group partition the group uniformly. The first row of the standard array contains C itself. The first column of the standard array contains the minimum weight elements from each coset. These are called *coset leaders*. Each entry in the table is the sum of the codeword on the top of its column and its coset leader. Since each pair of distinct codewords has Hamming distance at least d, each pair of elements in the same row also has Hamming distance at least d. It is a basic fact from coding theory that all elements on the same row of the standard array have the same syndrom and different rows have different syndroms.

We conclude this section by an important theorem. Again, the reader is refered to [3] for a proof.

Theorem 3 *If H is an $(n-m) \times n$ matrix where any $d-1$ columns of H are linearly independent and there exist d linearly dependent columns in H, then H is the parity check* *matrix of an* $[n, m, d]$*-code.*

3 Main Results

Lemma 1 *If* k *is even, let* $t = \frac{k}{2}$ *we have*

$$\chi_{\bar{k}}(n) \geq \sum_{i=0}^{t} \binom{n}{i} + \frac{1}{\left\lfloor \frac{n}{t+1} \right\rfloor} \binom{n}{t} \left(\frac{n-t}{t+1} - \left\lfloor \frac{n-t}{t+1} \right\rfloor \right)$$

If k *is odd, let* $t = \frac{k-1}{2}$ *we have*

$$\chi_{\bar{k}}(n) \geq 2 \left(\sum_{i=0}^{t} \binom{n-1}{i} + \frac{1}{\left\lfloor \frac{n-1}{t+1} \right\rfloor} \binom{n-1}{t} \left(\frac{n-1-t}{t+1} - \left\lfloor \frac{n-1-t}{t+1} \right\rfloor \right) \right)$$

Proof: Given a valid coloring of the n-cube with parameters n and k using m colors, let $S_i, 1 \leq i \leq m$ be the set of vertices which were colored i. Clearly for each i, S_i forms an $(n, |S_i|, d)$-code where $d \geq k+1$. With the note that $A(n, d)$ is a decreasing function in d, we have

$$2^n = \sum_{i=1}^{m} |S_i| \leq \sum_{i=1}^{m} A(n, k+1) = mA(n, k+1)$$

Thus, inparticular we have $\chi_{\bar{k}}(n) \geq \frac{2^n}{A(n,k+1)}$. When k is even, let $k = 2t$ then by theorem 2 we get

$$\chi_{\bar{k}}(n) \geq \sum_{i=0}^{t} \binom{n}{i} + \frac{1}{\left\lfloor \frac{n}{t+1} \right\rfloor} \binom{n}{t} \left(\frac{n-t}{t+1} - \left\lfloor \frac{n-t}{t+1} \right\rfloor \right)$$

When k is odd, let $k = 2t+1$ and combining theorems 1 and 2 gives us

$$\begin{aligned} \chi_{\bar{k}}(n) &\geq \frac{2^n}{A(n, k+1)} \\ &= \frac{2^n}{A(n, 2t+2)} \\ &= \frac{2^n}{A(n-1, 2t+1)} \\ &\geq 2 \left(\sum_{i=0}^{t} \binom{n-1}{i} + \frac{1}{\left\lfloor \frac{n-1}{t+1} \right\rfloor} \binom{n-1}{t} \left(\frac{n-1-t}{t+1} - \left\lfloor \frac{n-1-t}{t+1} \right\rfloor \right) \right) \square \end{aligned}$$

Lemma 2 *Let* $\left(\!\binom{n}{m}\!\right)$ *denotes* $\sum_{i=0}^{m}\binom{n}{i}$. *Then we have*

$$\chi_{\bar{k}}(n) \leq 2^{\lfloor \log_2(\binom{n-1}{k-1}) \rfloor + 1} \text{ when } k \text{ is even}$$

$$\chi_{\bar{k}}(n) \leq 2^{\lfloor \log_2(\binom{n-2}{k-2}) \rfloor + 2} \text{ when } k \text{ is odd}$$

Proof: Let C be an $[n, m, k+1]$-code. As we have noticed in the previous section, every two elements on the same row of the standard array of C are at least $k+1$ apart. Thus, coloring each row of of C's standard array by one separate color would give us a valid coloring. The number of colors used is 2^{n-m} – the number of rows of C's standard array. Consequently, one way to obtain a good coloring of the n-cube is to find a linear $[n, m, k+1]$ code with as large an m as possible. Moreover, by theorem 3 we can construct a linear $[n, m, d]$ code by trying to build its parity check matrix H, which is an $(n-m) \times n$ matrix with the property that d is the largest number such that any $d-1$ columns of H are linearly independent and there exist d dependent columns. Also, since all elements of a coset of the code (a row of its standard array) have the same syndrom, we can use H to color each vector $x \in V_2(n)$ with $synd(x) = xH^T$.

Let

$$p = \left\lfloor \log_2 \left(1 + \binom{n-1}{1} + \binom{n-1}{2} + \dots \binom{n-1}{d-2} \right) \right\rfloor + 1 = \left\lfloor \log_2 \left(\left(\!\binom{n-1}{d-2}\!\right) \right) \right\rfloor + 1$$

then clearly we have

$$\binom{n-1}{1} + \binom{n-1}{2} + \dots \binom{n-1}{d-2} < 2^p - 1$$

Now, we describe a procedure to construct a $p \times n$ parity check matrix H by trying to choose the column vectors of H. The first column vektor can be any non-zero vector. Suppose we already had a set V of i vectors so that any $d-1$ of them are linearly independent. The $(i+1)^{th}$ vector can be picked as long as it is not in the span of any $d-2$ vectors in V. In otherwords, since we're working over the field $\mathbb{F}_2$, the new vector can't be the sum of any $d-2$ or less vectors in V. The total number of *invalid* vector is at most $\binom{i}{1} + \binom{i}{2} + \dots \binom{i}{d-2}$

(this is an increasing function in i). Consequently, as long as $\binom{i}{1} + \binom{i}{2} + \dots \binom{i}{d-2} < 2^p - 1$ then we can still add a new column into H.

As we've noticed,

$$\binom{n-1}{1} + \binom{n-1}{2} + \dots \binom{n-1}{d-2} < 2^p - 1$$

so we can choose n column vectors of H. This bound in coding thery literature is a special case of the *Gilbert-Varshamov Bound* on the existence of linear codes.

The linear code C whose parity check matrix is H has minimum distance at least d (and size $|C| = 2^{n-p}$). The number of rows of the standard array of C is 2^p.

For our problem of looking for an upper bound of $\chi_{\bar{k}}(n)$, we want $d = k+1$. The linear code C constructed gives a valid coloring using 2^p colors, so

$$\chi_{\bar{k}}(n) \leq 2^p = 2^{\lfloor \log_2 (1 + \binom{n-1}{1} + \binom{n-1}{2} + \dots \binom{n-1}{k-1}) \rfloor + 1} = 2^{\lfloor \log_2 ((\binom{n-1}{k-1})) \rfloor + 1}$$

This inequality holds regardless of k being odd or even and thus it proves our lemma for the even k case. However, when k is odd we are able to do better than that.

We notice that if we add an even parity bit to each vector of $V_2(n-1)$ then we get half of $V_2(n)$. Adding an odd parity bit would give us the other half. When k is odd, we just proved that we can color the $(n-1)$-cube using $a = 2^{\lfloor \log_2 ((\binom{n-2}{k-2})) \rfloor + 1}$ colors so that if two vertices have the same color then their distance is at least k. From this, we can obtain a coloring of the n-cube as follows. We first add an even parity bit to each vertex of the $(n-1)$-cube, color them using a colors, and then add an odd parity bit and color them using a completely different set of a colors. This is clearly a coloring of the n-cube using $2a = 2^{\lfloor \log_2 ((\binom{n-2}{k-2})) \rfloor + 2}$ colors. What remained to be shown is that this coloring is valid with parameters n and k.

For x being any vertex of the n-cube, let x' be the vector obtained from x by deleting the parity bit added. By the way we constructed the coloring, if two vertices x and y of the n-cube have the same color then $d(x', y') \geq k$, and the same type of parity bit (even or odd) was added to them to get x and y. It is clear that if $d(x', y') \geq k+1$, then

$d(x,y) \geq k+1$. If $d(x',y') = k$, then since k is odd, x' and y' must have had different bits added to; consequently, $d(x,y) = k+1$. In sum, if two vertices x and y of the n-cube have the same color then $d(x,y) \geq k+1$, and so we had a valid coloring with parameters n and k.

Lemma 1 and 2 can be summarized by the following theorem.

Theorem 4 *Let $t = \lfloor \frac{k}{2} \rfloor$ and $(\binom{n}{m})$ denotes $\sum_{i=0}^{m} \binom{n}{i}$. Then, when k is even, we have*

$$\sum_{i=0}^{t} \binom{n}{i} + \frac{1}{\left\lfloor \frac{n}{t+1} \right\rfloor} \binom{n}{t} \left(\frac{n-t}{t+1} - \left\lfloor \frac{n-t}{t+1} \right\rfloor \right) \leq \chi_{\bar{k}}(n) \leq 2^{\left\lfloor \log_2 \left(\left(\binom{n-1}{k-1} \right) \right) \right\rfloor + 1}$$

and when k is odd, we have

$$2 \left(\sum_{i=0}^{t} \binom{n-1}{i} + \frac{1}{\left\lfloor \frac{n-1}{t+1} \right\rfloor} \binom{n-1}{t} \left(\frac{n-1-t}{t+1} - \left\lfloor \frac{n-1-t}{t+1} \right\rfloor \right) \right) \leq \chi_{\bar{k}}(n) \leq 2^{\left\lfloor \log_2 \left(\left(\binom{n-2}{k-2} \right) \right) \right\rfloor + 2}$$

Note that since

$$2^{\left\lfloor \log_2 \left(\left(\binom{n-1}{2-1} \right) \right) \right\rfloor + 1} = 2^{\lfloor \log_2 n \rfloor + 1} = 2^{\lceil \log_2 (n+1) \rceil}$$

and

$$2^{\left\lfloor \log_2 \left(\left(\binom{n-2}{3-2} \right) \right) \right\rfloor + 2} = 2^{\lfloor \log_2 (n-1) \rfloor + 2} = 2^{\lceil \log_2 n \rceil + 1}$$

inequalities (1) and (2) are dirrect consequences of this theorem.

4 Concluding Remarks

The key to get a good coloring is to find the parity check matrix H when k is even. As can be seen, the proof of theorem 4 implicitly gave us an algorithm to construct H, but it is still not very constructive. However, in the case $k = 2$ (and thus in case $k = 3$) we can explicitly construct H. To see this, consider the Hamming code $H_2(r)$, which is a $[2^r - 1, 2^r - 1 - r, 3]$ code. Its parity check matrix $H(r,2)$ has dimensions $r \times (2^r - 1)$. Let $r = \lceil \log_2(n+1) \rceil$, then $2^r - 1 \geq n$. So, if we remove the last $2^r - 1 - n$ columns of $H(r,2)$, then we get a parity check matrix of an $[n, n - \lceil \log_2(n+1) \rceil, 3]$ code. This code gives us a coloring of the

n-cube with parameters n and 2 using $2^{\lceil \log_2(n+1) \rceil}$ colors. This proves the upper bound of (1).

Besides the *Johnson bound* we used, other known upper bounds of $A(n, d)$ might give us better lower bound of $\chi_{\bar{k}}(n)$ such as the *Plotkin bound*, the *Elias bound* and the *Linear Programming bound*. However, applying these bounds breaks the problem into various cases and doesn't give us a significantly better result.

References

[1] P.-J. Wan, "Near-optimal conflict free channel set assignments for an optical cluster-based hypercube network," *Journal of Combinatorial Optimization*, pp. 179–186, 1997.

[2] D. S. Kim, D.-Z. Du, and P. M. Pardalos, "A coloring problem on the n-cube." Manuscript, 1999.

[3] S. Roman, *Coding and Information Theory.* New York: Springer-Verlag, 1992.