NEW RESULTS ON THE THEORY OF MORPHOLOGICAL FILTERS BY RECONSTRUCTION

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Abstract—This paper treats the problem of establishing bounds for the morphological filter by reconstruction class. Morphological filters by reconstruction, which are composed of openings and closings by reconstruction, are useful filters for image processing because they do not introduce discontinuities. The main contributions of this paper are: (a) To establish when the combination of openings by reconstruction (or, respectively, of closings by reconstruction) is an opening by reconstruction (respectively a closing by reconstruction). (b) To establish, for any filter by reconstruction, upper and lower bounds that are, respectively, a closing by reconstruction and an opening by reconstruction. In addition, the paper investigates certain aspects of filters by reconstruction that possess a robustness property called strong property. Some dual and equivalent forms are introduced for a family of multi-level filters recently introduced. A significant side-result is to determine some instances of connected openings composed by openings and closings by reconstruction that are *not* openings by reconstruction (similarly for closings).

1. INTRODUCTION

Mathematical morphology[1–5] concerns the application of set theory concepts to image analysis. Filters by reconstruction form an important class of morphological connected filters[6–8] that have proven to be very useful for image processing.[9, 10] The reason is that they cannot introduce discontinuities and, therefore, they preserve the shapes observed in an input image. In addition, we can mention that there exist very efficient algorithms based on waiting queues for the computation of filters by reconstruction.[11–13]

The problems of establishing the conditions of bound existence and of determining the expressions for their computation are major topics of the morphological filtering theory. In the literature, bounds on morphological filters have been studied and proposed.[14, 15] In Reference (16), a general theory on morphological bounds was proposed.

The problem addressed by our paper is different from that treated in the above-mentioned works. We will investigate the bound computation problem within the filter by reconstruction class. Thus, the bounds in which we are interested are filters by reconstruction and, in particular, openings and closings by reconstruction (which are the elementary pieces of the filter by reconstruction class). The two important contributions of our paper are the following:

- To establish when the combination of openings by reconstruction (or, respectively, of closings by reconstruction) is an opening by reconstruction (respectively a closing by reconstruction).
- To establish, for any filter by reconstruction, upper and lower bounds that are, respectively, a closing by reconstruction and an opening by reconstruction.

Regarding the first item, this problem was addressed in Reference (6), where some cases were studied. This paper completes those results and establishes all cases.

The second item has not previously been treated in the literature to our knowledge. Our work will show that there always exist such bounds for any filter by reconstruction (besides of course trivial cases). The computation of these bounds by some simple rules will also be described. Besides presenting the expressions corresponding to fine bounds, this work will present some coarser bounds that can be useful since they are extremely easy to compute. Some preliminary results of this work were presented in Reference (17).

In addition, this paper investigates the strong property (14) possessed by some morphological filters, in particular, by some morphological filters by reconstruction. The motivation of this study is the "classical" result that establishes that a strong filter can be expressed as the sequential composition of an opening γ followed by a closing φ, or *vice versa.* Certain dual forms for a family of multi-level filters are introduced, which complete some previous results of Reference (8). An important *side-result* is to determine that openings (or, respectively, closings) composed by openings and closings by reconstruction are not necessarily an opening (respectively a closing) by reconstruction. This can be deduced after studying the filter behavior on connected components.[18]

Section 2 provides some background on mathematical morphology. Composition laws for openings and closings by reconstruction are treated in Section 3. The filter by reconstruction bounds are presented in Section 4. The strong property is discussed in Section 5, where some dual and equivalent forms of strong filters are introduced.

2. BACKGROUND

2.1. Basic definitions

A *minimal* set of basic notions on mathematical morphology can be the following. General references are References (1–5, 15, 19–21).

- Mathematical morphology deals with increasing mappings defined on a complete latice.[22, 4] In a complete lattice there exists an ordering relation, and two basic operations called infimum and supremum (denoted by $\bigwedge$ and $\bigvee$, respectively).[22, 4]
- A transformation ψ is increasing if and only if it preserves ordering.
- A transformation ψ is increasing if and only if it preserves ordering.
- A transformation ψ is idempotent if and only if $\psi\psi = \psi$.
- A transformation ψ is a morphological filter if and only if it is increasing and idempotent.
- An opening (often denoted by γ) is an antiextensive morphological filter.
- A closing (often denoted by φ) is an extensive morphological filter.

In all theoretical expressions in this paper, we will be working on the lattice $\mathscr{P}(E)$, where E is a given set of points called *space* and $\mathscr{P}(E)$ denotes the set of all subsets of E (i.e. $\mathscr{P}(E) = \{A: A \subseteq E\}$). In other words, inputs and outputs will be supposed to be sets or, equivalently, binary functions. In this lattice, the sup $\bigvee$ and the inf $\bigwedge$ operations are the set union $\bigcup$ and the set intersection $\bigcap$ operations, while the order relation is the set inclusion relation $\subseteq$. Even though we will work on the lattice $\mathscr{P}(E)$, results are extendable for gray-level functions by means of the so-called flat operators.[3, 19, 23–25]

2.2. Connected filters

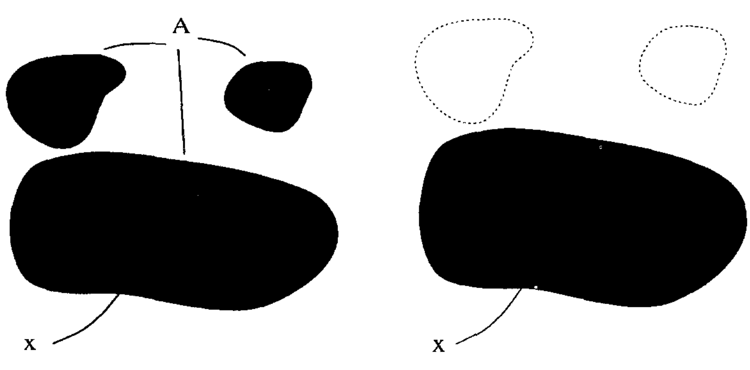
Connected operators ensure that if two points x, y in E are connected in A or in A^c (foreground and background are regarded symmetrically), then the pair x, y will be connected either in the output set or in the complement of the output set. This forces connected operators to process grains (connected components of the foreground) and pores (connected components of the background) in an all-or-nothing way. If a grain is to be removed (i.e. the grain is modified) then all its component points will be removed. Similarly for pores: either they are filled or they are left unchanged.

Connectivity is normally introduced in mathematical morphology by means of the point opening γ_x. The opening γ_x is the operation that extracts the connected component which a point x belongs to References (26, 6). An example is shown in Fig. 1.

Definition 1. Let E be a space equipped with γ_x and T a complete lattice. The flat zones of a function $f: E \to T$ are defined as the (largest) connected components of points $x \in E$ with the same function value.

Definition 2. An operator ψ is *connected* if and only if it extends the flat zones of the input function.

Definition 2 implies that each flat zone of the input function (image) is contained in a flat zone of the



(a) Input set A (in black) (b) $\gamma_x(A)$

Fig. 1. Connected component extraction. The opening $\gamma_x(A)$ extracts the connected component of A to which x belongs.

output function. Therefore, discontinuities cannot be introduced by flat operators, and shapes are well preserved. For the binary case, an equivalent definition of connected operator is that in Reference (6), which applies only to binary morphology: an operator $\psi : \mathscr{P}(E) \rightarrow \mathscr{P}(E)$ is said to be connected if and only if, for any set A, both set subtractions $A \backslash \psi(A)$ and $\psi(A) \backslash A$ are formed exclusively by connected components of A or of its complement A^c.

Clearly, the class of connected operators is closed under the operations sup, inf and the sequential composition of connected operators.[6]

2.3. *Elementary connected filters: Openings and closings by reconstruction*

This section defines an important group of connected filters, the so-called openings and closings by reconstruction, which are the elementary pieces of the filter by reconstruction class.[6, 10, 8] Openings and closings by reconstruction are defined by means of the concepts of *trivial opening* $\gamma_\circ$ and *trivial closing* $\varphi_\circ$.[26]

Definition 3. Let E be any space. An opening $\gamma_\circ : \mathscr{P}(E) \rightarrow \mathscr{P}(E)$ is a *trivial opening* if for all input sets $A \in \mathscr{P}(E)$:

$$\gamma_\circ(A) = \begin{cases} A, & \text{if } A \text{ satisfies an increasing criterion,} \\ \varnothing, & \text{if } A \text{ does not satisfy the increasing criterion.} \end{cases}$$

Definition 4. Let E be any space. A closing $\varphi_\circ : \mathscr{P}(E) \rightarrow \mathscr{P}(E)$ is a *trivial closing* if for all $A \in \mathscr{P}(E)$

$$\varphi_\circ(A) = \begin{cases} E & \text{if } A \text{ satisfies an increasing criterion,} \\ A & \text{if } A \text{ does not satisfy the increasing criterion.} \end{cases}$$

Definition 5. Let E be a space equipped with γ_x. An opening $\tilde{\gamma} : \mathscr{P}(E) \rightarrow \mathscr{P}(E)$ (or respectively a closing $\tilde{\varphi} : \mathscr{P}(E) \rightarrow \mathscr{P}(E)$) is an *opening by reconstruction* (respectively *closing by reconstruction*) if and only if

$$\tilde{\gamma} = \bigvee_{x \in E} \gamma_\circ \gamma_x \left(\text{respectively } \tilde{\varphi} = \bigwedge_{x \in E} \varphi_\circ \varphi_x \right),$$

where $\gamma_\circ$ is a trivial opening (respectively $\varphi_\circ$ is a trivial closing).

Thus, the output of an opening by reconstruction $\tilde{\gamma}$ performed on an input set A is the set formed by all connected components of A that satisfy the increasing criterion of the trivial opening $\gamma_\circ$ that is associated with $\tilde{\gamma}$.

From the definitions of $\tilde{\gamma}$ and $\tilde{\varphi}$, it is clear that (1) the fact that $\tilde{\gamma}$ and $\tilde{\varphi}$ are connected implies that $\tilde{\gamma}$, which is anti-extensive, exclusively removes grains and that $\tilde{\varphi}$, which is extensive, exclusively fills pores; and (2) $\tilde{\gamma}$ treats each grain and $\tilde{\varphi}$ treats each pore *independently* from the rest of grains and pores, respectively, of the input set.[8]

In this paper we will not discuss implementation issues. Let us mention that connected operators are normally implemented by means of the so-called *geodesic operators*, which are described in References (27, 13, 21).

2.4. *Filter by reconstruction definition*

In our work the filter by reconstruction class has openings $\tilde{\gamma}$ and closings $\tilde{\varphi}$ by reconstruction as its only building pieces.

Definition 6. Any composition of openings $\tilde{\gamma}$ and closings $\tilde{\varphi}$ by reconstruction under the inf, the sup or the sequential composition operators that is idempotent is called a *filter by reconstruction.*

Notice that not all compositions of $\tilde{\gamma}$ and $\tilde{\varphi}$ are filters. (Some examples are given in Reference 8.)

3. COMPOSITION LAWS OF OPENINGS AND CLOSINGS BY RECONSTRUCTION

It is well known that in general the compositions of openings (or respectively closings) is not an opening (respectively a closing). Only the sup of openings is again an opening, and the inf of closings is a closing.[14]

The following new theorem completes the previous results and determines the trivial opening or trivial closing associated with the resulting operation.

Theorem 1. Let $\tilde{\gamma}_a$ and $\tilde{\gamma}_b$ be two members of $\mathscr{C}_{\tilde{\gamma}}$ whose associated trivial openings are $\gamma_{\circ_a}$ and $\gamma_{\circ_b}$. Then,

(1) $\tilde{\gamma}_a \tilde{\gamma}_b$,

(2) $\tilde{\gamma}_b \tilde{\gamma}_a$,

(3) $\tilde{\gamma}_a \bigwedge \tilde{\gamma}_b$ and

(4) $\tilde{\gamma}_a \bigvee \tilde{\gamma}_b$

are all openings by reconstruction whose associated trivial openings are, in cases (1)–(3), equal to $\gamma_{\circ_a} \bigwedge \gamma_{\circ_b}$, and in case (4) equal to $\gamma_{\circ_a} \bigvee \gamma_{\circ_b}$, where $\gamma_{\circ_a}$ and $\gamma_{\circ_b}$ denote, respectively, the trivial openings associated with $\tilde{\gamma}_a$ and $\tilde{\gamma}_b$.

Proof. Let $A \in \mathscr{P}(E)$. Let us consider the case $\tilde{\gamma}_a \tilde{\gamma}_b$ (the other cases are similar). In $\tilde{\gamma}_b(A)$, the opening $\tilde{\gamma}_b$ has removed the grains of A that do not satisfy the increasing criterion of the trivial opening $\gamma_{\circ_b}$ associated with $\tilde{\gamma}_b$. And in $\tilde{\gamma}_a \tilde{\gamma}_b(A)$, the opening $\tilde{\gamma}_a$ removes the grains of $\tilde{\gamma}_b(A)$ (which are grains of A[8]) that do not satisfy the increasing criterion of the trivial opening $\gamma_{\circ_a}$ associated with $\tilde{\gamma}_a$. Therefore, $\tilde{\gamma}_a \tilde{\gamma}_b(A)$ contains only all grains of A that satisfy both increasing criterion associated $\Rightarrow \tilde{\gamma}_a \tilde{\gamma}_b$ is an opening by reconstruction whose associated trivial opening is $\gamma_{\circ_a} \bigwedge \gamma_{\circ_b}$. □

Cases (a) and (b) (sequential composition) were studied in Reference (6), where it was established that

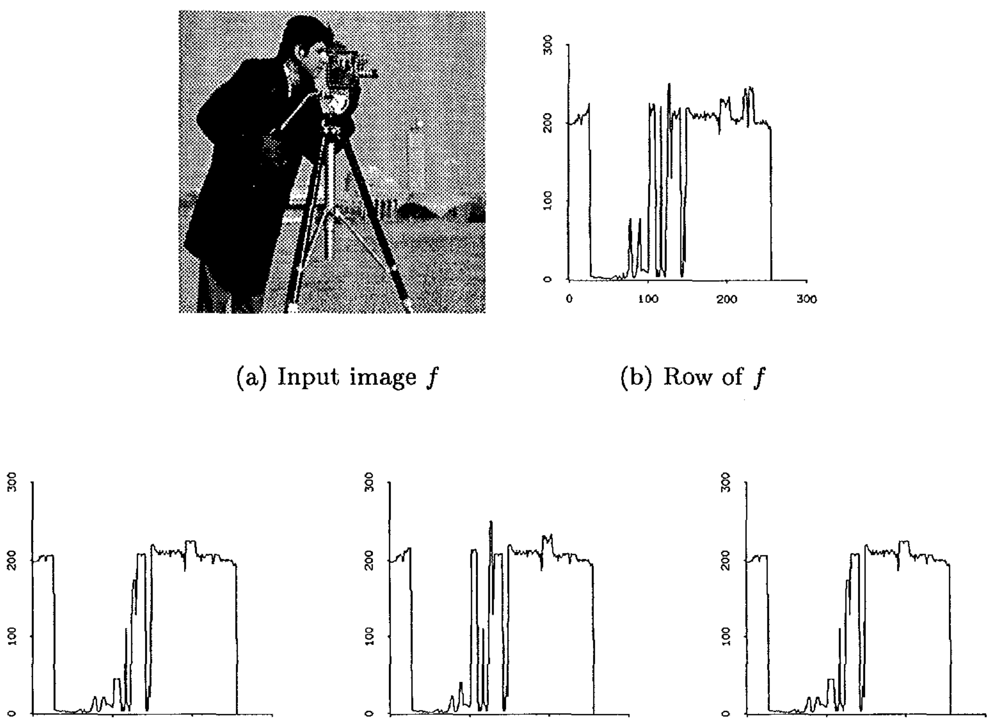
$\tilde{\gamma}_a\tilde{\gamma}_b$ and $\tilde{\gamma}_b\tilde{\gamma}_a$ are openings by reconstruction (in fact, identical). Theorem 1 completes those results and establishes the trivial opening or trivial closing associated with the resulting operation.

Obviously, Theorem 1 applies dually to closings by reconstruction. If $\tilde{\varphi}_a$ and $\tilde{\varphi}_b$ be two members of $\mathscr{C}_{\tilde{\varphi}}$ whose associated trivial closings are $\varphi_{\circ_a}$ and $\varphi_{\circ_b}$, then (1) $\tilde{\varphi}_a\tilde{\varphi}_b$, (2) $\tilde{\varphi}_a\tilde{\varphi}_b$, (3) $\tilde{\varphi}_a \bigvee \tilde{\varphi}_b$, and (4) $\tilde{\varphi}_a \bigwedge \tilde{\varphi}_b$ are all closings by reconstruction whose associated trivial closings are, in cases (1)–(3), equal to $\varphi_{\circ_a} \bigvee \varphi_{\circ_b}$ and, in case (4), equal to $\varphi_{\circ_a} \bigwedge \varphi_{\circ_b}$.

Example 1. Assume we operate on the $\mathscr{P}(E)$ lattice, where E is a subset of $\mathbf{Z}^2$. Let $\tilde{\gamma}_a$ be an opening by reconstruction that removes those grains whose area is less than 10 pixels and whose x-direction projection length is less than 5 pixels. Let $\tilde{\gamma}_b$ be an opening by reconstruction that removes those grains whose area is less than 10 pixels and whose y-direction projection length is less than 5 pixels.

- We have that (1) $\tilde{\gamma}_a\tilde{\gamma}_b$, (2) $\tilde{\gamma}_a\tilde{\gamma}_b$, (3) $\tilde{\gamma}_a \bigwedge \tilde{\gamma}_b$ are all the opening by reconstruction that removes those grains whose area is less than 10 pixels and that either (a) its x-direction projection length or (b) its y-direction projection length is less than 5 pixels. That is, the trivial opening associated with the resulting opening is the inf of the trivial openings associated with $\tilde{\gamma}_a$ and $\tilde{\gamma}_b$.
- On the other hand, $\tilde{\gamma}_a \bigvee \tilde{\gamma}_b$ is the opening by reconstruction that removes those grains whose area is less than 10 pixels and that both its x-direction projection length and its y-direction projection length are less than 5 pixels. Therefore the trivial opening associated with the resulting opening is the sup of the trivial openings associated with $\tilde{\gamma}_a$ and $\tilde{\gamma}_b$.

For openings and closings that are not by reconstruction, the results of Theorem 1 are true only in the restricted case when there exists an ordering among the openings (or respectively closings). (That is, when the openings and closings constitute respectively a granulometry and an antigranulometry.[2] As Theorem 1 states, for openings and closings by reconstruction such a requirement is not necessary. Figure 2 gives an example of an opening by reconstruction that is the infimum of two openings by reconstruction $\tilde{\gamma}_a$ and $\tilde{\gamma}_b$ between which there exists no ordering (i.e. $\tilde{\gamma}_a \not\leq \tilde{\gamma}_b \not\leq \tilde{\gamma}_a$). In this example $\tilde{\gamma}_a$ uses as increasing criterion a 1-D horizontal erosion of size 17×1, and $\tilde{\gamma}_b$ uses as increasing criterion a 1-D vertical erosion of size 1×17.



(a) Input image f (b) Row of f

(c) Row of $\tilde{\gamma}_a(f)$ (d) Row of $\tilde{\gamma}_b(f)$ (e) Row of $(\tilde{\gamma}_a \bigwedge \tilde{\gamma}_b)(f)$

Fig. 2. Infimum of openings by reconstruction. *Note*: $\tilde{\gamma}_a$ uses as increasing criterion a 1-D horizontal erosion of size 17×1; $\tilde{\gamma}_b$ uses as increasing criterion a 1-D vertical erosion of size 1×17. The space is equipped with eight connectivity.

It follows from Theorem 1 that $\tilde{\gamma}_a\tilde{\gamma}_b = \tilde{\gamma}_b\tilde{\gamma}_a$ (as presented in Reference (6)), i.e. the self-composition of elements of $\mathscr{C}_{\tilde{\gamma}}$ is commutative. We can state that each of the classes $\mathscr{C}_{\tilde{\gamma}}$ and $\mathscr{C}_{\tilde{\varphi}}$ forms a semi-group for the sup, the inf, and the sequential composition operations.

Corollary 1. Both the class of openings by reconstruction $\mathscr{C}_{\tilde{\gamma}}$ and the class of closings by reconstruction $\mathscr{C}_{\tilde{\varphi}}$ are closed under the sup the inf and the self-composition operations.

Notice that a composition of members of both families $\mathscr{C}_{\tilde{\gamma}}$ and $\mathscr{C}_{\tilde{\varphi}}$ is not, in general, a member of $\mathscr{C}_{\tilde{\gamma}}$ or $\mathscr{C}_{\tilde{\varphi}}$. For example, the sequential composition of an opening and a closing by reconstruction ($\tilde{\varphi}\,\tilde{\gamma}$ and $\tilde{\gamma}\,\tilde{\varphi}$) are neither an opening nor a closing (except trivial cases when $\tilde{\varphi}$ or $\tilde{\gamma}$ is the identity operator I). We have, however, the following theorem.

Theorem 2. Any sequential composition of elements of $\mathscr{C}_{\tilde{\gamma}}$ and $\mathscr{C}_{\tilde{\varphi}}$ is an idempotent operator.

Proof. From Theorem 1, any sequential composition ψ of $\mathscr{C}_{\tilde{\gamma}}$ and $\mathscr{C}_{\tilde{\varphi}}$ can be written as an alternated sequential composition of openings and closings by reconstruction, i.e. $\psi = \tilde{\varphi}_a\tilde{\gamma}_a \ldots \tilde{\varphi}_z\tilde{\gamma}_z$. Let $A \in \mathscr{P}(E)$, and let us consider the set $\tilde{\varphi}_a\tilde{\gamma}_a \ldots \tilde{\varphi}_z\tilde{\gamma}_z(A)$. Because (a) the openings $\tilde{\gamma}_a, \ldots ,\tilde{\gamma}_z$ have removed all grains of A that do not satisfy the increasing criteria of the trivial openings associated with, and (b) the closings $\tilde{\varphi}_a, \ldots ,\tilde{\varphi}_z$ cannot but extend grains, there are no grains in $\tilde{\varphi}_a\tilde{\gamma}_a \ldots \tilde{\varphi}_z\tilde{\gamma}_z(A)$ that can be removed by $\tilde{\gamma}_a, \ldots ,\tilde{\gamma}_z$. Reasoning in a similar way for pores, we deduce that there are no pores in $\tilde{\varphi}_a\tilde{\gamma}_a \ldots \tilde{\varphi}_z\tilde{\gamma}_z(A)$ that can be filled by $\tilde{\varphi}_a, \ldots ,\tilde{\varphi}_z$. Therefore, $\psi\psi(A) = \tilde{\varphi}_a\tilde{\gamma}_a \ldots \tilde{\varphi}_z\tilde{\gamma}_z(\tilde{\varphi}_a\tilde{\gamma}_a \ldots \tilde{\varphi}_z\tilde{\gamma}_z(A)) = \tilde{\varphi}_a\tilde{\gamma}_a \ldots \tilde{\varphi}_z\tilde{\gamma}_z(A) = \psi(A)$. □

Theorem 2 completes a previous result in Reference (6) that studied alternating compositions of members of granulometries and antigranulometries.

It can be observed that Theorem 1 can have significant implications concerning implementation issues because equivalences between fundamentally different ways to implement the same operator are established. For example, from Theorem 1 we have that

$$\tilde{\gamma}_a\tilde{\gamma}_b\tilde{\gamma}_c\tilde{\gamma}_d = \tilde{\gamma}_a \bigwedge \tilde{\gamma}_b \bigwedge \tilde{\gamma}_c \bigwedge \tilde{\gamma}_d. \tag{1}$$

Whereas the computation of the left-hand side of expression (1) is sequential (for example it is necessary to wait for the result of $\tilde{\gamma}_d$ before applying $\tilde{\gamma}_c$), the computation nature of the right-hand side is clearly parallel. This could be an important issue in the case that the opening by reconstruction of expression (1) is intended to be implemented in a parallel computer: the computation of the four component openings $\tilde{\gamma}_a$, $\tilde{\gamma}_b$, $\tilde{\gamma}_c$ and $\tilde{\gamma}_d$ could be performed simultaneously.

4. NEW BOUNDS ON FILTERS BY RECONSTRUCTION

Given a filter by reconstruction, it is important to know whether it can be bounded by an opening and a closing by reconstruction. The following theorem answers this question.

Theorem 3. Let ψ be a filter by reconstruction. We have that:

(a) The operator obtained substituting all closings by reconstruction (or respectively all openings by reconstruction) in ψ by the identity operator I is an opening by reconstruction $\tilde{\gamma}_\psi$ (respectively a closing by reconstruction $\tilde{\varphi}_\psi$).
(b) The filter by reconstruction ψ is bounded by $\tilde{\gamma}_\psi$ and $\tilde{\varphi}_\psi$:

$$\tilde{\gamma}_\psi \le \psi \le \tilde{\varphi}_\psi.$$

Proof. Let ψ' be the operator obtained by substituting any closing by reconstruction in ψ by the identity operator I. Clearly, $\psi' \le \psi$ and ψ' is an opening by reconstruction because it is a composition of openings by reconstruction (from Theorem 1). Similarly, if ψ'' is the operator obtained by substituting any opening by reconstruction in ψ by the identity operator I, then $\psi'' \ge \psi$ and ψ'' is a closing by reconstruction. Therefore, $\psi' = \tilde{\gamma}_\psi \le \psi \le \tilde{\varphi}_\psi = \psi''$. □

Note: Theorem 3 also applies when ψ is not idempotent.

The following example illustrates the application of Theorem 3 for computing the bounds of a (quite complex) filter by reconstruction.

Example 2. Let ψ be equal to $((\tilde{\varphi}_a\tilde{\gamma}_b\tilde{\varphi}_c \bigvee \tilde{\gamma}_d) \bigwedge \tilde{\varphi}_e\tilde{\gamma}_f)\tilde{\varphi}_g\tilde{\gamma}_h\tilde{\varphi}_i$. The operator ψ is bounded by the following opening $\tilde{\gamma}_\psi$ and closing $\tilde{\varphi}_\psi$ by reconstruction.

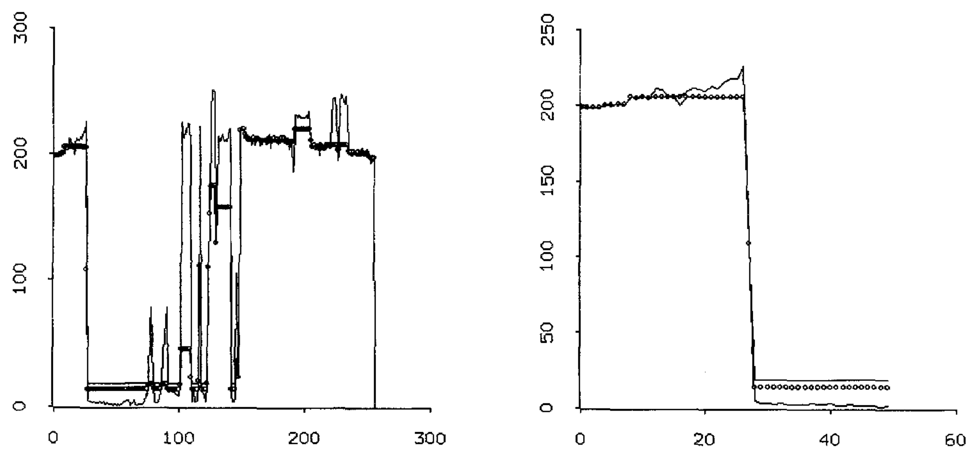
- $\tilde{\gamma}_\psi = ((I\tilde{\gamma}_b I \bigvee \tilde{\gamma}_d) \bigwedge I\tilde{\gamma}_f)I\tilde{\gamma}_h I = ((\tilde{\gamma}_b \bigvee \tilde{\gamma}_d) \bigwedge \tilde{\gamma}_f)\tilde{\gamma}_h$,
- $\tilde{\varphi}_\psi = ((\tilde{\varphi}_a I \tilde{\varphi}_c \bigvee I) \bigwedge \tilde{\varphi}_e I)\tilde{\varphi}_g I \tilde{\varphi}_i = ((\tilde{\varphi}_a\tilde{\varphi}_c) \bigwedge \tilde{\varphi}_e)\tilde{\varphi}_g\tilde{\varphi}_i$.

That is, $\tilde{\gamma}_\psi = ((\tilde{\gamma}_b \bigvee \tilde{\gamma}_d) \bigwedge \tilde{\gamma}_f)\tilde{\gamma}_h \le \psi \le ((\tilde{\varphi}_a\tilde{\varphi}_c) \bigwedge \tilde{\varphi}_e)\tilde{\varphi}_g\tilde{\varphi}_i = \tilde{\varphi}_\psi$.

In Example 2, $\tilde{\gamma}_\psi = ((\tilde{\gamma}_b \bigvee \tilde{\gamma}_d) \bigwedge \tilde{\gamma}_f)\tilde{\gamma}_h$ and $((\tilde{\varphi}_a\tilde{\varphi}_c) \bigwedge \tilde{\varphi}_e)\tilde{\varphi}_g\tilde{\varphi}_i = \tilde{\varphi}_\psi$ are, respectively, an opening and a closing by reconstruction (Theorem 1).

Figure 3 shows the bounds computed as indicated by Theorem 3 corresponding to the filter by reconstruction $\tilde{\gamma}_b\tilde{\varphi}\tilde{\gamma}_a$, where $\tilde{\gamma}_a$ and $\tilde{\gamma}_b$ are openings by reconstruction and $\tilde{\varphi}$ is a closing by reconstruction. *Note*: $\tilde{\gamma}_a$ uses as increasing criterion a 1-D horizontal erosion of size 33×1; $\tilde{\gamma}_b$ uses as increasing criterion a 1-D vertical erosion of size 1×33; and $\tilde{\varphi}$ uses as increasing criterion a square dilation of size 33×33.

There are possibly other morphological filters (i.e. increasing and idempotent operators) that are finest bounds than those given in Theorem 3 (see Reference (14) for a study of the morphological filter lattice).



(a) Plot of row of $\tilde{\varphi}(f)$, $\tilde{\gamma}_b\tilde{\varphi}\tilde{\gamma}_a(f)$ and $\tilde{\gamma}_b\tilde{\gamma}_a(f)$

(b) Detail of (a)

Fig. 3. Bounds. For the input image shown in Fig. 2(a), part (a) displays $\tilde{\gamma}_b\tilde{\varphi}\tilde{\gamma}_a(f)$ (little circles) and its two corresponding upper and lower bounds: $\tilde{\varphi}(f)$ and $\tilde{\gamma}_b\tilde{\gamma}_a(f)$. Part (b) shows a detail of (a). *Note*: $\tilde{\gamma}_a$ uses as increasing criterion a 1-D horizontal erosion of size 33×1; $\tilde{\gamma}_b$ uses as increasing criterion a 1-D vertical erosion of size 1×33; $\tilde{\varphi}$ uses as increasing criterion a square dilation of size 33×33. The space is equipped with eight connectivity.

However, notice that the bounds given by Theorem 3 *are*, respectively, an opening and a closing by reconstruction.

Two coarser, but extremely easy to compute, bounds that are, respectively, an opening and a closing by reconstruction are the following. For a filter by reconstruction ψ, let us call $\{\tilde{\gamma}_a, \tilde{\gamma}_b, \dots\}$ and $\{\tilde{\varphi}_a, \tilde{\varphi}_b, \dots\}$ be, respectively, the set of openings and closings by reconstruction that compose ψ. Then, $\bigwedge\{\tilde{\gamma}_a, \tilde{\gamma}_b, \dots\}$ and $\bigvee\{\tilde{\varphi}_a, \tilde{\varphi}_b, \dots\}$ are, respectively, an opening and a closing by reconstruction that bound ψ:

$$\bigwedge\{\tilde{\gamma}_a, \tilde{\gamma}_b, \dots\} \leq \psi \leq \bigvee\{\tilde{\varphi}_a, \tilde{\varphi}_b, \dots\},$$

where $\bigwedge\{\tilde{\gamma}_a, \tilde{\gamma}_b, \dots\}$ (or respectively $\bigvee\{\tilde{\varphi}_a, \tilde{\varphi}_b, \dots\}$) are $\tilde{\gamma}_a\bigwedge\tilde{\gamma}_b\bigwedge\dots$ (respectively $\tilde{\varphi}_a\bigvee\tilde{\varphi}_b\bigvee\dots$).

Example 3. Let ψ be equal to $((\tilde{\varphi}_a\tilde{\gamma}_b\tilde{\varphi}_c\bigvee\tilde{\gamma}_d)\bigwedge\tilde{\varphi}_e\tilde{\gamma}_f)\tilde{\varphi}_g\tilde{\gamma}_h\tilde{\varphi}_i$ (as in Example 2).

Two "coarse" openings and closings that bound ψ are:

- $\tilde{\gamma}' = \tilde{\gamma}_b\bigwedge\tilde{\gamma}_d\bigwedge\tilde{\gamma}_f\bigwedge\tilde{\gamma}_h$,
- $\tilde{\varphi}' = \tilde{\varphi}_a\bigvee\tilde{\varphi}_c\bigvee\tilde{\varphi}_e\bigvee\tilde{\varphi}_g\bigvee\tilde{\varphi}_i$.

Clearly, those bounds are less fine than those computed in Example 2, i.e.

$$\begin{aligned}\tilde{\gamma}' &= \tilde{\gamma}_b\bigwedge\tilde{\gamma}_d\bigwedge\tilde{\gamma}_f\bigwedge\tilde{\gamma}_h \leq \tilde{\gamma}_\psi = ((\tilde{\gamma}_b\bigvee\tilde{\gamma}_d)\bigwedge\tilde{\gamma}_f)\tilde{\gamma}_h \leq \psi \\ &\leq ((\tilde{\varphi}_a\tilde{\varphi}_c)\bigwedge\tilde{\varphi}_e)\tilde{\varphi}_g\tilde{\varphi}_i \\ &\leq \tilde{\varphi}' = \tilde{\varphi}_a\bigvee\tilde{\varphi}_c\bigvee\tilde{\varphi}_e\bigvee\tilde{\varphi}_g\bigvee\tilde{\varphi}_i.\end{aligned}$$

Notice that, in Example 3 we have that $\tilde{\gamma}_b\bigwedge\tilde{\gamma}_d\bigwedge\tilde{\gamma}_f\bigwedge\tilde{\gamma}_h = \tilde{\gamma}_b\tilde{\gamma}_d\tilde{\gamma}_f\tilde{\gamma}_h$.

5. STRONG FILTERS BY RECONSTRUCTION

5.1. The strong-property: Definitions

Matheron[28, 29, 14] has investigated the following expressions for increasing mappings ψ: $\psi(I\bigwedge\psi)$ and $\psi(I\bigvee\psi)$. (*Note*: all concepts presented in this section have been established by Matheron.) Using the fact that for increasing operators ψ and ψ_i, $\psi(\bigwedge_i\psi_i) \leq \bigwedge_i(\psi\psi_i)$ and $\psi(\bigvee_i\psi_i) \geq \bigvee_i(\psi\psi_i)$, it can be shown that

$$\psi(I\bigwedge\psi) \leq \psi\bigwedge\psi\psi \leq \psi, \tag{2}$$

$$\psi(I\bigvee\psi) \geq \psi\bigvee\psi\psi \geq \psi. \tag{3}$$

However, there exist some mappings for which the first or the second (or both) of the previous sets of inequalities (2) and (3) is an equality.

Definition 7. Let E be any space, and let ψ: $\mathscr{P}(E) \to \mathscr{P}(E)$ be an increasing mapping.

- ψ is an *overfilter* (or respectively an *underfilter*) if and only if $\psi\psi \geq \psi$ (respectively $\psi\psi \leq \psi$).
- ψ is an $\bigwedge$*-overfilter* (or respectively a $\bigvee$*-underfilter*) if and only if $\psi = \psi(I\bigwedge\psi)$ (respectively $\psi = \psi(I\bigvee\psi)$).
- ψ is an $\bigwedge$*-filter* (or respectively a $\bigvee$*-filter*) if and only if ψ is an $\bigwedge$-overfilter and an underfilter (respectively a $\bigvee$-underfilter and an overfilter).

Next, the important concept of *strong filter* is defined. As stated in Corollary 2 below, strong filters are robust in the sense that input variations (such as, for example, noise) within certain boundaries cause no variation in the output.

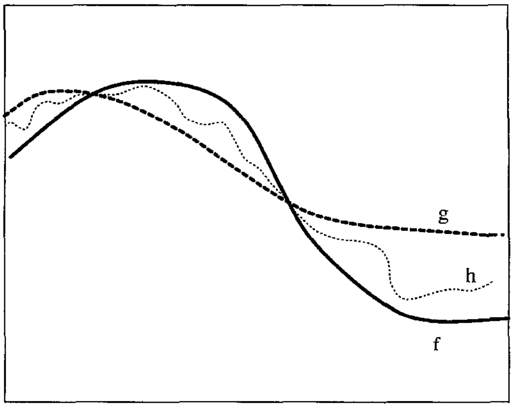


Fig. 4. Strong filter example. If f is an input function and $g = \psi(f)$, where ψ is a strong filter, for any function h between f and g it is true that $\psi(h) = \psi(f) = g$.

Definition 8. A filter ψ is *strong* if and only if ψ is both an $\bigwedge$-filter and a $\bigvee$-filter, i.e., $\psi = \psi(I \bigwedge \psi) = \psi(I \bigvee \psi)$.

Corollary 2. Let ψ be a filter from $\mathscr{P}(E)$ to $\mathscr{P}(E)$. If ψ is strong, then for all $A, B \in \mathscr{P}(E)$: $A \bigwedge \psi(A) \leq B \leq A \bigvee \psi(A) \Rightarrow \psi(A) = \psi(B)$.

Corollary 2 can be interpreted as follows: if A is an uncorrupted input signal, then a strong filter ψ computes the same output for corrupted signals whose values are within the boundaries $A \bigwedge \psi(A)$ and $A \bigvee \psi(A)$. Figure 4 illustrates this for functions.

It can be observed that $\varphi\gamma$ (or respectively $\gamma\varphi$) is an $\bigwedge$-filter (respectively $\bigvee$-filter). In fact the implication in the other sense is also true; i.e. all mappings that are $\bigwedge$-filters (or respectively $\bigvee$-filters) can be expressed in the form of $\varphi\gamma$ (respectively $\gamma\varphi$).

5.2. *Dual and equivalent forms for strong filters by reconstruction*

As commented in the previous section, Matheron established that an $\bigwedge$-filter (or respectively a $\bigvee$-filter) can be expressed in the form of $\varphi\gamma$ (respectively $\gamma\varphi$).[14]

The following "classical" theorem establishes the strong property of alternating filters by reconstruction.

Theorem 4. Let $\tilde{\gamma}$ and $\tilde{\varphi}$ be, respectively, an opening and a closing by reconstruction. Then, the filters $\tilde{\varphi}\tilde{\gamma}$ and $\tilde{\gamma}\tilde{\varphi}$ are strong.[26, 6]

(*Note*: Theorem 4 is a smaller (and simpler) version of that in References (26, 6).)

Since $\tilde{\varphi}\tilde{\gamma}$ and $\tilde{\gamma}\tilde{\varphi}$ are strong (i.e. they are both an $\bigwedge$-filter and a $\bigvee$-filter), we can expect to find an equivalent expression for them in the dual form of, respectively, $\gamma\varphi$ and $\gamma\varphi$. A well-known result is the following: ψ is a strong filter if

$$\psi = (I \bigvee \psi)\,(I \bigwedge \psi) = (I \bigwedge \psi)\,(I \bigvee \psi). \tag{4}$$

Therefore, if the alternating filter $\tilde{\varphi}\tilde{\gamma}$ is strong, then we know that

$$\tilde{\varphi}\tilde{\gamma} = (I \bigvee \tilde{\varphi}\tilde{\gamma})\,(I \bigwedge \tilde{\varphi}\tilde{\gamma}) = (I \bigwedge \tilde{\varphi}\tilde{\gamma})\,(I \bigvee \tilde{\varphi}\tilde{\gamma}), \tag{5}$$

where $(I \bigvee \tilde{\varphi}\tilde{\gamma})$ is a closing, and $(I \bigwedge \tilde{\varphi}\tilde{\gamma})$ is an opening. As expected, we can express $\tilde{\varphi}\tilde{\gamma}$ as the composition of either a closing followed by an opening, or an opening followed by a closing.

In fact, from expression (5) we can get

$$\tilde{\varphi}\tilde{\gamma} = \tilde{\varphi}(I \bigvee \tilde{\varphi}\tilde{\gamma})\,(I \bigwedge \tilde{\varphi}\tilde{\gamma}) = \tilde{\varphi}(I \bigwedge \tilde{\varphi}\tilde{\gamma}). \tag{6}$$

Some questions arise then:

(a) Expressions (5) and (6) give us a way to decompose the strong filter $\tilde{\varphi}\tilde{\gamma}$. Can we obtain a similar result concerning more complicated families of filters composed of alternating filters by reconstruction?
(b) It is well known that $I \bigwedge \tilde{\varphi}\tilde{\gamma}$ is an opening. Is in fact $\tilde{\gamma}$ equal to $(I \bigwedge \tilde{\varphi}\tilde{\gamma})$? Furthermore, is the opening $(I \bigwedge \tilde{\varphi}\tilde{\gamma})$ an opening *by reconstruction*?

About question (a), we will answer affirmatively by obtaining a similar form concerning a multi-level family of strong filters recently introduced in References (10, 7, 8). First, Theorem 5 presents this family; then, the new Theorem 6 establishes the dual and equivalent form.

Theorem 5. Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in \{1, \ldots, n\}$, be, respectively, a granulometry and an antigranulometry by reconstruction. Then $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$ and $\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i$ are strong filters.[8]

Figure 5 shows a simple one-dimensional example of this type of filter. A gray-level case is displayed in Fig. 6.

Theorem 6. Let $\{\tilde{\gamma}_i\}$ and $\{\tilde{\varphi}_i\}$, where $i \in \{1, \ldots n\}$, be, respectively, a granulometry and an antigranulometry by reconstruction. Then

$$\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i = \tilde{\varphi}_{i_0}\left(I \bigwedge \left(\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i\right)\right), \quad i_0 \in \{1, \ldots, n\}.$$

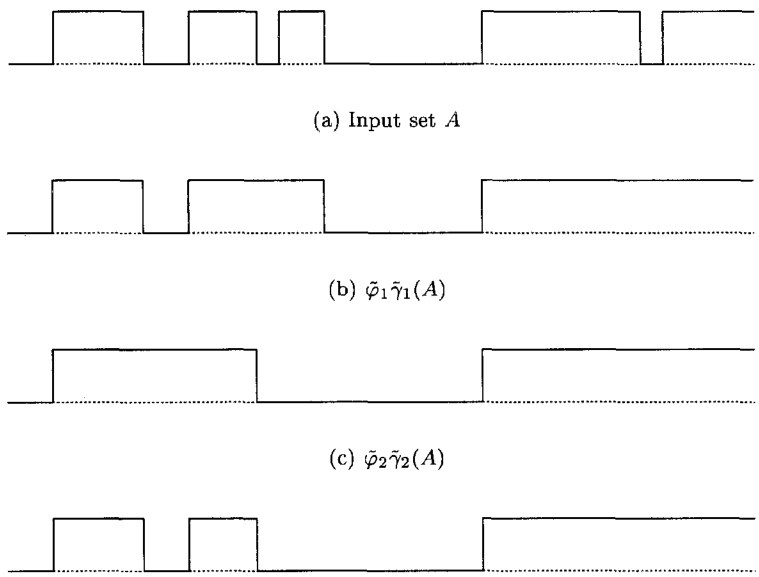
Proof. We have that $\tilde{\varphi}_{i_0}(I \bigwedge (\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)) = \tilde{\varphi}_1 \bigwedge (\tilde{\varphi}_{i_0} (\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i)) = \tilde{\varphi}_{i_0} \bigwedge (\bigwedge_{i=1}^{n} \tilde{\varphi}_{i_0}\tilde{\varphi}_i\tilde{\gamma}_i) = \tilde{\varphi}_{i_0} \bigwedge (\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i) = \bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$, where $i_0 \in \{1, \ldots, \mathrm{n}\}$.

(*Note*: some commutation properties of closings by reconstruction presented in References (7, 8) (and summarized in Table 1) have been used.) □

The corresponding result of Theorem 6 for the dual family $\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i$ is:

$$\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i = \tilde{\gamma}_{i_0}\left(I \bigvee \left(\bigvee_{i=1}^{n} \tilde{\gamma}_i\tilde{\varphi}_i\right)\right), \quad \text{where } i_0 \in \{1, \ldots, n\}.$$

Theorem 6 permits to express the filter $\bigwedge_{i=1}^{n} \tilde{\varphi}_i\tilde{\gamma}_i$ as a closing followed by an opening. *Note*: the operator



(a) Input set A

(b) $\tilde{\varphi}_1\tilde{\gamma}_1(A)$

(c) $\tilde{\varphi}_2\tilde{\gamma}_2(A)$

(d) $\bigwedge_{i\in\{1,2\}} \tilde{\varphi}_i\tilde{\gamma}_i(A)$

Fig. 5. $\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$: one-dimensional example.

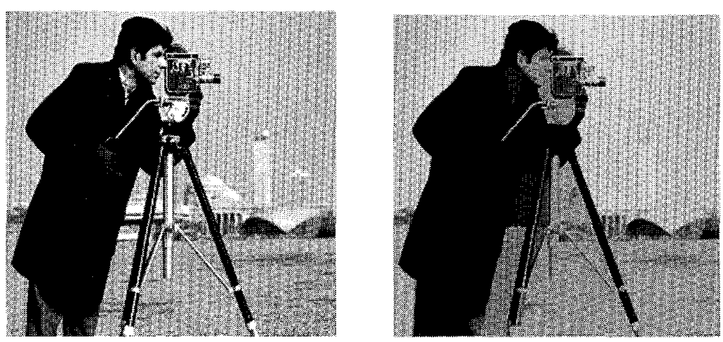


Fig. 6. $\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$: gray-level example.

Table 1. Filters by reconstruction: commutation properties. Expressions in the second half of the table are dual of those in the first half (from Reference (8))

$\tilde{\varphi}(\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i) = \bigwedge_{i=1}^{n}\tilde{\varphi}\tilde{\varphi}_i\tilde{\gamma}_i$
$\tilde{\varphi}_{i_0}\tilde{\gamma}_{i_0}(\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i) = \tilde{\varphi}_{i_0}(\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i) = \bigwedge_{i=1}^{n}\tilde{\varphi}_{i_0}\tilde{\varphi}_i\tilde{\gamma}_i = \bigwedge_{i=i_0}^{n}\tilde{\varphi}_i\tilde{\gamma}_i, \quad i_0\in\{1,\dots,n\}$
$\tilde{\varphi}(\bigvee_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i) = \bigvee_{i=1}^{n}\tilde{\varphi}\tilde{\varphi}_i\tilde{\gamma}_i = (\bigvee_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i)\bigvee\tilde{\varphi}\tilde{\varphi}_1\tilde{\gamma}_1.$
$\tilde{\gamma}(\bigvee_{i=1}^{n}\tilde{\gamma}_i\tilde{\varphi}_i) = \bigvee_{i=1}^{n}\tilde{\gamma}\tilde{\gamma}_i\tilde{\varphi}_i$
$\tilde{\gamma}_{i_0}\tilde{\varphi}_{i_0}(\bigvee_{i=1}^{n}\tilde{\gamma}_i\tilde{\varphi}_i) = \tilde{\gamma}_{i_0}(\bigvee_{i=1}^{n}\tilde{\gamma}_i\tilde{\varphi}_i) = \bigvee_{i=1}^{n}\tilde{\gamma}_{i_0}\tilde{\gamma}_i\tilde{\varphi}_i = \bigvee_{i=i_0}^{n}\tilde{\gamma}_i\tilde{\varphi}_i, \quad i_0\in\{1,\dots,n\}$
$\tilde{\gamma}(\bigwedge_{i=1}^{n}\tilde{\gamma}_i\tilde{\varphi}_i) = \bigwedge_{i=1}^{n}\tilde{\gamma}\tilde{\gamma}_i\tilde{\varphi}_i = (\bigwedge_{i=1}^{n}\tilde{\gamma}_i\tilde{\varphi}_i)\bigwedge\tilde{\gamma}\tilde{\gamma}_1\tilde{\varphi}_1.$

$(I\bigwedge(\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i))$ is an opening (from the fact that $\bigwedge_{i=1}^{n}\tilde{\varphi}_i\tilde{\gamma}_i$ is a strong filter).

Let us study the questions in item (b) above. We will obtain the important (and possibly surprising) finding that not *any opening* (*or respectively closing*) *composed of openings and closings by reconstruction is itself an opening by reconstruction* (*respectively closing by reconstruction*). We will arrive to this result after some simple manipulation of known results.

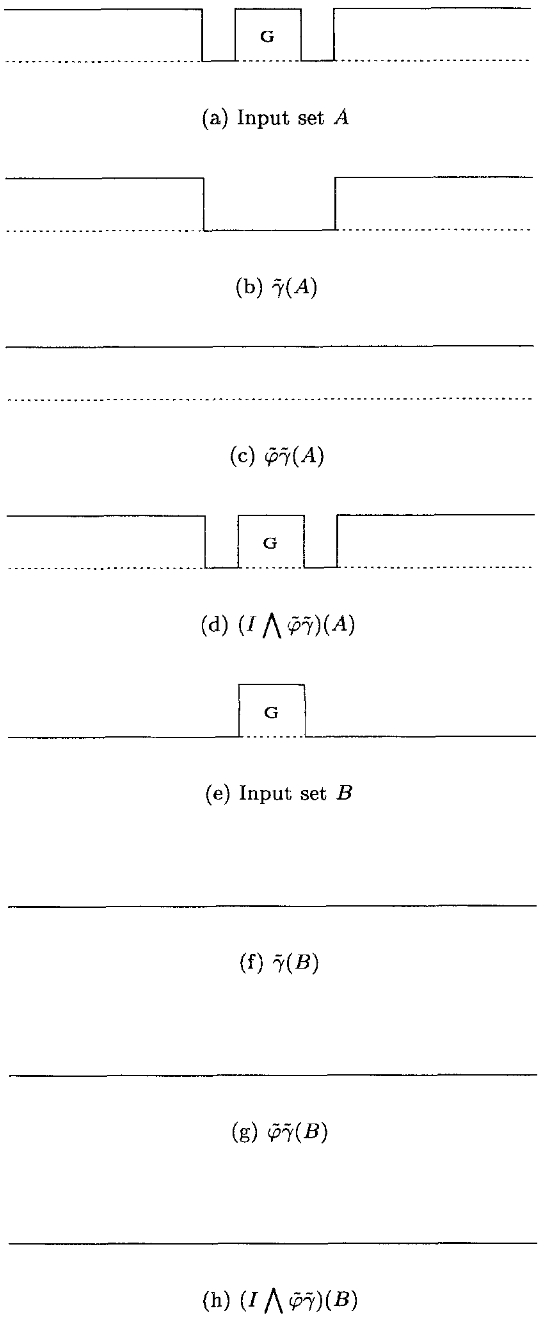


Fig. 7. Effect on grains of $I \bigwedge \tilde{\varphi}\tilde{\gamma}$: one-dimensional example. Let $\tilde{\gamma}$ remove the central grain G in A and B (identical grain). If $\tilde{\varphi}$ fills the resulting pore in $\tilde{\gamma}(A)$ (see part (b)) but not the one in $\tilde{\gamma}(B)$ (see part (f)), we have that G is included in $(I \bigwedge \tilde{\varphi}\tilde{\gamma})$ (A) (part (d)) but is not included in $(I \bigwedge \tilde{\varphi}\tilde{\gamma})$ (B) (part (h)).

We know (from expressions (5) and (6)) that

$$\tilde{\varphi}\tilde{\gamma} = (I \bigvee \tilde{\varphi}\tilde{\gamma})\,(I \bigwedge \tilde{\varphi}\tilde{\gamma}) = \tilde{\varphi}(I \bigvee \tilde{\varphi}\tilde{\gamma})(I \bigwedge \tilde{\varphi}\tilde{\gamma})$$
$$= \tilde{\varphi}(I \bigwedge \tilde{\varphi}\tilde{\gamma}).$$

The questions in item (b) were whether $\tilde{\gamma}$ is equal to $I \bigwedge \tilde{\varphi}\tilde{\gamma}$, and whether (more generally) the opening $I \bigwedge \tilde{\varphi}\tilde{\gamma}$ is an opening by reconstruction. The answers to both questions are "no". We will study and answer the second question using a counterexample. Obviously, if the answer to the second question is "no", this implies that the answer to the first one is negative as well.

Let us study the two cases shown in Fig. 7, where two different input sets A and B are applied to the opening $I \bigwedge \tilde{\gamma}\tilde{\varphi}$. Notice that both input sets possess the same grain $G(G \subseteq A$, and $G \subseteq B)$. In fact, the set B is just the grain G. Let us assume that the opening by reconstruction $\tilde{\gamma}$ removes the grain G. Then the grain G is not either in $\tilde{\gamma}(A)$ or in $\tilde{\gamma}(B)$ (and in fact $\tilde{\gamma}(B) = \varnothing$). Let us assume that $\tilde{\varphi}$ fills the pore in $\tilde{\gamma}(A)$ but not the pore in $\tilde{\gamma}(B)$ (which is the whole space and of course larger than that in $\tilde{\gamma}(A)$). We have that $G \subseteq \tilde{\varphi}\tilde{\gamma}(A)$ but that $G \nsubseteq \tilde{\varphi}\tilde{\gamma}(B)$, and $G \subseteq (I \bigwedge \tilde{\varphi}\tilde{\gamma})\,(A)$ but $G \nsubseteq (I \bigwedge \tilde{\varphi}\tilde{\gamma})\,(B)$.

Therefore, we obtain the result that $I \bigwedge \tilde{\varphi}\tilde{\gamma}$ does not process each grain independently of other grains, i.e., it is not a connected-component local filter (using the terms in Reference 18). This implies $I \bigwedge \tilde{\varphi}\tilde{\gamma}$ is not an opening by reconstruction, since openings by reconstruction (by definition) process each grain independently from other grains. Similarly, we could prove that $I \bigvee \tilde{\varphi}\tilde{\gamma}$ is not a closing by reconstruction. These results also apply to the dual filters: $I \bigwedge \tilde{\gamma}\tilde{\varphi}$ and $I \bigvee \tilde{\gamma}\tilde{\varphi}$ are not, respectively, an opening and a closing by reconstruction.

6. CONCLUSION

This paper has investigated the theory of morphological filters by reconstruction and has provided some new theoretical results. First, the composition laws concerning openings and closings by reconstruction have been presented. This result has been used for computing new bounds of the morphological filter by reconstruction class. Some examples have shown that morphological expressions containing openings and closings by reconstruction can be manipulated in a relatively easy manner. Nevertheless, the theoretical nature of those results, it can be observed that the filter bound computation can be applied to image simplification applications and, in particular, to determine the simplification dynamic range.

In addition, this paper has investigated the strong property possessed by some morphological filters and, in particular, by filters by reconstruction. Strong filters can be expressed as the sequential composition of an opening γ followed by a closing φ, or *vice versa*. Starting with a dual form of alternating filters by reconstruction, its extension to a multi-level family of strong filters recently introduced has been obtained. An important side result has been to establish a distinction between openings composed by openings and closings by reconstruction and the opening by reconstruction concept. Obviously, it is not true that any connected opening is necessarily an opening by reconstruction. However, as this paper has shown, neither is true that any opening composed of openings and closings by reconstruction is an opening by reconstruction (similarly for closings). An example of a connected opening composed exclusively of openings and closings by reconstruction that does not process grains and pores independently from the rest of grains and pore has been given.

LIST OF SYMBOLS

E	a space of points
$\mathscr{P}(E)$	power set of E
A, B	sets
A^c	complement of A
x, y	points
$\cup$, $\cap$, $\setminus$	set union, set intersection and set difference, respectively
T	lattice
$\leq$, $\bigwedge$, $\bigvee$	order relation (less than or equal to), inf operation and sup operation, respectively
ψ	operator
I, I^c	identity operator and complementation operator, respectively
γ, $\tilde{\gamma}$, φ, $\tilde{\varphi}$	opening, opening by reconstruction, closing and closing by reconstruction, respectively
γ_x, φ_x	point opening and its dual closing, respectively
$\gamma_\circ$, $\varphi_\circ$	trivial opening and trivial closing, respectively
$\{\gamma_i\}$, $\{\varphi_i\}$	granulometry and antigranulometry, respectively
$\{\gamma_a, \dots, \gamma_z\}$, $\{\varphi_a, \dots, \varphi_z\}$	arbitrary set of openings and closings, respectively (that is, there is no *a priori* relation among the openings and the closings—unlike in a granulometry and in an antigranulometry)
Z, **R**	the set of integers and the set of real numbers, respectively

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