PERGAMON

Pattern Recognition 33 (2000) 251–261

PATTERN RECOGNITION
THE JOURNAL OF THE PATTERN RECOGNITION SOCIETY

www.elsevier.com/locate/patcog

# Point-based projective invariants

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Received 23 January 1998; accepted 3 February 1999

## Abstract

The paper deals with features of a 2-D point set which are invariant with respect to a projective transform. First, projective invariants for five-point sets, which are simultaneously invariant to the projective transform and to permutation of the points, are derived. They are expressed as functions of five-point cross-ratios. Then, the invariants for more than five points are derived. The algorithm for searching the correspondence between the points of two 2-D point sets is presented. The algorithm is based on the comparison of two projective and permutation invariants of five-tuples of the points. The best-matched five tuples are then used for the computation of the projective transformation and that with the maximum of corresponding points is used. Stability and discriminability of the features and behavior of the searching algorithm are demonstrated by numerical experiments. 

*Keywords:* Projective transform; Point set matching; Point-based invariants; Projective invariants; Permutation invariants; Registration; Control points

## 1. Introduction

One of the important tasks in image processing and computer vision is a recognition of objects on images captured under different viewing angles. However, this problem cannot be solved in a general case [1]. Nevertheless, if we restrict ourselves to planar objects only, then the distortion between two frames can be described by *projective transform* (sometimes called *perspective projection*)

$$x' = (a_0 + a_1x + a_2y)/(1 + c_1x + c_2y),$$

$$y' = (b_0 + b_1x + b_2y)/(1 + c_1x + c_2y), \tag{1}$$

where $x$ and $y$ are the coordinates in the first frame and $x'$ and $y'$ are the coordinates in the second one.

Feature-based recognition of such objects requires features invariant to projective transform (1). Several different approaches to this problem have been published in recent works. One of them is based on the assumption that the non-linear term of the perspective projection is relatively small and thus the projective transform can be approximated by an affine transform. This assumption is true, if the distance from the sensor to the object is much greater than the size of the object. In such cases, various affine invariants can be applied such as moment invariants [2,3] or Fourier descriptors [4,5].

However, in some cases the projection cannot be approximated by the affine transform and therefore the use of exact projective invariants is required. The invariants, which have been developed for this purpose, can be categorized into two groups: differential invariants and point-based ones.

Differential invariants are applicable only if the object boundary is a smooth continuous curve. A set of invariants based on boundary derivatives up to the sixth order was presented by Weiss [6]. Unfortunately, these invariants are not defined for such important curves as straight lines or conics. Weiss's invariants are numerically unstable because of the high-order derivatives. To overcome this difficulty, several improvements were presented [7–9].

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PII: S0031-3203(99)00049-7

The second group of invariants is defined on point sets [10], on sets composed both from points and straight lines [11,12] and on triangle pairs [13]. A detailed survey of the point-based methods can be found [14].

Another problem to be solved is to establish the correspondence between two point sets, which are projectively deformed. To calculate the invariants, we have to order those sets somehow. The solution when the points are vertices of a polygon has been published [15]. Another solution is to use features also invariant to the order or labeling of the points. Five-point projective and permutation invariants are presented [2,16]. This approach is also used in this paper.

The plane transformed by projective transform (1) contains a straight line

$$1 + c_1x + c_2y = 0, \tag{2}$$

which is not mapped into the second plane (more precisely it is mapped into infinity) and which divides the plane into two parts. If all elements of our point set lie in one half-plane, then some additional theorems about topology of the set hold for the transform, e.g. the convex hull is preserved during the transform in that case. This fact can be used to derive invariants with lower computational complexity [17].

This paper deals with a general case of the projective transform, when the points can lie in both parts of the plane. The convex hull is not preserved under the transform and all possibilities of the positions of the points must be taken into account. The only assumption is that the points do not lie directly on straight line (2).

A projective invariant can be defined for at least five points. The simplest one is a five-point cross-ratio

$$\varrho(1, 2, 3, 4, 5) = \frac{P(1, 2, 3)P(1, 4, 5)}{P(1, 2, 4)P(1, 3, 5)}, \tag{3}$$

where $P(A, B, C)$ is the area of the triangle with vertices $A$, $B$ and $C$.

The point No. 1 is included in all four triangles and it is called the common point of the cross-ratio.

Reiss [2] proposes to use the median of all possible values of $\varrho$. A more precise description of the relations between various cross-ratio values under permutations of the given points can be found [16].

After the correspondence between the individual points in both sets has been established, we use them as the control points for image-to-image registration. However, there are often some points having no counterpart in the other image. An approach to solve this problem can be found [18], but that method becomes unstable if the number of the "wrong" points increases.

The goal of this paper is to derive projective and permutation invariants of point sets. Five-point projective and permutation invariants are derived in Section 2, they are generalized for more than five points in Section 3 and the sets with wrong points are discussed in Section 4. Experiments showing the numerical properties of the invariants as well as their usage for image registration are shown in Section 5.

## 2. Five-point permutation invariants

### 2.1. The derivation of the invariants

First we derive permutation invariants by the simplest way and then more detailed analysis will be performed. The main principle is to use addition or multiplication (or another symmetric function) of all possible values of the projective invariants over the permutations of the points. The order of terms and factors is only changed during permutations, but the result stays invariant.

To obtain permutation invariants, we can employ various functions of cross-ratio (3). Reiss [2] used the function $\varrho + \varrho^{-1}$, which is unstable near zero. If some triplet of five points in Eq. (3) is collinear, then the function $\varrho + \varrho^{-1}$ is infinite. Thus the more suitable function is $\psi = 2/(\varrho + \varrho^{-1}) = 2\varrho/(\varrho^2 + 1)$. If $\varrho$ or $\varrho^{-1}$ is zero, then the function $\psi$ is zero.

The function $\varrho$ can have only three distinct values during permutations of four points, therefore the functions:

$$\begin{aligned} F'_{+}(1, 2, 3, 4, 5) &= \psi(1, 2, 3, 4, 5) + \psi(1, 2, 3, 5, 4) \\ &\quad + \psi(1, 2, 4, 5, 3), \\ F'_{\cdot}(1, 2, 3, 4, 5) &= \psi(1, 2, 3, 4, 5) \cdot \psi(1, 2, 3, 5, 4) \\ &\quad \cdot \psi(1, 2, 4, 5, 3) \end{aligned} \tag{4}$$

are invariant to the choice of labeling of the last four points, but the point No. 1 must be common at all cross-ratios. To obtain full invariance to the choice of labeling, we must alternate all five points as common ones:

$$\begin{aligned} I_{s_1,s_2}(1, 2, 3, 4, 5) = {} & F_{s_1}(1, 2, 3, 4, 5) s_2 F_{s_1}(2, 3, 4, 5, 1) \\ & s_2 F_{s_1}(3, 4, 5, 1, 2) s_2 F_{s_1}(4, 5, 1, 2, 3) \\ & s_2 F_{s_1}(5, 1, 2, 3, 4), \end{aligned} \tag{5}$$

where $s_1$ and $s_2$ are either sign $+$ or $\cdot$.

The set of $n$ points has $2n$ degrees of freedom and the projective transform has eight parameters. Therefore, the set can have only

$$m = 2n - 8 \tag{6}$$

independent invariants to the projective transform. That is why only two of the four invariants $I_{+}, I_{\cdot +}, I_{+ \cdot}$ and $I_{\cdot\cdot}$ can be independent.

## *2.2. The roots of the invariants*

Lenz and Meer [16] dealt with the five-point projective and permutation invariants in detail. They discovered that if the common point stays the same and the other points are permuted, then the values of the cross-ratios are

$$\varrho_1 = \varrho, \quad \varrho_2 = \frac{1}{\varrho}, \quad \varrho_3 = 1 - \varrho, \quad \varrho_4 = \frac{1}{1-\varrho},$$

$$\varrho_5 = \frac{\varrho}{\varrho - 1}, \quad \varrho_6 = \frac{\varrho - 1}{\varrho}. \tag{7}$$

If we construct a function $F(\varrho)$, which has the same value for all these values of $\varrho(\varrho_1, \varrho_2, \ldots, \varrho_6)$, it is invariant to the permutation of four points. If we change the common point, we receive another value of the cross-ratio, let us say $\sigma$, and the function

$$K(\varrho, \sigma) = F(\varrho) + F(\sigma) + F\left(\frac{\varrho}{\sigma}\right) + F\left(\frac{\varrho - 1}{\sigma - 1}\right) + F\left(\frac{\varrho(\sigma - 1)}{\sigma(\varrho - 1)}\right) \tag{8}$$

is a five-point projective and permutation invariant.

As our study implies, if the function $F(\varrho)$ is constructed as the quotient of two polynomials and $\varrho$ is its root, then each of the values $\varrho_1, \varrho_2, \ldots, \varrho_6$ must be its root. We can consider it in the form

$$F(\varrho) = \frac{P_1(\varrho)}{P_2(\varrho)}, \tag{9}$$

$$P_1(\varrho) = (\varrho - \varrho_1)(\varrho - \varrho_2)(\varrho - \varrho_3)(\varrho - \varrho_4) \times (\varrho - \varrho_5)(\varrho - \varrho_6) \tag{10}$$

and similarly for $P_2(\varrho)$ (when all roots differ from zero).

It is advantageous if $F(\varrho)$ is defined for any real $\varrho$. Thus, $P_2(\varrho)$ should not have real roots. Two such invariants are proposed [16]

$$F_{14} = \frac{2\varrho^6 - 6\varrho^5 + 9\varrho^4 - 8\varrho^3 + 9\varrho^2 - 6\varrho + 2}{\varrho^6 - 3\varrho^5 + 3\varrho^4 - \varrho^3 + 3\varrho^2 - \varrho 3 + 1}, \tag{11}$$

$$F_{15} = \frac{(\varrho^2 - \varrho + 1)^3}{\varrho^6 - 3\varrho^5 + 5\varrho^3 - 3\varrho + 1}. \tag{12}$$

The following Theorem describes the properties of the roots of $P_2(\varrho)$.

**Theorem 1.** *If the roots of* $P_2(\varrho)$ *are imaginary, then they lie on the following curves*:

$$a_1^2 + b_1^2 = 1,$$

$$(a_2 - 1)^2 + b_2^2 = 1,$$

$$a_3 = \tfrac{1}{2}, \tag{13}$$

*where there are the following relations between the roots*:

$$a_2 = 1 - a_1, b_2 = b_1, b_3 = \frac{b_1}{(a_1 - 1)^2 + b_1^2} = \frac{b_2}{a_2^2 + b_2^2}, \tag{14}$$

*where* $a_i$ *and* $b_i$, $i = 1, 2, 3$ *are real and imaginary parts of the roots.*

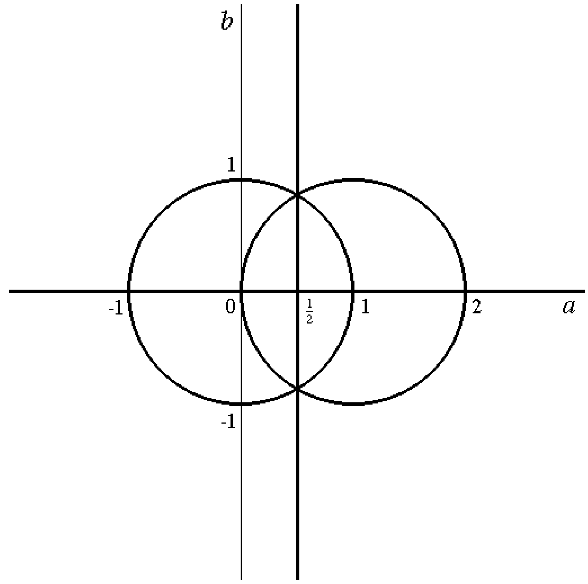


Fig. 1. Illustration of Theorem 1. The circles and the straight line show possible positions of the roots of the invariants in the complex plane ($a$ is real part and $b$ imaginary part).

The theorem is illustrated in Fig. 1.

**Proof**. If we express $P_2(\varrho)$ in the form

$$P_2(\varrho) = (\varrho - a_1 - b_1\mathrm{i})(\varrho - a_1 + b_1\mathrm{i})(\varrho - a_2 - b_2\mathrm{i}) \times (\varrho - a_2 + b_2\mathrm{i})(\varrho - a_3 - b_3\mathrm{i})(\varrho - a_3 + b_3\mathrm{i}),$$

$$\mathrm{i} = \sqrt{-1}, \tag{15}$$

then there are $6! = 720$ possibilities of assignment between $a_1 \pm b_1\mathrm{i}$, $a_2 \pm b_2\mathrm{i}$, $a_3 \pm b_3\mathrm{i}$ and $\varrho_1$–$\varrho_6$ in Eq. (7). If we use the assignment

$$a_1 + b_1\mathrm{i} = \varrho_1, \tag{16}$$

$$a_1 - b_1\mathrm{i} = \varrho_2, \tag{17}$$

$$a_2 + b_2\mathrm{i} = \varrho_6, \tag{18}$$

$$a_2 - b_2\mathrm{i} = \varrho_3, \tag{19}$$

$$a_3 + b_3\mathrm{i} = \varrho_4 \tag{20}$$

and

$$a_3 - b_3 i = \varrho_5, \tag{21}$$

then from Eqs. (7) and (16)

$$\varrho_2 = \frac{1}{\varrho_1} = \frac{1}{a_1 + b_1 i} = \frac{a_1 - b_1 i}{a_1^2 + b_1^2} \tag{22}$$

and from Eq. (17)

$$a_1 = \frac{a_1}{a_1^2 + b_1^2}, \qquad -b_1 = \frac{-b_1}{a_1^2 + b_1^2}. \tag{23}$$

Therefore the first two roots must lie on the circle $a_1^2 + b_1^2 = 1$. From Eq. (7)

$$\varrho_6 = \frac{\varrho_1 - 1}{\varrho_1} = \frac{a_1 - 1 + b_1 i}{a_1 + b_1 i} = \frac{a_1^2 - a_1 + b_1^2 + b_1 i}{a_1^2 + b_1^2} \tag{24}$$

and

$$\varrho_3 = 1 - \varrho_1 = 1 - a_1 - b_1 i \tag{25}$$

from Eqs. (18), (23) and (24)

$$a_2 = \frac{a_1^2 - a_1 + b_1^2}{a_1^2 + b_1^2} = 1 - a_1, \quad b_2 = \frac{b_1}{a_1^2 + b_1^2} = b_1 \tag{26}$$

and from Eqs. (19) and (26)

$$a_2 = 1 - a_1, b_2 = b_1, \tag{27}$$

therefore the second two roots must lie on the circle $(a_2 - 1)^2 + b_2^2 = 1$. From Eq. (7)

$$\varrho_4 = \frac{1}{1 - \varrho_1} = \frac{1}{1 - a_1 - b_1 i} = \frac{1 - a_1 + b_1 i}{(1 - a_1)^2 + b_1^2} \tag{28}$$

and

$$\varrho_5 = \frac{\varrho_1}{\varrho_1 - 1} = \frac{a_1 + b_1 i}{a_1 - 1 + b_1 i} = \frac{a_1^2 - a_1 + b_1^2 - b_1 i}{(1 - a_1)^2 + b_1^2}, \tag{29}$$

from Eqs. (20), (23) and (28)

$$a_3 = \frac{1 - a_1}{(1 - a_1)^2 + b_1^2} = \frac{1}{2}, \quad b_3 = \frac{b_1}{(1 - a_1)^2 + b_1^2} = \frac{b_2}{a_2^2 + b_2^2} \tag{30}$$

and from Eqs. (21) and (29)

$$a_3 = \frac{a_1^2 - a_1 + b_1^2}{(1 - a_1)^2 + b_1^2} = \frac{1}{2}, \quad b_3 = \frac{b_1}{(1 - a_1)^2 + b_1^2} = \frac{b_2}{a_2^2 + b_2^2} \tag{31}$$

therefore the third two roots must lie on the straight line $a_3 = \frac{1}{2}$.

We cannot investigate all other 719 possibilities because of insufficient space. The number of cases can be reduced significantly, if we consider mutual relations among the roots only. Thus, we have to deal with $5! = 120$ cases only. The other 600 cases are just permutations. We treated all of the 120 cases and we proved that each individual case falls into one of the following categories:

1. The result is some permutation of the previous case.
2. The result is only a finite set of values, typically $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.
3. The case has no solution.

Thus, there is no other solution and the theorem has been proven. □

Our invariants have the form in this context

$$F'_+ = \frac{2\varrho}{\varrho^2 + 1} + \frac{2(1 - \varrho)}{(1 - \varrho)^2 + 1} + \frac{2\varrho/(\varrho - 1)}{\varrho^2/(\varrho - 1)^2 + 1}$$
$$= 2\frac{\varrho^6 - 3\varrho^5 + 3\varrho^4 - \varrho^3 + 3\varrho^2 - 3\varrho + 1}{2\varrho^6 - 6\varrho^5 + 11\varrho^4 - 12\varrho^3 + 11\varrho^2 - 6\varrho + 2},$$

$$F'_{\cdot} = \frac{(-8)\varrho^2(1 - \varrho)^2}{(1 + \varrho^2)(2 - 2\varrho + \varrho^2)(1 - 2\varrho + 2\varrho^2)}$$
$$= \frac{-8\varrho^2(1 - \varrho)^2}{2\varrho^6 - 6\varrho^5 + 11\varrho^4 - 12\varrho^3 + 11\varrho^2 - 6\varrho + 2}. \tag{32}$$

The choice of the invariant has one degree of freedom, we can choose one root of the denominator on some curve from Fig. 1, other roots must be defined by Theorem 1 and the numerator defines the range of values of the invariant. Since both $F'_+$ and $F'_{\cdot}$ have the same denominator (roots $1 \pm i$, $\pm i$ and $0.5 \pm 0.5i$), it is suitable to change one of them. The other can be $(\varrho^2 - \varrho + 1)^3$ with roots $0.5 \pm i\frac{\sqrt{3}}{2}$. Then, if we want the range of values of the consequential invariants $I_{++}$ and $I_{\cdot +}$ from 0 to 1, our invariants will be

$$F_+ = \frac{8}{5}\frac{\varrho^2(1 - \varrho)^2}{(\varrho^2 - \varrho + 1)^3}$$

$$F_{\cdot} = \frac{3\varrho^2(1 - \varrho)^2}{2\varrho^6 - 6\varrho^5 + 11\varrho^4 - 12\varrho^3 + 11\varrho^2 - 6\varrho + 2} \tag{33}$$

and relations to the original invariants will be $F_+ = 16(\frac{1}{5} - 1/(6 - F'_+))$ and $F_{\cdot} = -\frac{3}{8}F'_{\cdot}$.

### 2.3. The normalization of the invariants

The invariants $I_1 = I_{++}$ and $I_2 = I_{\cdot +}$ corresponding to the functions $F_+$ and $F_{\cdot}$ utilize the feature space ineffectively (see Fig. 2).

A better result can be reached by the sum and the difference (see Fig. 3)

$$I'_1 = (I_1 + I_2)/2, \quad I'_2 = (I_1 - I_2 + 0.006)\cdot 53, \tag{34}$$

but the best utilization of the given range is reached by subtraction and division by the polynomials (see Fig. 4)

$$I''_1 = I'_1, \quad I''_2 = \frac{1 - I'_2 + p(I'_1)}{d(I'_1)}. \tag{35}$$

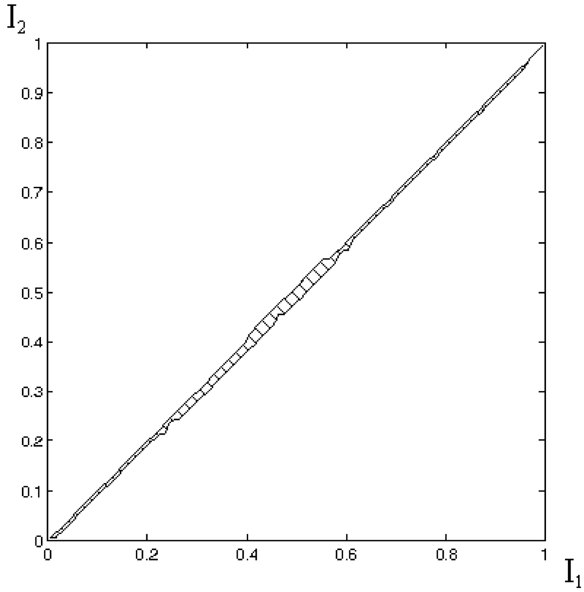


Fig. 2. Possible range of values of the invariants $I_1, I_2$. It was acquired numerically by computing invariants for all combinations of five points with integer coordinates from 0 to 511.

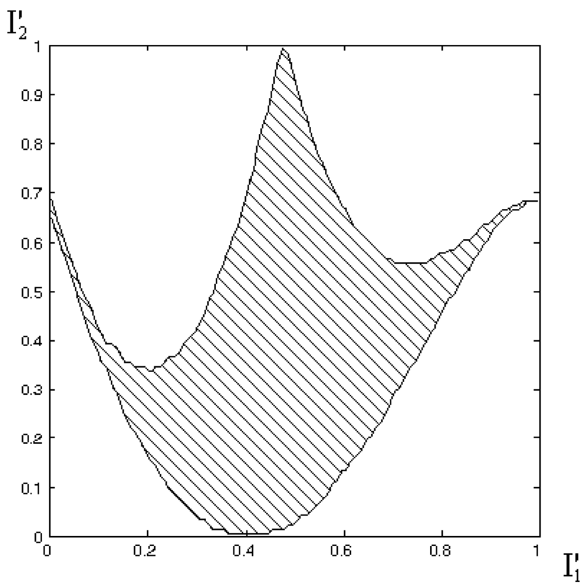


Fig. 3. Possible range of values of the invariants $I'_1, I'_2$.

Exact coefficients of the polynomials $p(I'_1)$ and $d(I'_1)$ are shown in Appendix B. This normalization is not necessary, but it makes possible to use a simpler classifier.

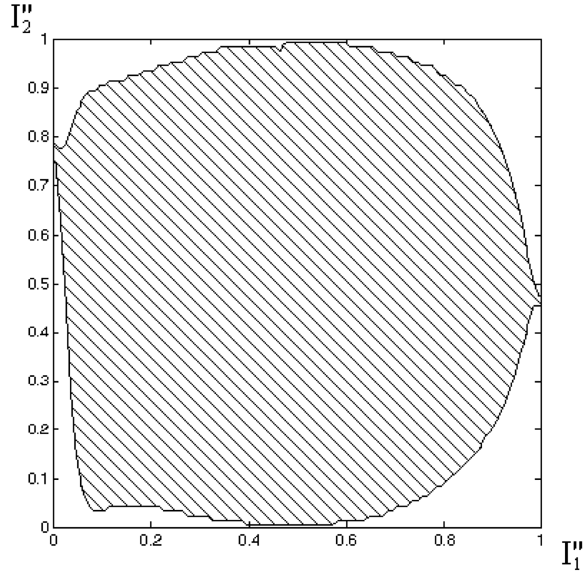


Fig. 4. Possible range of values of the invariants $I''_1, I''_2$.

## 3. Invariants for more than five points

There is a number of approaches to the problem of the generalization of the invariants from the previous section to $n$ points ($n > 5$). One of them, yielding good experimental results, consists in summation of powers of the invariants $I''_1, I''_2$ over all possible combinations of 5 from $n$ points $C^n_5$.

**Theorem 2.**

$$I_{1,k} = \sum_{Q \in C^n_5} I''^k_1(Q), \quad I_{2,k} = \sum_{Q \in C^n_5} I''^k_2(Q),$$

$$k = 1, 2, \dots n-4 \tag{36}$$

*are projective and permutation invariants of a set of n points.*

**Proof.** $I''_1$ and $I''_2$ are projective invariants (see Eq. (3)) and an arbitrary function of invariants is also invariant (if it does not depend on the parameters of the transform), therefore $I_{1,k}$ and $I_{2,k}$ are also projective invariants. $I''_1$ and $I''_2$ are also 5-point permutation invariants and summation over all combinations guarantees permutation invariance of the $I''_1$ and $I''_2$. □

The number of these invariants is chosen as $2n - 8$ according to Eq. (6). The computing complexity is approximately $\binom{n}{5}T$, i.e. $O(n^5)$, where $T$ is the computing complexity of one five-point invariant. However, the number of the terms is very high and that is why a normalization of the results is suitable. To preserve the values inside acceptable intervals, we can use

$$I'_{s,k} = \sqrt[k]{1\Big/\binom{n}{5}\sum_{Q\in C_5^n} I''^k_s(Q)}, \tag{37}$$

where $s = 1$ or 2.

Another, perhaps more sophisticated, normalization is the following. We can consider the five-point invariants $I''_1, I''_2$ as independent random variables with uniform distribution in the interval from 0 to 1. Then the distribution function $F_k(x)$ of the $k$th power of the invariant is

$$\begin{aligned} F_k(x) &= 0 && \text{from } -\infty \text{ to } 0, \\ F_k(x) &= x^{1/k} && \text{from } 0 \text{ to } 1, \\ F_k(x) &= 1 && \text{from } 1 \text{ to } \infty \end{aligned} \tag{38}$$

with the mean value $\mu_k = 1/(1+k)$ and the variance $\sigma_k^2 = k^2/(1+k)^2(1+2k)$. The number of terms in sum (36) is relatively high and the Central Limit Theorem implies that the distribution of the sum is approximately Gaussian, its mean value is the sum of its mean values $\mu_k$ and the variance is the sum of the variances $\sigma_k^2$. The given range is the best utilized in case of uniform distribution of the resulting invariants and therefore we can normalize the invariants with Gaussian distribution function

$$G(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} \mathrm{e}^{-(\xi-\mu)^2/2\sigma^2}\,\mathrm{d}\xi \tag{39}$$

with the mean value $\mu = \binom{n}{5}/(1+k)$, the variance $\sigma^2 = \binom{n}{5}k^2/(1+k)^2(1+2k)$ and the standard deviation $\sigma = k/(1+k)\sqrt{\binom{n}{5}/(1+2k)}$

$$I''_{sk} = G(I_{s,k};\mu,\sigma). \tag{40}$$

An approximate substitution of the function $G$ is used in practice.

## 4. Point matching

The problem we are dealing with in this section can be formulated as follows. Let us have two-point sets selected from two projectively deformed images. The sets can include some wrong points, i.e. points without a counterpart in the other set. We look for the parameters of the projective transform to register the images. We deal with the methods, which do not consider the image functions, but only the positions of the points.

### 4.1. Full search algorithm

The simplest algorithm is the full search of all possibilities of the correspondence. We can examine each four points from the input image against each four points from the reference image. If we have $n$ points in the input image and $\ell$ points in the reference one, we must examine $\binom{n}{4}\binom{\ell}{4}4!$ possibilities.

The examination means the computation of the projective transform, the transformation of the points in the input image and judgment of the quality of the transform. We performed this judgment in the following way. The two nearest points are found and removed and again two nearest points from the remaining ones are found. The search is complete when the distance between the nearest points exceeds the suitable threshold. The number of corresponding points is used as the criterion of the quality of the transform. If the number is the same, the distance of the last two points is used. The best transform according to this criterion is used as the solution.

The threshold must correspond to the precision of the selection of the points. The threshold 5 pixels proved its suitability in usual situations.

If $n$ and $\ell$ are approximately the same and high, this algorithm has the computing complexity $O(n^{11})$ and in our experience it is too time consuming.

### 4.2. Pairing algorithm by means of the projective and permutation invariants

We can compute the distance in the feature space between invariants of each five points from the input image against invariants of each five points in the reference image. Nevertheless, it was found that wrong five points often match one another randomly, this false match can be better than the correct one and we must search not only the best match, but also each good match.

We carried out experiments with a number of searching algorithms. We consider the following as the best one. We find the first $b$ best matches and the full search algorithm from the previous section is applied on each pair of five tuples corresponding each match. The number $b$ was chosen as $\binom{\max(n,\ell)}{5}$, but this number is not critical. Note: the total number of pairs of five tuples is $\binom{n}{5}\binom{\ell}{5}$.

In Ref. [19] the convex hull constraint is proposed. It is based on the assumption that the sets lie in one half-plane of Eq. (2) and that the projective transform preserves the position of points on or inside the convex hull. Then the pairs of five-tuples with the different number of points on the convex hull need not be considered. As was written in the introduction, we consider the general case of the projective transform and therefore this constraint is not used. In the same work the idea of partial search is proposed. The authors randomly chose about one-fifteenth of all pairs and tried to search them only. They found the decrease of reliability relatively small. This constraint can be used in our algorithm too, but the following numerical experiment used the general algorithm without this constraint, because the amount of time saved is relatively small.

## 5. Numerical experiments

How do we investigate the stability and discriminability of the invariants? Let us imagine the following situation. We have two similar sets of points and we would like to recognize one from the other, but one of them can be distorted by the noise in its coordinates and we would like the noise not to influence the recognition.

The following numerical experiment was carried out to observe the behavior of invariants (37) and (40) in this situation.

Let us have three sets of 11 points. One of them was created by a pseudo-random generator. The point coordinates are uniformly distributed from 0 to 511. The second set was created from the first one by moving one point a relatively large distance. The coordinates of the movement are also defined by a pseudo-random generator, but with Gaussian distribution. The third set was created from the first one by adding small noise to all point coordinates. The noise is independent with zero-mean Gaussian distribution and with gradually increasing standard deviation. The standard deviation $\sigma_1$ of the movement in the second set increased from 0 to 190 with the step 10 and the standard deviation $\sigma_2$ of the noise in the third set increased from 0 to 9.5 with the step 0.5. The distances $d$ between original and noisy set in the space of the invariants were computed.

Since one experiment would not be representative enough, 10, 20, 100 and 1000 experiments were gradually carried out for each standard deviation. A curve of dependency of the distance on the noise standard deviation was acquired as the average of 1000 experiments, because

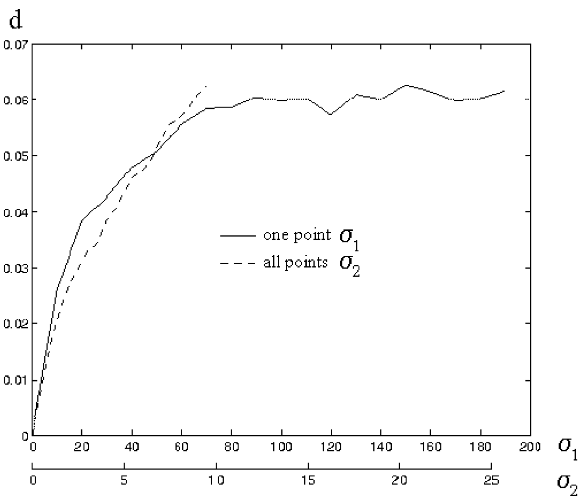


Fig. 5. The distance between the first and second sets (solid line) and between the first and third sets (dashed line) in the Euclidean space of the invariants normalized by the average and root as a function of the noise standard deviation.

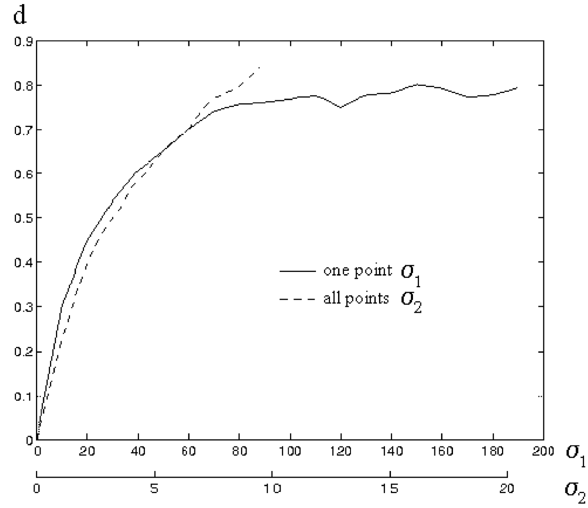


Fig. 6. The distance between the first and second sets (solid line) and between the first and third sets (dashed line) in the Euclidean space of the invariants normalized by the Gaussian distribution function as a function of the noise standard deviation.

the average of fewer values was too dependent on the concrete realization of the noise. The result of invariants (37) normalized by the average and root is given in Fig. 5.

The scale of the horizontal axis is different for both cases so the curves were as similar as possible, more precisely, the area of the square of the difference between them was minimized. The ratio of the scales is 7.36; it means if two sets differ by only one point, then the distance of the points must be at least approximately 7.36 times greater than the noise standard deviation to be possible to distinguish both sets. In another words, if the ratio of the standard deviations is 7.36 and their value is such that the dashed line is under the solid one, the sets will be recognized correctly. If the ratio increases, the limit of correct recognition increases too, but if the noise standard deviation is greater than approximately 9, i.e. about 2% of the coordinate range, then the sets cannot be distinguished at all, because the dashed line is always above the solid one.

The results of invariants (40) normalized by the Gaussian distribution function are given in Fig. 6.

The result is similar to the previous case, the ratio of the scales is 11.18, that means a little bit worse discriminability. The main difference is the scale on the vertical axis, which is about twice larger. It means these invariants utilize the given range better.

The second experiment demonstrates using the pairing algorithm by means of the projective and permutation invariants. A cut of a Landsat Thematic Mapper image of north-east Bohemia (surroundings of the town Trutnov) from 29 August 1990 ($256 \times 256$) was used as the

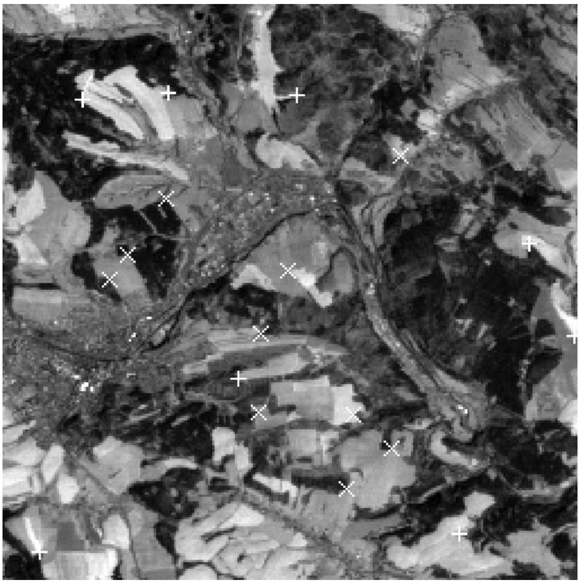

Fig. 7. The satellite image used as reference one ( × the control points with counterparts in the input image, + the points without counterparts).

reference image (see Fig. 7) and an aerial image from 1984 ($180 \times 256$) with a relatively strong projective distortion was used as the input one (see Fig. 8).

Sixteen points were selected in the input image (see their coordinates in Table 1), 18 points were selected in the reference one (see their coordinates in Table 2) and 10 points in both images had counterparts in the other image (numbers 1–10 in the input correspond to numbers 9–18 in the reference). The first $\binom{18}{5} = 8568$ best matches was examined and the 476th one was correct. All 10 pairs of control points were found, the distance of the tenth pair was 2.23 pixels. The result is shown in Fig. 9. The final parameters of the transform were computed from all 10 control points by means of the least-square method. The deviations on the control points were from 0.17 to 1.71 pixels, the average was 0.94 pixels.

The time of the search of the best matches was about an hour and a half on the workstation HP 9000/700 and the time of the examination of all 8568 matches was about two hours and a half, but the correct 476th match was found in 8 min.

The method supposes plane point sets, that is satisfied on these images only approximately. In our experience, if the height of the flight is significantly greater than altitude differences between hills and valleys, then the influence of the terrain causes only small perturbations of point coordinates. Owing to the robustness of the algorithm, we can handle those cases satisfactorily.

Fig. 8. The aerial image used as the input one ( × the control points with counterparts in the reference image, + the points without counterparts).

Table 1
The coordinates of the points marked by × and + in the input image in Fig. 8

| No. | $x$ | $y$ |
|---|---|---|
| 1 | 35 | 42 |
| 2 | 233 | 106 |
| 3 | 104 | 166 |
| 4 | 253 | 147 |
| 5 | 16 | 243 |
| 6 | 202 | 235 |
| 7 | 73 | 39 |
| 8 | 130 | 40 |
| 9 | 55 | 111 |
| 10 | 176 | 67 |
| 11 | 172 | 197 |
| 12 | 152 | 215 |
| 13 | 47 | 122 |
| 14 | 72 | 86 |
| 15 | 126 | 118 |
| 16 | 114 | 146 |
| 17 | 113 | 181 |
| 18 | 155 | 182 |

Table 2
The coordinates of the points marked by $\times$ and $+$ in the reference image in Fig. 7

| No. | $x$ | $y$ |
|---|---|---|
| 1 | 22 | 26 |
| 2 | 161 | 27 |
| 3 | 117 | 189 |
| 4 | 64 | 214 |
| 5 | 8 | 31 |
| 6 | 50 | 15 |
| 7 | 90 | 52 |
| 8 | 61 | 73 |
| 9 | 35 | 114 |
| 10 | 96 | 142 |
| 11 | 65 | 143 |
| 12 | 14 | 116 |
| 13 | 35 | 153 |
| 14 | 145 | 149 |
| 15 | 139 | 51 |
| 16 | 161 | 75 |

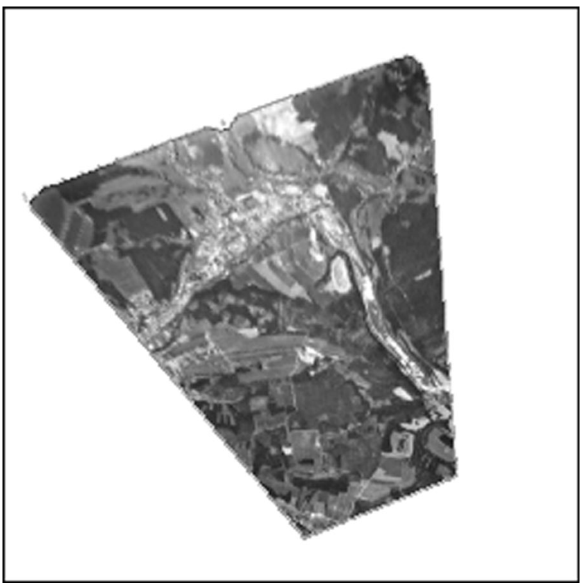

Fig. 9. The registered image.

## 6. Conclusion

The roots of the polynomials in the five-point projective and permutation invariants have one degree of freedom. We can choose one of them and the others must lie symmetrically on certain curves. The normalization of these invariants is suitable for improving numerical stability of following computations with them. The normalization of the invariants for more than five points is also suitable, because then they can be used for recognition in Euclidean feature space without any additional weights. The normalization by the Gaussian distribution function is suitable in case of less noise for better distinguishing of the sets.

We can use the invariants also for registration of the images by means of control points. If the affine and simpler transforms can be used for approximation of the distortion between the images, other methods are suitable. In case of strong projective distortion between the images, the described algorithm is one of the possible solutions of the task. The minimum number of corresponding pairs is six and correspondence between point sets with less pairs of points cannot be found principally. In case of six corresponding pairs only once the wrong correspondence was found during tens of experiments, in case of more than six corresponding pairs no error was found. It means that in case of more than six corresponding pairs the hope of a successful result is very high.

## 7. Summary

The paper deals with features of a point set which are invariant with respect to a projective transform. First, projective invariants for five-point sets, which are simultaneously invariant to the projective transform and to permutation of the points, are derived. They are expressed as functions of five-point cross-ratios. The roots of the polynomials in the five-point projective and permutation invariants have one degree of freedom. We can choose one of them and the others must lie symmetrically on certain curves. The normalization of these invariants is suitable for improving the numerical stability of following computations with them.

The invariants for more than five points are derived. The normalization of the invariants for more than five points is also suitable, because then they can be used for recognition in Euclidean feature space without any additional weights. The normalization by the Gaussian distribution function is suitable in case of less noise for better distinguishing of the sets else the normalization by the average and the root should be used.

The algorithm for searching the correspondence between the points of two 2-D point sets is presented. The algorithm is based on the comparison of two projective and permutation invariants of five-tuples of the points. The best-matched five-tuples are then used for the computation of the projective transformation and that with the maximum of corresponding points is used. Stability and discriminability of the features and behavior of the searching algorithm are demonstrated by numerical experiments.

## Acknowledgements

This work has been supported by the Grant Nos. 102/98/P069 and No. 102/96/1694 of the Grant Agency of the Czech Republic.

## Appendix A

Sometimes a task on how to save and load information about combinations to and from a memory may be required to be solved. We have got the combinations of $k$ elements from n and we can save this information in the following way:

$a = 0$

for $i_1 := 0;\ i_1 < n$

for $i_2 := i_1 + 1;\ i_2 < n$

$\vdots$

for $i_k := i_{k-1} + 1;\ i_k < n$

$\{m[a] :=$ information $(i_1, i_2, \dots, i_k)$

$a := a + 1\}$

When information about $k$-tuple $(i_1, i_2, \dots, i_k)$ is required, we need to compute the address $a$ from the $k$-tuple. If we sort the indices by size so it holds

$i_1 < i_2 < \cdots < i_k,$

then this address can be computed as

$$a = \binom{n}{k} - 1 + \sum_{j=1}^{k} \sum_{m=0}^{k-j+1} (-1)^{m+1} \binom{n}{k-j-m+1} \binom{i_j + m}{m}. \tag{41}$$

## Appendix B

$$p(I_1') = 10.110488 \cdot I_1''^6 - 27.936483 \cdot I_1'^5 + 31.596612 \cdot I_1'^4 - 16.504259 \cdot I_1'^3 - 0.32251158 \cdot I_1' + 3.0473587 \cdot I_1' - 0.66901966.$$

If $I_1' < 0.475$ then

$$d(I_1') = 17.575974 \cdot I_1'^4 - 16.423212 \cdot I_1'^3 + 9.111527 \cdot I_1'^2 - 0.43942294 \cdot I_1' + 0.016542258$$

else

$$d(I_1') = 3.9630392 \cdot I_1'^4 - 13.941518 \cdot I_1'^3 + 21.672754 \cdot I_1'^2 - 17.304971 \cdot I_1' + 5.6198814.$$

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