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GLOBAL INSTABILITY IN EXPERIMENTAL GENERAL EQUILIBRIUM: THE SCARF EXAMPLE

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Abstract

Scarf (1960) proposed a market environment and a model of dynamic adjustment in which the standard tatonnement price adjustment process orbits around, rather than converges to, the competitive equilibrium. Hirota (1981) characterized the price paths by the configuration of endowments. We explore the predictions of Scarf's model in a non-tatonnement experimental double auction. We find that the average transaction prices in each period do follow the path predicted by the Scarf and Hirota models. When the model predicts prices will converge to the competitive equilibrium, our data converge; when the model predicts prices will orbit, our data orbit the equilibrium, and in the direction predicted by the model. Moreover, we observe a weak tendency for prices within a period to follow the path predicted by the model.

Global Instability in Experimental General Equilibrium: The Scarf Example

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1 Introduction[1]

This paper is motivated by two issues. The first is the nature of the equilibration process in multiple, interdependent, continuous, double auction markets. Not only are such markets in widespread use, their study has been of special scientific value. It has been known for years that this particular form of industrial organization creates an effective price discovery mechanism. Prices typically result near the competitive equilibrium. However, exactly how this price discovery takes place, the process itself, is unknown. The second motivating issue is the extent to which models based on tatonnement help with

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understanding this process. Almost all general equilibrium, if not almost all economics, is based on the tatonnement tool. The tool itself was developed to help theorists abstract from the complexity caused by disequilibrium trading patterns. In practice, economic systems are then analyzed as if the mechanism of price discovery were tatonnement even when it is obviously not. To what extent are tatonnement models reliable when applied in such a manner? Existing experimental results suggest that such models are more reliable than might be believed.

A natural way to study the dynamics of price discovery is to focus on stability, or more specifically, on instability. It is within an unstable system that the forces at work to move prices and quantities are most clearly identifiable. The classic paper by Scarf (1960) uses a tatonnement model to produce some striking predictions about the types of behavior a multiple market system might exhibit. Using a three commodity example he constructs an exchange environment in which price convergence never takes place. Specifically, under certain distributions of endowments the prices exhibited by a tatonnement process travel only in closed orbits. Hirota (1981) characterized the price movements in terms of endowments. The combined theoretical result is that prices can move in a systematic fashion without equilibration. The direction of the orbit can be reversed by a change in the endowments. Or, by simply choosing a different endowment, the tatonnement system will converge to the competitive equilibrium.

Curiosity alone demands that one ask about the type of behavior that takes place in an economy such as the one produced by Scarf, even if the mechanism is not tatonnement. Price movement without equilibration has never before been observed in experiments. In fact, systematic price movements tend to be inconsistent with bodies of theory that hold that prices must exist in an arbitrage free configuration. Certainly this is not the case for an orbit. However, experimental evidence exists that suggests that the Scarf environment might produce unusual behavior. It has been decisively demonstrated that nontatonnement markets can be unstable.[2] That is, it is

[2]Three concepts of instability have been investigated. The first was the cobweb model, which does not seem to receive empirical support from experiments [Johnson and Plott (1989)]. Markets tend to stabilize near the competitive equilibrium even though the possibility of a supply lag, with resulting cycles, exists. Two additional concepts of stability are found in the literature. The Marshallian concept is based on quantity changes and the Walrasian concept is based on price changes. The Marshallian concept becomes the appropriate model when the markets contain a Marshallian externality [Plott and

known that the laws of movement (or attraction) in continuous markets are such that prices and quantities can move away from an equilibrium. Interestingly enough, the conditions under which such instabilities are observed are derived using of a tatonnement model and those conditions are exactly when strong income effects exist of the type produced by complements.

Thus, more than curiosity suggests that one should take a careful look at the Scarf world. Instability has been observed under conditions similar to those of the Scarf environment and the phenomena are predicted by tatonnement models. Of course, the Scarf world is a major generalization from what has been studied previously, from two commodities to three and from simple instability to well defined paths.

The substance of this paper is that in the Scarf environment a market system exhibits properties that can readily be interpreted as having been predicted by the Scarf model. An interpretation is required because the Scarf model is not developed in operational terms. The paper is divided into six sections. In the next section, we briefly introduce the nature of a tatonnement model and our adaptation of the Scarf and Hirota model. We explain how the design induces unusual equilibrium properties. In doing this we use price normalization rather than the unit sphere normalization used by Scarf and Hirota. The alternative model is more easily used in conjunction with data. In Section 3 we explain our experimental design and the treatments we use. Section 4 addresses some problems with fitting differential equations to our data. Section 5 contains the results of our study: Section 5.2 focuses on movement of prices from one period to the next, and Section 5.3 focuses on movement of prices within periods. Finally, Section 6 summarizes our results.

Appendix A provides a careful exposition of our adaptation of Scarf's model, proving its stability properties under price normalization as opposed to the unit sphere normalization.

George (1992)]. The Walrasian concept is the foundation for an appropriate model if the preferences have strong income effects [Plott and Smith (1999)]. Interestingly, the first reported attempt to study the adjustment path of a double auction lead to the rejection of the Walrasian model but that early study was incapable of addressing the issue of stability and the alternative model accepted in the study has not been generalized to multiple markets [Smith (1965)].

2 The Nature of Tatonnement

The concept of tatonnement was invented by Walras (1874) to address the problem of disequilibrium trades. Walras was well aware that the equations describing a general equilibrium system would be mathematically altered with each trade out of equilibrium. He therefore postulated a price discovery system that involves no such trades. In a tatonnement system a disinterested auctioneer announces a price p_k in each market k. It is assumed that all agents in the system respond with optimal quantities given those prices. The auctioneer then observes the excess demands, $E_k(p)$, the quantity demanded in each market minus the quantity supplied. If the quantity demanded in each market is exactly equal to the quantity supplied then the system stops and trade takes place at the announced price. If the quantity demanded in each market does not equal the quantity supplied in the market then no trade takes place and the auctioneer announces a new price. The new price announced depends on the excess demands observed. Typically the theoretical adjustment process follows a form such as

$$\frac{dp_k}{dt} = \lambda_k E_k(p) \quad \forall\ k = 1, ..., l \tag{1}$$

In other words, price adjustments are simply proportional to excess demands, $E_k(p)$, where $\lambda_k > 0$ is a scaling vector. If excess demands are positive the prices go up, and if supply is greater than demand price goes down. As indicated by the formula, if excess demands are zero then prices do not change; the system has attained equilibrium.

It is important to notice that time is continuous in the model. Prices are moving continuously in response to the excess demand at each instant. By contrast, actual market activities take place in discrete instances in real time. Typically, in a double auction bids and asks are tendered in real time according to the decision of agents, as are accepted bids and offers. After a period in which trading takes place in an experiment the markets close and profits are allocated. Then, parameters are reset and the experience is repeated, similar to a series of trading days, each like the one before. Thus, a substantial disconnect exists between the concept of time in the tatonnement model and the timing of events in an actual market.

The markets we create have a period structure where endowments are restored at the beginning of each period. Trading takes place within a period and those trades can be at different prices, depending on the bids and asks

and pattern of acceptances. Thus, there are within-period shifts in holdings which could cause deviations from predicted price adjustment paths. On the other hand, the period-to-period activity can reflect beliefs of the market agents acquired by previous trades and by the trading activity of others. In the analysis that follows price movements within periods and also across periods price movements will be studied. As it turns out, it is the across period price movements that are most accurately captured by the Scarf model.

2.1 Theoretical Model

In Scarf's formulation of a nonconvergent economy, the assignment of endowments can be manipulated so that the Walrasian tatonnement dynamic predicts clockwise orbits, counterclockwise orbits, or a globally stable competitive equilibrium.

In the economy we study, there are three commodities, X, Y and Z. The total amount of each commodity in the economy is given by ω_x, ω_y and ω_z. There are three types of agents, each type desiring a pair of commodities as perfect complements: type 1 does not want the first commodity and has a utility function proportional to $min\{y/\omega_y, z/\omega_z\}$; type 2 does not want the second commodity and has a utility function proportional to $min\{x/\omega_x, z/\omega_z\}$; and type 3 does not want commodity 3 and has a utility function proportional to $min\{x/\omega_x, y/\omega_y\}$.

There are two specifications of the price adjustment process proposed by Scarf. In Scarf's original case, prices move proportional to excess demand, which we will call the Absolute Price Adjustment model. In a specification frequently used in macroeconomics, the derivative is scaled by the price level. We will call this second specification the Price-Scaled Adjustment model. Both models will be reviewed in the two major sections below.

2.2 Absolute Price Adjustment

The Walrasian price adjustment process assumes that prices move proportional to excess demands. Appendix A.1 demonstrates that, given these utility functions, and the price normalization as opposed to the unit sphere normalization, the absolute price adjustment process is described by the differential equations

$$\frac{dp_x}{dt} = \lambda_x \left[\frac{\omega_x M^2(p_x, p_y)}{p_x\omega_x + \omega_z} + \frac{\omega_x M^3(p_x, p_y)}{p_x\omega_x + p_y\omega_y} - \omega_x \right] \quad (2)$$

$$\frac{dp_y}{dt} = \lambda_y \left[\frac{\omega_y M^1(p_x, p_y)}{p_y\omega_y + \omega_z} + \frac{\omega_y M^3(p_x, p_y)}{p_x\omega_x + p_y\omega_y} - \omega_y \right] \quad (3)$$

where M^i is the value of the endowment of an agent of type i at current prices. From any initial price $(p_x(0), p_y(0))$, this system prescribes the sequence of prices which would be called out by the Walrasian auctioneer.

Figure 1a shows convergent price adjustment paths from a few initial price pairs and $\lambda_x = \lambda_y = 1$. Note that each path approaches the Walrasian equilibrium, but not always in a straight line. Appendix A.2.2 proves this system of equations is globally stable if each agent is endowed with all of the commodity (s)he does not want.

Figure 1b shows examples of orbits paths for two different values of λ, and Figure 1c projects an orbit into price-time space.

Appendix A.2.1 shows that if the agents are each endowed with all of exactly one of the commodities she desires, Equations 2 and 3 prescribe a stable center. A stable center is an adjustment pattern where, if the price starts at the equilibrium, the system will stay there. However, if the starting price is not the equilibrium, prices will move around the equilibrium in a closed orbit.

2.2.1 Liapunov Function

Assessing the stability of a system of differential equations requires determining whether or not prices are getting "closer" to the equilibrium over time. A Liapunov function provides the appropriate notion of distance by which to determine whether prices are getting closer to the equilibrium over time. If the derivative of the Liapunov function is negative, the system is converging; if it is positive, the system is diverging; if it is zero, the system is moving in closed orbits.

The different treatments in this experiment are analyzed using different Liapunov functions.[3] In Appendix A.2.1, it is shown that, for the orbiting treatments, the system of Equations 2 and 3 can be integrated explicitly,

[3]We were unable to identify a Liapunov function which applies to both the convergent and orbiting treatments.

giving a function for the closed orbits around the equilibrium in which prices move. The function takes on a constant value everywhere on the orbit, with higher values further from the equilibrium. This function can be normalized by having the equilibrium value of the function subtracted to give a function with value zero at equilibrium; this normalized version is a Liapunov function for the orbiting treatments.

For notational compactness, write

$$\mathcal{X} = [\frac{1}{2}p_x(t)^2 - (\frac{\omega_z}{\omega_x})^2 \ln(p_x(t))] - [\frac{1}{2}(p_x^*)^2 - (\frac{\omega_z}{\omega_x})^2 \ln(p_x^*)] \tag{4}$$
$$\mathcal{Y} = [\frac{1}{2}p_y(t)^2 - (\frac{\omega_z}{\omega_y})^2 \ln(p_y(t))] - [\frac{1}{2}(p_y^*)^2 - (\frac{\omega_z}{\omega_y})^2 \ln(p_y^*)]$$

where (p_x^*, p_y^*) is the equilibrium price pair. Then the Liapunov function for the orbiting treatments is given by

$$V(p_x(t), p_y(t)) = \frac{1}{\lambda_x}\mathcal{X} + \frac{1}{\lambda_y}\mathcal{Y}. \tag{5}$$

Appendix A.2.2 derives a Liapunov function for the convergent cases. For notational compactness, write

$$\mathcal{X} = \omega_x[\frac{1}{3}p_x(t)^3 - (\frac{\omega_z}{\omega_x})^2 p_x(t) + \frac{2}{3}(\frac{\omega_z}{\omega_x})^3] \tag{6}$$
$$\mathcal{Y} = \omega_y[\frac{1}{3}p_y(t)^3 - (\frac{\omega_z}{\omega_y})^2 p_y(t) + \frac{2}{3}(\frac{\omega_z}{\omega_y})^3]$$

Then

$$V(p_x(t), p_y(t)) = \frac{1}{\lambda_x}\mathcal{X} + \frac{1}{\lambda_y}\mathcal{Y} \tag{7}$$

is a Liapunov function for the system of Equations 2 and 3.

2.3 Price-scaled Price Adjustment

An alternative specification of the price adjustment process hold that prices move proportional to excess demands, but scaled by the price level. In this case, the price adjustment process is described by the differential equations

$$\frac{\frac{dp_x}{dt}}{p_x} = \lambda_x \left[\frac{\omega_x M^2(p_x, p_y)}{p_x\omega_x + \omega_z} + \frac{\omega_x M^3(p_x, p_y)}{p_x\omega_x + p_y\omega_y} - \omega_x \right] \tag{8}$$

$$\frac{\frac{dp_y}{dt}}{p_y} = \lambda_y \left[\frac{\omega_y M^1(p_x, p_y)}{p_y\omega_y + \omega_z} + \frac{\omega_y M^3(p_x, p_y)}{p_x\omega_x + p_y\omega_y} - \omega_y \right] \tag{9}$$

where M^i is the value of the endowment of an agent of type i at current prices. This system is interpreted exactly as Equations 2 and 3.

Appendix A.3.2 proves this system of equations is globally stable if each agent is endowed with all of the commodity she does not want.

Appendix A.3.1 shows that if the agents are each endowed with all of exactly one of the commodities she desires, Equations 2 and 3 prescribe a stable center.

2.3.1 Liapunov Function

The stability of this system can also be analyzed using a Liapunov function. As with the absolute model, a different Liapunov function is used for the convergent and orbiting treatments.

Appendix A.3.1 shows how the system of Equations 8 and 9 can be integrated to give a function for the orbits in which prices move. Like in the absolute model, this function can be translated so it takes on the value zero at equilibrium and larger values further from equilibrium. For notational compactness write

$$\begin{aligned} \mathcal{X} &= [p_x(t) + (\frac{\omega_z}{\omega_x})^2 \frac{1}{p_x(t)}] - [p_x^* + (\frac{\omega_z}{\omega_x})^2 \frac{1}{p_x^*}] \\ \mathcal{Y} &= [p_y(t) + (\frac{\omega_z}{\omega_y})^2 \frac{1}{p_y(t)}] - [p_y^* + (\frac{\omega_z}{\omega_y})^2 \frac{1}{p_y^*}] \end{aligned} \tag{10}$$

where (p_x^*, p_y^*) is the equilibrium price pair. Then the Liapunov function for the orbiting treatments is given by

$$V(p_x(t), p_y(t)) = \frac{1}{\lambda_x}\mathcal{X} + \frac{1}{\lambda_y}\mathcal{Y}. \tag{11}$$

Appendix A.3.2 derives a Liapunov function for the convergent treatment. For notational compactness, write

$$\mathcal{X} = \omega_x[(\frac{1}{2}p_x(t)^2 - (\frac{\omega_z}{\omega_x})^2 \ln(p_x(t))) - (\frac{1}{2}(p_x^*)^2 - (\frac{\omega_z}{\omega_x})^2 \ln(p_x^*))] \tag{12}$$

$$\mathcal{Y} = \omega_y[(\frac{1}{2}p_y(t)^2 - (\frac{\omega_z}{\omega_y})^2 \ln(p_y(t))) - (\frac{1}{2}(p_y^*)^2 - (\frac{\omega_z}{\omega_y})^2 \ln(p_y^*))].$$

Then

$$V(p_x(t), p_y(t)) = \frac{1}{\lambda_x}\mathcal{X} + \frac{1}{\lambda_y}\mathcal{Y} \tag{13}$$

is the Liapunov function for the convergent treatment.

3 Experimental Design

The parameters used in the experiments are in Table 1. Two elements of the Scarf and Hirota model's predictions are the focus of the research. First, do prices orbit when the model predicts they will orbit and do prices converge when the model predicts they will converge? Second, if orbits are observed, do prices move along the orbits in the direction the model predicts? To answer these questions three treatments outlined in Table 1 are used.

In all treatments the competitive equilibrium is the same but the dynamics predicted by the model differ. Agents are of three types. Preferences and endowments for each type are listed for each treatment. Each agent is endowed only with one of the items that contributes to the agent's payoff. For example, under Treatment I, Type 1 people like Y and Z, and are all endowed with 20 units of Y. Under the endowments listed as Treatment I the model predicts counter clockwise price orbits. Clockwise orbits are predicted under Treatment II. Price convergence is predicted for Treatment III. The table also lists the equilibrium values that are the same for all treatments.

The sequence of experimental conditions are in Table 2. Experiments included five agents of each type, for a total of 15 subjects in each experiment. We ran experiments in Caltech's Laboratory for Experimental Economics and Political Science between July, 1998 and May, 1999. Subjects were Caltech undergraduate and graduate students, many of whom had participated in unrelated experiments, but who did not necessarily have any training in economics. No subject participated in more than one of the inexperienced trials.

Trading was conducted using MUDA, a standard software for conducting experimental multiple unit double auction markets. Types were assigned to computers prior to subjects' entering the laboratory, and subjects selected a computer (with its monitor off) when they arrived; type assignment was

effectively random. Subject payments averaged \$29, with a range of \$21 to \$35 for a three hour experiment.

At the beginning of each experiment the instructions in Appendix B were read aloud, and the MUDA software was explained to the subjects. There was then a practice trading period for subjects to get used to the software. Then payoff tables inducing preferences were passed out and the first trading period then began. In the inexperienced sessions, the first period was 15 minutes long, the second, third and fourth were 12 minutes long, and the remaining periods were each ten minutes long. In the experienced sessions, the first period was 12 minutes long, the second, third and fourth were ten minutes long and the remaining periods were eight minutes long. The inexperienced sessions had between eight and 11 periods, and the experienced sessions lasted 15 or 16 periods.

4 Econometric Methodology and Data Analysis

The experiments provide two kinds of data, transaction by transaction data and period average prices. When using average period prices the concept of a price pair at an instant of time, t, has a natural interpretation as the pair of average prices that exist in a period. That is, the average X transaction price in a period with the average Y transaction price in the same period to generate observations in (p_x, p_y)-space. We analyze this period-to-period data using Liapunov functions, as well as numerical solutions to Equations 2 and 3.

The raw, transaction by transaction data are separate time series of X transaction prices and Y transaction prices. Because transactions in the two markets do not take place at the same time, these series are not synchronized. That is, we do not have observations in the (p_x, p_y)-space in which the theory makes predictions. We make an assumption to synchronize the two markets and analyze the transformed data using Liapunov functions.

4.1 Fitting the Liapunov Function

As discussed above, the Liapunov function can be used to determine whether or not prices are converging. The Liapunov functions for the convergent and unstable treatments under both the Absolute Price Adjustment and the

Price-Scaled Adjustment model specifications of the price adjustment process can be expressed as

$$V(p_x(t), p_y(t)) = \frac{1}{\lambda_x}\mathcal{X} + \frac{1}{\lambda_y}\mathcal{Y} \tag{14}$$

where λ_x and λ_y are unknown parameters of the system. The analytical derivatives of this function can be studied to determine the analytical stability properties of the system. To determine whether or not the data are converging, the evolution of the value of the function at observed price pairs can be studied. Specifically, to determine if prices are getting closer to an equilibrium we would like to be able to use the model

$$V(p_x(t), p_y(t)) = \alpha + \beta t \tag{15}$$

where β is a notion of the derivative of the data: a positive β corresponds to divergence, a negative β to convergence, and a β not distinguishable from zero to orbiting. Unfortunately, this model is complicated by the unknown parameters, λ_x and λ_y, in the Liapunov function.

To solve this problem we estimate the following model,

$$\mathcal{Y} = \gamma_0 + \gamma_1(-\mathcal{X}) + \gamma_2 t \tag{16}$$

where $\gamma_0 = \lambda_y \cdot \alpha$ is the constant, $\gamma_1 = \lambda_y/\lambda_x$ and $\gamma_2 = \lambda_y \cdot \beta$. It is not a concern that we cannot identify coefficients separate from λ_y: λ_x, λ_y and the constant are nuisance parameters, and we care only about the sign of β, not its value. Because Scarf's model requires that the λ_ks be strictly positive, we can infer that the sign of γ_2 is the sign of β.[4] Therefore, we can test whether or not our data are converging by testing whether γ_2 is significantly positive (diverging), negative (converging) or neither (orbiting). The t data, either the trade number for the trade-to-trade data or period number for the period-to-period data, is normalized to range between zero and one by dividing each t by the largest value it takes on within a period or within an experiment respectively. This means the γ_0 can be interpreted as the initial value of the Liapunov function and $\gamma_0 + \gamma_2$ is the value of the Liapunov function at the end of the series.

[4] We do not explicitly impose this restriction in our estimation.

4.2 Fitting Numerical Solutions

While the model specified in the previous section is simple, we do lose some information looking only at the analytical solution to Equations 2 and 3. The Liapunov functions do not capture the different predictions of the clockwise and counterclockwise treatments. This problem can be overcome by considering numerical solutions to the system of differential equations.

The numerical solution model has four parameters, a p_x starting point, a p_y starting point, a λ_x/λ_y ratio,[5] and an amount of abstract time, T. This abstract time parameterizes the path along which prices evolve; it is the time with respect to which the differential equations are specified. T is the amount of this time which transpires during a period.[6] For instance, if an orbit had a period of $t = 10$ and the data were explained by half of that orbit, then T would be estimated to equal 5. We estimate the starting points because forcing the data to start at the first point could unfairly penalize a model on data with an outlier first period; the model above implicitly allows for this by using all the data points.

These four parameters identify a portion of a path with a particular shape and starting point. Predictions for successive periods are taken to be evenly spaced, in t, along the path. Simulated annealing is used to identify the combination of parameters which maximizes the likelihood of the simultaneous p_x and p_y errors under a bivariate normal distribution.

The direction of the path along an orbit is indicated by the sign of T. If T is positive, $T > 0$, then the path is counterclockwise. If T is negative, $T < 0$, then the direction is clockwise. This feature is central to the testing.

4.3 A Straight-Line

The model in Section 4.2 does not have a naturally nested alternative model, in either the orbiting or convergent treatments. To gauge how well it is performing, we compare it to a model that predicts that price will follow a straight line from the starting point to the competitive equilibrium. In the convergent treatments both models predict convergence, but along slightly different paths.

[5]The parameters λ_x and λ_y cannot be identified in price space.

[6]The time T bears no relation (one that we understand, anyway) to either the amount of clock time that has passed during the experiment or the number of trades that have occurred, two frequently used measures of time in experimental market analysis.

The objective function for the straight line model is the same as in Section 4.2, but the predictions are generated along a straight line. We define $T = 0$ at the starting point and $T = 1$ at equilibrium. Thus, if the data actually converge, we would expect to see T estimated about 1. If $T > 1$, the prediction overshoots the equilibrium; if $T < 0$, it is moving away from equilibrium, and if $0 < T < 1$, then the data move along the path toward equilibrium, but do reach it. We can compare the relative likelihoods, using standard techniques to account for the difference in the number of parameters, to determine which of the two models explain the data better.

5 Results

5.1 Data Overview

Figure 2 presents the time series of contract prices of the 4/12/99conv and 5/19/99ccw sessions. The time series of X prices and equilibrium are indicated in grey, and the time series for Y is in black. The convergent session's time series of 4/12/99conv has features common to standard double auction markets. The market prices start away from equilibrium and move slowly toward it. Importantly, they rarely cross over the equilibrium price for their respective market, and never for more than a fraction of a period.

This is not the case for the counterclockwise session of 5/19/99. Several times, in periods 3 and 12 in the X market and periods 7 and 14 in the Y market, the time series of prices passes through the equilibrium value and moves away from equilibrium for several periods. In simple markets with standard preferences, passing through the equilibrium price seldom occurs and when it does, the overshooting quickly corrects itself. The sustained movements away from equilibrium in this experiment suggests there is something very unusual about the Scarf environment.

That important features of these data are captured by the Scarf model is most apparent in (p_x, p_y)-space when dealing with average period prices. Figures 3, 4 and 5 present average price per period data from the counterclockwise, clockwise and convergent treatments respectively. Each experiment's first period average price is labeled with the experiment's date, and consecutive periods are connected with lines. Following lines away from the dated starting point gives an idea of how prices moved during the experiment.

These data have three important features that we interpret as predicted

by the Scarf model. First, convergence occurs in the convergent treatment, but not in the clockwise and counterclockwise treatments. Thus, the Scarf model is an accurate indicator of whether or not the market will converge. Second, in the nonconvergent treatments, data appear to be systematically orbiting the equilibrium rather than moving randomly about the price space, so the model accurately predicts the nature of the nonconvergent behavior. Finally, when we do observe orbits, the movement is in the direction (clockwise or counterclockwise) predicted by the model.

In the counterclockwise treatment shown in Figure 3, the Scarf model predicts that prices should move counterclockwise around the equilibrium, and not converge. The 7/13/98, 11/24/98, 1/12/99, 2/19/99 and 5/11/99 experiments all clearly move in the counterclockwise direction from their starting points; the one experiment which does not is 1/22/99, which does not really move anywhere. Furthermore, the 7/13/98, 11/24/98, 1/12/99, and 5/11/99 experiments do not move noticeably closer to the equilibrium in later periods, instead appearing to move in closed orbits around the equilibrium. Even the 2/19/99 experiment, which does eventually converge, begins by moving away from equilibrium along a counterclockwise path.

In the clockwise treatment, both experiments move clockwise from their starting points. Moreover, both time series pass through an equilibrium price in at least one market, a move away from equilibrium consistent with orbiting behavior. In this treatment, too, the data are consistent with the Scarf model's prediction of the existence of an orbit and its direction.

Both experiments in the convergent treatment begin away from the equilibrium and move toward it, but there are important details to notice about the convergence path in relation to the model. The Scarf model predicts that, if p_x begins below 40, then p_y will drop below 20 and rise as p_y approaches equilibrium. The time series reflect this detail of the predicted convergence path: the 4/14/99 experiment starts with $p_y > 20$, but p_y drops below the equilibrium price as p_x increases until both rise to meet the equilibrium; in 4/12/99, p_y begins below 20, but it drops further below 20 before rising to approach the equilibrium.

The remainder of this section attempts to summarize these pictures statistically.

5.2 Statement of Results: Average Period Prices

The first result tells us that the impression gained from the figures is correct with respect to the conditions under which convergence is observed.

Result 1 *When the Scarf model predicts nonconvergence, we observe nonconvergence; when the Scarf model predicts convergence, we observe convergence.*

Support: Table 3 presents the least squares estimates of Equation 16, corrected for first-order serially correlated residuals using the Prais-Winsten algorithm, and White-corrected standard errors.[7] In six of the eight orbiting treatments, the hypothesis that $\beta = 0$ is not rejected in either specification. This means that the value of the Liapunov function is not changing over time, suggesting that the data are orbiting the Walrasian equilibrium in these six cases. Perhaps surprisingly, this is also true of 2/19/99ccw, which actually does converge. This suggests that the many periods moving away from the equilibrium as if along an orbit is of greater importance than the few periods' (possibly random) movement toward equilibrium.

The remaining two cases do not provide strong evidence against the Scarf model, however. In 11/24/99ccw, β is estimated to be significantly positive in both specifications, consistent with divergence, as opposed to convergence. Looking at the 11/24/99ccw time series in Figure 3a, the first periods appear to be on a very round orbit, while the latter periods appear to be following a much flatter one. Since the linear model estimates a λ ratio between the high λ_y/λ_x ratio which would appear to fit the first periods well and the low λ_y/λ_xratio which would fit the last few periods best. The net result is that, for any fixed λ ratio, the data are moving from orbits which are close to equilibrium to those which are further away; this is divergence.

In the other experiment where Scarf's model predicts orbiting, 4/19/99cw, β is significantly less than zero in both specifications, consistent with convergence. Looking at the time series in Figure 4, this data does not provide strong support for convergence. The time series begins very close to the equilibrium, and remains close; the fit of this data is driven by random fluctuations in the data. This explanation is reinforced by the rejection of the maintained hypothesis that λ_y/λ_x is positive in this regression.

[7] The reported R^2 compare the model variance with a constant only model. However, a constant-only model is not nested in the Prais-Winsten transformed data. Therefore, the R^2 can be less than zero (*).

The maintained hypothesis that $\lambda_y/\lambda_x > 0$ is statistically rejected, by both specifications, in one other experiment, 1/22/99ccw. Looking at the time series in Figure 3b reveals that this experiment began close to the equilibrium and made no systematic move toward or away from it.

Therefore, of the eight experiments in which Scarf's model predicts orbiting, in none is there a tendency toward convergence: two begin close to equilibrium and remain there, one diverges, and five move in clear orbits around the equilibrium.

Scarf's model predicts convergence in two experiments. In one, 4/12/99conv, β is significantly less than zero in both specifications. This means that the value of the Liapunov function is decreasing over time, consistent with convergence. In the other, 4/14/99conv, β is not significantly different from zero.

A closer look at the predicted paths suggests that we should not be surprised that the convergent datasets did not get closer to equilibrium. The equilibrium is reached only in the limit, but the limit is approached very slowly. If we imagine the Walrasian auctioneer making one adjustment per period of abstract time, we can get some sense of the slowness predicted by numerically solving the Equations 2 and 3. If the starting point is (15,25), then the prices are (24.988,18.660) after two adjustments, (39.113,19.736) after 16, (39.946,19.983) after 32 and (39.954,19.985) after 64. At this point, adjustments are so small that they exceed machine precision, suggesting that the remaining approach is very difficult. One can imagine that, in the discretized price space of this experiment, this difficulty in approaching equilibrium would be reflected by data making lots of small jumps near the equilibrium but never reaching it. Looking at Figure 5, this is exactly what the convergent treatment data look like. They start away from the equilibrium, quickly move closer, but getting within two units is very difficult, as prices vibrate short of the equilibrium, without ever getting there or overshooting. This vibration causes the sign of the derivative to fluctuate, although movements are small. These fluctuations lead to ambiguous results, although the pictures clearly indicate the data are converging.

Overall, these data suggest that Scarf's model provides considerable leverage on whether or not a non-tatonnement market system is going to converge. In the orbiting treatments, none of the eight experiments are clearly converging as the model predicts. In the convergent treatment, one of the experiments clearly converges, while the other begins close to the equilibrium and remains there without approaching it. Taken together, these results suggest

that Scarf's model of tatonnement price adjustment can shed light on the convergence properties of a market system, even with a non-tatonnement market mechanism.

Having established that the data from the orbiting treatment are in fact orbiting the equilibrium, the next question is whether or not the direction of the orbits is the direction predicted. Because the Liapunov function does not provide information about the direction of an orbit, we must use a different model to determine whether the movement along the orbits is in the direction predicted by the Scarf model. To test the direction the data are moving, we numerically integrate Equations 2 and 3 in the way discussed in Section 4.2.

Table 4 presents maximum likelihood estimates and goodness of fit statistics for the numerically solved Scarf model and the straight line model. To help compare models, two statistics are presented, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC).[8] These statistics can be compared directly and used for model calibration; they are designed to reach a maximum value at an optimal tradeoff between improvement of fit and additional parameters, even when models are non-nested as is the case here.

The first thing to note is that these estimates are consistent with the previous two results. In the orbiting treatments, the AIC selects the Scarf model over the straight line model in every experiment; the BIC selects the Scarf model in all but the 4/19/99cw experiment. This suggests that the data are better predicted by an orbiting path than one which converges.

Are the orbiting cases going in the right direction? Having confirmed that the data are in fact orbiting the equilibrium, the next question is whether or not the direction of the orbits is the direction predicted by the Scarf model.

Result 2 *The directions of the orbits are as predicted by the Scarf model.*

Support: The Scarf model estimated in Table 4 predicts that T should be positive in the the counterclockwise treatments and negative in the clockwise

[8]The AIC is the total in-sample log-likelihood minus the number of model parameters. It is widely used for model comparison, but not motivated by any optimality considerations. The BIC is the total in-sample log-likelihood minus the half the number of model parameters times the natural log of the number of observations. The BIC can be interpreted as follows: if model i has a higher BIC than j, then $\exp\{-(BIC_i - BIC_j)\}$ is an approximation to the posterior odds ratio, $\Pr(i)/\Pr(j)$, of a Bayesian observer with equal priors (Carlin and Louis, 1996).

treatments. The estimates bear this out, as every experiment is consistent with this prediction.

Are the particular paths of the convergent cases consistent with the Scarf model? Having established that the data from the convergent treatments are actually converging, we can now ask a more precise question about how they are converging: are they converging along a simple straight line path, or are they following exactly the path predicted by the Scarf model?

Result 3 *The Scarf model predicts better than a straight-line convergence model in the convergent case.*

Support: The evidence from Table 4 points weakly to the paths predicted by the Scarf model. The AIC selects the Scarf model in both convergent experiments, but the BIC, the stronger of the criteria, selects the straight line model in the 4/14/99conv experiment. This is not surprising because the 4/14/99conv data do not converge strongly. Taken together, these estimates suggest that our data not only converge, but they follow the indirect path prescribed by the Scarf model rather than a direct path to the equilibrium.

5.3 Within-Period Price Adjustments

The results in the previous section examined the movement of the mean trade prices in each period. The section concluded that the period-to-period path was well-predicted by the Scarf model. The next natural question is whether or not the Scarf model predicts trade-to-trade price movements within a period.

In general, this is an exceptionally difficult question to answer. The primary reason is that trade-to-trade data are not observed in the same space as the model makes predictions. The Scarf model makes predictions about comovements in (p_x, p_y)-space, but we typically only observe a trade in one market at a time. This means that when we observe a trade in the X market at p_x, we do not know what the corresponding p_y is, and therefore cannot calculate excess demand or formulate a prediction from the Scarf model, without special assumptions.

To generate price pairs, we assume that the price in the inactive market is equal to the price of the last trade in the market; for instance, if we observe an X trade at p_x, we assume the corresponding p_y is the price at which Y last traded, although it did not in fact occur contemporaneously.

Result 4 *Trade-to-trade price paths are not consistent with the Scarf model.*

Support: Table 5 presents the Prais-Winsten estimates of Equation 16, corrected for first-order serially correlated residuals, and White-corrected standard errors. . In neither of the experiments for which the Scarf model predicts convergence do the trade-to-trade data give evidence of convergence. For the 4/12/99conv the β suggests divergence and for the 4/14/99conv the β is not significantly different from zero for either specification, suggesting an orbit. In the experiments in which Scarf's model predicts orbiting, both specifications conclude that six are orbiting and two are diverging.

Without a compelling distinction between the orbiting and convergent treatments, these data do not support the predictions of Scarf's model in the trade-to-trade price paths. However, the fact that orbiting is observed suggests that a closer look at the convergent cases might be productive. One wonders if different methods of dealing with the data might reveal convergence in those cases in which the model suggests that convergence will take place. In particular, the study of Figure 2a suggests that the reason that the model shows little support for the convergent cases is because in the later periods prices begin a period so close to the equilibrium that there is little room for further movement. Thus, the model does not reflect movement toward the equilibrium in the convergent cases because in the later periods there is little movement at all. This beginning is followed by some rather large "spikes" near the end of a period. However, for now we must leave the question as a paradox, perhaps to be answered by different econometric methodology.

6 Summary of Conclusions

The conclusions are easy to state. The Scarf model does a surprisingly good job of predicting features of the markets we created. The Scarf model is a type of idealized adjustment process in which disequilibrium trades do not take place and prices change in the smooth fashion described by a continuous tatonnement process. By contrast, the markets we study are characterized by bids, asks, and trades that take place at prices that jump around from trade to trade. Clearly the assumptions of the idealized tatonnement model are not satisfied by our markets. Nevertheless, a correspondence between the predictions of the model and the actual market behavior is there.

When the Scarf model predicts convergence to the competitive equilibrium the markets are observed converging across periods. When the model predicts that prices will move in counterclockwise orbits, such orbits are observed in the period to period average prices. When the model predicts that the prices will move in clockwise orbits, again, the period to period average prices tend to move in that direction.

Looking for similar results for within-period price adjustments is considerably more difficult because we do not observe points in (p_x, p_y)-space. The timing of trades is not simultaneous in our markets, so price pairs are not observed unless additional assumptions are made. Therefore, our results in the within-period domain are statistically weaker, and provide little support for Scarf-like movement on trade-to-trade price paths.

Obviously, the ability of the tatonnement model to predict prominent features of market behavior under circumstances in which it was generally believed to be inapplicable calls for some theoretical speculation. Among periods, the endowments are refreshed, so prices (or expectations about prices based on history) can evolve separate from endowments. Across periods expectations might act much like a tatonnement mechanism. On the other hand, within periods, holdings change as trades are made, meaning this environment is missing a critical feature of the tatonnement mechanism. These data suggest the Scarf model does not explain these price paths, so exactly what explains these adjustments remains a mystery, and an excellent avenue for future research.

The exercise forcefully exhibits a major conclusion. Markets are subject to laws of adjustment. There is a process of movement that may not involve a stationary state and the absence of arbitrage opportunities. Our understanding of the price discovery process should focus on the nature of those laws. This research suggests that the classical tatonnement tool may be more useful in understanding these laws than was previously believed.

Of course, the implications of the results are at this point only a subject for speculation. In a sense, the Scarf example suggests that if the Samuelsonian dynamic exists in markets, then general equilibrium must be concerned about stability. That is clearly the case given the data produced so far. Obviously, additional market instruments, such as options, or other instruments that help remove arbitrage possibilities from time paths might serve to induce stability and convergence. But, those issues are matters for further research.

Type	$U^{type}(x, y, z)$	Endowment $(\omega_x^s, \omega_y^s, \omega_z^s)$
Treatment I: Counterclockwise orbits		
1	40 $\min\{y/20, z/400\}$	(0, 20, 0)
2	40 $\min\{x/10, z/400\}$	(0, 0, 400)
3	40 $\min\{x/10, y/20\}$	(10, 0, 0)
Treatment II: Clockwise orbits		
1	40 $\min\{y/20, z/400\}$	(0, 0, 400)
2	40 $\min\{x/10, z/400\}$	(10, 0, 0)
3	40 $\min\{x/10, y/20\}$	(0, 20, 0)
Treatment III: Globally Stable		
1	40 $\min\{y/20, z/400\}$	(10, 0, 0)
2	40 $\min\{x/10, z/400\}$	(0, 20, 0)
3	40 $\min\{x/10, y/20\}$	(0, 0,400)
Equilibrium Values		
1	\$2.50 (U=20)	(0,10,200)
2	\$2.50 (U=20)	(5,0,200)
3	\$2.50 (U=20)	(5,10,0)

Table 1: Utility functions and endowments for each type of agent for the three treatments.

Treatment	Date	Periods	Experienced
I. Counter-clockwise (ccw)	7/13/98	8	No
	11/24/98	10	No
	1/12/99	10	No
	1/22/99	10	No
	2/19/22	16	Yes
	5/11/99	15	Yes
II. Clockwise (cw)	4/19/99	10	No
	4/20/99	10	No
III. Convergent (conv)	4/12/99	11	No
	4/14/99	11	No

Table 2: The treatment, date, level of experience and number of periods for each session.

	Absolute $\dot{p}$				Relative $\dot{p}/p$			
	λ_y/λ_x	$\lambda_y \cdot \beta$ Period	$\lambda_y \cdot \alpha$ Constant	R^2	λ_y/λ_x	$\lambda_y \cdot \beta$ Period	$\lambda_y \cdot \alpha$ Constant	R^2
7/13/98ccw	0.12	87.37	175.86	0.936	0.76	7.41	16.14	0.915
	4.50	1.49	17.62		4.24	1.22	16.20	
11/24/98ccw	0.05	279.37	151.29	0.898	0.47	10.26	26.48	0.951
	6.63	4.63	6.68		11.33	2.60	15.84	
1/12/99ccw	-0.01	-6.64	18.18	*	0.02	-0.10	0.84	*
	-0.17	-0.25	1.22		0.27	-0.08	1.47	
1/22/99ccw	-0.12	1.34	-4.12	0.558	-0.35	0.10	-0.29	0.533
	-2.20	0.24	-0.68		-2.05	0.29	-0.76	
2/19/99ccw	-0.06	6.60	7.39	0.026	-0.14	0.22	0.30	0.038
	-0.89	0.19	0.94		-0.96	0.17	1.06	
5/11/99ccw	0.20	12.44	74.12	0.354	0.49	-0.19	4.66	0.561
	3.70	0.19	1.77		4.74	-0.12	3.39	
4/19/99cw	-1.23	-39.13	24.01	0.672	-1.91	-1.58	1.01	0.703
	-2.08	-3.50	3.08		-2.40	-3.66	3.19	
4/20/99cw	0.04	0.98	121.83	*	0.11	-0.55	4.31	*
	0.44	0.01	0.96		1.13	-0.10	1.22	
4/12/99conv	0.04	-5639.5	5327.24	0.458	0.06	-313.79	293.55	0.459
	1.98	-3.38	3.84		1.98	-3.42	3.86	
4/14/99conv	-0.00	2439.43	1183.63	0.171	0.02	146.31	64.27	0.215
	-0.03	1.07	0.74		0.12	1.11	0.71	

Table 3: Prais-Winsten least squares estimates of the Liapunov function model in Equation 16 on period-to-period data. The t-statistics are presented below each estimate.

	LL	$p_x(0)$	$p_y(0)$	λ_x/λ_y	T	AIC	BIC
Scarf Model							
7/13/98ccw	-38.77	16.87	10.08	5.84	7.29	-45.77	-46.05
11/24/98ccw	-70.77	3.63	25.38	457.44	2.25	-77.77	-78.83
1/12/99ccw	-44.22	43.67	15.67	16.93	9.97	-51.22	-52.27
1/22/99ccw	-37.94	50.77	18.12	521.75	20.92	-44.94	-46.00
2/19/99ccw	-80.23	42.95	19.54	1301.20	22.50	-87.23	-89.94
5/11/99ccw	-68.79	23.46	14.87	4.63	23.18	-75.79	-78.26
4/19/99cw	-39.29	39.20	23.59	0.53	-20.11	-46.29	-47.68
4/20/99cw	-61.41	23.22	23.07	2.18	-24.47	-68.41	-69.80
4/12/99conv	-38.89	28.99	20.26	1040.47	81.01	-45.89	-47.28
4/14/99conv	-40.39	35.23	21.97	0.08	15.19	-47.39	-48.78
Straight-line Model							
7/13/98ccw	-53.07	31.02	21.35		6.85	-60.07	-59.30
11/24/98ccw	-82.04	0.00	18.57		3.72	-89.04	-88.95
1/12/99ccw	-59.10	52.76	18.91		2.36	-66.10	-66.01
1/22/99ccw	-39.54	49.08	18.15		0.16	-46.54	-46.45
2/19/99ccw	-92.55	56.00	23.82		0.99	-99.55	-100.87
5/11/99ccw	-106.66	50.37	12.29		1.86	-113.66	-114.78
4/19/99cw	-40.15	40.07	24.99		0.74	-47.15	-47.35
4/20/99cw	-76.49	19.08	20.38		1.99	-83.49	-83.68
4/12/99conv	-40.43	27.61	17.00		0.93	-47.43	-47.62
4/14/99conv	-41.16	36.98	18.61		-0.38	-48.16	-48.35

Table 4: Parameter estimates and goodness of fit statistics for the numerical Scarf model and the straight line model. Estimated moments of the bivariate normal error distribution are not reported.

	Absolute $\dot{p}$				Relative $\dot{p}/p$			
	λ_y/λ_x	$\lambda_y \cdot \beta$ Trade #	$\lambda_y \cdot \alpha$ Constant	R^2	λ_y/λ_x	$\lambda_y \cdot \beta$ Trade #	$\lambda_y \cdot \alpha$ Constant	R^2
7/13/98ccw	0.02	-27.74	195.46	0.078	0.22	0.19	19.10	0.015
	9.48	-1.18	18.69		6.82	0.03	9.00	
11/24/98ccw	0.00	157.77	119.92	0.016	0.04	8.25	13.69	0.013
	0.72	1.67	2.49		2.42	1.34	5.10	
1/12/99ccw	0.00	24.4	15.46	0.019	0.00	1.05	0.72	0.013
	2.90	2.98	5.58		4.37	2.62	5.93	
1/22/99ccw	0.00	-5.65	17.68	0.002	0.00	-2.08	2.44	0.002
	1.94	-0.46	2.45		1.84	-1.76	2.21	
2/19/99ccw	-0.03	6.43	20.61	0.005	-0.04	0.26	0.83	0.006
	-1.05	0.33	2.17		-0.94	0.41	2.49	
5/11/99ccw	0.05	56.52	52.91	0.026	0.26	3.50	3.24	0.039
	4.22	3.12	6.77		7.13	3.18	7.03	
4/19/99cw	-0.01	-19.6	32.35	0.006	0.00	-0.72	1.51	0.001
	-0.45	-1.29	3.12		-0.44	-1.15	2.82	
4/20/99cw	0.00	34.15	154.67	0.005	0.00	8.68	3.90	0.001
	0.49	0.27	2.99		0.07	0.91	1.80	
4/12/99conv	-0.01	4721.64	965.19	0.008	-0.02	164.35	101.97	0.008
	-1.73	2.21	1.10		-1.92	1.92	2.93	
4/14/99conv	0.00	4875.48	3938.56	0.001	0.00	-138.26	429.28	0.001
	0.23	0.51	1.22		0.90	-0.32	2.29	

Table 5: Prais-Winsten least squares estimates of the Liapunov function model in Equation 16 on trade-to-trade data. The t-statistics are presented below each estimate.

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A Appendix: Theory of Price Adjustment Differential Equations of a Generalized Scarf Model

This appendix proves the stability properties of the different treatments. The first section builds on Hirota's (1981) generalization of the model in Scarf's original (1960) paper to allow for different treatments. The next section demonstrates stability and instability for different initial endowments for the absolute adjustment model presented in the main text. The third section demonstrates stability and instability for different initial endowments for the price-scaled adjustment model presented in the main text. The final section shows that the prices orbit in the direction stated in the main text.

A.1 Notation and Setup

This section generalizes Scarf's (1960) model of global instability in a Walrasian competitive market. Scarf's economy has three commodities, say X, Y, and Z. There are three types of consumers, 1, 2, and 3, who respectively have the following utility functions and initial holdings:

$$\begin{aligned}
U^1(x^1, y^1, z^1) &= \min\left\{\frac{y^1}{\omega_y}, \frac{z^1}{\omega_z}\right\}, \ (\omega_x^1, \omega_y^1, \omega_z^1) = (1-\alpha-\beta)\omega_x, \alpha\omega_y, \beta\omega_z); \\
U^2(x^2, y^2, z^2) &= \min\left\{\frac{x^2}{\omega_x}, \frac{z^2}{\omega_z}\right\}, \ (\omega_x^2, \omega_y^2, \omega_z^2) = (\beta\omega_x, \gamma\omega_y, (1-\beta-\gamma)\omega_z); \\
U^3(x^3, y^3, z^3) &= \min\left\{\frac{x^3}{\omega_x}, \frac{y^3}{\omega_y}\right\}, \ (\omega_x^3, \omega_y^3, \omega_z^3) = (\alpha\omega_x, (1-\alpha-\gamma)\omega_y, \gamma\omega_z)
\end{aligned}$$

where $\omega_x > 0, \omega_y > 0, \omega_z > 0, \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ \alpha+\beta \leq 1,\ \beta+\gamma \leq 1$, and $\alpha + \gamma \leq 1$. Individual i's endowment of commodity j is denoted by ω_j^i, and let $\omega_j = \omega_j^1 + \omega_j^2 + \omega_j^3$.

The parameters α, β and γ identify each agent's endowment. Scarf considered the case where $\alpha = 1$ and $\beta = \gamma = 0$. The generalization to other initial endowments follows Hirota (1981). However, this economy differs from his in the normalization of prices. Hirota as well as Scarf considered prices where $p_x^2 + p_y^2 + p_z^2 = 3$, so that prices were confined to a sphere. This analysis considers the case where prices are identified by setting $p_z = 1$.

Given this price normalization, the income functions of the consumers are given by

$$\begin{aligned} M^1(p_x,\ p_y) &= p_x(1-\alpha-\beta)\omega_x + p_y\alpha\omega_y + \beta\omega_z, \\ M^2(p_x,\ p_y) &= p_x\beta\omega_x + p_y\gamma\omega_y + (1-\beta-\gamma)\omega_z, \\ M^3(p_x,\ p_y) &= p_x\alpha\omega_x + p_y(1-\alpha-\gamma)\omega_y + \gamma\omega_z. \end{aligned}$$

With these incomes, the individual demand functions are given by:

$$\begin{aligned} y^1(p_x,\ p_y) &= \frac{\omega_y M^1(p_x,\ p_y)}{p_y\omega_y + \omega_z}, z^1(p_x,\ p_y) = \frac{\omega_z M^1(p_x,\ p_y)}{p_y\omega_y + \omega_z} \\ x^2(p_x,\ p_y) &= \frac{\omega_x M^2(p_x,\ p_y)}{p_x\omega_x + \omega_z},\ z^2(p_x,\ p_y) = \frac{\omega_z M^2(p_x,\ p_y)}{p_x\omega_x + \omega_z} \\ x^3(p_x,\ p_y) &= \frac{\omega_x M^3(p_x,\ p_y)}{p_x\omega_x + p_y\omega_y},\ y^3(p_x,\ p_y) = \frac{\omega_y M^3(p_x,\ p_y)}{p_x\omega_x + p_y\omega_y} \end{aligned}$$

and $x^1(p_x, p_y) = y^2(p_x, p_y) = z^3(p_x, p_y) = 0$. Therefore, the aggregate excess demand functions are:

$$\begin{aligned} E^x(p_x,\ p_y) &= \frac{\omega_x M^2(p_x,\ p_y)}{p_x\omega_x + \omega_z} + \frac{\omega_x M^3(p_x,\ p_y)}{p_x\omega_x + p_y\omega_y} - \omega_x \\ E^y(p_x,\ p_y) &= \frac{\omega_y M^1(p_x,\ p_y)}{p_y\omega_y + \omega_z} + \frac{\omega_y M^3(p_x,\ p_y)}{p_x\omega_x + p_y\omega_y} - \omega_y \\ E^z(p_x,\ p_y) &= \frac{\omega_z M^1(p_x,\ p_y)}{p_y\omega_y + \omega_z} + \frac{\omega_z M^2(p_x,\ p_y)}{p_x\omega_x + \omega_z} - \omega_z \end{aligned}$$

The system is in equilibrium when there is no excess demand. This happens when

$$\frac{M^2(p_x,\ p_y)}{p_x + 1} = \frac{M^1(p_x,\ p_y)}{p_y + 1} = \frac{M^3(p_x,\ p_y)}{p_x + p_y} = \frac{1}{2}.$$

The point $(p_x, p_y) = (\omega_z/\omega_x, \omega_z/\omega_y)$ is the unique equilibrium which meets the above equations if and only if $4\gamma(1-\alpha-\beta) - (1-2\alpha)(1-2\beta) \neq 0$. The assumption is made when appropriate.

A.2 The Absolute Adjustment Specification

The original specification considered by Scarf held that prices adjust in proportion to excess demand. That is, the adjustment paths of prices are described by the following system of simultaneous differential equations:

$$\dot{p}_x = \lambda_x E_x(p_x, p_y) \tag{1}$$
$$\dot{p}_y = \lambda_y E_y(p_x, p_y) \tag{2}$$

where $\lambda_x > 0$ and $\lambda > 0$.

A.2.1 Instability

The equilibrium described above is a stable center when $\alpha+\beta = 1$, and $\gamma = 0$. In this case, the aggregate excess demand functions are:

$$E^x(p_x,\ p_y) = \frac{\omega_x(p_x(1-\alpha)\omega_x + \alpha\omega_z)}{p_x\omega_x + \omega_z} + \frac{\omega_x(p_x\alpha\omega_x + p_y(1-\alpha)\omega_y)}{p_x\omega_x + p_y\omega_y} - \omega_x$$
$$E^y(p_x,\ p_y) = \frac{\omega_y(p_y\alpha\omega_y + (1-\alpha)\omega_z)}{p_y\omega_y + \omega_z} + \frac{\omega_y(p_x\alpha\omega_x + p_y(1-\alpha)\omega_y)}{p_x\omega_x + p_y\omega_y} - \omega_y$$

The dynamic system of Equations (1) and (2) reduce to:

$$\begin{aligned}\frac{\dot{p}_x}{\lambda_x\omega_x} &= \frac{p_x(1-\alpha)\omega_x\omega_x + \alpha\omega_z}{p_x\omega_x + \omega_z} + \frac{p_x\alpha\omega_x + p_y(1-\alpha)\omega_y}{p_x\omega_x + p_y\omega_y} - 1 \\ &= \frac{p_x\omega_x(1-2\alpha)(p_y\omega_y - \omega_z)}{(p_x\omega_x + \omega_z)(p_x\omega_x + p_y\omega_y)}\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\dot{p}_y}{\lambda_y\omega_y} &= \frac{p_y\alpha\omega_y + (1-\alpha)\omega_z}{p_y\omega_y + \omega_z} + \frac{p_x\alpha\omega_x + p_y(1-\alpha)\omega_y}{p_x\omega_x + p_y\omega_y} - 1 \\ &= \frac{p_y\omega_y(2\alpha-1)(p_x\omega_x - \omega_z)}{(p_y\omega_y + \omega_z)(p_x\omega_x + p_y\omega_y)}\end{aligned} \tag{4}$$

Both Equation (3) and Equation (4) can be manipulated to find an expression which is equal to $(1-2\alpha)(p_y\omega_y - \omega_z)(p_x\omega_x - \omega_z)/(p_x\omega_x + p_y\omega_y)$. Subtracting these two expressions gives

$$\begin{aligned}\frac{\dot{p}_x}{\lambda_x p_x(\omega_x)^2}(p_x\omega_x + \omega_z)(p_x\omega_x - \omega_z) + \frac{\dot{p}_y}{\lambda_y p_y(\omega_y)^2}(p_y\omega_y + \omega_z)(p_y\omega_y - \omega_z) &= 0 \\ \frac{\dot{p}_x}{\lambda_x}\left(p_x - \left(\frac{\omega_z}{\omega_x}\right)^2\frac{1}{p_x}\right) + \frac{\dot{p}_y}{\lambda_y}\left(p_y - \left(\frac{\omega_z}{\omega_y}\right)^2\frac{1}{p_y}\right) &= 0\end{aligned}$$

This equation can be integrated with respect to t to give a function which describes the paths along which prices adjust, the solutions to the system of Equations (3) and (4). In particular,

$$\frac{1}{\lambda_x}\left(\frac{1}{2}(p_x)^2 - \left(\frac{\omega_z}{\omega_x}\right)^2 \log p_x\right) + \frac{1}{\lambda_y}\left(\frac{1}{2}(p_y)^2 - \left(\frac{\omega_z}{\omega_y}\right)^2 \log p_y\right) = C$$

where C is the constant value of the level surface on which (p_x, p_y) lies.

The stability properties of the equilibrium can be studied by examining this solution to system of Equations (3) and (4). Define the function $f : \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ by

$$f(p_x, p_y) = \frac{1}{\lambda_x}\left(\frac{1}{2}(p_x)^2 - \left(\frac{\omega_z}{\omega_x}\right)^2 \log p_x\right) + \frac{1}{\lambda_y}\left(\frac{1}{2}(p_y)^2 - \left(\frac{\omega_z}{\omega_y}\right)^2 \log p_y\right).$$

Checking the first derivatives confirms that the unique critical point of this system is the equilibrium posited above.

$$\begin{aligned} f_x(p_x, p_y) &= \frac{1}{\lambda_x p_x}\left(p_x + \left(\frac{\omega_z}{\omega_x}\right)\right)\left(p_x - \left(\frac{\omega_z}{\omega_x}\right)\right) \\ f_x(p_x, p_y) &= 0 \Leftrightarrow p_x = \frac{\omega_z}{\omega_x} \end{aligned}$$

$$\begin{aligned} f_y(p_x, p_y) &= \frac{1}{\lambda_y p_y}\left(p_y + \left(\frac{\omega_z}{\omega_y}\right)\right)\left(p_y - \left(\frac{\omega_z}{\omega_y}\right)\right) \\ f_y(p_x, p_y) &= 0 \Leftrightarrow p_y = \frac{\omega_z}{\omega_y} \end{aligned}$$

The matrix of second derivatives characterizes the stability of the equilibrium.

$$\begin{aligned} f_{xx}(p_x, p_y) &= \frac{1}{\lambda_x}\left(1 + \left(\frac{\omega_z}{\omega_x}\right)^2 \frac{1}{(p_x)^2}\right) \\ &> 0, \\ f_{yy}(p_x, p_y) &= \frac{1}{\lambda_y}\left(1 + \left(\frac{\omega_z}{\omega_y}\right)^2 \frac{1}{(p_y)^2}\right) \\ &> 0, \\ f_{xy}(p_x, p_y) &= f_{yx}(p_x, p_y) = 0 \end{aligned}$$

This gives that $f_{xx}(p_x, p_y)f_{yy}(p_x, p_y) - f_{xy}(p_x, p_y)f_{yx}(p_x, p_y) > 0$. Hence, f is a strictly convex function which takes the global minimum at $(\omega_z/\omega_x, \omega_z/\omega_y)$. It also satisfies the boundary condition that $f(p_x, p_y) \to \infty$ as either $p_x \to 0$ or $p_y \to 0$. The isoquants of f are therefore ellipse-like closed curves with $(\omega_z/\omega_x, \omega_z/\omega_y)$ the center. Thus, the solution to the system of Equations (3) and (4) is a stable center, with the particular closed orbit depending on the initial point $(p_x(0), p_y(0))$.

A.2.2 Stability

If $\alpha = \beta = 0$,and $\gamma = 1$ in Equations (1) and (2), the equilibrium is globally stable.

In this case, the aggregate excess demand functions are:

$$\begin{aligned} E^x(p_x,\ p_y) &= \frac{p_y\omega_x\omega_y}{p_x\omega_x + \omega_z} + \frac{\omega_x\omega_z}{p_x\omega_x + p_y\omega_y} - \omega_x; \\ E^y(p_x,\ p_y) &= \frac{p_x\omega_x\omega_y}{p_y\omega_y + \omega_z} + \frac{\omega_y\omega_z}{p_x\omega_x + p_y\omega_y} - \omega_y. \end{aligned}$$

Hence, the price adjustment system is:

$$\frac{\dot{p}_x}{\lambda_x} = \frac{p_y\omega_x\omega_y}{p_x\omega_x + \omega_z} + \frac{\omega_x\omega_z}{p_x\omega_x + p_y\omega_y} - \omega_x; \tag{5}$$

$$\frac{\dot{p}_y}{\lambda_y} = \frac{p_x\omega_x\omega_y}{p_y\omega_y + \omega_z} + \frac{\omega_y\omega_z}{p_x\omega_x + p_y\omega_y} - \omega_y. \tag{6}$$

This system is not easily integrable as in the unstable case. Its stability properties are demonstrated using Liapunov's Second Method (Boyce and DiPrima, 1992; Arrow and Hahn, 1971) in which an auxiliary function, a Liapunov function, is used to provide insight into the solution to the system without explicit knowledge of the solution itself.

Let g be the real-valued function on $\mathbb{R}_{++}$ defined by $g(v) = (v)^3/3 - (k)^2 v + 2(k)^3$ for each $v > 0$, where k is a positive constant. Then it takes the minimum 0 at $v = k$. We can thus see that

$$\frac{1}{3}(p_x)^3 - \left(\frac{\omega_z}{\omega_x}\right)^2 p_x + \frac{2}{3}\left(\frac{\omega_z}{\omega_x}\right)^3 \geq 0 \ \forall p_x > 0$$

with equality if and only if $p_x = \omega_z/\omega_x$, and

$$\frac{1}{3}(p_y)^3 - \left(\frac{\omega_z}{\omega_y}\right)^2 p_y + \frac{2}{3}\left(\frac{\omega_z}{\omega_y}\right)^3 \geq 0 \ \forall p_y > 0$$

with equality if and only if $p_y = \omega_z/\omega_y$.

Consider the real-valued function D on $\mathbb{R}^2_{++}$ defined by

$$\begin{aligned} D(p_x, p_y) &= \frac{\omega_x}{\lambda_x}\left[\frac{1}{3}(p_x)^3 - \left(\frac{\omega_z}{\omega_x}\right)^2 p_x + \frac{2}{3}\left(\frac{\omega_z}{\omega_x}\right)^3\right] \\ &\quad + \frac{\omega_y}{\lambda_y}\left[\frac{1}{3}(p_y)^3 - \left(\frac{\omega_z}{\omega_y}\right)^2 p_y + \frac{2}{3}\left(\frac{\omega_z}{\omega_y}\right)^3\right] \end{aligned}$$

for each $(p_x, p_y) \in \mathbb{R}^2_{++}$. Note that $D(p_x, p_y) \geq 0$ for each $(p_x, p_y) \in \mathbb{R}^2_{++}$, and $D(p_x, p_y) = 0$ if and only if $(p_x, p_y) = (\omega_z/\omega_x, \omega_z/\omega_y)$. Therefore, the following function $V(t)$ is a Liapunov function for the system of Equations (5) and (6):

$$\begin{aligned} V(t) &= D(p_x(t), p_y(t)) \\ &= \frac{\omega_x}{\lambda_x}\left[\frac{1}{3}(p_x(t))^3 - \left(\frac{\omega_z}{\omega_x}\right)^2 p_x(t) + \frac{2}{3}\left(\frac{\omega_z}{\omega_x}\right)^3\right] \\ &\quad + \frac{\omega_y}{\lambda_y}\left[\frac{1}{3}(p_y(t))^3 - \left(\frac{\omega_z}{\omega_y}\right)^2 p_y(t) + \frac{2}{3}\left(\frac{\omega_z}{\omega_y}\right)^3\right] \end{aligned}$$

where t denotes time.

To establish global stability, it is sufficient to show that the value of V decreases as t increases until (p_x, p_y) reaches $(\omega_z/\omega_x, \omega_z/\omega_y)$ regardless of where (p_x, p_y) starts.

$$\begin{aligned} \dot{V} &= \frac{\omega_x}{\lambda_x}\left((p_x)^2 - \left(\frac{\omega_z}{\omega_x}\right)^2\right)\dot{p}_x + \frac{\omega_y}{\lambda_y}\left((p_y)^2 - \left(\frac{\omega_z}{\omega_y}\right)^2\right)\dot{p}_y \\ &= \frac{1}{\lambda_x p_x \omega_x}(p_x\omega_x + \omega_z)(p_x\omega_x - \omega_z)\,\dot{p}_x + \frac{1}{\lambda_y p_y \omega_y}(p_y\omega_y + \omega_z)(p_y\omega_y - \omega_z)\,\dot{p}_y \end{aligned}$$

$$
\begin{aligned}
&= (p_x\omega_x+\omega_z)(p_x\omega_x-\omega_z)\left(\frac{p_y\omega_y}{p_x\omega_x+\omega_z}+\frac{\omega_z}{p_x\omega_x+p_y\omega_y}-1\right)\\
&\quad +(p_y\omega_y+\omega_z)(p_y\omega_y-\omega_z)\left(\frac{p_x\omega_x}{p_y\omega_y+\omega_z}+\frac{\omega_z}{p_x\omega_x+p_y\omega_y}-1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (p_x\omega_x+\omega_z)(p_x\omega_x-\omega_z)\left(\frac{p_y\omega_y(p_y\omega_y-\omega_z)-(p_x\omega_x+\omega_z)(p_x\omega_x-\omega_z)}{(p_x\omega_x+\omega_z)(p_x\omega_x+p_y\omega_y)}\right)\\
&\quad +(p_y\omega_y+\omega_z)(p_y\omega_y-\omega_z)\left(\frac{p_x\omega_x(p_x\omega_x-\omega_z)-(p_y\omega_y+\omega_z)(p_y\omega_y-\omega_z)}{(p_y\omega_y+\omega_z)(p_x\omega_x+p_y\omega_y)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{p_x\omega_x+p_y\omega_y}\left[-2\omega_z(p_x\omega_x-\omega_z)(p_y\omega_y-\omega_z)-(p_x\omega_x-p_y\omega_y)^2(p_x\omega_x+p_y\omega_y)\right]\\
&= -\frac{\omega_z^3}{p_x\omega_x+p_y\omega_y}\left[2(\frac{p_x\omega_x}{\omega_z}-1)(\frac{p_y\omega_y}{\omega_z}-1)+\left(\frac{p_x\omega_x}{\omega_z}-\frac{p_y\omega_y}{\omega_z}\right)^2\left(\frac{p_y\omega_y}{\omega_z}+\frac{p_y\omega_y}{\omega_z}\right)\right]
\end{aligned}
$$

The sign of $\dot{V}$ is determined by

$$
2(\frac{p_x\omega_x}{\omega_z}-1)(\frac{p_y\omega_y}{\omega_z}-1)+\left(\frac{p_x\omega_x}{\omega_z}-\frac{p_y\omega_y}{\omega_z}\right)^2\left(\frac{p_y\omega_y}{\omega_z}+\frac{p_y\omega_y}{\omega_z}\right).
$$

For every $a \geq 0$, and every $b \geq 0$,

$$
2(a-1)(b-1)+(a-b)^2(a+b) \geq 0,
$$

with equality if and only if $a = b = 1$.[1]

[1] Here is a proof:
We have

$$
\begin{aligned}
&2(a-1)(b-1)+(a-b)^2(a+b)\\
=\ &a^3+b^3+2-a^2b-ab^2+2ab-2a-2b.
\end{aligned}
$$

Let $k = a+b$, then

$$
2(a-1)(b-1)+(a-b)^2(a+b)
$$

Hence, $\dot{V} \leq 0$ with equality if and only if $p_x\omega_x = \omega_z = p_y\omega_y$ (i.e., $(p_x,\ p_y) = (\omega_z/\omega_x, \omega_z/\omega_y)$). Therefore, $V(t)$ is strictly decreasing everywhere except at the equilibrium, so for all starting price pairs, the prices will converge to the equilibrium. This means the system of Equations (5) and (6) is globally stable.

A.3 Relative Price Adjustment

While microeconomists typically use the absolute price adjustment specification considered by Scarf, macroeconomists often assume that price adjustments are proportional to the price level. In this economy, this assumption means prices adjust according to the following simultaneous differential equations:

$$\begin{aligned} &= a^3 + (k-a)^3 + 2 - a^2(k-a) - a(k-a)^2 + 2a(k-a) - 2k \\ &= (4k-2)a^2 + (2k-4k^2)a + k^3 - 2k + 2 \\ &= 4(k-\frac{1}{2})(a-\frac{k}{2})^2 + \frac{1}{2}(k-2)^2. \end{aligned}$$

Define the real-valued function h on $[0, k]$ by

$$h(a) = 4(k-\frac{1}{2})(a-\frac{k}{2})^2 + \frac{1}{2}(k-2)^2$$

for each $a \in [0, k]$.

(i) If $0 \leq k < 1/2$, then f takes the minimum $k^3 - 2k + 2$ at $a = 0$ and $a = k$. Since $0 \leq k < 1/2$, the derivative $3k^2 - 2$ is negative and the infimum of $k^3 - 2k + 2$ is $(1/2)^3 - 2(1/2) + 2 > 0$, so that $k^3 - 2k + 2 > 0$. Hence, for each $a \in [0, k]$,

$$h(a) \geq k^3 - 2k + 2 \geq \left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right) + 2 > 0.$$

(ii) If $k = 1/2$, then for each $a \in [0, k]$,

$$h(a) = \frac{1}{2}\left(\frac{3}{2}\right)^2 > 0.$$

(iii) If $k > 1/2$, then f takes the minimum $(k-2)^2/2$ at $a = k/2$. Note that $(k-2)^2/2 \geq 0$ with equality if and only if $k = 2$. Since $a = k/2$, we have

$$k = 2 \text{ if and only if } a = 1 = b.$$

By (i), (ii), and (iii), we complete the proof.

$$\frac{\dot{p}_x}{p_x} = \lambda_x E^x(p_x, p_y) \tag{7}$$

$$\frac{\dot{p}_y}{p_y} = \lambda_y E^y(p_x, p_y) \tag{8}$$

where $\lambda_x > 0$ and $\lambda_y > 0$.

A.3.1 Instability

As with the absolute specification, the system of Equations (7) and (8) is globally unstable if $\alpha + \beta = 1$, and $\gamma = 0$.

Following the same method to integrate the system as in the absolute adjustment specification, the system can be written

$$\begin{aligned}\frac{\dot{p}_x}{\lambda_x \omega_x p_x} &= \frac{p_x(1-\alpha)\omega_x + \alpha\omega_z}{p_x\omega_x + \omega_z} + \frac{p_x\alpha\omega_x + p_y(1-\alpha)\omega_y}{p_x\omega_x + p_y\omega_y} - 1 \\ &= \frac{p_x\omega_x(1-2\alpha)(p_y\omega_y - \omega_z)}{(p_x\omega_x + \omega_z)(p_x\omega_x + p_y\omega_y)}\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\dot{p}_y}{\lambda_y \omega_y p_x} &= \frac{p_y\alpha\omega_y + (1-\alpha)\omega_z}{p_y\omega_y + \omega_z} + \frac{p_x\alpha\omega_x + p_y(1-\alpha)\omega_y}{p_x\omega_x + p_y\omega_y} - 1 \\ &= \frac{p_y\omega_y(2\alpha-1)(p_x\omega_x - \omega_z)}{(p_y\omega_y + \omega_z)(p_x\omega_x + p_y\omega_y)}.\end{aligned} \tag{10}$$

Both Equation (9) and Equation (10) can be manipulated to find an expression which is equal to $(1-2\alpha)(p_y\omega_y - \omega_z)(p_x\omega_x - \omega_z)/(p_x\omega_x + p_y\omega_y)$. Subtracting these two expressions gives

$$\begin{aligned}\frac{\dot{p}_x}{\lambda_x (p_x)^2 (\omega_x)^2}(p_x\omega_x + \omega_z)(p_x\omega_x - \omega_z) + \frac{\dot{p}_y}{\lambda_y (p_y)^2 (\omega_y)^2}(p_y\omega_y + \omega_z)(p_y\omega_y - \omega_z) &= 0 \\ \frac{\dot{p}_x}{\lambda_x}\left(1 - \left(\frac{\omega_z}{\omega_x}\right)^2 \frac{1}{(p_x)^2}\right) + \frac{\dot{p}_y}{\lambda_y}\left(1 - \left(\frac{\omega_z}{\omega_y}\right)^2 \frac{1}{(p_y)^2}\right) &= 0\end{aligned}$$

This equation can be directly integrated with respect to t to give a function which describes the paths along which prices adjust, the solution to the system of Equations (9) and (10). In particular,

$$\frac{1}{\lambda_x}\left(p_x + \left(\frac{\omega_z}{\omega_x}\right)^2 \frac{1}{p_x}\right) + \frac{1}{\lambda_y}\left(p_y + \left(\frac{\omega_z}{\omega_y}\right)^2 \frac{1}{p_y}\right) = C$$

where C is the constant value of the level surface on which (p_x, p_y) lies.

The stability properties of the equilibrium can be studied by examining this solution to the system of Equations (9) and (10). Define the function $f : \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ by

$$f(p_x, p_y) = \frac{1}{\lambda_x}\left(p_x + \left(\frac{\omega_z}{\omega_x}\right)^2 \frac{1}{p_x}\right) + \frac{1}{\lambda_y}\left(p_y + \left(\frac{\omega_z}{\omega_y}\right)^2 \frac{1}{p_y}\right).$$

Checking the first derivatives confirms that the unique critical point of this system is the equilibrium posited above.

$$\begin{aligned} f_x(p_x, p_y) &= \frac{1}{\lambda_x (p_x)^2}\left(p_x + \left(\frac{\omega_z}{\omega_x}\right)\right)\left(p_x - \left(\frac{\omega_z}{\omega_x}\right)\right) \\ f_x(p_x, p_y) &= 0 \Leftrightarrow p_x = \frac{\omega_z}{\omega_x} \end{aligned}$$

$$\begin{aligned} f_y(p_x, p_y) &= \frac{1}{\lambda_y (p_y)^2}\left(p_y + \left(\frac{\omega_z}{\omega_y}\right)\right)\left(p_y - \left(\frac{\omega_z}{\omega_y}\right)\right) \\ f_y(p_x, p_y) &= 0 \Leftrightarrow p_y = \frac{\omega_z}{\omega_y} \end{aligned}$$

The matrix of second derivatives characterizes the stability of the equilibrium.

$$\begin{aligned} f_{xx}(p_x, p_y) &= \frac{2}{\lambda_x (p_x)^3}\left(\frac{\omega_z}{\omega_x}\right)^2 \\ &> 0 \\ f_{yy}(p_x, p_y) &= \frac{2}{\lambda_y (p_y)^3}\left(\frac{\omega_z}{\omega_y}\right)^2 \\ &> 0, \\ f_{xy}(p_x, p_y) &= f_{yx}(p_x, p_y) = 0 \end{aligned}$$

Since $f_{xx}(p_x,p_y)f_{yy}(p_x,p_y) - f_{xy}(p_x,p_y)f_{yx}(p_x,p_y) > 0$, f is a strictly convex function which takes the global minimum at $(\omega_z/\omega_x, \omega_z/\omega_y)$. It also satisfies the boundary condition that $f(p_x,p_y) \to \infty$ as either $p_x \to 0$ or $p_y \to 0$. The isoquants of f are therefore ellipse-like closed curves with $(\omega_z/\omega_x, \omega_z/\omega_y)$ the center. Thus, the solution to the system of Equations (9) and (10) is a stable center, with the particular closed orbit depending on the initial point $(p_x(0), p_y(0))$.

A.3.2 Stability

If $\alpha = \beta = 0$,and $\gamma = 1$ in Equations (7) and (8), the equilibrium is globally stable.

In this case, the aggregate excess demand functions are:

$$\begin{aligned} E^x(p_x,\ p_y) &= \frac{p_y\omega_x\omega_y}{p_x\omega_x + \omega_z} + \frac{\omega_x\omega_z}{p_x\omega_x + p_y\omega_y} - \omega_x; \text{ and} \\ E^y(p_x,\ p_y) &= \frac{p_x\omega_x\omega_y}{p_y\omega_y + \omega_z} + \frac{\omega_y\omega_z}{p_x\omega_x + p_y\omega_y} - \omega_y. \end{aligned}$$

Hence, the price adjustment system is:

$$\frac{\dot{p}_x}{p_x} = \lambda_x \left[\frac{p_y\omega_x\omega_y}{p_x\omega_x + \omega_z} + \frac{\omega_x\omega_z}{p_x\omega_x + p_y\omega_y} - \omega_x \right]; \tag{11}$$

$$\frac{\dot{p}_y}{p_y} = \lambda_y \left[\frac{p_x\omega_x\omega_y}{p_y\omega_y + \omega_z} + \frac{\omega_y\omega_z}{p_x\omega_x + p_y\omega_y} - \omega_y \right]. \tag{12}$$

As in the absolute adjustment specification, this system is not as easily integrable as the unstable case, so its stability properties are demonstrated using Liapunov's Second Method.

Let g be the real-valued function on $\mathbb{R}_{++}$ defined by $g(v) = \frac{1}{2}(v)^2 - (k)^2 \log v$ for each $v > 0$, where k is a positive constant. Then it takes the minimum $\frac{1}{2}(k)^2 - (k)^2 \log k$ at $v = k$. We can thus see that

$$\frac{1}{2}(p_x)^2 - \left(\frac{\omega_z}{\omega_x}\right)^2 \log p_x \geq \frac{1}{2}\left(\frac{\omega_z}{\omega_x}\right)^2 - \left(\frac{\omega_z}{\omega_x}\right)^2 \log\left(\frac{\omega_z}{\omega_x}\right) \ \forall p_x > 0$$

with equality if and only if $p_x = \omega_z/\omega_x$, and

$$\frac{1}{2}(p_y)^2 - \left(\frac{\omega_z}{\omega_y}\right)^2 \log p_y \geq \frac{1}{2}\left(\frac{\omega_z}{\omega_y}\right)^2 - \left(\frac{\omega_z}{\omega_y}\right)^2 \log\left(\frac{\omega_z}{\omega_y}\right) \quad \forall p_y > 0$$

with equality if and only if $p_y = \omega_z/\omega_y$. Consider the real-valued function D on $\mathbb{R}^2_{++}$ defined by

$$\begin{aligned} D(p_x, p_y) &= \frac{\omega_x}{\lambda_x}\left[\left(\frac{1}{2}(p_x)^2 - \left(\frac{\omega_z}{\omega_x}\right)^2 \log p_x\right) - \left(\frac{1}{2}\left(\frac{\omega_z}{\omega_x}\right)^2 - \left(\frac{\omega_z}{\omega_x}\right)^2 \log\left(\frac{\omega_z}{\omega_x}\right)\right)\right] \\ &\quad + \frac{\omega_y}{\lambda_y}\left[\left(\frac{1}{2}(p_y)^2 - \left(\frac{\omega_z}{\omega_y}\right)^2 \log p_y\right) - \left(\frac{1}{2}\left(\frac{\omega_z}{\omega_y}\right)^2 - \left(\frac{\omega_z}{\omega_y}\right)^2 \log\left(\frac{\omega_z}{\omega_y}\right)\right)\right] \end{aligned}$$

for each $(p_x, p_y) \in \mathbb{R}^2_{++}$. Note that $D(p_x, p_y) \geq 0$ for each $(p_x, p_y) \in \mathbb{R}^2_{++}$, and $D(p_x, p_y) = 0$ if and only if $(p_x, p_y) = (\omega_z/\omega_x, \omega_z/\omega_y)$. Therefore, the following function $V(t)$ is a Liapunov function for the system of Equations (11) and (12):

$$\begin{aligned} V(t) &= D(p_x(t), p_y(t)) \\ &= \frac{\omega_x}{\lambda_x}\left[\left(\frac{1}{2}(p_x(t))^2 - \left(\frac{\omega_z}{\omega_x}\right)^2 \log p_x(t)\right) - \left(\frac{1}{2}\left(\frac{\omega_z}{\omega_x}\right)^2 - \left(\frac{\omega_z}{\omega_x}\right)^2 \log\left(\frac{\omega_z}{\omega_x}\right)\right)\right] \\ &\quad + \frac{\omega_y}{\lambda_y}\left[\left(\frac{1}{2}(p_y(t))^2 - \left(\frac{\omega_z}{\omega_y}\right)^2 \log p_y(t)\right) - \left(\frac{1}{2}\left(\frac{\omega_z}{\omega_y}\right)^2 - \left(\frac{\omega_z}{\omega_y}\right)^2 \log\left(\frac{\omega_z}{\omega_y}\right)\right)\right] \end{aligned}$$

where t denotes time.

To establish global stability, it is sufficient to show that the value of V decreases as t increases until (p_x, p_y) reaches $(\omega_z/\omega_x, \omega_z/\omega_y)$ regardless of where (p_x, p_y) starts. The first derivative is given by

$$\begin{aligned} \dot{V} &= \frac{\omega_x}{\lambda_x}\left(p_x - \left(\frac{\omega_z}{\omega_x}\right)^2 \frac{1}{p_x}\right)\dot{p}_x + \frac{\omega_y}{\lambda_y}\left(p_y - \left(\frac{\omega_z}{\omega_y}\right)^2 \frac{1}{p_y}\right)\dot{p}_y \\ &= \frac{1}{\lambda_x p_x \omega_x}(p_x\omega_x + \omega_z)(p_x\omega_x - \omega_z)\,\dot{p}_x + \frac{1}{\lambda_y p_y \omega_y}(p_y\omega_y + \omega_z)(p_y\omega_y - \omega_z)\,\dot{p}_y \end{aligned}$$

$$
\begin{aligned}
&= (p_x\omega_x + \omega_z)(p_x\omega_x - \omega_z)\left(\frac{p_y\omega_y}{p_x\omega_x + \omega_z} + \frac{\omega_z}{p_x\omega_x + p_y\omega_y} - 1\right) \\
&\quad +(p_y\omega_y + \omega_z)(p_y\omega_y - \omega_z)\left(\frac{p_x\omega_x}{p_y\omega_y + \omega_z} + \frac{\omega_z}{p_x\omega_x + p_y\omega_y} - 1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (p_x\omega_x + \omega_z)(p_x\omega_x - \omega_z)\left(\frac{p_y\omega_y(p_y\omega_y - \omega_z) - (p_x\omega_x + \omega_z)(p_x\omega_x - \omega_z)}{(p_x\omega_x + \omega_z)(p_x\omega_x + p_y\omega_y)}\right) \\
&\quad +(p_y\omega_y + \omega_z)(p_y\omega_y - \omega_z)\left(\frac{p_x\omega_x(p_x\omega_x - \omega_z) - (p_y\omega_y + \omega_z)(p_y\omega_y - \omega_z)}{(p_y\omega_y + \omega_z)(p_x\omega_x + p_y\omega_y)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{p_x\omega_x + p_y\omega_y}\left[-2\omega_z(p_x\omega_x - \omega_z)(p_y\omega_y - \omega_z) - (p_x\omega_x - p_y\omega_y)^2(p_x\omega_x + p_y\omega_y)\right] \\
&= -\frac{\omega_z^3}{p_x\omega_x + p_y\omega_y}\left[2(\frac{p_x\omega_x}{\omega_z} - 1)(\frac{p_y\omega_y}{\omega_z} - 1) + \left(\frac{p_x\omega_x}{\omega_z} - \frac{p_y\omega_y}{\omega_z}\right)^2\left(\frac{p_y\omega_y}{\omega_z} + \frac{p_y\omega_y}{\omega_z}\right)\right].
\end{aligned}
$$

The sign of $\dot{V}$ is determined by

$$
2(\frac{p_x\omega_x}{\omega_z} - 1)(\frac{p_y\omega_y}{\omega_z} - 1) + \left(\frac{p_x\omega_x}{\omega_z} - \frac{p_y\omega_y}{\omega_z}\right)^2\left(\frac{p_y\omega_y}{\omega_z} + \frac{p_y\omega_y}{\omega_z}\right).
$$

For every $a \geq 0$, and every $b \geq 0$,

$$
2(a-1)(b-1) + (a-b)^2(a+b) \geq 0,
$$

with equality if and only if $a = b = 1$. Hence, $\dot{V} \leq 0$ with equality if and only if $p_x\omega_x = \omega_z = p_y\omega_y$ (i.e., $(p_x,\ p_y) = (\omega_z/\omega_x, \omega_z/\omega_y)$). Therefore, $V(t)$ is strictly decreasing everywhere except at the equilibrium, so for all starting price pairs, the prices will converge to equilibrium. This means that the system of Equations (11) and (12) is globally stable.

A.4 Directions of the Limit Cycles

Having established that Scarf's model predicts prices will orbit around the competitive equilibrium when $\gamma = 0$, this section considers the direction around the orbits in which prices move.

For all $\alpha + \beta = 1$, p_x moves according to Equation (3) in the absolute adjustment model and Equation (9) in the relative adjustment models. Since endowments must be nonnegative and prices and λ_x must be positive, the sign of both equations is determined by

$$(1 - 2\alpha)(p_y\omega_y - \omega_z). \tag{13}$$

The second term is zero when $p_y = \omega_z/\omega_y$; it is positive when p_y is above the equilibrium price, and negative if p_y is below the equilibrium price. The first term is positive when $\alpha < \frac{1}{2}$. Therefore, when $\alpha < \frac{1}{2}$, p_x is increasing when p_y is above the equilibrium price and negative when p_x is below the equilibrium price. The direction of adjustment reverses when $\alpha > \frac{1}{2}$ and the sign of the first term changes.

For all $\alpha + \beta = 1$, p_y moves according to Equation (4) in the absolute adjustment model and Equation (10) in the relative adjustment models. Since endowments must be nonnegative and prices and λ_x must be positive, the sign of both equations is determined by

$$(2\alpha - 1)(p_x\omega_x - \omega_z). \tag{14}$$

The second term is zero when $p_x = \omega_z/\omega_x$; it is positive when p_x is above the equilibrium price, and negative if p_x is below the equilibrium price. The first term is positive when $\alpha > \frac{1}{2}$. Therefore, when $\alpha > \frac{1}{2}$, p_y is increasing when p_x is above the equilibrium price and negative when p_y is below the equilibrium price. The direction of adjustment reverses when $\alpha < \frac{1}{2}$ and the sign of the first term changes.

In the counterclockwise treatment, where $\alpha = 1$, these results combine to indicate a counterclockwise orbit. The price space can be divided into quadrants, split by the line $p_x = \omega_z/\omega_x$ and the line $p_y = \omega_z/\omega_y$. In the quadrant where $p_x < \omega_z/\omega_x$ and $p_y < \omega_z/\omega_y$, p_x is increasing and p_y is decreasing; where $p_x < \omega_z/\omega_x$ and $p_y > \omega_z/\omega_y$, p_x is decreasing and p_y is decreasing; where $p_x > \omega_z/\omega_x$ and $p_y < \omega_z/\omega_y$, p_x is increasing and p_y is increasing; where $p_x > \omega_z/\omega_x$ and $p_y > \omega_z/\omega_y$, p_x is decreasing and p_y is increasing. This pattern establishes the prices are adjusting around the orbits in a counterclockwise direction. That the prices orbit clockwise when $\alpha = 0$ follows symmetrically.

Figure 1a: Sample Convergence Paths (λ=1)

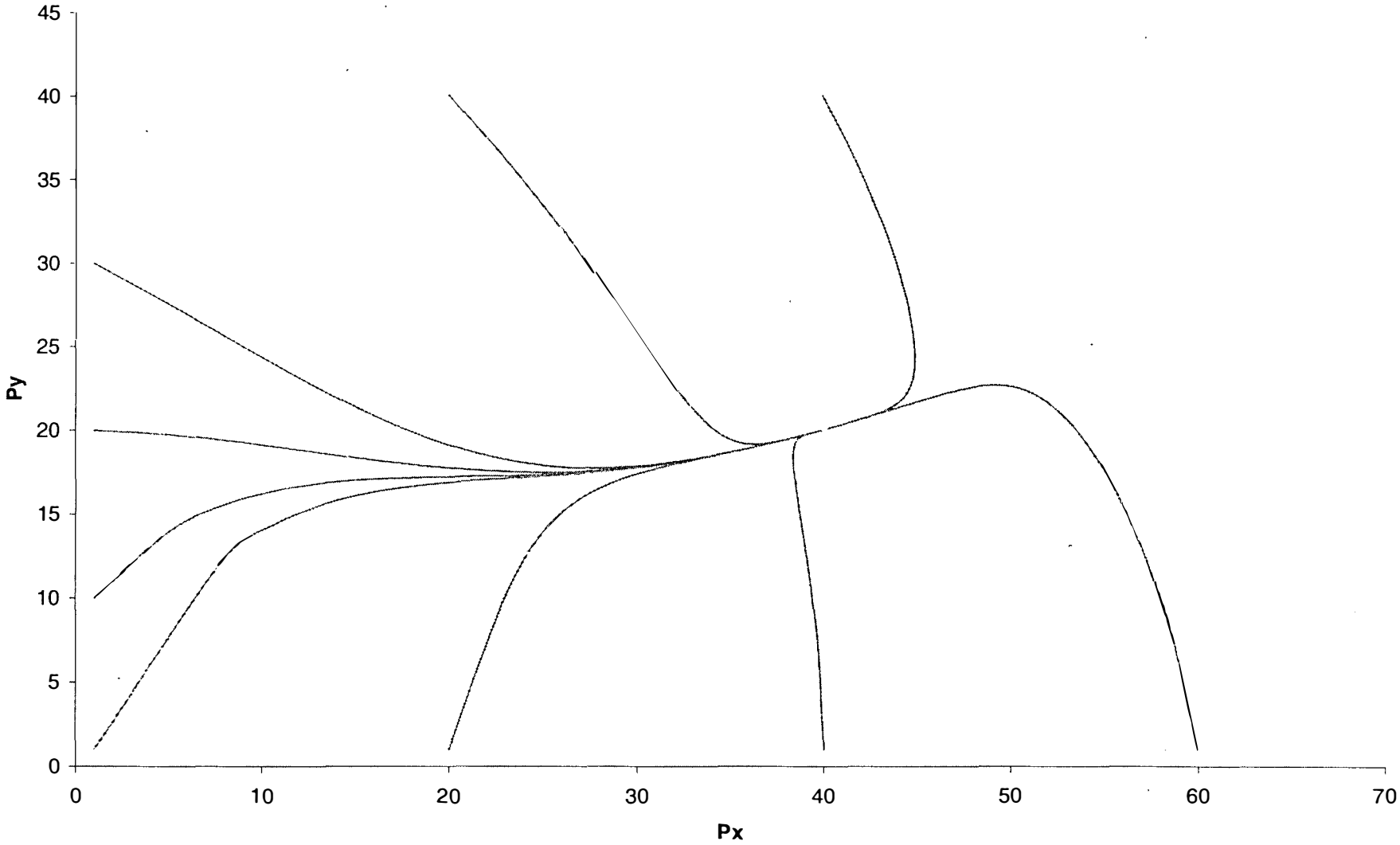


Figure 1b: Sample Orbits

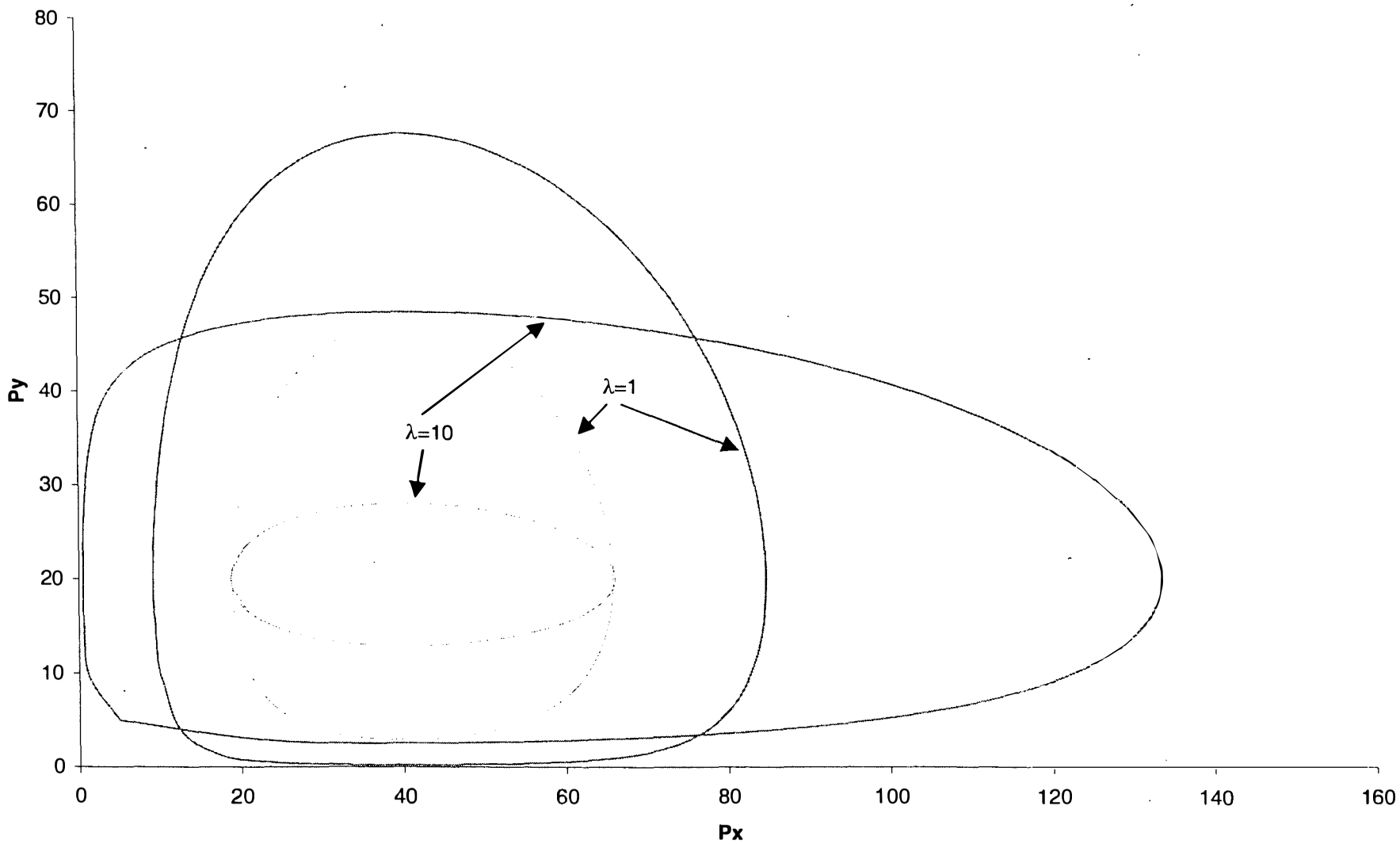


Figure 1c: Price Adjustment Paths vs. Time

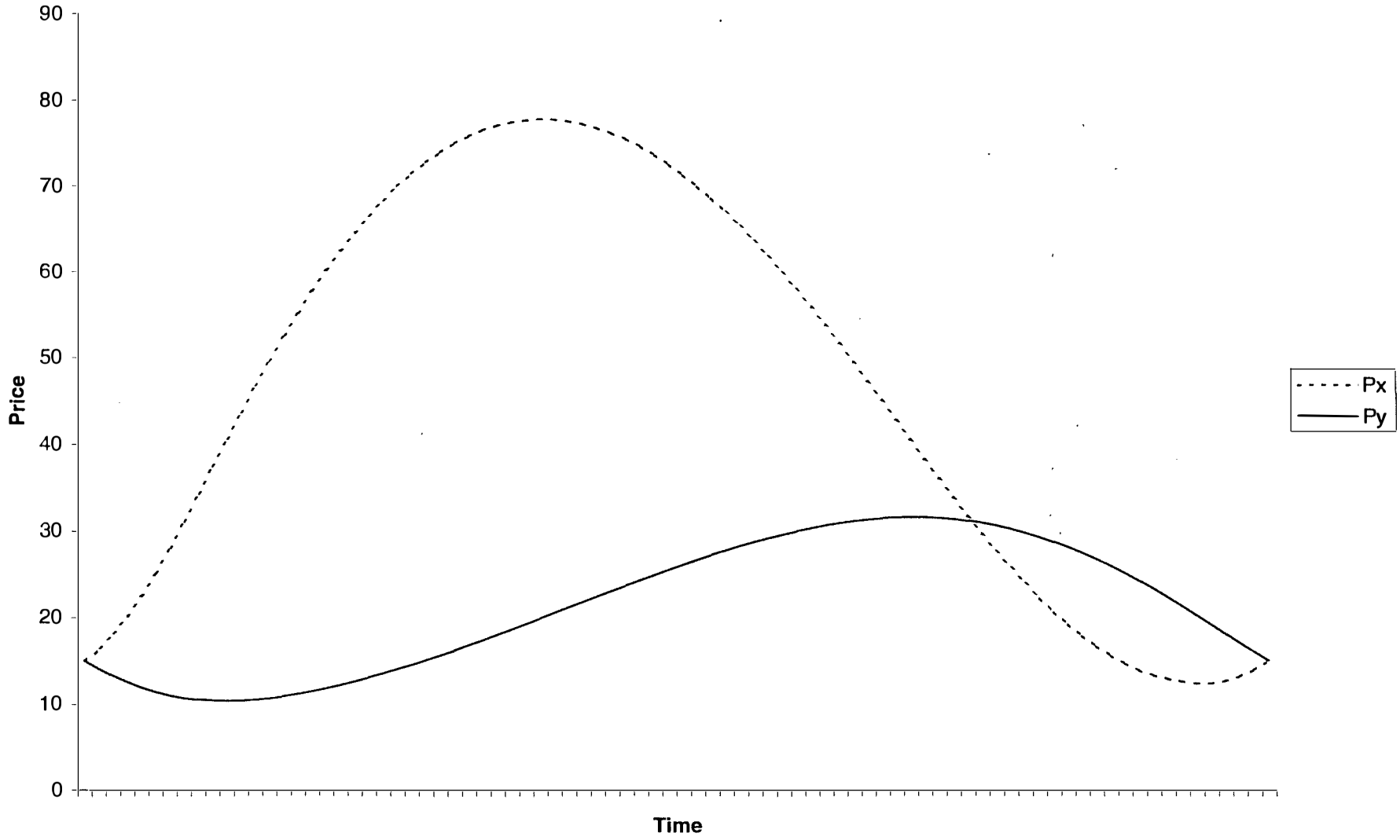


Figure 2a: 4/12/99 Convergent

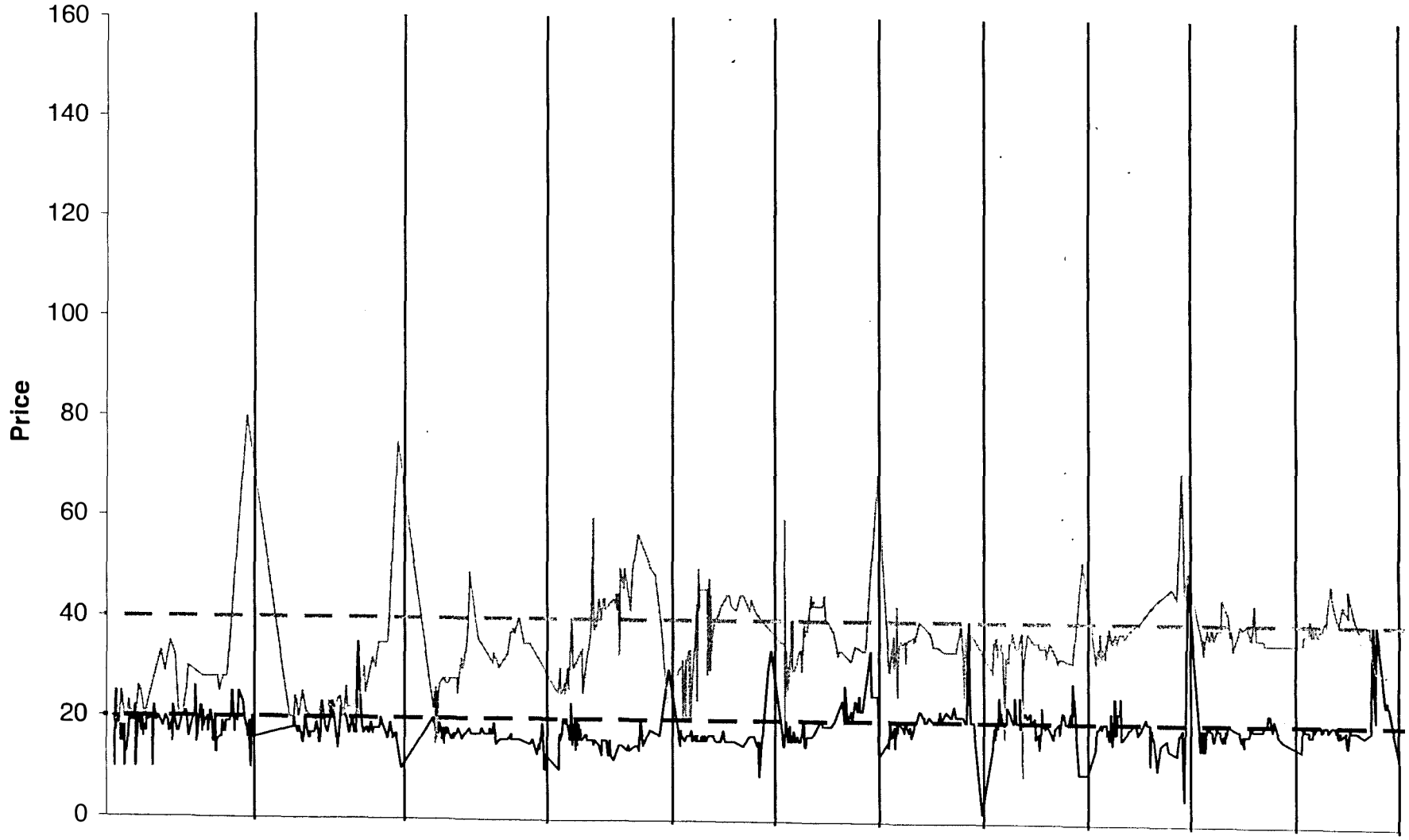


Figure 2b: 5/11/99 CCW-Experienced

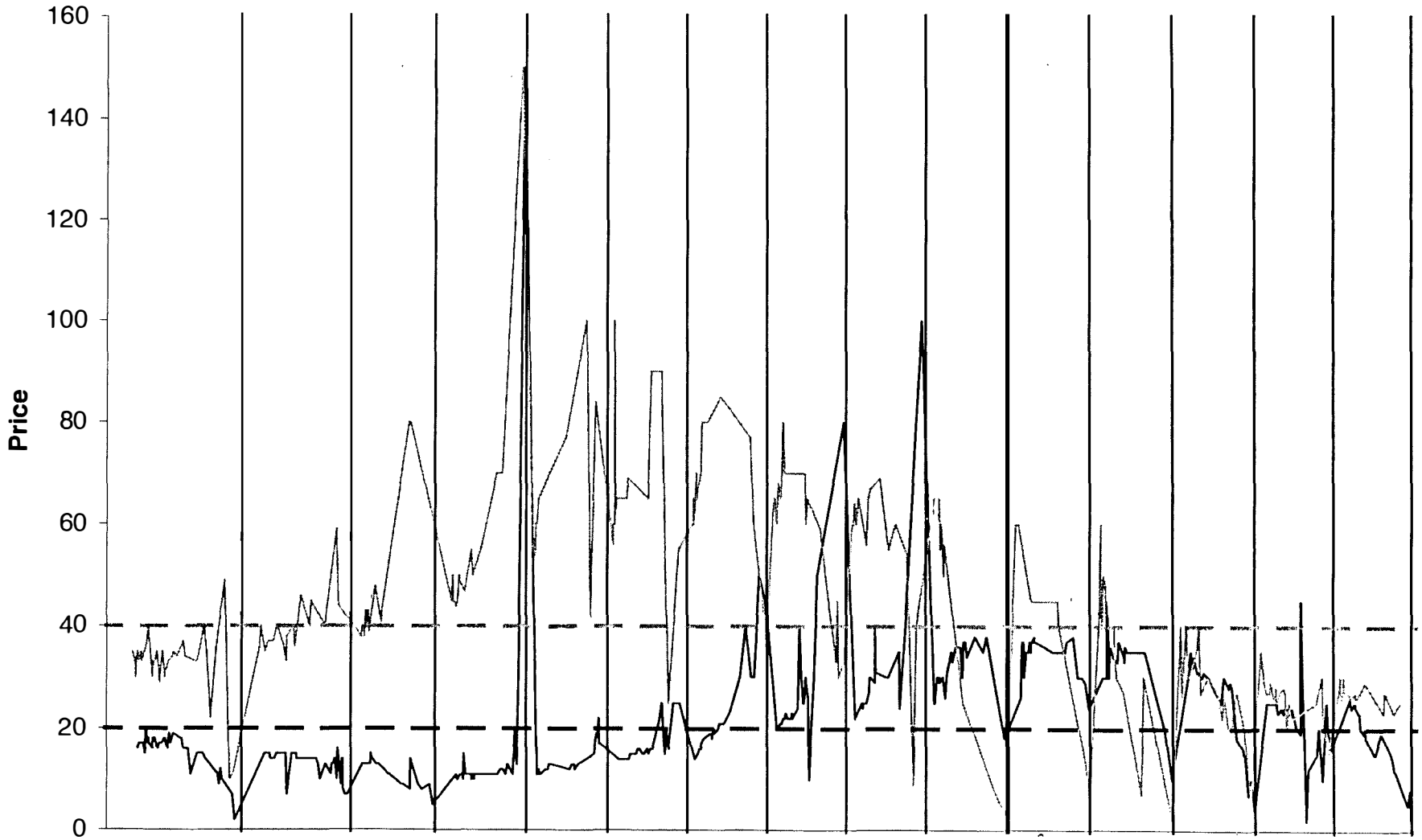


Figure 3a: Counterclockwise Experiments

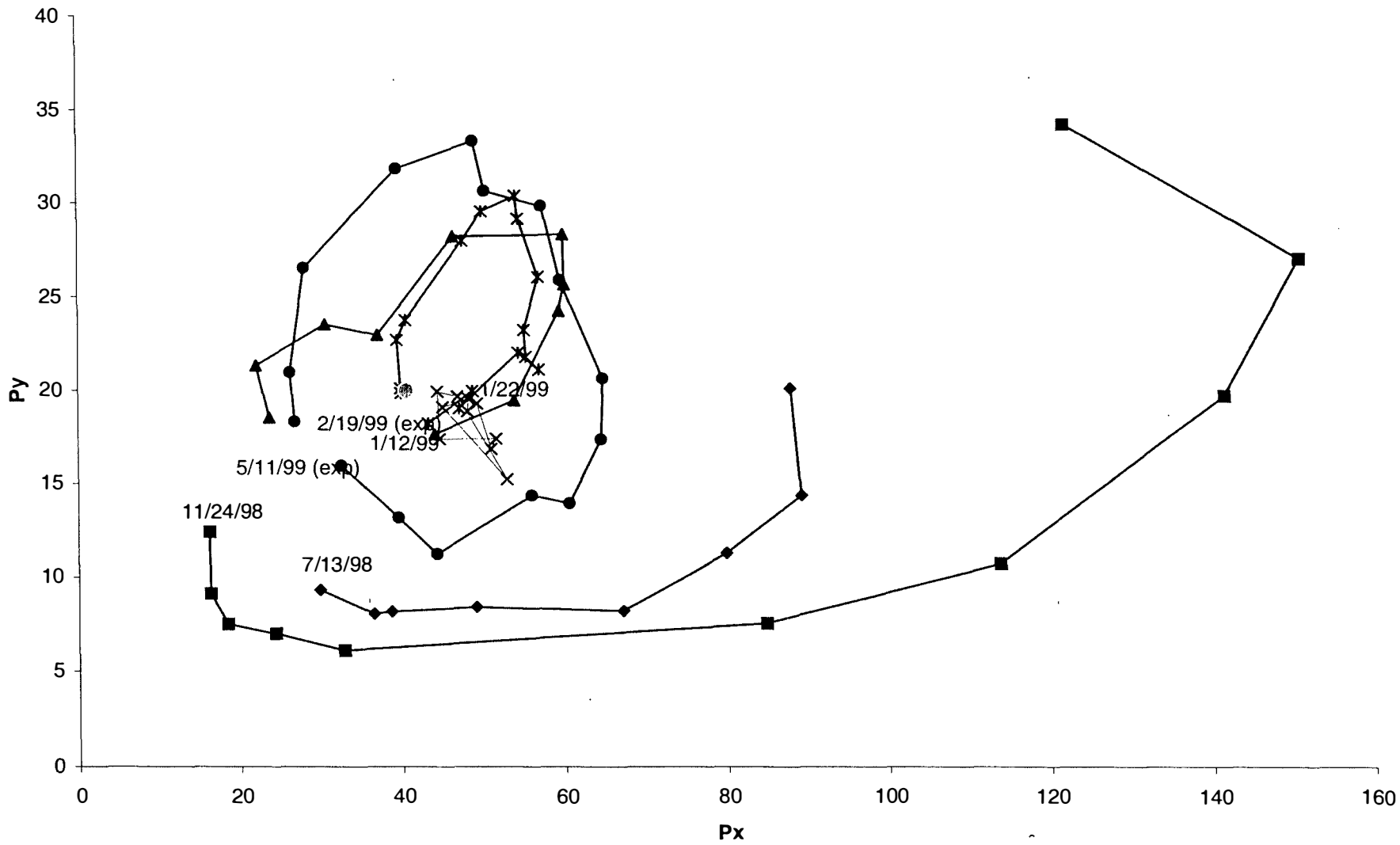


Figure 3b: Counterclockwise Experiments (Detail)

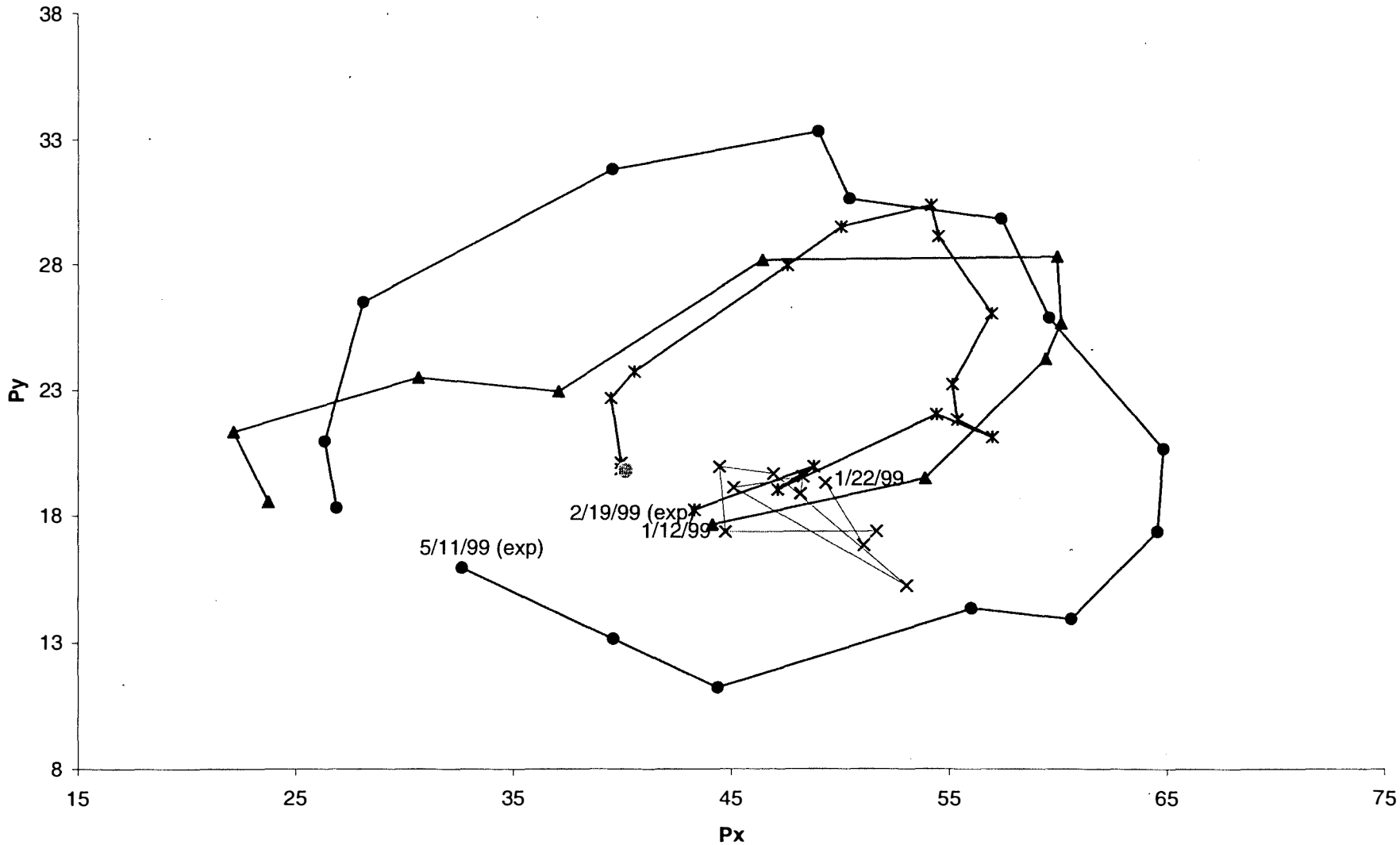


Figure 4: Clockwise Experiments

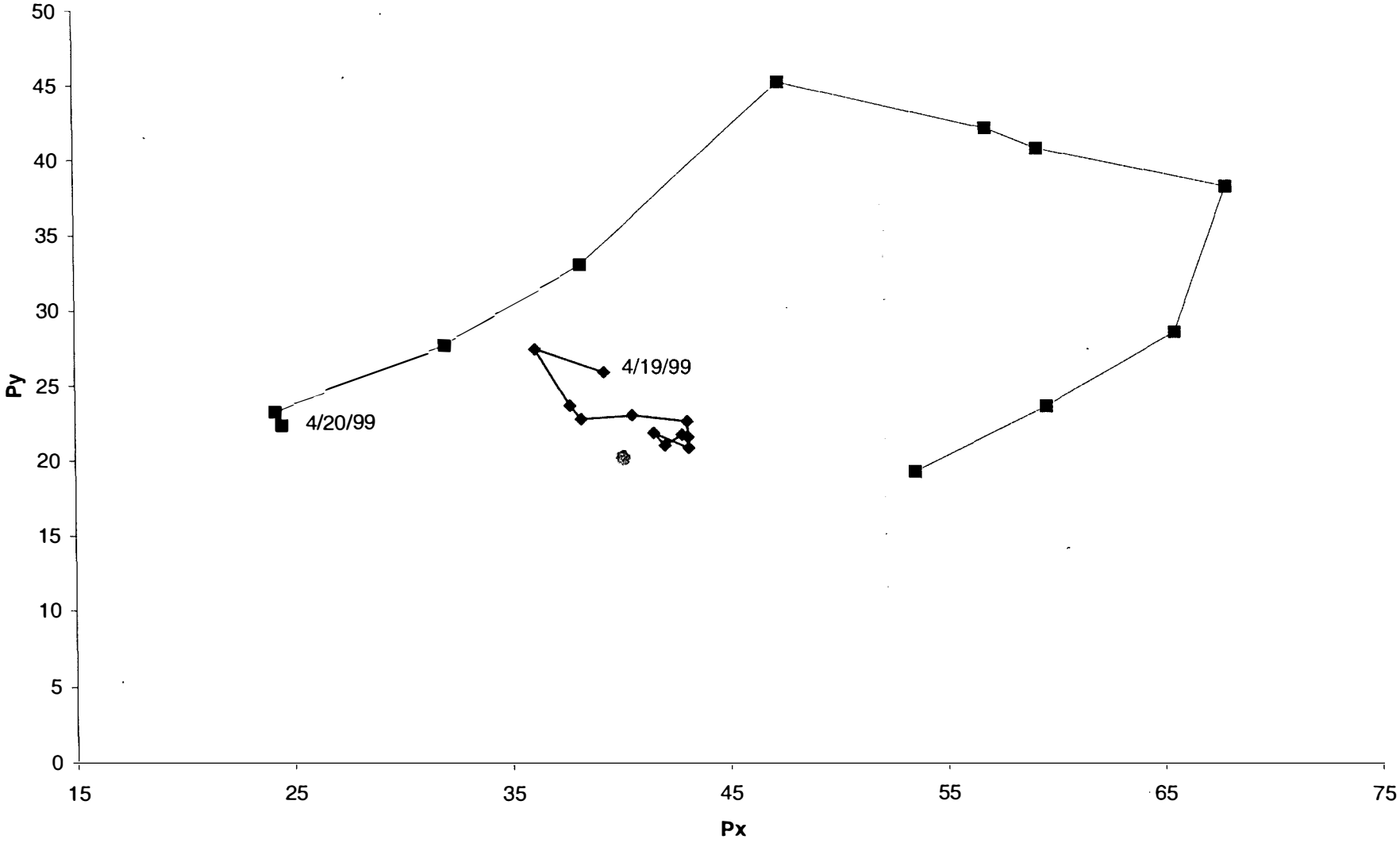
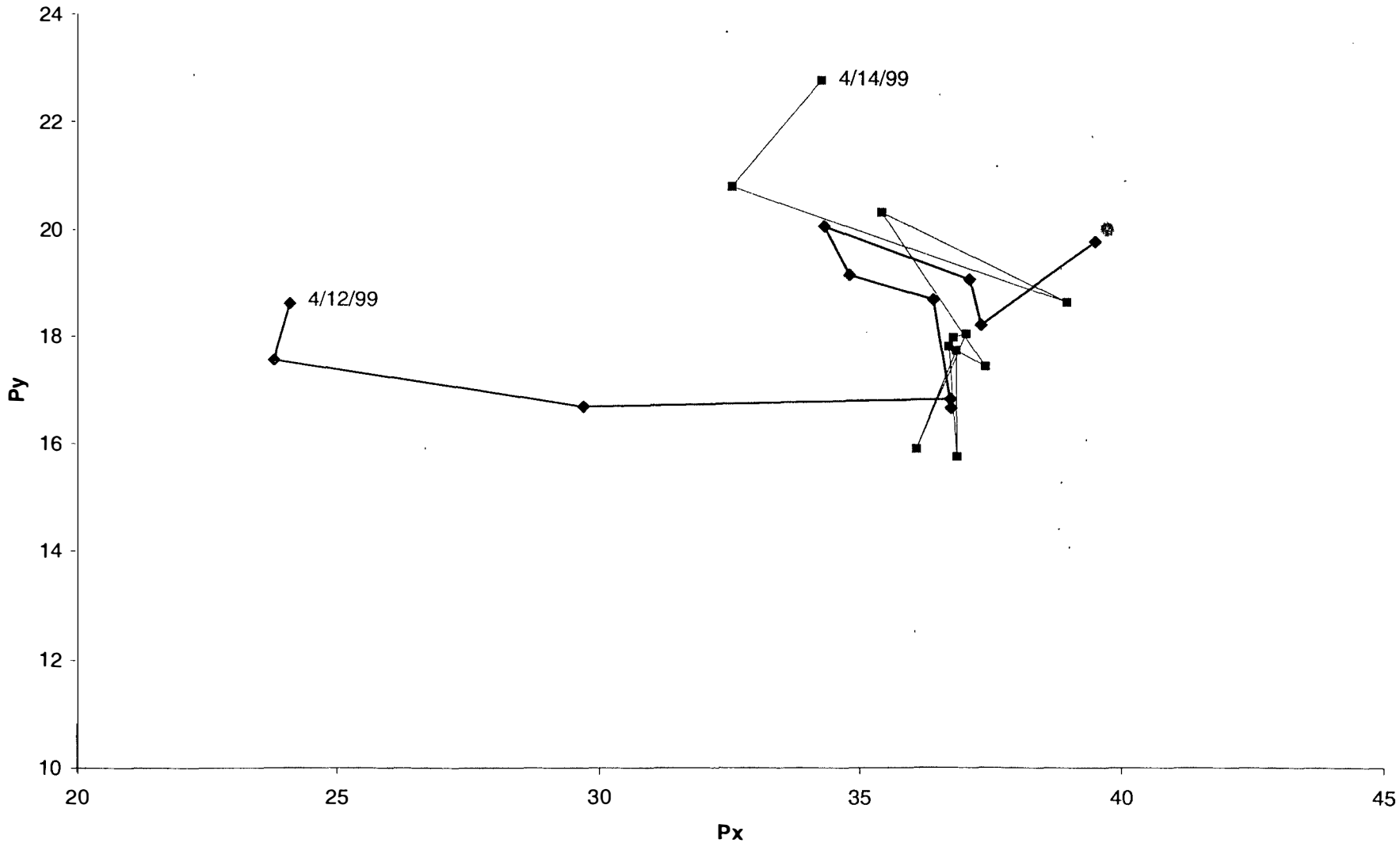


Figure 5: Convergent Experiments



1 Instructions

This is an experiment in the economics of market decision-making. The instructions are simple, and if you follow them carefully and make good decisions, you might earn a considerable amount of money that will be paid to you in cash. In this experiment, we are going to conduct a market in which you will make decisions to buy, sell or hold three different "commodities," called X, Y and "francs" in a sequence of trading periods. Attached to the instructions you will find a sheet labeled "Payoff Table" which will help determine the value to you of any decisions you might make. YOU ARE NOT TO REVEAL THE INFORMATION ON THIS SHEET TO ANYONE. It is your own private information.

Commodity X is traded in market X, and commodity Y is traded in market Y. The currency used in these two markets is the third commodity, francs. All prices are quoted in terms of francs. Your final payoff, however, will be in terms of dollars, which depends upon your end of period "inventories" of X and Y, and your inventory of francs, listed as "cash on hand."

1.1 The Payoff Tables

You will find a table labeled "PAYOFF TABLE" in your folder. You will also find a figure representation of the table. Your dollar payoff is determined at the end of each period by the number of units of X, Y, and francs that you hold as inventory at the end of a period. The dollar amounts can depend upon all three. In some cases the value is determined by two of the variables and the value of the third commodity can be added to the value created by the other two. It is important to note the third commodity might have no value at all, in which case you may not want to hold any of it if you can trade it for something of value.

The values listed in the following EXAMPLE payoff table generally have nothing to do with the values in your actual payoff tables, but they should help you read the actual table. In the EXAMPLE TABLE, values depend upon the final holding of only two of the commodities α, β while the third, γ, is worth nothing. Of course α and β can be any pair of the three commodities (X, Y and francs). Suppose that you have decided to hold 8 units of α, 14 units of β, and 4 units of γ. Your dollar payoff in this case is determined as follows. Notice first that the 4 units of the third commodity are worth

nothing in this case, so these units contribute nothing to your dollar payoff. Now, find the row corresponding to 8 units of α and the column corresponding to 14 units of β. Read the entry for the total payoff, 800. This would be your dollar payoff for that period. If you held 9 units of α and 14 units of β, your total payoff would be 1200 instead of 800.

The value per units of γ is listed at the bottom of the table. If this is not zero, then it should be added to the computations from the table to get your period payoff.

In some payoff tables, the payoffs are not continuous. Namely, it might take several units of a commodity to reach a higher level of payment. That is, **the payoff chart requires that the full number of units be achieved before the higher payoff is attained and units that fall short of that might add nothing to your payoff**. The interval payoffs indicate the effect on your payoff of α one-interval change in holdings. For example, given 14 units of β, the 8th unit of α adds nothing to your payoff but the 9th adds 400 (= 1200 - 800) to your payoff.

The RECORD SHEET is filled out as follows. Suppose that 8 units of α, 14 units of β, and 4 units of g are held. Suppose further that α represents X, β represents Y, and γ is francs. Column 1 would show 4 francs inventory holding (cash on hand). Column 2 would show 8 units of X inventory holding, and column 3 would show 14 units of Y inventory holding. Column 4 would show a payoff of 800 for the period.

The figure attached after your payoff table is a graphical representation of the table. For the EXAMPLE, the units of α are on the horizontal axis, the units of β are on the vertical axis, and the units of γ are shown at the bottom of the figure. The value shown there would be added to the value computed from the figure for the other two. Of course, if the third commodity were worth nothing to you, then holding it would add nothing to your total.

1.2 Endowments

At the beginning of each period of the experiment, you will be given a one time endowment of [ ] francs cash on hand, [ ] units of X, and [ ] units of Y. You can see them on your screen.

1.3 How the System Works

In order to buy anything, you must have enough cash in francs on hand. Unless you are endowed with francs at the beginning of the period, you can only acquire francs by selling something. You must sell one thing in order to get the francs to buy the other. If you do not have enough cash to buy units in market X, you must then sell some of your units in market Y in order to obtain cash to buy in market X. How many units of commodities an individual would want to sell or how many units of francs the individual would want to retain depends upon the individual.

1.4 Time and the End of the Experiment

The market system is organized as follows. The market will open in a series of trading periods, each of which lasts for [ ] minutes. The last period will be announced after it has closed.

1.5 Some Notes

No talking;
No "flashing"(i.e., rapid cancellation);
No advantage to grabbing typos;
Beware of "sliding" (i.e., low bids (high asks) when sellers (buyers) are rapidly accepting);
Beware of "switching" (e.g., bids of 1 for 5 units when people have been bidding 5 for 1 unit).

17 - 20	1200	1200	1200	1200	1400	1400
13 - 16	800	800	800	1200	1200	1400
9 - 12	600	600	800	800	1200	1200
5 - 8	200	600	600	800	800	1200
1 - 4	200	200	600	600	800	1200
0	0	200	200	600	800	1200
β / α	**0**	**1 - 4**	**5 - 8**	**9 - 12**	**13 - 16**	**17 - 20**

(value of γ) = 0 per unit

Table 1: Payoff Table

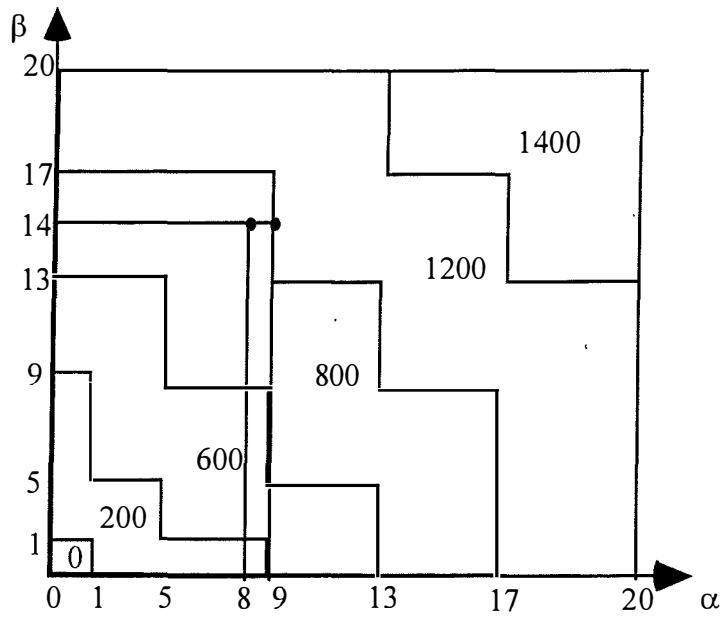


Figure B.1: Graphical representation of the Payoff Table.