# MULTINOMIAL IDENTITIES ARISING FROM FREE PROBABILITY THEORY 

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#### Abstract

We prove a family of new identities fulfilled by multinomial coefficients, which were conjectured by Dykema and Haagerup. Our method bases on a study of the, so-called, triangular operator T by the means of the free probability theory.


## 1. Introduction

1.1. Overview. In order to answer some questions in the theory of operator algebras Dykema and Haagerup started investigation of the, so-called, triangular operator T [DH01]. Currently there are many different descriptions of this operator: in terms of random matrices, in terms of free probability theory and a purely combinatorial one (and we will recall them in the following).

Dykema and Haagerup conjectured that the moments of this operator fulfill

$$
\begin{equation*}
\phi\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n}\right]=\frac{n^{n k}}{(n k+1)!} \tag{1}
\end{equation*}
$$

for any $k, n \in \mathbb{N}$. By using the combinatorial description of T Dykema and Yan [DY01] showed that this conjecture would imply an infinite family of identities for multinomial coefficients. For example, for $n=2$ the conjecture is equivalent to a well-known identity (in the following we will be taking sums over nonnegative integers):

$$
\begin{equation*}
2^{2 k}=\sum_{p+q=k}\binom{2 p}{p}\binom{2 q}{q} \tag{2}
\end{equation*}
$$

[^0]while for $n=3$ is equivalent to the following one, not known before:
\[

$$
\begin{align*}
& 3^{3 k}=\sum_{p+q=k}\binom{3 p}{p, p, p}\binom{3 q}{q, q, q}+  \tag{3}\\
&+3 \sum_{\substack{p+q+r=k-1 \\
r^{\prime}+q^{\prime}=r+q+1 \\
p^{\prime \prime}+r^{\prime \prime}=p+r+1}}\binom{2 p+p^{\prime \prime}}{p, p, p^{\prime \prime}}\binom{2 q+q^{\prime}}{q, q, q^{\prime}}\binom{r+r^{\prime}+r^{\prime \prime}}{r, r^{\prime}, r^{\prime \prime}}
\end{align*}
$$
\]

The complication of the formula grows superexponentially with $n$ and already for $\mathrm{n}=4$ it becomes very complicated:

$$
\begin{align*}
& 4^{4 k}=\sum_{p+q=k}\binom{4 p}{p, p, p, p}\binom{4 q}{q, q, q, q}+  \tag{4}\\
& +8 \sum_{\substack{p+q+r=k-1 \\
p^{\prime}+q^{\prime}=\boldsymbol{p}+q+1 \\
p^{\prime \prime}+q^{\prime \prime}=p+q+1 \\
q^{\prime \prime \prime}+r^{\prime \prime \prime}=q+r+1}}\binom{2 p+p^{\prime}+p^{\prime \prime}}{p, p, p^{\prime}, p^{\prime \prime}}\binom{q+q^{\prime}+q^{\prime \prime}+q^{\prime \prime \prime}}{q, q^{\prime}, q^{\prime \prime}, q^{\prime \prime \prime}}\binom{3 r+r^{\prime \prime \prime}}{r, r, r, r^{\prime \prime \prime}}+ \\
& +4 \sum_{\substack{p+q^{\prime}++^{\prime}=k-1 \\
p+q^{\prime \prime}+r^{\prime \prime}=k-1 \\
p^{\prime \prime \prime}+q^{\prime \prime \prime}=p+q^{\prime}+1 \\
p^{\prime \prime \prime}+q^{\prime \prime \prime}=p+q^{\prime \prime}+1}}\binom{2 p+p^{\prime \prime \prime}+p^{\prime \prime \prime \prime}}{p, p, p^{\prime \prime \prime}, p^{\prime \prime \prime \prime}}\binom{q^{\prime}+q^{\prime \prime \prime}}{q^{\prime}, q^{\prime \prime \prime}}\binom{q^{\prime \prime}+q^{\prime \prime \prime \prime}}{q^{\prime \prime}, q^{\prime \prime \prime \prime}} \times \\
& \times\binom{ 2 r^{\prime \prime}}{r^{\prime \prime}, r^{\prime \prime}}\binom{2 r^{\prime}}{r^{\prime}, r^{\prime}}\binom{q^{\prime}+q^{\prime \prime}+q^{\prime \prime \prime}+q^{\prime \prime \prime \prime}+2 r^{\prime}+2 r^{\prime \prime}+2}{q^{\prime}+q^{\prime \prime \prime}+2 r^{\prime}+1, q^{\prime \prime}+q^{\prime \prime \prime \prime}+2 r^{\prime \prime}+1}+ \\
& +8 \sum_{\substack{p+q+r+s=k-2 \\
q^{\prime}+r^{\prime}=q+r+s+2 \\
p^{\prime \prime}+r^{\prime \prime}=p+q+r+2}}\binom{2 p}{p, p}\binom{q+q^{\prime}}{q, q^{\prime}}\binom{r+r^{\prime \prime}}{r, r^{\prime \prime}}\binom{2 s}{s, s} \times \\
& \times\binom{ 3 p+p^{\prime \prime}+2 q+q^{\prime}+2}{2 p+q+q^{\prime}+1, p+q+1, p^{\prime \prime}}\binom{2 r+r^{\prime}+r^{\prime \prime}+3 s+2}{r+r^{\prime \prime}+2 s+1, r+s+1, r^{\prime}} .
\end{align*}
$$

The main result of this article is the proof of the conjecture (11). Since this conjecture was originally formulated in the language of the free probability theory, it should not be a great surprise that also its proof is formulated in this language. However, in order to make this article as comprehensible as possible for the wide audience, we will use only combinatorial methods of free probability and include all necessary notions.

This article is organized as follows. In the remaining part of Sect. 1 we present briefly the operator T from the random matrix and operator theoretic point of view. In Sect. 2 we present some rudimentary concepts from the
probability theory, in particular the notion of the generalized circular element. In Sect. 3we define the triangular operator T as a certain generalized circular elements and we prove the conjecture (11). In Sect. 4 we show how to express moments of T in terms of multinomial coefficients and hence to get Eq. (2)-(4) (and an infinite number of other identities). Finally, Sect. [5] is devoted to some technical proofs.
1.2. Triangular operator T-motivations. This section is independent of the rest of this article and can be skipped by a reader interested only in the combinatorics. We hope, however, that a broader view presented here might be interesting even for mathematicians not involved in operator theory.

Von Neumann algebras are algebras of bounded operators on a Hilbert space $\mathcal{H}$, containing identity operator and $A^{\star}$ when they contain $A$ and closed in strong-operator topology [KR97a, KR97b]. Especially interesting are type $\mathrm{II}_{1}$ factors $\mathcal{A}$, i.e. von Neumann algebras equipped with a unique normalized faithful tracial state $\phi: \mathcal{A} \rightarrow \mathbb{C}$.

There are many important open questions concerning type $\mathrm{II}_{1}$ factors, for example the famous invariant subspace conjecture asking if for every $x \in \mathcal{A}$ (which is not a multiple of identity operator) there exists a closed invariant subspace $\mathcal{K} \subset \mathcal{H}$ which is nontrivial $(\mathcal{K} \neq\{0\}$ and $\mathcal{K} \neq \mathcal{H})$ and such that the orthogonal projection $\pi_{\mathcal{K}}: \mathcal{H} \rightarrow \mathcal{K}$ fulfills $\pi_{\mathcal{K}} \in \mathcal{A}$. There are many partial results concerning this question and the most recent one is due to Haagerup [Haa01]; he shows that such an invariant subspace exists if the operator $x$ can be appoximated in a certain way by finite-dimensional matrices (or, strictly speaking, if $\mathcal{A}$ is embeddable into $\mathrm{R}^{\omega}$, the ultrapower of the hyperfinite factor) and if the eigenvalues of $x$ are not all equal (or, strictly speaking, if the Brown measure of $x$ is not supported in a single point). This result restricts strongly the class of possible counterexamples for the invariant subspace conjecture and suggests us to study quasinilpotent operators, for which the assumptions of Haagerup's theorem are not fulfilled.

The triangular operator T of Dykema and Haagerup arose as a natural candidate for such a counterexample. The distribution of $T$ was defined originally [DH01] as the limit of distribution of random matrices $\mathrm{T}_{\mathrm{N}}$ :

$$
\begin{equation*}
\phi\left(\mathrm{T}^{s_{1}} \cdots \mathrm{~T}^{s_{n}}\right)=\lim _{\mathrm{N} \rightarrow \infty} \frac{1}{\mathrm{~N}} \mathbb{E} \operatorname{Tr}_{\mathrm{N}}^{s_{1}} \cdots \mathrm{~T}_{\mathrm{N}}^{s_{n}} \tag{5}
\end{equation*}
$$

for any $n \in \mathbb{N}$ and $s_{1}, \ldots, s_{n} \in\{1, \star\}$, where

$$
\mathrm{T}_{\mathrm{N}}=\left[\begin{array}{ccccc}
\mathrm{t}_{1,1} & \mathrm{t}_{1,2} & \cdots & \mathrm{t}_{1, n-1} & \mathrm{t}_{1, n} \\
0 & \mathrm{t}_{2,2} & \cdots & \mathrm{t}_{2, n-1} & \mathrm{t}_{2, n} \\
\vdots & & \ddots & \vdots & \vdots \\
& & & t_{n-1, n-1} & t_{n-1, n} \\
0 & & \cdots & 0 & t_{n, n}
\end{array}\right]
$$

and $\left(\mathrm{t}_{\mathrm{i}, \mathrm{j}}\right)_{1 \leq i \leq j \leq \mathrm{N}}$ are independent centered Gaussian random variables with variance $\frac{1}{\mathrm{~N}}$.

When this article was nearly finished Dykema and Haagerup announced that they had proved existence of nontrivial hyperinvariant subspaces of the operator T. Their proof uses Theorem 2 from this article.

## 2. OPERATOR-VALUED FREE PROBABILITY THEORY

2.1. Free probability. Free probability theory was initiated by Voiculescu in order to answer some old questions in the theory of operator algebras [VDN92, Voi95, Voi00], but it soon evolved into an exciting self-standing theory with many links to other fields, to mention only the theory of random matrices [Voi91], theory of representations of groups of permutations $S_{n}$ [Bia98] and theoretical physics [SN99]. Free probability has also its combinatorial aspect connected with the so-called noncrossing partitions [Spe97].

Many questions concerning large random matrices can be easily reformulated and answered in the framework of operator-valued free probability; this is also the case of the triangular operator T . One can show that the original definition (5) is equivalent to our favourite definition of T as a generalized circular operator which will be presented in Section 3 [Shl96, Shl98]. However, since in this article we do not need this equivalence, we skip the proof.

In this section we recall briefly some combinatorial aspects the operatorvalued free probability. More information can be found in papers [Spe97 Spe98].
2.2. Operator-valued probability space. A triple $(\mathcal{B} \subseteq \mathcal{A}, \mathbb{E})$ is called an operator-valued probability space if $\mathcal{A}$ is a unital $\star$-algebra, $\mathcal{B} \subseteq \mathcal{A}$ is a unital $\star$-subalgebra, and $\mathbb{E}: \mathcal{A} \rightarrow \mathcal{B}$ is a conditional expectation, i.e. $\mathbb{E}$ is linear, satisfies $\mathbb{E}(1)=1$ and $\mathbb{E}\left(b_{1} x b_{2}\right)=b_{1} \mathbb{E}(x) b_{2}$ for every $b_{1}, b_{2} \in \mathcal{B}$ and $x \in \mathcal{A}$.
2.3. Noncrossing pair partitions. If $X$ is a finite, ordered set, we denote by $\mathrm{NC}_{2}(\mathrm{X})$ the set of all noncrossing pair partitions of $X$ [Spe98, Kre72].

A noncrossing pair partition $\pi=\left\{Y_{1}, \ldots, Y_{n}\right\}$ of $X$ is a decomposition of $X$ into disjoint two-element sets:

$$
X=Y_{1} \cup \cdots \cup Y_{n}, \quad Y_{i} \cap Y_{j}=\emptyset \quad \text { if } i \neq j
$$

which has the additional property that for $Y_{i}=\{a, c\}$ and $Y_{j}=\{b, d\}$ it cannot happen that $\mathrm{a}<\mathrm{b}<\mathrm{c}<\mathrm{d}$.

We say that the sets $Y_{1}, \ldots, Y_{n}$ are the lines of the pair partition $\pi=$ $\left\{\mathrm{Y}_{1}, \ldots, \mathrm{Y}_{n}\right\}$. It is useful to describe pair partitions graphically by connecting elements of the same line by an arc.

Example. There are exactly two noncrossing pair partitions of the set $\{1,2,3,4\}$ and these are $\{\{1,4\},\{2,3\}\}$ represented by 1234 and $\{\{1,2\},\{3,4\}\}$ represented by 1234 . On the other hand the pair partition $\{\{1,3\},\{2,4\}\}$ is crossing as it can be seen on its graphical representation 1234.

以
2.4. Nested evaluation. For the purpose of this section we shall forget that $\mathcal{A}$ is an algebra and that $\mathcal{B} \subseteq \mathcal{A}$. We will assume only that $\mathcal{B}$ is an algebra and that $\mathcal{A}$ is a $\mathcal{B}$-bimodule.

Let there be given a bilinear map $\mathrm{k}: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}$. We will denote

$$
a_{1} \bullet_{k} a_{2}=\kappa\left(a_{1}, a_{2}\right)
$$

and assume that $\mathrm{\kappa}$ is such that

$$
\begin{align*}
\left(b a_{1}\right) \bullet_{k} a_{2} & =b\left(a_{1} \bullet_{\kappa} a_{2}\right),  \tag{6}\\
\left(a_{1} b\right) \bullet_{k} a_{2} & =a_{1} \bullet_{\kappa}\left(b a_{2}\right),  \tag{7}\\
a_{1} \bullet_{\kappa}\left(a_{2} b\right) & =\left(a_{1} \bullet_{k} a_{2}\right) b \tag{8}
\end{align*}
$$

for any $b \in \mathcal{B}$ and $a_{1}, a_{2} \in \mathcal{A}$.
Noncrossing pair partitions of the set $\{1,2, \ldots, 2 n\}$ (or, equivalently, of the set $\left\{a_{1}, \ldots, a_{2 n}\right\}$ ) can be identified with ways of writing brackets in the product $a_{1} a_{2} \ldots a_{2 n}$ in such a way that inside each pair of brackets there are exactly two factors $a_{i}$ and $a_{j}$ (and possibly some other nested brackets). To be more explicit: each line $Y=\left\{a_{i}, a_{j}\right\}$ of a pair partition corresponds to a certain pair of brackets: the opening bracket and its closing counterpart. The two factors contained in between this pair of brackets are exactly $a_{i}$ and $a_{j}$.

## Example.

$$
a_{1} a_{2} a_{3} a_{4} a_{5} a_{6} \text { corresponds to }\left(a_{1} a_{2}\right)\left(a_{3} a_{4}\right)\left(a_{5} a_{6}\right)
$$

$a_{1} a_{2} a_{3} a_{4} a_{5} a_{6}$ corresponds to $\left(a_{1} a_{2}\right)\left(a_{3}\left(a_{4} a_{5}\right) a_{6}\right)$,

$a_{1} a_{2} a_{3} a_{4} a_{5} a_{6}$ corresponds to $\left(a_{1}\left(a_{2} a_{3}\right)\left(a_{4} a_{5}\right) a_{6}\right)$,

$a_{1} a_{2} a_{3} a_{4} a_{6}$ corresponds to $\left(a_{1}\left(a_{2}\left(a_{3} a_{4}\right) a_{5}\right) a_{6}\right)$.


In order to evaluate such a product with brackets we will use the following rule: in order to multiply two elements of $\mathcal{A}$ we use the product $\bullet_{\kappa}$; in order to multiply two elements of $\mathcal{B}$ or an element of $\mathcal{B}$ and an element of $\mathcal{A}$ we use the standard multiplication. In this way for any noncrossing pair partition $\pi \in \mathrm{NC}_{2}(\{1, \ldots, 2 n\})$ we have defined a multilinear map $\kappa_{\pi}: \mathcal{A}^{2 n} \rightarrow \mathcal{B}$.

## Example.

$$
\begin{aligned}
& \kappa(\underbrace{a_{1}}, a_{2}, \underbrace{a_{3}}, a_{4}, \underbrace{a_{5}}, a_{6})=\left(a_{1} \bullet_{\kappa} a_{2}\right)\left(a_{3} \bullet_{\kappa} a_{4}\right)\left(a_{5} \bullet_{\kappa} a_{6}\right)= \\
& \kappa\left(a_{1}, a_{2}\right) \kappa\left(a_{3}, a_{4}\right) \kappa\left(a_{5}, a_{6}\right), \\
& \kappa\left(a_{1}, a_{2}, a_{3}, a_{4}, a_{5}, a_{6}\right)=\left(a_{1} \bullet_{\kappa} a_{2}\right)\left(a_{3} \bullet_{\kappa}\left(a_{4} \bullet_{\kappa} a_{5}\right) a_{6}\right)= \\
& \kappa\left(a_{1}, a_{2}\right) \kappa\left(a_{3}, \kappa\left(a_{4}, a_{5}\right) a_{6}\right), \\
& \kappa\left(a_{1}, a_{2}, a_{3}, a_{4}, a_{5}, a_{6}\right)=a_{1} \bullet_{\kappa}\left(a_{2} \bullet_{\kappa} a_{3}\right)\left(a_{4} \bullet_{\kappa} a_{5}\right) a_{6}= \\
& \checkmark \vee \kappa\left(a_{1}, \kappa\left(a_{2}, a_{3}\right) \kappa\left(a_{4}, a_{5}\right) a_{6}\right) \text {, } \\
& \kappa\left(a_{1}, a_{2}, a_{3}, a_{4}, a_{5}, a_{6}\right)=a_{1} \bullet_{\kappa}\left(a_{2} \bullet_{\kappa}\left(a_{3} \bullet_{\kappa} a_{4}\right) a_{5}\right) a_{6}= \\
& \checkmark \kappa\left(a_{1}, \kappa\left(a_{2}, \kappa\left(a_{3}, a_{4}\right) a_{5}\right) a_{6}\right) \text {. }
\end{aligned}
$$

2.5. Generalized circular elements. Let an operator-valued probability space $(\mathcal{B} \subseteq \mathcal{A}, \mathbb{E})$ by given. We say that $\mathrm{T} \in \mathcal{A}$ is a generalized circular element if there exists a bilinear function $\mathrm{\kappa}: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}$ (called variance of $T$ ) which fulfills (6)-(8) and such that
$\mathbb{E}\left(b_{1} T^{s_{1}} b_{2} T^{s_{2}} \cdots b_{2 n} T^{s_{2 n}}\right)=\sum_{\pi \in \mathrm{NC}_{2}(\{1, \ldots, 2 n\})} \kappa_{\pi}\left(b_{1} T^{s_{1}}, b_{2} T^{s_{2}}, \ldots, b_{2 n} T^{s_{2 n}}\right)$,

$$
\begin{equation*}
\mathbb{E}\left(b_{1} T^{s_{1}} b_{2} T^{s_{2}} \cdots b_{2 n+1} T^{s_{2 n+1}}\right)=0 \tag{10}
\end{equation*}
$$

for every $b_{1}, \ldots, b_{2 n+1} \in \mathcal{B}$ and $s_{1}, \ldots, s_{2 n+1} \in\{1, \star\}$.

## 3. The main result

3.1. Triangular operator $T$. Let $\mathcal{B}=\mathbb{C}[x]$ be the $\star$-algebra of polynomials of one variable with multiplication defined to be the usual multiplication of polynomials and let $(\mathcal{B} \subset \mathcal{A}, \mathbb{E})$ be an operator-valued probability space.

The Dykema-Haagerup triangular operator $\mathrm{T} \in \mathcal{A}$ is defined to be the generalized circular element with the variance $\kappa$ given by

$$
\left\{\begin{align*}
{\left[k\left(T, b T^{\star}\right)\right](x) } & =\int_{x}^{1} b(t) d t  \tag{11}\\
{\left[k\left(T^{\star}, b T\right)\right](x) } & =\int_{0}^{x} b(t) d t \\
{[k(T, b T)](x) } & =0 \\
{\left[k\left(T^{\star}, b T^{\star}\right)\right](x) } & =0
\end{align*}\right.
$$

for any $b \in \mathcal{B}$.
From the following on we shall assume that the algebra $\mathcal{A}$ is generated by the algebra $\mathcal{B}$ and operators $T$ and $T^{\star}$.
3.2. Automorphism $\alpha$. Let us consider a linear automorphism of the algebra $\mathcal{A}$ defined on generators by

$$
\begin{gathered}
\alpha(\mathrm{T})=\mathrm{T}^{\star}, \quad \alpha\left(\mathrm{T}^{\star}\right)=\mathrm{T}, \\
{[\alpha(\mathrm{~b})](\mathrm{x})=\mathrm{b}(1-\mathrm{x}) \quad \text { for any } \mathrm{b} \in \mathcal{B} .}
\end{gathered}
$$

Proposition 1. For any $a \in \mathcal{A}$ we have

$$
\mathbb{E}[\alpha(a)]=\alpha[\mathbb{E}(a)]
$$

Proof. It is easy to see that the defining relations (11) give

$$
\kappa\left(\alpha\left(T^{s_{1}}\right), \alpha(b) \alpha\left(T^{s_{2}}\right)\right)=\alpha\left(\kappa\left(T^{s_{1}}, b T^{s_{2}}\right)\right)
$$

for any $s_{1}, s_{2} \in\{1, \star\}$ and $b \in \mathcal{B}$ and, by induction, that

$$
\kappa_{\pi}\left(\alpha\left(b_{1} T^{s_{1}}\right), \ldots, \alpha\left(b_{n} T^{s_{n}}\right)\right)=\alpha\left(\kappa\left(b_{1} T^{s_{1}}, \ldots, b_{n} T^{s_{n}}\right)\right)
$$

for any $s_{1}, \ldots, s_{n} \in\{1, \star\}, b_{1}, \ldots, b_{n} \in \mathcal{B}$ and $\pi \in \operatorname{NC}_{2}(\{1, \ldots, n\})$. Finally, we use the definition (9).
3.3. Scalar-valued distribution of $T$. We define a state $\phi: \mathcal{A} \rightarrow \mathbb{C}$ as follows: for $b \in \mathcal{B}$ we put

$$
\phi(b)=\int_{0}^{1} b(x) d x
$$

while for a general $a \in \mathcal{A}$ we put

$$
\phi(a)=\phi(\mathbb{E}(a)) .
$$

Remark. Alternative defintions of the state $\phi$ on the $\star$-algebra generated by T can be found in articles by Dykema and Haagerup [DH01] and by Dykema and Yan [DY01], as well as the proof that $\phi$ is indeed a tracial state.
3.4. Proof of Dykema-Haagerup conjecture. The following theorem will be our main tool in the proof of Dykema-Haagerup conjecture. However, since its proof is a bit technical, we postpone it to Sect. [5] We will use the convention that $\mathrm{T}^{1}=\mathrm{T}$ while $\mathrm{T}^{-1}=\mathrm{T}^{\star}$.

Theorem 2. Let $s_{1}, \ldots, s_{2 m} \in\{1, \star\}$. Then $\mathbb{E}\left(T^{s_{1}} \ldots T^{s_{2 m}}\right)$ is a polynomial of degree m and hence $\frac{\mathrm{d}^{m}}{\mathrm{~d} x^{m}} \mathbb{E}\left(\mathrm{~T}^{\mathrm{s}_{1}} \cdots \mathrm{~T}^{s_{2 m}}\right)$ can be identified with a real number.

Secondly, for any $\mathrm{n} \in \mathbb{N}$ we have

$$
\begin{align*}
& \frac{d^{n}}{d x^{n}} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2 m}}\right)=\sum_{\substack{0=j_{0}<j_{1}<\cdots \\
\cdots \ll j_{2 n}<j_{2 n+1}=2 m+1}}  \tag{12}\\
& \left(\frac{d^{n}}{d x^{n}} \mathbb{E}\left(T^{s_{j}} T^{s_{j}} \cdots T^{s_{j_{2 n}}}\right)\right) \prod_{0 \leq r \leq 2 m} \mathbb{E}\left(T^{s_{j}+1} T^{s_{j_{r}+2}} \cdots T^{s_{j_{r+1}-1}}\right) .
\end{align*}
$$

In the above sum the nonvanishing terms are obtained only for sequences $\left(\mathrm{j}_{\mathrm{k}}\right)$ such that $\mathrm{s}_{\mathrm{j}_{\mathrm{r}}+1}+\mathrm{s}_{\mathrm{j}_{\mathrm{r}}+2}+\cdots+\mathrm{s}_{\mathrm{j}_{\mathrm{r}+1}-1}=0$ for every $0 \leq \mathrm{r} \leq 2 \mathrm{~m}$.
Example. Due to (10) the only nonvanishing terms for $\mathfrak{m}=3$ and $\mathfrak{n}=1$ are

$$
\begin{aligned}
& \frac{d}{d x} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{6}}\right)=\left(\frac{d}{d x} \mathbb{E}\left(T^{s_{1}} T^{s_{2}}\right)\right) \mathbb{E}\left(T^{s_{3}} \cdots T^{s_{6}}\right)+ \\
&\left(\frac{d}{d x} \mathbb{E}\left(T^{s_{1}} T^{s_{4}}\right)\right) \mathbb{E}\left(T^{s_{2}} T^{s_{3}}\right) \mathbb{E}\left(T^{s_{5}} T^{s_{6}}\right)+\left(\frac{d}{d x} \mathbb{E}\left(T^{s_{1}} T^{s_{6}}\right)\right) \mathbb{E}\left(T^{s_{2}} \cdots T^{s_{5}}\right)+ \\
&\left(\frac{d}{d x} \mathbb{E}\left(T^{s_{3}} T^{s_{4}}\right)\right) \mathbb{E}\left(T^{s_{1}} T^{s_{2}}\right) \mathbb{E}\left(T^{s_{5}} T^{s_{6}}\right)+\left(\frac{d}{d x} \mathbb{E}\left(T^{s_{3}} T^{s_{6}}\right)\right) \mathbb{E}\left(T^{s_{1}} T^{s_{2}}\right) \mathbb{E}\left(T^{s_{4}} T^{s_{5}}\right)+ \\
&\left(\frac{d}{d x} \mathbb{E}\left(T^{s_{5}} T^{s_{6}}\right)\right) \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{4}}\right)
\end{aligned}
$$

Theorem 3. Let $s_{1}, \ldots, s_{2 m} \in\{1,-1\}$ be such that $s_{1}, s_{1}+s_{2}, \ldots, s_{1}+$ $\cdots+s_{2 m-1} \leq 0$ and $s_{1}+\cdots+s_{2 m}=0$. Then

$$
\frac{\mathrm{d}^{\mathrm{m}}}{\mathrm{dx} x^{\mathrm{m}}} \mathbb{E}\left(\mathrm{~T}^{s_{1}} \cdots \mathrm{~T}^{s_{2 m}}\right)=1
$$

Let $\mathrm{s}_{1}, \ldots, \mathrm{~s}_{2 \mathrm{~m}} \in\{1,-1\}$ be such that $\mathrm{s}_{1}, \mathrm{~s}_{1}+\mathrm{s}_{2}, \ldots, \mathrm{~s}_{1}+\cdots+\mathrm{s}_{2 \mathrm{~m}-1} \geq 0$ and $s_{1}+\cdots+s_{2 m}=0$. Then

$$
\frac{\mathrm{d}^{\mathrm{m}}}{\mathrm{~d} x^{\mathrm{m}}} \mathbb{E}\left(\mathrm{~T}^{s_{1}} \cdots \mathrm{~T}^{s_{2 m}}\right)=(-1)^{m}
$$

Proof. We shall prove the first part of the theorem by induction (the proof of the second part is analogous and we skip it).

Let us compute $\frac{\mathrm{d}^{\mathrm{m}-1}}{\mathrm{~d} x^{\mathrm{m}-1}} \mathbb{E}\left(\mathrm{~T}^{s_{1}} \cdots \mathrm{~T}^{s_{2 m}}\right)$ from Theorem 2. It is easy to observe that it yields that

$$
\begin{aligned}
& \frac{d^{m-1}}{d x^{m-1}} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2 m}}\right)= \\
& \sum_{1 \leq k \leq 2 m-1}\left(\frac{d^{m-1}}{d x^{m-1}} \mathbb{E}\left(T^{s_{1}} T^{s_{2}} \cdots T^{s_{k-1}} T^{s_{k+2}} T^{s_{k+3}} \cdots T^{s_{2} m}\right)\right) \mathbb{E}\left(T^{s_{k}} T^{s_{k+1}}\right)
\end{aligned}
$$

For nonzero summands the inductive hypothesis can be applied and hence

$$
\frac{d^{m-1}}{d x^{m-1}} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2} m}\right)=\sum_{1 \leq k \leq 2 m-1} \mathbb{E}\left(T^{s_{k}} T^{s_{k+1}}\right)
$$

From (11) follows that if $s_{k}=s_{k+1}$ then $\mathbb{E}\left(T^{s_{k}} T^{s_{k+1}}\right)=0$. Since the sequence $s_{1}, \ldots, s_{2 m}$ must begin with -1 and end with 1 , hence for some $l \geq 0$ there are exactly $l$ values of the index $1 \leq k<2 \mathrm{~m}$ such that $\left(s_{\mathrm{k}}, s_{\mathrm{k}+1}\right)=(1,-1)$ and exactly $l+1$ values of the index $1 \leq \mathrm{k}<2 \mathrm{~m}$ such that $\left(s_{k}, s_{k+1}\right)=(-1,1)$. Therefore

$$
\frac{d^{m}}{d x^{m}} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2 m}}\right)=l \frac{d}{d x} \mathbb{E}\left(T^{1} T^{-1}\right)+(l+1) \frac{d}{d x} \mathbb{E}\left(T^{-1} T^{1}\right)=1
$$

Remark. We leave to the reader the proof of the following: suppose that for $m \in \mathbb{N}$ we have $s_{1}, \ldots, s_{2 m}, s_{1}^{\prime}, \ldots, s_{2 m}^{\prime} \in\{1,-1\}$ such that $s_{1}+\cdots+$ $s_{2 m}=s_{1}^{\prime}+\cdots+s_{2 m}^{\prime}=0$ and that for every $1 \leq k \leq 2 m$ the sums $s_{1}+\cdots+s_{k}$ and $s_{1}^{\prime}+\cdots+s_{k}^{\prime}$ have the same sign (to be precise: $\left(s_{1}+\cdots+\right.$ $\left.\left.s_{k}\right)\left(s_{1}^{\prime}+\cdots+s_{k}^{\prime}\right) \geq 0\right)$. Then

$$
\frac{d^{m}}{d x^{m}} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2 m}}\right)=\frac{d^{m}}{d x^{m}} \mathbb{E}\left(T^{s_{1}^{\prime}} \cdots T^{s_{2 m}^{\prime}}\right)
$$

Proposition 4. Let $s_{1}, \ldots, s_{2 m} \in\{1, \star\}$. If $s_{1}=1$ or $s_{2 m}=\star$ then

$$
\left.\mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2 m}}\right)\right|_{\chi=1}=0
$$

Proof. From defining relations (11) we obtain that for any $b \in \mathcal{B}$ and $s \in\{1, \star\}$ we have $\left.\mathbb{E}\left(\mathrm{Tb}^{s}\right)\right|_{x=1}=\left.\mathbb{E}\left(\mathrm{T}^{s} \mathrm{~T}^{\star}\right)\right|_{x=1}=0$. Now observe that in the evaluation of a nested product $\kappa_{\pi}\left(T^{s_{1}}, \ldots, T^{s_{2 m}}\right)$ one of the above expressions must appear (if observations are not successful we refer to Eq. (21).

Theorem 5. For every $k, n \in \mathbb{N}, x \in \mathbb{R}$ and $0 \leq m \leq k-1$

$$
\left.\begin{array}{rl}
\frac{d^{m}}{d x^{m}} & \mathbb{E}
\end{array}\left(\mathrm{~T}^{k}\left(\mathrm{~T}^{\star}\right)^{k}\right)^{n}\right]\left.\right|_{x=1}=0, \quad \begin{aligned}
& \frac{d^{k}}{d x^{k}} \mathbb{E}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n}\right](x)=(-1)^{k} \mathbb{E}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n-1}\right](x-1)
\end{aligned}
$$

Remark. Equations (13) and (14) allow us to find $\mathbb{E}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n}\right]$ uniquely if $\mathbb{E}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n-1}\right]$ is known. In particular, for $k=1$ equation (14) coincides with the defining relation of Abel polynomials (up to normalisation), hence

$$
\mathbb{E}\left[\left(T T^{\star}\right)^{n}\right](1-x)=\mathbb{E}\left[\left(T^{\star} T\right)^{n}\right](x)=\frac{1}{n!} A_{n}(x)=\frac{x(x-n)^{n-1}}{n!}
$$

Proof. We use Theorem 2 in order to compute $\left.\frac{d^{m}}{d x^{m}} \mathbb{E}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n}\right]\right|_{x=1}$. Proposition 4 implies that nonzero terms will be obtained only for sequences

$$
\begin{aligned}
& \mathfrak{j}_{1}=1, j_{2}=2, \ldots, j_{k}=k<j_{k+1}<j_{k+2}<\cdots<j_{2 m-k}< \\
& <j_{2 m-k+1}=2 k n-k+1, j_{2 m-k+2}=2 k n-k+2, \ldots, j_{2 m}=2 k n .
\end{aligned}
$$

It has twofold implications. Firstly, since the above condition cannot be fulfilled for $m<k$, the first part of the theorem follows.

Secondly, for every $m \geq 0$

$$
\left.\frac{d^{m+k}}{d x^{m+k}} \mathbb{E}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n}\right]\right|_{x=1}=\left.(-1)^{k+m} \frac{d^{m}}{d x^{m}} \mathbb{E}\left[\left(\left(T^{\star}\right)^{k} T^{k}\right)^{n-1}\right]\right|_{x=1}
$$

by applying Eq. (12) and Theorem 3 to both sides of the equation (we leave it as an exercise to the reader to verify that assumptions of Theorem 3 are fulfilled for nonvanishing summands). It follows that

$$
\frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{dx}^{k}} \mathbb{E}\left[\left(\mathrm{~T}^{\mathrm{k}}\left(\mathrm{~T}^{\star}\right)^{\mathrm{k}}\right)^{\mathrm{n}}\right](1+x)=(-1)^{\mathrm{k}} \mathbb{E}\left[\left(\left(\mathrm{~T}^{\star}\right)^{\mathrm{k}} T^{\mathrm{k}}\right)^{n-1}\right](1-x)
$$

since both sides of the above equation are polynomials, all derivatives of which coincide in $x=0$.

Now observe that

$$
\begin{aligned}
&(-1)^{k} \mathbb{E}\left[\left(\left(T^{\star}\right)^{k} T^{k}\right)^{n-1}\right](1-x)=(-1)^{k} \mathbb{E} {\left[\alpha\left(\left(\left(T^{\star}\right)^{k} T^{k}\right)^{n-1}\right)\right](x)=} \\
&(-1)^{k} \mathbb{E}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n-1}\right](x)
\end{aligned}
$$

which finishes the proof.
Remark. Let $k_{1} \geq k_{2} \geq \cdots \geq k_{n} \geq 1$. By $T^{k_{1}}\left(T^{\star}\right)^{k_{2}} T^{k_{3}} \cdots$ we denote the alternating product of powers of T and $\mathrm{T}^{\star}$ the last factor of which is equal to $\mathrm{T}^{k_{n}}$ (for n odd) or equal to $\left(\mathrm{T}^{\star}\right)^{k_{n}}$ (for $n$ even). We leave to the reader the proof of the following: if $0 \leq m \leq k_{1}-1$ then

$$
\begin{aligned}
& \left.\frac{d^{m}}{d x^{m}} \mathbb{E}\left[\left(T^{k_{1}}\left(T^{\star}\right)^{k_{2}} T^{k_{3}} \cdots\right)\left(\cdots\left(T^{\star}\right)^{k_{3}} T^{k_{2}}\left(T^{\star}\right)^{k_{1}}\right)\right]\right|_{x=1}=0 \\
& \frac{d^{k_{1}}}{d x^{k_{1}}} \mathbb{E}\left[\left(T^{k_{1}}\left(T^{\star}\right)^{k_{2}} T^{k_{3}} \cdots\right)\left(\cdots\left(T^{\star}\right)^{k_{3}} T^{k_{2}}\left(T^{\star}\right)^{k_{1}}\right)\right](x)= \\
& (-1)^{k_{1}} \mathbb{E}\left[\left(T^{k_{2}}\left(T^{\star}\right)^{k_{3}} T^{k_{4}} \cdots\right)\left(\cdots\left(T^{\star}\right)^{k_{4}} T^{k_{3}}\left(T^{\star}\right)^{k_{2}}\right)\right](x-1) .
\end{aligned}
$$

Corollary 6. Let $\mathrm{k}, \mathrm{n} \in \mathbb{N}$. For every $\mathrm{x}<1$ we have

$$
\begin{equation*}
\mathbb{E}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n}\right](x)=\operatorname{vol}\left\{\left(x_{1}, \ldots, x_{n k}\right) \in \mathbb{R}^{n k}:\right. \tag{15}
\end{equation*}
$$

$$
\left.x<x_{1}<x_{2}<\cdots<x_{\mathrm{kn}} \text { and } x_{k}<1 \text { and } x_{2 k}<2, \cdots, \text { and } x_{n k}<n\right\} .
$$

## Furthermore

$$
\phi\left[\left(T^{\mathrm{k}}\left(\mathrm{~T}^{\star}\right)^{\mathrm{k}}\right)^{\mathrm{n}}\right]=\operatorname{vol} \mathrm{V}_{\mathrm{k}, n},
$$

where

$$
\begin{aligned}
& V_{k, n}=\left\{\left(x_{0}, x_{1}, \ldots, x_{n k}\right) \in \mathbb{R}^{n k+1}: 0<x_{0}<x_{1}<\cdots<x_{k n}\right. \text { and } \\
&\left.x_{k}<1 \text { and } x_{2 k}<2, \cdots, \text { and } x_{n k}<n\right\} .
\end{aligned}
$$

Proof. From Theorem 5 it follows that

$$
\begin{aligned}
& \mathbb{E}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n}\right](x)= \\
& \quad \int_{x}^{1} \int_{x_{1}}^{1} \cdots \int_{x_{k-1}}^{1} \mathbb{E}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n-1}\right]\left(x_{k}-1\right) d x_{1} d x_{2} \cdots d x_{k}
\end{aligned}
$$

and since the volume in (15) can be written as an iterated integral, the first part of the Lemma follows by induction.

The second part of the Corollary is a direct consequence of the first one.

Theorem 7 (Conjecture of Dykema and Haagerup). For any $k, n \in \mathbb{N}$ we have

$$
\phi\left[\left(T^{\mathrm{k}}\left(\mathrm{~T}^{\star}\right)^{\mathrm{k}}\right)^{\mathrm{n}}\right]=\frac{\mathrm{n}^{\mathrm{nk}}}{(\mathrm{nk}+1)!}
$$

Proof. We denote

$$
\begin{aligned}
& S_{k, n}=\left\{\mathbf{x}=\left(x_{0}, \ldots, x_{n k}\right) \in \mathbb{R}^{n k+1}:\right. \\
& \left.\quad 0<x_{0}<\cdots<x_{n k}<n, \quad x_{0}, \ldots, x_{n k} \notin \mathbb{Z}\right\}
\end{aligned}
$$

and for $l \in\{0,1, \ldots, n-1\}$ and $\mathbf{x} \in S_{k, n}$ we define $s_{l}(\mathbf{x})+\frac{n k+1}{n}$ to be the number of elements of the tuple $\mathbf{x}$ which are also elements of the interval $(l, l+1)$ :

$$
s_{l}(\mathbf{x})=\overline{\overline{\left\{x_{0}, \ldots, x_{n k}\right\} \cap[l, l+1]}}-\frac{n k+1}{n} .
$$

So defined numbers fulfill

$$
s_{0}(\mathbf{x})+\cdots+s_{n-1}(\mathbf{x})=0 .
$$

For $\mathbf{x} \in S_{k, n}$ it is easy to see that $\mathbf{x} \in V_{k, n}$ if and only if

$$
s_{0}(\mathbf{x}), s_{0}(\mathbf{x})+s_{1}(\mathbf{x}), \ldots, s_{0}(\mathbf{x})+\cdots+s_{n-2}(\mathbf{x})>0
$$

For any integer number $\mathfrak{m}$ and $\mathbf{x} \in S_{k, n}$ we define $m+\mathbf{x} \in S_{k, n}$ to be the increasing sequence such that the set of values of $m+\mathbf{x}$ is equal to

$$
\left\{m+x_{0} \bmod n, m+x_{1} \bmod n, \ldots, m+x_{n k} \bmod n\right\}
$$

We recall that if $y \in \mathbb{R}$ then $y \bmod n$ is defined to be the $y^{\prime}$ such that $y^{\prime}-y$ is a multiple of $n$ and $0 \leq y^{\prime}<n$.

Note that $s_{l}(m+\mathbf{x})=s_{(l-m) \bmod n}(\mathbf{x})$ and hence the sequences

$$
\left(s_{0}(m+\mathbf{x}), \ldots, s_{n-1}(m+\mathbf{x})\right)_{m=0,1,2, \ldots, n-1}
$$

are cyclic rotations of each other.
From Raney lemma Ran60] it follows that for every $\mathbf{x} \in S_{k, n}$ there exists exactly one $m \in\{0,1, \ldots, n-1\}$ such that $m+\mathbf{x} \in V_{k, n}$; in order to be self-contained we indicate the proof. Indeed, it is the $m$ for which the sum $s_{1}(\mathbf{x})+\cdots+s_{m}(\mathbf{x})$ takes its minimal value. Such index $m$ is unique, since if $m_{1} \neq m_{2}$ then

$$
\left(s_{1}(\mathbf{x})+\cdots+s_{m_{1}}(\mathbf{x})\right)-\left(s_{1}(\mathbf{x})+\cdots+s_{m_{2}}(\mathbf{x})\right)
$$

cannot be integer and hence

$$
s_{1}(\mathbf{x})+\cdots+s_{\mathfrak{m}_{1}}(\mathbf{x}) \neq s_{1}(\mathbf{x})+\cdots+s_{\mathfrak{m}_{2}}(\mathbf{x}) .
$$

We have proved that $S_{k, n}$ is a disjoint sum of the sets ( $m+$ $\left.V_{k, n}\right)_{m=0,1, \ldots, n-1}$. Since all these sets have the same volume, we have

$$
n \operatorname{vol} V_{k, n}=\frac{n^{n k+1}}{(n k+1)!}
$$

which finishes the proof after application of Corollary 6
Remark. In this article we define $\mathcal{B}$ to be an algebra of polynomials on the whole real line. From the viewpoint of the theory of large random matrices developed by Shlyakhtenko [Sh196, Shl98] it does not have much sense, since only the values of functions on the interval $[0,1]$ have a nice interpretation. Nevertheless, our proof of Dykema-Haagerup conjecture uses extensively such senseless objects. It is a small mystery that we still do not understand.

Remark. By refining the above proof one can show the following: for every sequence $l_{1} \geq l_{2} \geq \cdots \geq l_{n}$ it is possible to find a close formula for a family of moments

$$
\phi\left[\left(\mathrm{T}^{k+l_{1}}\left(\mathrm{~T}^{\star}\right)^{k+l_{2}} \mathrm{~T}^{k+l_{3}} \cdots\right)\left(\cdots\left(T^{\star}\right)^{k+l_{3}} T^{k+l_{2}}\left(T^{\star}\right)^{k+l_{1}}\right)\right] .
$$

For example,

$$
\begin{gathered}
\phi\left(T^{n+1}\left(T^{\star}\right)^{n} T^{n}\left(T^{\star}\right)^{n+1}\right)=\frac{1}{2}\left(\frac{2^{2 n+2}}{(2 n+2)!}-\frac{1}{[(n+1)!]^{2}}\right), \\
\phi\left(T^{n+1}\left(T^{\star}\right)^{n} T^{n}\left(T^{\star}\right)^{n} T^{n}\left(T^{\star}\right)^{n+1}\right)=\frac{3^{3 n+1}}{(3 n+2)!}-\frac{1}{n![(n+1)!]^{2}}, \\
\phi\left(T^{n+2}\left(T^{\star}\right)^{n} T^{n}\left(T^{\star}\right)^{n} T^{n}\left(T^{\star}\right)^{n+2}\right)= \\
\frac{3^{3 n+2}}{(3 n+3)!}-\frac{2^{2 n}}{(n+2)!(2 n+1)!}-\frac{2^{2 n+3}}{3(n+1)!(2 n+2)!}+\frac{1}{3[(n+1)!]^{3}} .
\end{gathered}
$$

We leave to the reader to apply methods from Sect. 4 to the left hand sides of the above equations and obtain even more multlinomial identities similar to (2)-(4).

## 4. FORMULAS FOR $\phi\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n}\right]$ INVOLVING MULTINOMIAL <br> COEFFICIENTS

For any fixed n it is possible to write a (possibly very complicated) formula for $\phi\left[\left(\mathrm{T}^{k}\left(\mathrm{~T}^{\star}\right)^{k}\right)^{n}\right]$ which involves multinomial coefficients. The first step is to enumerate all noncrossing pair partitions of the tuple

with a property that each line connects either T with $\mathrm{T}^{\star}$ or $\mathrm{T}^{\star}$ with T (see the example below).

Secondly, for $\alpha, \beta \geq 0$ we define a function $f_{\alpha, \beta} \in \mathcal{B}$ by

$$
f_{\alpha, \beta}(x)=\frac{x^{\alpha}(1-x)^{\beta}}{\alpha!\beta!}
$$

It is not difficult to show the following lemma.
Lemma 8. For every $\alpha, \beta, \alpha^{\prime}, \beta^{\prime}, p \geq 0$ we have

$$
\begin{aligned}
& f_{\alpha, \beta} f_{\alpha^{\prime}, \beta^{\prime}}=\binom{\alpha+\alpha^{\prime}}{\alpha}\binom{\beta+\beta^{\prime}}{\beta} f_{\alpha+\alpha^{\prime}, \beta+\beta^{\prime}}, \\
& k(\overbrace{\underbrace{p}, f_{\alpha, \beta} T^{\star}}^{\text {lines }})=\sum_{0 \leq k \leq \alpha}\binom{k+p}{p} f_{\alpha-k, \beta+k+p}, \\
& k(\overbrace{T^{*}, f_{\alpha, \beta} T}^{p \text { lines }})=\sum_{0 \leq k \leq \alpha}\binom{k+p}{p} f_{\alpha+k+p, \beta-k},
\end{aligned}
$$

$$
\begin{aligned}
& \phi(\kappa(\overbrace{\int^{T}, f_{\alpha, \beta} T_{j}^{\star}}^{p \text { lines }}))=\frac{\binom{p+\alpha}{\alpha}}{(\alpha+\beta+p+1)!}, \\
& \phi(\kappa(\overbrace{\overbrace{T^{\star}}^{p}, f, \beta}^{\text {lines }}))=\frac{\binom{p+\beta}{\beta}}{(\alpha+\beta+p+1)!},
\end{aligned}
$$

where in order to save space and keep the notation simple, instead of


The third and last step to find the formula for $\phi\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n}\right]$ is a direct application of Lemma 8 to compute the nested product $\kappa_{\pi}\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{n}\right]$.

The last two steps can also be replaced by a more combinatorial procedure of counting orderings on certain graphs [DH01].

Example. We have that for $n=3$ every noncrossing pair partition (with the additional property mentioned above) of the tuple (16) is in the form

with $p+q=k$ or in one of the forms



with $\mathrm{p}+\mathrm{q}+\mathrm{r}=\mathrm{k}-1$.
We leave it as an exercise to the reader to apply Lemma 8 and check that $\phi\left[\left(T^{k}\left(T^{\star}\right)^{k}\right)^{3}\right]$ is indeed equal to the right hand side of (3).

## 5. Technical Results

The main result of this Section is the proof of Theorem 2
Proposition 9. If T is a generalized circular element, $\mathrm{m} \geq 0$ and $s_{1}, \ldots, s_{2 m} \in\{1, \star\}$ then

$$
\begin{align*}
& \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2 m}}\right)=\sum_{k \geq 1} \sum_{\substack{1=i_{1}<i_{2}<\ldots \\
\cdots<i_{k+1}=2 m+1}} \prod_{1 \leq r \leq k}  \tag{21}\\
& \kappa\left(T^{s_{i}}, \mathbb{E}\left(T^{s_{i+1}} T^{s_{i_{r}+2}} \cdots T^{s_{i_{r+1}-3}} T^{s_{i_{r+1}-2}}\right) T^{s_{i_{r+1}-1}}\right) .
\end{align*}
$$

Remark. We can treat Proposition 9 as an alternative recursive definition of the generalized circular elements.

Example. If T is a generalized circular element then for any $s_{1}, \ldots, s_{6} \in$ $\{1, \star\}$ we have

$$
\begin{aligned}
& \mathbb{E}\left(T^{s_{1}} T^{s_{2}} T^{s_{3}} T^{s_{4}} T^{s_{5}} T^{s_{6}}\right)= \\
& \kappa\left(T^{s_{1}}, \mathbb{E}\left(T^{s_{2}} T^{s_{3}} T^{s_{4}} T^{s_{5}}\right) T^{s_{6}}\right)+\kappa\left(T^{s_{1}}, T^{s_{2}}\right) \kappa\left(T^{s_{3}}, \mathbb{E}\left(T^{s_{4}} T^{s_{5}}\right) T^{s_{6}}\right)+ \\
& \kappa\left(T^{s_{1}}, \mathbb{E}\left(T^{s_{2}} T^{s_{3}}\right) T^{s_{4}}\right) \kappa\left(T^{s_{5}}, T^{s_{6}}\right)+\kappa\left(T^{s_{1}}, T^{s_{2}}\right) \kappa\left(T^{s_{3}}, T^{s_{4}}\right) \kappa\left(T^{s_{5}}, T^{s_{6}}\right) .
\end{aligned}
$$

Proof. Observe that for $\pi \in \mathrm{NC}_{2}(\{1, \ldots, 2 \mathrm{~m}\})$ we can distinguish certain lines in $\pi$ which will be called outer lines. This name can be easily justified by the example below, where the outer lines were plotted with a bold line. To be precise: we define $i_{1}=1$ and we inductively define $i_{n}$ by the requirement that the index $i_{n}-1$ is joined by a line in $\pi$ with the index $\mathfrak{i}_{n-1}$. In this way we have defined a tuple $1=\mathfrak{i}_{1}<\mathfrak{i}_{2}<\cdots<\mathfrak{i}_{\mathrm{k}+1}=2 \mathrm{~m}+1$ and the outer lines in $\pi$ are exacly $\left\{i_{1}, i_{2}-1\right\},\left\{i_{2}, i_{3}-1\right\}, \ldots,\left\{i_{\mathrm{k}}, i_{\mathrm{k}+1}-1\right\}$.

Furthermore, for $1 \leq r \leq k$ we define $\pi_{r} \in \mathrm{NC}_{2}\left(\left\{\mathfrak{i}_{r}+1, \mathfrak{i}_{r}+2, \ldots, \mathfrak{i}_{r+1}-\right.\right.$ 2\}) by $\pi_{r}=\left\{\{a, b\} \in \pi: i_{r}+1 \leq a, b \leq i_{r+1}-2\right\}$. Lines of noncrossing pair partitions $\pi_{1}, \ldots, \pi_{k}$ have a nice graphical interpretation as inner lines of the partition $\pi$ [BLS96].

Example. For a noncrossing pair partition $\pi$ given by

we have $\mathfrak{i}_{1}=1, \mathfrak{i}_{2}=7, \mathfrak{i}_{3}=9, \mathfrak{i}_{4}=13$. The outer lines are: $\{1,6\}$, $\{7,8\},\{9,12\}$ and are drawn with a bold line. We have $\pi_{1}=\{\{2,3\},\{4,5\}\}$, $\pi_{2}=\emptyset, \pi_{3}=\{\{10,11\}\}$.

Now we see that the sum over $\pi$ in (9) can be replaced by the sum over $k \geq 1$ (i.e. the number of outer lines of $\pi$ ), over indices $1=\mathfrak{i}_{1}<\mathfrak{i}_{2}<$ $\cdots<\mathfrak{i}_{\mathrm{k}+1}=2 \mathrm{~m}+1$ (i.e. the positions of the outer lines of $\pi$ ) and over $\pi_{1}, \ldots, \pi_{k}$ such that $\pi_{r} \in \mathrm{NC}_{2}\left(\left\{\mathfrak{i}_{r}+1, \mathfrak{i}_{r}+2, \ldots, \mathfrak{i}_{r+1}-2\right\}\right)$ (i.e. the inner lines of $\pi$ ):

$$
\begin{aligned}
& \mathbb{E}\left(T^{s_{1}} \ldots T^{s_{2 m}}\right)= \\
& \sum_{k \geq 1} \sum_{i_{1}<\cdots<i_{k+1}} \kappa(\underbrace{\left.T^{s_{i_{1}}}, \sum_{\pi_{1}} \kappa_{\pi_{1}}\left(T^{s_{i_{1}+1}}, \ldots, T^{s_{i_{2}-2}}\right) T^{s_{i_{i_{2}}-1}}, \sum_{\pi_{k}} \kappa_{\pi_{k}}\left(T^{s_{i_{k}}+1}, \ldots, T^{s_{i_{k+1}-2}}\right) T^{s_{i_{k+1}-1}}\right) .}
\end{aligned}
$$

Since by definition

$$
\sum_{\pi_{r}} \kappa_{\pi_{r}}\left(T^{s_{i_{r}+1}}, \ldots, T^{s_{i_{r+1}}-2}\right)=\mathbb{E}\left(\mathrm{T}^{s_{i_{r}+1}} \cdots \mathrm{~T}^{s_{i_{r+1}}-2}\right)
$$

for any $1 \leq r \leq k$, the second part of the proposition follows.

Proof of Theorem 2 Let us consider the case $n=1$. We apply the Leibnitz rule to the right-hand side of Eq. (21):

$$
\begin{aligned}
& \frac{d}{d x} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2} m}\right)=\sum_{k \geq 1} \sum_{\substack{1=i_{1}<i_{2}<\ldots \\
\cdots<i_{k+1}=2 m+1}} \sum_{1 \leq r^{\prime} \leq k} \\
& \left(\frac{d}{d x} \kappa\left(T^{s_{i_{r^{\prime}}}}, \mathbb{E}\left(T^{s_{i_{r^{\prime}}+1}} T^{s_{i_{r}+2}} \cdots T^{s_{i_{r^{\prime}+1}}-2}\right) T^{s_{i_{r^{\prime}+1}}-1}\right)\right) \times \\
& \quad \times \prod_{r \neq r^{\prime}} \kappa\left(T^{s_{i_{r}}}, \mathbb{E}\left(T^{s_{i_{r}+1}} T^{s_{i_{r}+2}} \cdots T^{s_{i_{r+1}-2}}\right) T^{s_{i_{r+1}-1}}\right) .
\end{aligned}
$$

We denote $j_{1}=\mathfrak{i}_{r^{\prime}}, \mathfrak{j}_{2}=\mathfrak{i}_{r^{\prime}+1}-1, k^{\prime}=r^{\prime}-1, k^{\prime \prime}=k-r^{\prime}, \mathfrak{i}_{s}^{\prime}=\mathfrak{i}_{s}$ for $1 \leq s \leq k^{\prime}+1$ and $i_{s}^{\prime \prime}=i_{s+r^{\prime}}$ for $1 \leq s \leq k^{\prime \prime}+1$. Hence
(22) $\frac{d}{d x} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2} m}\right)=\sum_{1 \leq j_{1}<j_{2} \leq 2 m}$
$\left(\sum_{k^{\prime} \geq 1} \sum_{\substack{1=i_{1}^{\prime}<\cdots \\ \cdots<i_{k^{\prime}+1}^{\prime}=j_{1}+1}} \prod_{1 \leq r \leq k^{\prime}} \kappa\left(T^{s_{i_{r}^{\prime}}}, \mathbb{E}\left(T^{s_{i_{r}^{\prime}+1}} \cdots T^{s_{i_{r+1}^{\prime}}-2}\right) T^{s_{i_{r+1}^{\prime}}^{\prime}-1}\right)\right) \times$ $\times\left(\frac{d}{d x} \kappa\left(T^{s_{j_{1}}}, \mathbb{E}\left(T^{s_{j_{1}+1}} T^{s_{j_{2}+2}} \cdots T^{s_{j_{2}-1}}\right) T^{s_{j_{2}}}\right)\right) \times$

$$
\times\left(\sum_{\substack{k^{\prime \prime} \geq 1 \\ \cdots}} \sum_{\substack{j_{2}+1=i_{i^{\prime \prime}}^{\prime \prime}<\cdots \\ i_{k}^{\prime \prime}+1}} \prod_{1 \leq r \leq k^{\prime \prime}} \kappa\left(T^{s_{i_{r}^{\prime \prime}}}, \mathbb{E}\left(T^{s_{i_{i_{r}^{\prime \prime}}+1}} \cdots T^{s_{i_{r+1}^{\prime \prime}}^{\prime \prime}}\right) \mathrm{T}^{s_{i_{r+1}^{\prime \prime}}^{\prime \prime}}\right)\right)
$$

Observe that the first and the third factor on the right-hand side of (22) correspond to the right-hand side of (21). The second factor can be simplified by an observation that (11) imply $\frac{d}{d x} k\left(T^{s_{j_{1}}}, \mathrm{bT}^{s_{j_{2}}}\right)=b \frac{\mathrm{~d}}{\mathrm{dx}} \kappa\left(\mathrm{T}^{s_{j_{1}}}, \mathrm{~T}^{s_{j_{2}}}\right)$ for any $b \in \mathcal{B}$. Hence

$$
\begin{aligned}
\frac{d}{d x} \mathbb{E}\left(T^{s_{1}} \cdots\right. & \left.T^{s_{2} m}\right)= \\
& \sum_{1 \leq j_{1}<j_{2} \leq 2 m}\left(\frac{d}{d x} \mathbb{E}\left(T^{s_{j_{1}}} T^{s_{j_{2}}}\right)\right) \times \\
& \times \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{1}-1}\right) \mathbb{E}\left(T^{s_{j_{1}+1}} \cdots T^{s_{j_{2}-1}}\right) \mathbb{E}\left(T^{s_{j_{2}+1}} \cdots T^{s_{2} m}\right)
\end{aligned}
$$

which finishes the proof of the case $n=1$.
For the general case we can use recursively (12) for $\mathfrak{n}=1$ and Leibnitz rule in order to compute $\frac{d^{n}}{d x^{n}} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2} m}\right)$. For example, for $n=2$ we
obtain three summands; for brevity we present below only one of them:

$$
\begin{aligned}
& \frac{d^{2}}{d x^{2}} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2 m}}\right)=\sum_{1 \leq j_{1}^{\prime}<j_{2}^{\prime} \leq 2 m}\left(\frac{d}{d x} \mathbb{E}\left(T^{s_{j_{1}^{\prime}}} T^{s_{j_{2}^{\prime}}}\right)\right) \times \\
& \times\left(\sum_{1 \leq j_{1}^{\prime \prime}<j_{2}^{\prime \prime}<j_{1}^{\prime}}\left(\frac{d}{d x} \mathbb{E}\left(T^{s_{j_{1}^{\prime \prime}}} T^{s_{j_{2}^{\prime \prime}}}\right)\right) \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{j_{1}^{\prime \prime}-1}}\right) \mathbb{E}\left(T^{s_{j_{1}^{\prime \prime}+1}} \cdots T^{s_{j_{2}^{\prime \prime}-1}}\right) \times\right. \\
& \left.\quad \times \mathbb{E}\left(T^{s_{j_{2}^{\prime \prime}+1}} \cdots T^{s_{j_{1}^{\prime}-1}}\right)\right) \mathbb{E}\left(T^{s_{j_{1}^{\prime}+1}} \cdots T^{s_{j_{2}^{\prime}-1}}\right) \mathbb{E}\left(T^{s_{j_{2}^{\prime}+1}} \cdots T^{s_{2 m}}\right)+\ldots
\end{aligned}
$$

and by renaming the indices we get

$$
\begin{aligned}
& \frac{\mathrm{d}^{2}}{\mathrm{dx} x^{2}} \mathbb{E}\left(\mathrm{~T}^{s_{1}} \cdots T^{s_{2 m}}\right)=\sum_{0=j_{0}<\cdots<j_{5}=2 m+1} \\
& \begin{aligned}
\left(2 \frac{d}{d x} \mathbb{E}\left(T^{s_{j_{1}}} T^{s_{j_{2}}}\right) \frac{d}{d x} \mathbb{E}\left(T^{s_{j_{3}}} T^{s_{j_{4}}}\right)\right. & \left.+\frac{d}{d x} \mathbb{E}\left(T^{s_{j_{1}}} T^{s_{j_{4}}}\right) \frac{d}{d x} \mathbb{E}\left(T^{s_{j_{2}}} T^{s_{j_{3}}}\right)\right) \times \\
& \times \prod_{0 \leq r \leq 4} \mathbb{E}\left(T^{s_{j_{r}+1}} T^{s_{j_{r}+2}} \cdots T^{s_{j_{r+1}-1}}\right) .
\end{aligned}
\end{aligned}
$$

In the general case we get

$$
\begin{align*}
& \frac{d^{n}}{d x^{n}} \mathbb{E}\left(T^{s_{1}} \cdots T^{s_{2 m}}\right)=  \tag{23}\\
& \quad \sum_{0=j_{0}<\cdots<j_{2 n+1}=2 m+1} c_{s_{j_{1}}, s_{j_{2}}}, \ldots, s_{j_{2 n}} \prod_{0 \leq r \leq 2 n} \mathbb{E}\left(T^{s_{j_{r}+1}} T^{s_{j_{r}+2}} \cdots T^{s_{j_{r+1}-1}}\right)
\end{align*}
$$

for some constant $\mathrm{c}_{\mathrm{s}_{1}, s_{j_{2}}}, \ldots, \mathrm{~s}_{\mathrm{j}_{2 n}}$ which does not depend on m . This constant can be easily evaluated by the observation that (23) gives in particular

$$
\frac{d^{n}}{d x^{n}} \mathbb{E}\left(T^{s_{j_{1}}} T^{s_{j_{2}}} \ldots T^{s_{j_{2 n}}}\right)=c_{s_{j_{1}}, s_{j_{2}}, \ldots, s_{j_{2 n}}}
$$

which finishes the proof.
The final remark about nonvanishing summands follows easily from the observation that if $s_{1}, \ldots, s_{n} \in\{1,-1\}$ and $s_{1}+\cdots+s_{n} \neq 0$ then every pair partition of the tuple $\left(\mathrm{T}^{s_{1}}, \ldots, \mathrm{~T}^{s_{n}}\right)$ must connect T with T or $\mathrm{T}^{\star}$ with $\mathrm{T}^{\star}$. The defining relation (11) implies that $\mathbb{E}\left(\mathrm{T}^{s_{1}} \cdots \mathrm{~T}^{s_{n}}\right)=0$.

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