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A novel adaptive fuzzy variable structure control for a class of nonlinear uncertain systems via backstepping

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Abstract

This paper presents a new design of adaptive fuzzy variable structure control to solve the traditional problem of model reference adaptive control (MRAC) for a class of single-input, single-output minimum-phase uncertain nonlinear systems via backstepping. Instead of taking the tedious coordinate transformation and yielding a "hard" high-gain controller, we introduce smooth B-spline-type membership functions into the controller so as to compensate for the uncertainties much "softer", i.e., in a much smoother and locally weighted manner. To be rigorous, it is shown that the stability of the closed-loop system can be assured and the tracking error can globally approach to an arbitrary preset dead-zone range. In order to demonstrate the effectiveness of the developed novel controller, an example is extensively simulated to provide quite satisfactory performance.

Keywords: Fuzzy control; Backstepping; B-spline; Adaptive control; Variable structure control

1. Introduction

Based on the differential geometrical approach, output feedback control problems of affine nonlinear systems subject to some coordinate-free conditions can be solved by adaptive controller design via a backstepping approach which can be employed to yield global asymptotical stability either through some linear parameterization process [8,9] or nonlinear one [10]. For more general and challenging problems, involving system uncertainties which are not leariable, the above-mentioned adaptive control may not achieve the control goal of model reference adaptive control (MRAC). Furthermore, it is well known that variable structure control

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(VSC) is invariant to disturbances and parametric uncertainties based on a simple realization [2]. However, discontinuous control in VSC is not realizable via backstepping approach based on Lyapunov stability theory employing the so-called "hard" high-gain control laws which yields undesirable chattering. This is because during the backstepping procedure, the foregoing non-smooth controller will harden the subsequent controllers, resulting in excessively large control gains. This kind of problem can be resolved by taking "soft" control laws by using smooth controller design with embedded arbitrarily small dead-zone range [3]. However, a systematic approach is lacking for enhancing the smoothness of the controllers.

On the other hand, fuzzy-rule-based modeling can now play a role as a so-called universal approximator [16,11], which can facilitate parameterization of vague system or uncertain systems. This motivates some researches on combining fuzzy control techniques with nonlinear control theories [1] such as adaptive fuzzy control [6] and fuzzy sliding mode control [14,4]. Besides, fuzzy control can also provide the smoothness by selecting smooth membership functions [7] such as Gaussian functions and splines (in particular, B-spline) with arbitrary order [15,17]. Furthermore, fuzzy control can approximate the variable structure control [12,13]. In this paper, an adaptive fuzzy variable structure controller employing smooth membership functions and backstepping concept is hereby systematically developed to yield improved tracking performance relative to that from the conventional high-gain controller. We choose the smooth B-spline basis functions as the membership functions in the paradigm of fuzzy approximation [5]. This choice is due to the fact that nonlinear functions whose feature are mostly strongly local can be represented by smooth and compact support [6]. Given such smooth B-spline-type membership functions, the proposed adaptive fuzzy variable structure controller with a dedicated structure can adaptively compensate for the system uncertainties, in a smooth and locally weighted manner, but not to evolve into a global "hard" high-gain controller [3].

The organization of the paper is as follows. Section 2 formulates the problem to be investigated and introduces the output-feedback variable structure control based on backstepping when all the necessary bounding functions can be evaluated. Section 3 introduces the output-feedback variable structure control based on backstepping when all the necessary bounding functions can be evaluated. In Section 4, we propose a novel adaptive fuzzy variable structure to solve the problems in the presence of system uncertainties while avoiding getting into a hard high-gain controller. Finally, Section 5 makes some concluding remarks.

2. Problem formulation

Consider an affine nonlinear system of the form as follows:

$$\dot{x} = f(x,\alpha) + g(x,\alpha)u,$$
$$y = h(x,\alpha) \tag{1}$$

where $x \in \Re^n$, $u \in \Re$, $y \in \Re$, and α ($\in \Omega_\alpha$: a compact set) is an unknown constant parameter vector which characterizes the smooth nonlinear function vectors f, g, and scalar h, satisfying $f(0,\alpha) = 0$, $g(0,\alpha) \neq 0$ and $h(0,\alpha) = 0$, $\forall \alpha \in \Omega_\alpha$.

If the nonlinear system described in (1) can satisfy some geometric coordinate-free conditions (defined in [9]), then there exists a coordinate transformation $z = T(x)$ such that the nonlinear system can be transformed into the following output-feedback form:

$$\dot{z} = Az + \Phi(y,\alpha) + b(\alpha)\delta(y)u,$$
$$y = c^{\mathrm{T}}z, \tag{2}$$

where

$$A=\begin{bmatrix}0&1&0&\dots&0\\0&0&1&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&1\\0&0&0&\dots&0\end{bmatrix},\qquad \Phi(y,\alpha)=\begin{bmatrix}\Phi_1(y,\alpha)\\\vdots\\\Phi_n(y,\alpha)\end{bmatrix},\qquad b(\alpha)=\begin{bmatrix}b_1(\alpha)\\\vdots\\b_n(\alpha)\end{bmatrix},\qquad c=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}.$$

Furthermore, system (1) is assumed with strong relative degree ρ [9]. Here, our control goal is to force the output y to follow a desired trajectory y_m, constructed by the following reference model:

$$\begin{aligned}\dot{z}_m&=A_c z_m+b_m r_m,\\ y_m&=c^{\mathrm{T}}z_m,\end{aligned}\tag{3}$$

where

$$A_c=\begin{bmatrix}-a_1&1&0&\dots&0\\-a_2&0&1&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\-a_{n-1}&0&0&\dots&1\\-a_n&0&0&\dots&0\end{bmatrix}$$

is a stable Hurwitz matrix with constant parameters $a_1,a_2,\dots,a_n$ and, constant vector $b_m=[0,\dots,b_{m\rho},\dots,b_{mn}]^{\mathrm{T}}$ and $r_m\in L_\infty$ is a bounded reference input, or $y_m=W_m(s)r_m$ with $W_m(s)=(b_{m\rho}s^{n-\rho}+\dots+b_{mn})/(s^n+a_1s^{n-1}+\dots+a_n)$ so that $y_m\in L_\infty$.

Given such tracking problem, we proceed with rearranging form (2) by the following:

$$\begin{aligned}\dot{z}&=Az-ay+ay+\Phi(y,\alpha)+b(\alpha)\delta u\\&=A_c z+ay+\Phi(y,\alpha)+b(\alpha)\delta u,\\ y&=c^{\mathrm{T}}z,\end{aligned}\tag{4}$$

where $a=[a_1,a_2,\dots,a_n]^{\mathrm{T}}$ and the input–output transfer function is defined as

$$W(s)=c^{\mathrm{T}}(sI-A_c)^{-1}b=\frac{b_\rho s^{n-\rho}+\dots+b_n}{s^n+a_1s^{n-1}+\dots+a_n}.\tag{5}$$

Furthermore, we define the error vector $e=z_m-z$, and then the error model can be derived as follows:

$$\begin{aligned}\dot{e}&=A_c e-ay-\Phi+b_m r_m-b\delta(y)u,\\ e_o&=y_m-y=c^{\mathrm{T}}e.\end{aligned}\tag{6}$$

Remark. Apparently, to force e_o approaching zero can realize our control goal. To achieve this goal, the intuitive control efforts to solve the tracking problem are either to cancel or to make the system robust to the nonlinear term $\Phi(y,\alpha)$ by properly designing the control input u through the input gain $b(\alpha)\delta(y)$. However, it is difficult to exactly construct the transformation $T(x)$, satisfying the above-mentioned coordinate-free conditions. Therefore, in this paper we propose the robust variable structure controller design to compensate for unknown nonlinear term Φ, instead directly cancelling Φ. In order to make the model tracking problem more tractable, we make the following reasonable assumptions.

Assumptions.

(A1) $b=[b_1,\ldots,b_n]^{\mathrm{T}}$ is a vector of Hurwitz coefficients of degree ρ, i.e., the associated polynomial

$$b_1 s^{n-1}+b_2 s^{n-2}+\cdots+b_n$$

is of degree $n-\rho$ (i.e., $b_1\neq 0$ if $\rho=1$ or $b_1=\cdots=b_{\rho-1}=0$, $b_\rho\neq 0$, if $\rho>1$) and is Hurwitz.

(A2) The sign of $b_\rho(\alpha)$ is known and constant for any $\alpha\in\Omega_\alpha$, $1\leqslant\rho\leqslant n$.

(A3) $\Phi(y,\alpha)$ can be expressed as a Taylor's series expansion in y for any $\alpha\in\Omega_\alpha$, i.e., $\Phi(y,\alpha)=\Phi(y_0,\alpha)+\sum_{n=1}^{\infty}[(1/n!)(\partial^{(n)}\Phi/\partial y^n)|_{y=y_0}(y-y_0)^n]$.

(A4) $\|\Phi(y,\alpha)-\Phi(y_0,\alpha)\|\leqslant l_\Phi(y-y_0,y_0,\alpha)|y-y_0|$, for some $l_\Phi(y-y_0,y_0,\alpha)\geqslant 0$.

Based on the former assumptions, $T(x)$ may not need to be known, but the control problems can still be formulated as a backstepping-based tracking problem.

To derive the backstepping-based error model, first we define a stable filter

$$W_f^{-1}(s)=\frac{1}{(s+\lambda_1)\cdots(s+\lambda_i)\cdots(s+\lambda_{\rho-1})}, \tag{7}$$

where $\rho>1$ (the case of $\rho=1$ can be intuitively derived), $\lambda_i>0$, characterizing the input–output relationship $\eta_1=W_f^{-1}(s)\delta(y)u$. Based on the filter (7) and the above assumptions, we can apply the tranformation developed by Marino and Tomei [10] to the error model (6) and yield the backstepping-based error model as follows (derived in Appendix A):

$$\dot{e}_o=\xi_1+\frac{d_2(\alpha)}{d_1(\alpha)}e_o-\phi(e_o,y_m,\alpha)-a_1y_m-d_1\eta_1, \tag{8}$$

where ξ_1 is a new auxiliary variable, $d=[d_1,\ldots,d_n]$ is a constant vector, defined in Appendix A, and

$$\phi(e_o,y_m,\alpha)=\Phi_1(y_m-e_o,\alpha)=\Phi_1(y,\alpha). \tag{9}$$

Example 2.1. Consider the system (see [10]) as follows:

$$\dot{x}_1=x_2+x_1^{\alpha},$$
$$\dot{x}_2=u,$$
$$y=x_1.$$

The system is a relative degree $\rho=2$ system and matches the nonlinear system (2). The desired model reference trajectory can be given as $y_m(t)=(1/(s^2+2s+1))r_m(t)$, and $r_m(t)$ is given as a step input, i.e., $r_m(t)=1$. A stable filter (7) is given as $1/(s+1)$ and we can derive the backstepping-based error model as follows:

$$\dot{e}_o=\xi_1+e_o-(y_m-e_o)^{\alpha}-2y_m-\eta_1,$$
$$\dot{\eta}_1=-\eta_1+u.$$

3. Output-feedback variable structure control via backstepping

It is well known that applying the controller design based on the backstepping procedure can solve the control problem with the error model (8) [10]. The concepts of the backstepping control are to first design the designated controller of the first desired filter state, η_1^*, which can guarantee that the output tracking error e_o given in Eq. (8) can approach zero, and then to design the designated controller of the second desired filter state, η_2^*, which can realize η_1 as η_1^* subject to the equation

$$\dot{\eta}_1 = -\lambda_1\eta_1 + \eta_2. \tag{10}$$

Similarly, the controller design back steps to the designated control input controller $\delta(y)u = \eta_\rho^*$ so that $\eta_{\rho-1}$ can approach $\eta_{\rho-1}^*$.

Now, we design a variable structure controller η_1^* as follows:

$$\eta_1^* = \operatorname{sgn}(d_1)k_{11}e_o + \operatorname{sgn}(d_1)k_{12}(t)\operatorname{sgn}(e_o) \tag{11}$$

with smooth functions $k_{11}(t)$ and $k_{12}(t)$ satisfying the following expressions:

$$k_{11}(t) \geqslant \frac{1}{|d_1|}\left(\left|\frac{d_2}{d_1}\right| + \frac{(\frac{1}{2} + \|P\|(\|\beta\| + l_\Phi))^2}{q_0} + \rho\varepsilon_1\right),$$

$$k_{12}(t) \geqslant \frac{1}{|d_1|}(\|\hat{\xi}\| + |\phi| + |a_1||y_m|), \tag{12}$$

where q_0, $\varepsilon_1 > 0$ are positive constants, P is a positive definite matrix defined in Appendix B, and $\|\hat{\xi}\|$ is constructed from the following dynamic equation:

$$\dot{\hat{\xi}} = \Gamma\hat{\xi} + \Psi(y_m,\alpha) + \Psi_r(y_m,r_m) \tag{13}$$

with initial conditions $\hat{\xi}(0) = \hat{\xi}_0$, where Γ, Ψ and Ψ_r are defined in Appendix B. Apparently, system (13) is BIBO stable, since $\Psi(y_m,\alpha)$ and $\Psi_r(y_m,r_m)$ are bounded for the bounded y_m, α, and r_m, resulting in $\hat{\xi} \in L_\infty^{n-1}$. This control law will be shown effective to the tracking control problem via the following proposition.

Proposition 3.1. *If $\eta_1 = \eta_1^*$ as given in Eq.* (11), *then the output tracking error of system* (6) *will be driven to zero globally and exponentially.*

Proof. See Appendix B.1. □

To realize control law (11), apparently, the switching function $\operatorname{sgn}(e_o)$ will render η_1^* to be discontinuous at $e_o = 0$. This fact often causes η_1^* to be unrealizable when it comes to design the subsequent designated controller η_2^*. To resolve this problem, it is straightforward to modify the previous controller by embedding a smooth compensator for a specified dead-zone range, such as a saturation-type compensator or a hyper tangent-type compensator. Then, this controller can be expressed as follows:

$$\eta_1^\dagger = \operatorname{sgn}(d_1)k_1(e_o,\alpha,\Delta_e) = \begin{cases} \operatorname{sgn}(d_1)[k_{11}(t)e_o + k_{12}(t)\operatorname{sgn}(e_o)] & \text{if } e_o \notin [-\Delta_e,\Delta_e], \\ k_{s1}(e_o,\alpha,\Delta_e) & \text{otherwise,} \end{cases} \tag{14}$$

where $k_{s1}(\cdot)$ is a smooth function to make $\eta_1^\dagger$ smooth, and $[-\Delta_e,\Delta_e]$ is a designated dead-zone range which can be arbitrarily set. Then, the following proposition is valid.

Proposition 3.2. *If the control law $\eta_1 = \eta_1^\dagger$ is given as in Eq.* (14), *then the tracking error of system* (16) *will be driven to the dead zone range $[-\Delta_e,\Delta_e]$ globally and exponentially.*

Proof. See Appendix B.2. □

However, in fact η_1 is driven by η_2 according to the first-order equation (10), and hence we define the difference between the desired smooth filter state $\eta_1^\dagger$ and the real filter state η_1 as $\widetilde{\eta_1} = \eta_1^\dagger - \eta_1$, so that its time derivative can be derived from (10) as follows:

$$\dot{\widetilde{\eta_1}} = \dot{\eta}_1^\dagger - \dot{\eta}_1 = \frac{\partial \eta_1^\dagger}{\partial e_o}\dot{e}_o - \lambda_1\widetilde{\eta_1} + \lambda_1\eta_1^\dagger - \eta_2. \tag{15}$$

As a result, the goal of the control law η_2 is apparently to force η_1 to achieve $\eta_1^\dagger$ and, hence, we derive the variable structure control η_2^* as follows:

$$\eta_2^* = \lambda_1\eta_1^\dagger + k_{21}(t)\widetilde{\eta_1} + k_{22}(t)\,\mathrm{sgn}(\widetilde{\eta_1}), \tag{16}$$

where smooth functions $k_{21}(\cdot)$ and $k_{22}(\cdot)$ satisfy the following conditions:

$$k_{21}(t) \geqslant |d_1|\left|\frac{\partial \eta_1^\dagger}{\partial e_o}\right| + \frac{(d_1 + (d_2/d_1)(\partial\eta_1^\dagger/\partial e_o))^2}{4\varepsilon_1} + \frac{1}{4\varepsilon_2}\left(\frac{\partial \eta_1^\dagger}{\partial e_o}\right)^2,$$

$$k_{22}(t) \geqslant \left|\frac{\partial \eta_1^\dagger}{\partial e_o}\right|\left(\|\widehat{\xi}\| + |\phi| + |a_1||y_m| + |d_1||\eta_1^\dagger| + \left|\frac{d_2}{d_1}\right|\Delta_e\right), \tag{17}$$

so that the following proposition will be valid.

Proposition 3.3. *If* $\eta_2 = \eta_2^*$ *is given as in* (16), *then the tracking error of system* (6) *will converge to the dead-zone range* $[-\Delta_e, \Delta_e]$ *globally and exponentially.*

Proof. See Appendix B.3. □

However, the controller given in (16) again faces the problem with discontinuity so that, similar to (14), we replace controller (16) with a smooth compensator with the dead-zone range $[-\Delta_{\eta_1}, \Delta_{\eta_1}]$ as follows:

$$\eta_2^\dagger = \lambda_1\eta_1^\dagger + k_2(e_o, \widetilde{\eta_1}, \alpha, \Delta_e, \Delta_{\eta_1}), \tag{18}$$

where

$$k_2(e_o, \widetilde{\eta_1}, \alpha, \Delta_e, \Delta_{\eta_1}) = \begin{cases} k_{21}(t)\widetilde{\eta_1} + k_{22}(t)\,\mathrm{sgn}(\widetilde{\eta_1}), & \widetilde{\eta_1} \notin [-\Delta_{\eta_1}, \Delta_{\eta_1}], \\ k_{s2}(e_o, \widetilde{\eta_1}, \alpha, \Delta_e, \Delta_{\eta_1}) & \text{otherwise} \end{cases} \tag{19}$$

with $k_{s2}(\cdot)$ being a smooth function in order to make $\eta_2^\dagger$ smooth. By the same token, the controller design back steps to the equation containing the real control input:

$$\dot{\eta}_{\rho-1} = -\lambda_{\rho-1}\eta_{\rho-1} + \delta(y)u. \tag{20}$$

However, unfortunately, the above controller, can realize its designated controller only up to the corresponding dead-zone ranges. For example, the controller $\eta_2^\dagger$ given in (18) can realize $\eta_1^\dagger$ given in (14) only up to the dead-zone range $[-\Delta_{\eta_1}, \Delta_{\eta_1}]$. This fact results in which the former proposition will no longer hold. Thus, we will require additional compensators to compensate for the backward dead-zone ranges, yielding the following set of designated controllers (desired filter states):

$$\eta_1^\dagger = \mathrm{sgn}(d_1)[k_1(e_o, \alpha, \Delta_e) + k_{\delta 1}e_o],$$
$$\eta_2^\dagger = \lambda_1\eta_1^\dagger + k_2(e_o, \widetilde{\eta_1}, \alpha, \Delta_e, \Delta_{\eta_1}) + k_{\delta 2}(e_o, \alpha, \Delta_e, \Delta_{\eta_1}, \Delta_{\eta_2})\widetilde{\eta_1},$$

$$\vdots$$

$$\eta^{\dagger}_{\rho-1} = \lambda_{\rho-2}\eta^{\dagger}_{\rho-2} + k_{\rho-1}(e_o, \tilde{\eta}_1, \ldots, \widetilde{\eta_{\rho-2}}, \alpha, \Delta_e, \Delta_{\eta_1}, \ldots, \Delta_{\eta_{\rho-2}})$$

$$+ k_{\delta\rho-1}(e_o, \tilde{\eta}_1, \ldots, \widetilde{\eta_{\rho-2}}, \alpha, \Delta_e, \Delta_{\eta_1}, \ldots, \Delta_{\eta_{\rho-1}})\widetilde{\eta_{\rho-2}},$$

$$\delta(y)u = \eta^{\dagger}_{\rho} = \lambda_{\rho-1}\eta^{\dagger}_{\rho-1} + k_{\rho}(e_o, \tilde{\eta}_1, \ldots, \widetilde{\eta_{\rho-1}}, \alpha, \Delta_e, \Delta_{\eta_1}, \ldots, \Delta_{\eta_{\rho-1}}), \tag{21}$$

where $k_3(\cdot), \ldots, k_\rho(\cdot)$ and $k_{\delta 1}(\cdot), \ldots, k_{\delta\rho-1}(\cdot)$ are the designated compensators to be defined in Appendix C, the tracking errors are defined as $\tilde{\eta}_2 = \eta^{\dagger}_2 - \eta_2, \ldots, \widetilde{\eta_{\rho-1}} = \eta^{\dagger}_{\rho-1} - \eta_{\rho-1}$ corresponding to the dead-zone ranges $[-\Delta_{\eta_2}, \Delta_{\eta_2}], \ldots, [-\Delta_{\eta_{\rho-1}}, \Delta_{\eta_{\rho-1}}]$, respectively, and

$$\tilde{\eta}_{j\Delta} = \begin{cases} \tilde{\eta}_j & \text{as } \tilde{\eta}_j < -\Delta_{\eta_j} \text{ or } \tilde{\eta}_j > \Delta_{\eta_j}, \\ 0 & \text{otherwise (i.e. } \tilde{\eta}_j \in [-\Delta_{\eta_j}, \Delta_{\eta_j}]), \end{cases} \tag{22}$$

so that

$$\dot{\tilde{\eta}}_{j\Delta} = \dot{\tilde{\eta}}_j \quad \text{for } \tilde{\eta}_{j\Delta} \neq 0 \text{ for } j = 1, \ldots, \rho - 1.$$

Then, the following theorem is valid.

Theorem 3.1. *If the control law $\delta(y)u = \eta^{\dagger}_{\rho}$ is given as in* (21), *then the system state in* (4) *is guaranteed to be bounded and the tracking error of system* (6) *will converge to the dead-zone range* $[-\Delta_e, \Delta_e]$ *globally and exponentially.*

Proof. See Appendix D. □

3.1. Computer simulation

In this subsection, we simulate controller (21) on the system introduced in the Example shown in Section 2. Here, $\alpha = 2$ is given. Simulation is run for the initial condition $x(0) = [0.5, 0]$ by Matlab and the control law can be easily derived from (12), (17), (21) and (63) by setting $\Delta_e = \Delta_{\eta_1} = 0.1$, $\varepsilon_1 = 0.5$, $\varepsilon_2 = 0.5$, $P = 1$, $q_0 = 1$, $\|\tilde{\xi}\| \leqslant 2$, $\|\beta\| = 1$, $l_\Phi \leqslant e_o^2 + \frac{1}{4} + 2$, as follows:

$$\eta^{\dagger}_1 = \left(e_o^2 + 6\tfrac{3}{4}\right) e_o + (6 + 2e_o^2)\frac{\tanh(\gamma_1 e_o)}{\tanh(\gamma_1 \Delta_e)},$$

$$u = \eta^{\dagger}_1 + \left[\frac{\partial \eta^{\dagger}_1}{\partial e_o} + \frac{(1 + \partial\eta^{\dagger}_1/\partial e_o)^2}{2} + 2\left(\frac{\partial \eta^{\dagger}_1}{\partial e_o}\right)^2\right]\tilde{\eta}_1 + \frac{\partial \eta^{\dagger}_1}{\partial e_o}\left[7 + 2e_o^2 + \Delta_e + \left(\frac{\eta^{\dagger}_1}{2}\right)^2\right]\frac{\tanh(\gamma_2 \tilde{\eta}_1)}{\tanh(\gamma_2 \Delta_{\eta_1})}, \tag{23}$$

where

$$\frac{\partial \eta^{\dagger}_1}{\partial e_o} = 3e_o^2 + 6\tfrac{3}{4} + 4e_o\frac{\tanh(\gamma_1 e_o)}{\tanh(\gamma_1 \Delta_e)} + \frac{(6 + 2e_o^2)}{\tanh(\gamma_1 \Delta_e)}\frac{4\gamma_1}{(\mathrm{e}^{\gamma_1 e_o} + \mathrm{e}^{-\gamma_1 e_o})^2} > 0$$

and $\gamma_1 = 10$, $\gamma_2 = 5$. Fig. 1(a) shows that the tracking errors cannot only converge into the dead-zone range but also approach zero. Fig. 1(b) shows that the system state x_2 can be bounded. Figs. 1(c) and (d) show that the designated filter controller $\eta^{\dagger}_1$ and the final control input u are smooth.

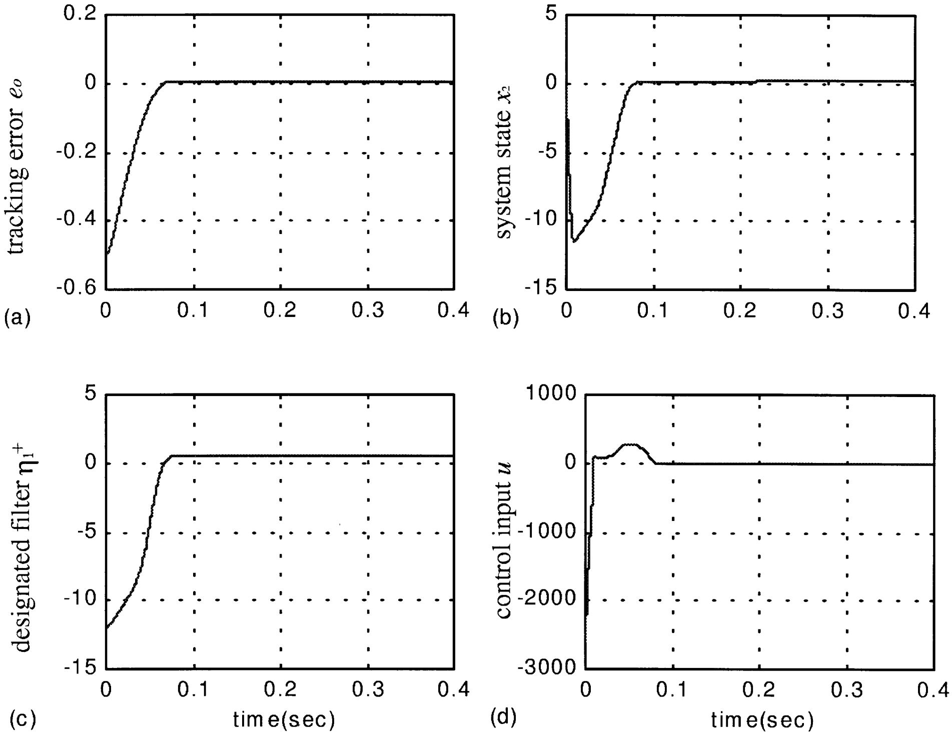

Fig. 1. The simulation results for the nonlinear controller.

Remark. While the given designated controllers are not smooth, then the subsequent designated controllers will be hardened with high gain since the controllers contain the differential terms, e.g., $\partial\eta_1^\dagger/\partial e_o$, $\partial\eta_2^\dagger/\partial e_o$, $\partial\eta_2^\dagger/\partial\tilde{\eta}_1,\ldots,\partial\eta_{\rho-1}^\dagger/\partial\widetilde{\eta_{\rho-2}}$ [3]. Besides, it is difficult to realize these designated controllers when considering the uncertainties of the controlled system. In the following section, we will propose an adaptive fuzzy variable structure control to solve the above-mentioned problem.

4. Adaptive fuzzy variable structure control

Consider a fuzzy controller $u_f = [u_{f_1},\ldots,u_{f_\rho}]^{\mathrm{T}}$, consisting of ρ multi-input single-output (MISO) fuzzy controllers, which are respectively characterized by

$$
\begin{aligned}
u_{f_1} &\triangleq u_{f_1}(w_1) : \Omega_{w1} \to \Re,\\
u_{f_2} &\triangleq u_{f_2}(w_1, w_2) : \Omega_{w1} \times \Omega_{w2} \to \Re,\\
&\vdots\\
u_{f_i} &\triangleq u_{f_i}(w_1,\ldots,w_i) : \Omega_{w1} \times \Omega_{w2} \times \cdots \times \Omega_{wi} \to \Re,\\
&\vdots\\
u_{f_\rho} &\triangleq u_{f_\rho}(w_1,\ldots,w_\rho) : \Omega_{w1} \times \cdots \times \Omega_{w\rho} \to \Re,
\end{aligned}
$$

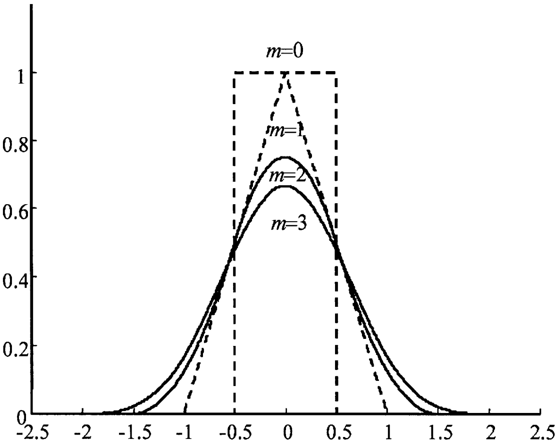

Fig. 2. The mth-order B-spline basis for $m = 0, 1, 2,$ and 3.

where $u_{f_i}(w_1, \ldots, w_i)$ is the ith fuzzy controller; $w = [w_1, \ldots, w_\rho]^T = [e_o, \widetilde{\eta_1}, \ldots, \widetilde{\eta_{\rho-1}}]^T$ and $w_1, \ldots, w_\rho$ are defined as input fuzzy variables; and $\Omega_{w1} \equiv [-\Upsilon\Delta_1, \Upsilon\Delta_1]$, $\Omega_{w2} \equiv [-\Upsilon\Delta_2, \Upsilon\Delta_2], \ldots, \Omega_{w\rho} \equiv [-\Upsilon\Delta_\rho, \Upsilon\Delta_\rho]$, with Υ being an arbitrarily large positive integer, and $\Delta_1, \ldots, \Delta_\rho$ being some positive real numbers. Here, each of the membership functions is given as an mth ($m \geqslant 2$)-order multiple dimension central B-spline function (as depicted in Fig. 2), the jth dimension of which is defined as follows:

$$N_{mj}(x) = \sum_{k=0}^{m+1} \frac{(-1)^k}{m!} \binom{m+1}{k} \left[\left(x + \left(\frac{m+1}{2} - k \right) \Delta_j \right)_+ \right]^m, \tag{24}$$

where we define the notation

$$x_+ := \max(0, x). \tag{25}$$

The mth-order B-spline type of membership function has the following properties:

- an $(m-1)$th order continuously differentiable function, i.e., $N_{mj}(x) \in C^{m-1}$;
- local compact support, i.e., $N_{mj}(x) \neq 0$ only for $x \in [-((m+1)/2)\Delta_j, ((m+1)/2)\Delta_j]$;
- $N_{mj}(x) > 0$ for $x \in (-((m+1)/2)\Delta_j, ((m+1)/2)\Delta_j)$;
- symmetric with respect to the center point (zero point);
- $\sum_{i_1=-\infty}^{\infty} \cdots \sum_{i_j=-\infty}^{\infty} N_{m1}(x - i_1\Delta_1) \cdots N_{mj}(x - i_j\Delta_j) = 1,\ \forall x \in \Re,\ j \in \mathscr{Z}^+$.

Based on the definition of the local compact support, the above property can be rewritten as

$$\sum_{i_1 \in I_{c1}(x)} \cdots \sum_{i_j \in I_{cj}(x)} N_{m1}(x - i_1\Delta_1) \cdots N_{mj}(x - i_j\Delta_j) = 1, \quad \forall x \in \Re,\ j \in \mathscr{Z}^+, \tag{26}$$

where $I_{cj}(x)$ is an integer set, defined as follows:

$$I_{cj}(x) \equiv \left\{ i\colon \frac{x}{\Delta_j} - \frac{m+1}{2} < i < \frac{x}{\Delta_j} + \frac{m+1}{2},\ i \in \mathscr{Z},\ j \in \mathscr{Z}^+ \right\}. \tag{27}$$

Then the membership functions for the jth fuzzy variable w_j are defined as follows:

$$\mu_{j_i}(w_j) = N_{mj}(w_j - i\Delta_j), \quad i = -\Upsilon, \dots, 0, \dots, \Upsilon \tag{28}$$

whose compact support is given as

$$\Omega_{w_{j_i}} = \left[\left(i - \frac{m+1}{2} \right) \Delta_j, \ \left(i + \frac{m+1}{2} \right) \Delta_j \right], \quad i = -\Upsilon, \dots, 0, \dots, \Upsilon, \tag{29}$$

which means that $w_j \in \operatorname{int}(\Omega_{w_{j_i}})$ implies that $\mu_{j_i}(w_j) > 0$. Apparently, we can get $\Omega_{w_j} \equiv \bigcup_{i \in \{-\Upsilon, \dots, \Upsilon\}} \Omega_{w_{j_i}} \equiv [-\Upsilon\Delta_j, \Upsilon\Delta_j]$. Besides, it is possible that $\Omega_{w_{j_i}} \cap \Omega_{w_{j_k}} \neq \emptyset$, for some $i \neq k$, i.e., w_j can simultaneously fall into several compact supports. It is interesting to note that the indices labeling those supports by the definition of (27) can be reexpressed as

$$\begin{aligned} I_{cj}(w_j) &\equiv \{ i\colon \ w_j \in \operatorname{int}(\Omega_{w_{j_i}}), \ i \in \mathscr{Z}, \ -\Upsilon \leqslant i \leqslant \Upsilon \} \\ &\equiv \{ i\colon \ \Omega_{w_{j_i}} \subset \Omega_{c_j}(w_j) \}, \end{aligned} \tag{30}$$

where $\Omega_{cj}(w_j)$ is the union set of those compact supports, defined as follows:

$$\Omega_{cj}(w_j) \equiv \bigcup_{i \in I_{cj}(w_j)} \Omega_{w_{j_i}}, \tag{31}$$

which means that $i \in I_{cj}(w_j)$ if and only if $w_j \in \Omega_{cj}(w_j)$.

As a general representation of the MISO fuzzy controller with center average defuzzifier, inference with product compositional operator, and singleton fuzzifier [16], we can represent the above fuzzy controllers as follows:

$$\begin{aligned} u_{f_1} &= \frac{\sum_{i_1=-\Upsilon}^{\Upsilon} \mu_{1 i_1}(w_1)\theta_{i_1}}{\sum_{i_1=-\Upsilon}^{\Upsilon} \mu_{1 i_1}(w_1)} = \sum_{i_1=-\Upsilon}^{\Upsilon} v_{i_1}(w_1)\theta_{i_1} = \sum_{i_1 \in I_{c1}(w_1)} v_{i_1}(w_1)\theta_{i_1} = \theta^{(1)^{\mathrm{T}}} v^{(1)}, \\ &\ \vdots \\ u_{f_k} &= \frac{\sum_{i_1=-\Upsilon}^{\Upsilon} \cdots \sum_{i_k=-\Upsilon}^{\Upsilon} \mu_{1 i_1}(w_1) \cdots \mu_{k i_k}(w_k)\theta_{i_1 i_2 \cdots i_k}}{\sum_{i_1=-\Upsilon}^{\Upsilon} \cdots \sum_{i_k=-\Upsilon}^{\Upsilon} \mu_{1 i_1}(w_1) \cdots \mu_{k i_k}(w_k)} \\ &= \sum_{i_1=-\Upsilon}^{\Upsilon} \cdots \sum_{i_k=-\Upsilon}^{\Upsilon} v_{i_1 \cdots i_k}(w_1, \dots, w_k)\theta_{i_1 \cdots i_k} \\ &= \sum_{i_1 \in I_{c1}(w_1)} \cdots \sum_{i_k \in I_{ck}(w_k)} v_{i_1 \cdots i_k}(w_1, \dots, w_k)\theta_{i_1 \cdots i_k} = \theta^{(k)^{\mathrm{T}}} v^{(k)}, \\ &\ \vdots \\ u_{f_\rho} &= \frac{\sum_{i_1=-\Upsilon}^{\Upsilon} \cdots \sum_{i_\rho=-\Upsilon}^{\Upsilon} \mu_{1 i_1}(w_1) \cdots \mu_{\rho i_\rho}(w_\rho)\theta_{i_1 i_2 \cdots i_\rho}}{\sum_{i_1=-\Upsilon}^{\Upsilon} \cdots \sum_{i_\rho=-\Upsilon}^{\Upsilon} \mu_{1 i_1}(w_1) \cdots \mu_{\rho i_\rho}(w_\rho)} \\ &= \sum_{i_1=-\Upsilon}^{\Upsilon} \cdots \sum_{i_\rho=-\Upsilon}^{\Upsilon} v_{i_1 \cdots i_\rho}(w_1, \dots, w_\rho)\theta_{i_1 \cdots i_\rho} \\ &= \sum_{i_1 \in I_{c1}(w_1)} \cdots \sum_{i_\rho \in I_{c\rho}(w_\rho)} v_{i_1 \cdots i_\rho}(w_1, \dots, w_\rho)\theta_{i_1 \cdots i_\rho} = \theta^{(\rho)^{\mathrm{T}}} v^{(\rho)}, \end{aligned} \tag{32}$$

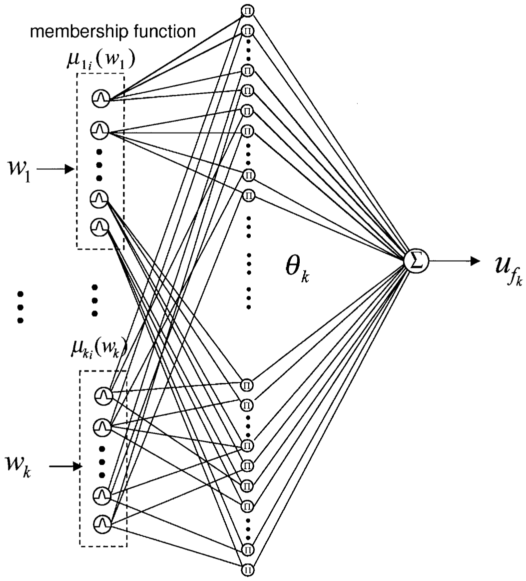

Fig. 3. The network implementation of fuzzy controllers.

where $i_1, \ldots, i_k, \ldots, i_\rho$ are integer indices, $v_{i_1 \cdots i_k}(w_1, \ldots, w_k)$ is the fuzzy basis function of the kth fuzzy controller associated with the indices $i_1 \cdots i_k$, defined as follows:

$$v_{i_1 \cdots i_k}(w_1, \ldots, w_k) = \frac{\mu_{1\,i_1}(w_1) \cdots \mu_{k\,i_k}(w_k)}{\sum_{i_1=-\Upsilon}^{\Upsilon} \cdots \sum_{i_k=-\Upsilon}^{\Upsilon} \mu_{1\,i_1}(w_1) \cdots \mu_{k\,i_k}(w_k)}$$

$$= \frac{\mu_{1\,i_1}(w_1) \cdots \mu_{k\,i_k}(w_k)}{\sum_{i_1 \in I_{c1}(w_1)} \cdots \sum_{i_k \in I_{ck}(w_k)} \mu_{1\,i_1}(w_1) \cdots \mu_{k\,i_k}(w_k)};$$

$\theta_{i_1 \cdots i_k}$ is the parameter of the kth fuzzy controller associated with the indices $i_1 \cdots i_k$, $v^{(k)}$ and $\theta^{(k)}$ are the vectors consisting of $v_{i_1 \cdots i_k}$ and $\theta_{i_1 \cdots i_k}$ for $i_1 = -\Upsilon, \ldots, \Upsilon, \ldots, i_k = -\Upsilon, \ldots, \Upsilon$, respectively. The proposed fuzzy controller (32) can be implemented as a network implementation as shown in Fig. 3 and the block diagram of the overall closed-loop system is depicted in Fig. 4. Furthermore, we realize the adaptive fuzzy variable structure control law as an integrated control consisting of the former fuzzy control vector u_f and a supervised control vector $\bar{\eta} = [\overline{\eta_1}, \ldots, \overline{\eta_\rho}]^{\mathrm{T}}$ as follows:

$$\text{if } w_1 \in \Omega_{w1}, \text{ then } \eta_1^\dagger = \operatorname{sgn}(d_1)(u_{f_1} + k_{f_1} w_1); \text{ otherwise } \eta_1^\dagger = \overline{\eta_1},$$

$$\text{if } w_2 \in \Omega_{w2}, \text{ then } \eta_2^\dagger = \lambda_1 \eta_1^\dagger + u_{f_2} + k_{f_2} w_2; \text{ otherwise } \eta_2^\dagger = \overline{\eta_2},$$

$$\vdots$$

$$\text{if } w_{\rho-1} \in \Omega_{w\rho-1}, \text{ then } \eta_{\rho-1}^\dagger = \lambda_{\rho-2} \eta_{\rho-2}^\dagger + u_{f_{\rho-1}} + k_{f_{\rho-1}} w_{\rho-1}; \text{ otherwise } \eta_{\rho-1}^\dagger = \overline{\eta_{\rho-1}},$$

$$\text{if } w_\rho \in \Omega_{w\rho}, \text{ then } \eta_\rho^\dagger = \lambda_{\rho-1} \eta_{\rho-1}^\dagger + u_{f_\rho} + k_{f_\rho} w_\rho; \text{ otherwise } \eta_\rho^\dagger = \overline{\eta_\rho}, \tag{33}$$

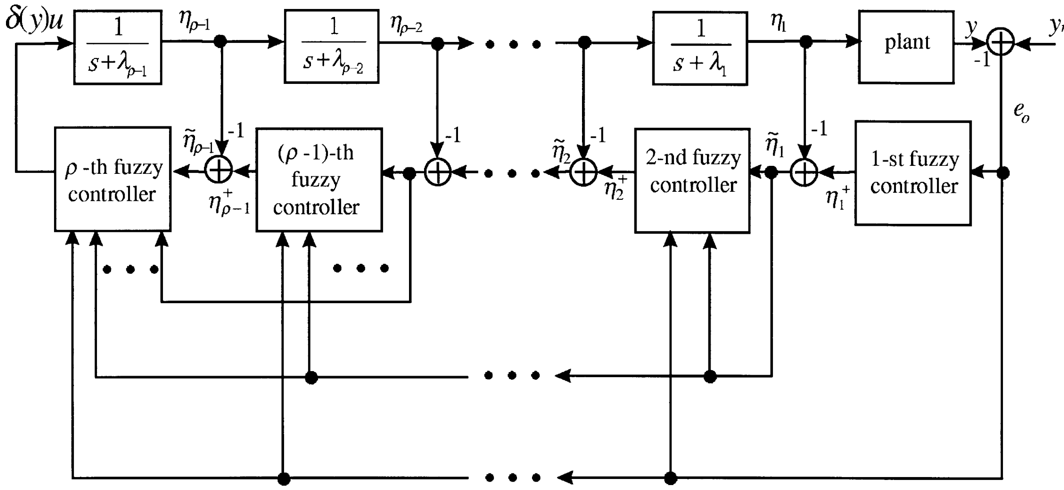

Fig. 4. The block diagram of the closed-loop system with $\rho \geqslant 2$.

where $k_{f_1}, \ldots, k_{f_\rho}$ are some positive constants and

$$\overline{\eta_1} = \mathrm{sgn}(d_1)\overline{k_1}(t),$$

$$\overline{\eta_2} = \lambda_1 \eta_1^{\dagger} + \overline{k_2}(t),$$

$$\vdots$$

$$\overline{\eta_\rho} = \lambda_{\rho-1}\eta_{\rho-1}^{\dagger} + \overline{k_\rho}(t), \tag{34}$$

where $\mathrm{sgn}(\overline{k_j}) = \mathrm{sgn}(w_j)$ and $|\overline{k_j}(t)| \geqslant |k_j(t) + k_{\delta j}(t)w_j|$, for $j = 1, \ldots, \rho$, with $k_j(t)$ and $k_{\delta j}(t)$ being defined in (14), (19), (61) and (63).

Remark. The supervised compensator $\bar{\eta}$ is to assure that if the jth input fuzzy variable $w_j \notin \Omega_{w_j}$, the integrated control can drive w_j into the compact set Ω_{w_j}, and, in turn, the jth fuzzy controller u_{f_j} can then be applied to derive w_j further into the prespecified dead-zone range $[-\Delta_{w_j}, \Delta_{w_j}]$ for $w_j \in \Omega_{w_j}$. Apparently, $\bar{\eta}$ is a high-gain compensator, and instead we can set the compact sets Ω_{w_j}'s to be sufficiently large to avoid applying $\bar{\eta}$ so frequently. Such strategy will however result in a trade-off which creates more fuzzy rules.

Here, the dead-zone ranges are defined as the compact supports of membership functions $\mu_{1\Upsilon}(w_1), \ldots, \mu_{\rho\Upsilon}(w_\rho)$, namely,

$$\Omega_{w_{10}} \equiv \left[-\frac{m+1}{2}\Delta_1, \frac{m+1}{2}\Delta_1\right] \equiv [-\Delta_{w_1}, \Delta_{w_1}] \equiv [-\Delta_e, \Delta_e],$$

$$\vdots$$

$$\Omega_{w_{\rho 0}} \equiv \left[-\frac{m+1}{2}\Delta_\rho, \frac{m+1}{2}\Delta_\rho\right] \equiv [-\Delta_{w_\rho}, \Delta_{w_\rho}] \equiv [-\Delta_{\eta_{\rho-1}}, \Delta_{\eta_{\rho-1}}]. \tag{35}$$

Then, the following proposition can be established.

Proposition 4.1. *If the control law* $\delta(y)u=\eta_\rho^\dagger$ *is given as in Eqs.* (32)–(34), *then there exist a class of the fuzzy controller vector* u_f *given as in* (32) *which can drive the tracking error of system* (6), w_1 (*i.e.*, e_o), *into the dead-zone range* $[-\Delta_{w_1},\Delta_{w_1}]$ *globally and exponentially.*

Proof. To prove this, we have to assure that u_f satisfies the two properties, namely, (a) $\mathrm{sgn}(u_{f_j})=\mathrm{sgn}(w_j)$, and (b) $|u_{f_j}|\geqslant|k_j+k_{\delta j}w_j|$, when $w_j\in\Omega_{w_j}\backslash\Omega_{w_{j0}}$, for $j=1,\ldots,\rho$, respectively, from variable structure control theory, by investigating especially Eqs. (14), (19), (61) and (63), presented in the previous section.

Proof of (a): From the definition I_{cj} in (27), when $w_j\in\Omega_{w_j}\backslash\Omega_{w_{j0}}$, it follows that

$$I_{cj}(w_j)\subset\{-\Upsilon,\ldots,-1\},\quad w_{j_\Delta}<0\ \left(\text{i.e. } w_j<-\frac{m+1}{2}\Delta_j\right),$$

$$I_{cj}(w_j)\subset\{1,\ldots,\Upsilon\},\quad w_{j_\Delta}>0\ \left(\text{i.e. } w_j>\frac{m+1}{2}\Delta_j\right) \tag{36}$$

for $j=1,\ldots,\rho$, respectively, where $\{-\Upsilon,\ldots,-1\}$ and $\{1,\ldots,\Upsilon\}$ are both integer sets. Then, the representation of the jth fuzzy controller can be rewritten as follows:

$$u_{f_j}(w_1,\ldots,w_j)=\begin{cases}\sum_{i_1=-\Upsilon}^{\Upsilon}\cdots\sum_{i_{j-1}=-\Upsilon}^{\Upsilon}\sum_{i_j=-\Upsilon}^{-1}v_{i_1\cdots i_j}(w_1,\ldots,w_j)\theta_{i_1\cdots i_j} & \text{as } w_{j_\Delta}<0,\\ \sum_{i_1=-\Upsilon}^{\Upsilon}\cdots\sum_{i_{j-1}=-\Upsilon}^{\Upsilon}\sum_{i_j=1}^{\Upsilon}v_{i_1\cdots i_j}(w_1,\ldots,w_j)\theta_{i_1\cdots i_j} & \text{as } w_{j_\Delta}>0.\end{cases}$$

Since $v_{i_1\cdots i_j}(w_1,\ldots,w_j)$ is always positive, the sign of u_{f_j} can be determined by $\theta_{i_1\cdots i_j}$'s. Hence, we set $\theta_{i_1\cdots i_j}<0$, for $-\Upsilon\leqslant i_k\leqslant\Upsilon$, $k\neq j$, $-\Upsilon\leqslant i_j\leqslant-1$ and $\theta_{i_1\cdots i_j}>0$, for $-\Upsilon\leqslant i_k\leqslant\Upsilon$, $k\neq j$, $1\leqslant i_j\leqslant\Upsilon$. As a result, we can conclude that $\mathrm{sgn}(u_{f_j})=\mathrm{sgn}(w_j)$, for $j=1,\ldots,\rho$.

Proof of (b): Given the following definitions for $j=1,\ldots,\rho$:

$$k_{j_{\max}}(w_1,\ldots,w_\rho)=\max\Bigg\{\sup_{x\in\Omega_{c1}(w_1)\times\cdots\times\Omega_{cj}(w_j)}|k_j(x,\alpha,\Delta_1,\ldots,\Delta_j)$$
$$+k_{\delta j}(x,\alpha,\Delta_1,\ldots,\Delta_j,\Delta_{j+1})x_j|,\ x\in\Re^j\Bigg\},$$

$$\theta_{j_{\min}}(w_1,\ldots,w_\rho)=\min\{|\theta_{i_1\cdots i_j}|,\ i_1\in I_{c1}(w_1),\ldots,i_j\in I_{cj}(w_j)\},$$

where $\Omega_{c1}(w_1),\ldots,\Omega_{c\rho}(w_\rho)$ are defined as in expression (31), $k_j(x,\alpha,\Delta_1,\ldots,\Delta_j)$ and $k_{\delta j}(x,\alpha,\Delta_1,\ldots,\Delta_j)$ are defined as in (14), (19), (61) and (63). Furthermore, by setting $\theta_{j_{\min}}\geqslant k_{j_{\max}}$, we will obtain the following inequality:

$$|\theta_{i_1\cdots i_j}|\geqslant\theta_{j_{\min}}\geqslant k_{j_{\max}}\quad\text{for } i_1\in I_{c1}(w_1),\ldots,i_j\in I_{cj}(w_j).$$

By virtue of the fact $\sum_{i_1=-\Upsilon}^{\Upsilon}\cdots\sum_{i_j=-\Upsilon}^{\Upsilon}v_{i_1\cdots i_j}=\sum_{i_1\in I_{c1}(w_1)}\cdots\sum_{i_j\in I_{cj}(w_j)}v_{i_1\cdots i_j}=1$ and in the cases of $w_{j_\Delta}\neq0$, for $j=1,\ldots,\rho$, we can derive the following result:

$$|u_{f_j}|=\sum_{i_1\in I_{c1}(w_1)}\cdots\sum_{i_j\in I_{cj}(w_j)}v_{i_1\cdots i_j}(w_1,\ldots,w_j)|\theta_{i_1\cdots i_j}|$$
$$\geqslant\sum_{i_1\in I_{c1}(w_1)}\cdots\sum_{i_j\in I_{cj}(w_j)}v_{i_1\cdots i_j}(w_1,\ldots,w_j)\theta_{j_{\min}}$$

$$= \theta_{j_{\min}}(w_1,\dots,w_j) \geqslant k_{j_{\max}}(w_1,\dots,w_j)$$
$$\geqslant |k_j(t) + k_{\delta j}(t) w_j| \quad \text{as } w_{j_\Delta} \neq 0$$

for $j=1,\dots,\rho$. Thus, from Theorem 3.1, we can conclude that $w_{1\Delta} \to 0$ as $t \to \infty$. $\square$

Now, define the optimal parameter vector of the jth fuzzy controller as follows:

$$\theta^{(j)*} = \operatorname{argmin}\left\{ \sup_{w_1 \in \Omega_{w1},\dots,w_{i_j-1} \in \Omega_{w_{i_j-1}}, w_j \in \Omega_{w_{i_j}} \setminus \Omega_{w_{j0}}} \sum_{i_1=-\Upsilon}^{\Upsilon} \cdots \sum_{i_{j-1}=-\Upsilon}^{\Upsilon} \sum_{i_j=-\Upsilon}^{\Upsilon} v_{i_1 \cdots i_j} \theta_{i_1 \cdots i_j} \operatorname{sgn}(w_j) \geqslant |k_j(t) + k_{\delta j}(t) w_j| \right\}. \tag{37}$$

It is, however, that $\theta^{(j)*}$ may not easily be available due to the complexity of $k_j(t)$ and $k_{\delta j}(t)$, $j=1,\dots,\rho$. Therefore, the following adaptive law to update the parameters vector $\theta^{(j)}$ will be necessary so that the tracking error can be driven toward the dead-zone range:

$$\dot{\theta}^{(1)} = \begin{cases} r d_1 v^{(1)}(w_1) w_{1\Delta} & \text{for } w_1 \in \Omega_{w1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\dot{\theta}^{(2)} = \begin{cases} r v^{(2)}(w_1, w_2) w_{2\Delta} & \text{for } w_1 \in \Omega_{w1}, w_2 \in \Omega_{w2}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\vdots$$

$$\dot{\theta}^{(\rho)} = \begin{cases} r v^{(\rho)}(w_1,\dots,w_\rho) w_{\rho\Delta} & \text{for } w_1 \in \Omega_{w1},\dots,w_\rho \in \Omega_{w\rho}, \\ 0 & \text{otherwise,} \end{cases} \tag{38}$$

where $w_{1\Delta} = e_{o\Delta}$, $w_{2\Delta} = \eta_{1\Delta},\dots,$ $w_{\rho\Delta} = \eta_{\rho-1_\Delta}$ are defined as in (56) and (23). Based on the control law in (32)–(34), and the adaptive law (38), the system can be shown to achieve appropriate output error convergence into a prespecified dead-zone range. This is summarized and proved in the following theorem.

Theorem 4.1. *If adaptive fuzzy variable structure control law is given as in Eqs.* (32)–(34) *with the adaptive law* (38), *then the output tracking error of system* (6) *will be driven to the dead zone range* $[-\Delta_{w1}, \Delta_{w1}]$ *globally and asymptotically.*

Proof. Define $\tilde{\theta}^{(j)} = \theta^{(j)*} - \theta^{(j)}$, and a new vector $\sigma \in \Re^\rho$ with the jth element defined as follows:

$$\sigma_j(w_1,\dots,w_j) = \begin{cases} 1 & \text{for } w_1 \in \Omega_{w_1},\dots,w_j \in \Omega_{w_j}, \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

Then, consider a Lyapunov function candidate

$$V_f = \tilde{\xi}^{\mathrm{T}} P \tilde{\xi} + \frac{1}{2} \sum_{j=1}^{\rho} w_{j\Delta}^2 + \frac{1}{2r} \sum_{j=1}^{\rho} \widetilde{\theta^{(j)}}^{\mathrm{T}} \widetilde{\theta^{(j)}}$$
$$= \tilde{\xi}^{\mathrm{T}} P \tilde{\xi} + \frac{1}{2} \sum_{j=1}^{\rho} \left[(1-\sigma_j) \left(w_{j\Delta}^2 + \frac{1}{r} \widetilde{\theta^{(j)}}^{\mathrm{T}} \widetilde{\theta^{(j)}} \right) + \sigma_j \left(w_{j\Delta}^2 + \frac{1}{r} \widetilde{\theta^{(j)}}^{\mathrm{T}} \widetilde{\theta^{(j)}} \right) \right]. \tag{40}$$

By applying Theorem 3.1 and Proposition 4.1, we can compute the time derivative of V_f for $w_{1\Delta} > 0$ as follows:

$$\begin{aligned}
\dot{V}_f &= \tilde{\xi}^{\mathrm{T}} P \dot{\tilde{\xi}} + \dot{\tilde{\xi}}^{\mathrm{T}} P \tilde{\xi} + \sum_{j=1}^{\rho} \left[(1-\sigma_j) w_{j_\Delta} \dot{w}_{j_\Delta} + \sigma_j \left(w_{j_\Delta} \dot{w}_{j_\Delta} + \frac{1}{r} \widetilde{\theta^{(j)}}^{\mathrm{T}} \dot{\widetilde{\theta^{(j)}}} \right) \right] \\
&\leqslant -\varepsilon_1 w_{1\Delta}^2 - \varepsilon_2 \|\tilde{\xi}\|^2 - \sum_{i=2}^{\rho} \varepsilon_{i+1} w_{i\Delta}^2 + \sigma_1 d_1 w_{1\Delta} v^{(1)} (\widetilde{\theta^{(1)}}^{\mathrm{T}} - \widetilde{\theta^{(1)}}^{\mathrm{T}}) \\
&\quad + \sum_{j=2}^{\rho} [\sigma_j w_{j_\Delta} v^{(j)} (\widetilde{\theta^{(j)}}^{\mathrm{T}} - \widetilde{\theta^{(j)}}^{\mathrm{T}})] \\
&\leqslant -\varepsilon_1 w_{1\Delta}^2 - \varepsilon_2 \|\tilde{\xi}\|^2 - \sum_{i=2}^{\rho} \varepsilon_{i+1} w_{i\Delta}^2
\end{aligned} \tag{41}$$

and

$$\dot{V}_f \leqslant -\varepsilon_2 \|\tilde{\xi}\|^2 - \sum_{i=2}^{\rho} \varepsilon_{i+1} w_{i\Delta}^2 + \left[\frac{\frac{1}{2} + \|P\|(\|\beta\| + l_\Phi)}{\sqrt{q_0}} w_1 \right]^2 + (\rho - 1)\varepsilon_1 w_1^2 \quad \text{for } w_{1\Delta} = 0.$$

It follows that w_{j_Δ}, $\theta^{(j)}$, $j = 1, \ldots, \rho$ and $\tilde{\xi}$ are bounded and hence, Eqs. (40) and (41) imply

$$\lim_{t\to\infty} \int_0^t -\frac{\mathrm{d}V_f(\tau)}{\mathrm{d}\tau} \,\mathrm{d}\tau = V_f(0) - \lim_{t\to\infty} V_f(t) < \infty \tag{42}$$

which results in

$$\lim_{t\to\infty} \int_0^t w_{1\Delta}^2(\tau) \,\mathrm{d}\tau < \infty. \tag{43}$$

By applying Barbalat's lemma, we obtain

$$\lim_{t\to\infty} |w_{1\Delta}| = 0. \qquad \square \tag{44}$$

Remark. The fuzzy controller (32) with the adaptive law (38) possesses the following advantages:

- *Locally weighted fuzzy controller*: Only rules supported by compact set Ω_{cj} are required to be updated, and hence, those rules are locally weighted.
- *Smooth fuzzy controller*: Apparently, the fuzzy controller (32) can behave as a smoother controller provided the differential terms $\partial u_{f_1}/\partial w_1$, $\partial u_{f_2}/\partial w_1, \ldots, \partial u_{f_\rho}/\partial w_\rho$ can be made small, which then requires that smoother membership functions are adopted. Thus, hardening the controllers with high gain in the backstepping procedure can be naturally avoided here, if we can choose the membership functions to be even smoother high-order B-spline functions.

4.1. Computer simulation

Consider the developed adaptive fuzzy variable structure control to be applied to the system described in Section 2, $\alpha = 2$ is assumed to be unknown. The fuzzy controller is synthesized as follows: the first fuzzy controller for the designated filter state η_1 takes the tracking error e_o as its single input variable w_1 whereas

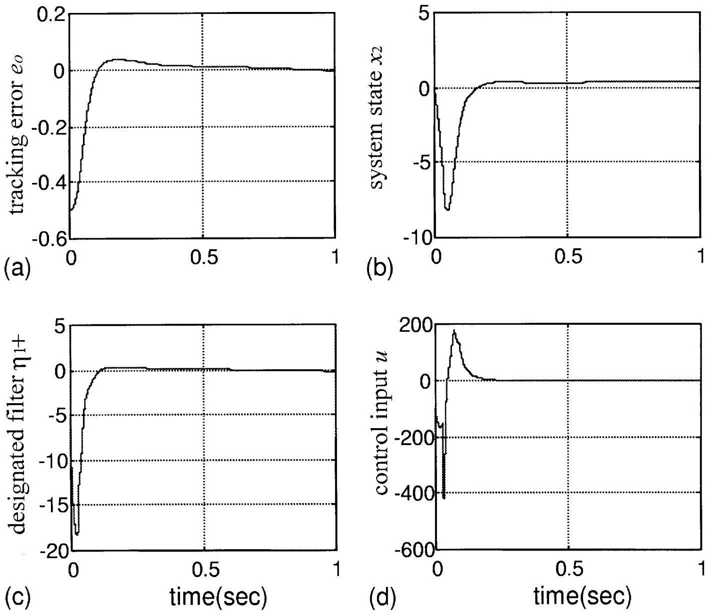

Fig. 5. The simulation results for the fuzzy controller.

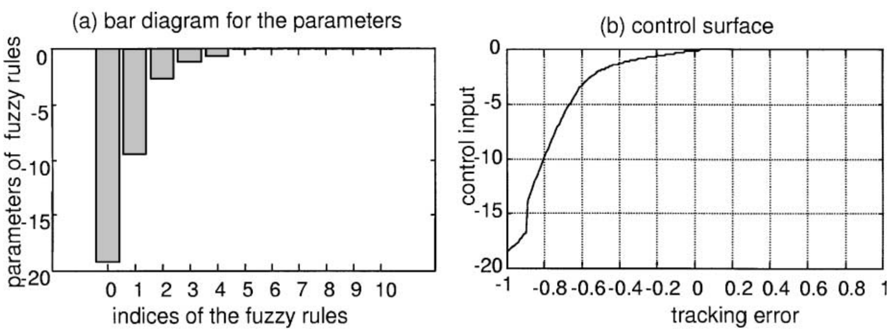

Fig. 6. The bar diagram for the parameters of the first fuzzy controller and control surface.

the second fuzzy controller takes the tracking error e_o and output of the first fuzzy controller as two input variables (w_1, w_2). The fuzzy rule number of the first fuzzy controller is equal to $2\Upsilon + 1 = 11$ and that of the second fuzzy controller is equal to $11 \times 11 = 121$. Moreover, $k_{f_1} = k_{f_2} = 10$ is assigned. Fig. 5 shows that the tracking errors can converge into the dead-zone range and the magnitude of the control input is smaller than Fig. 1(d). Fig. 6(a) shows a bar diagram for the final parameters of the first fuzzy controller after updating. Apparently, only a half of parameters of fuzzy rules had been updated since e_o almost stays within the region $e_o < 0$ during the task running. Fig. 6(b) shows the smooth control surface of the first fuzzy controller. Fig. 7(a) shows a three-dimensional (3d) bar diagram of the final parameters of the second fuzzy controller after updating and Fig. 7(b) shows a three-dimensional plot or smooth control surface of the first fuzzy controller.

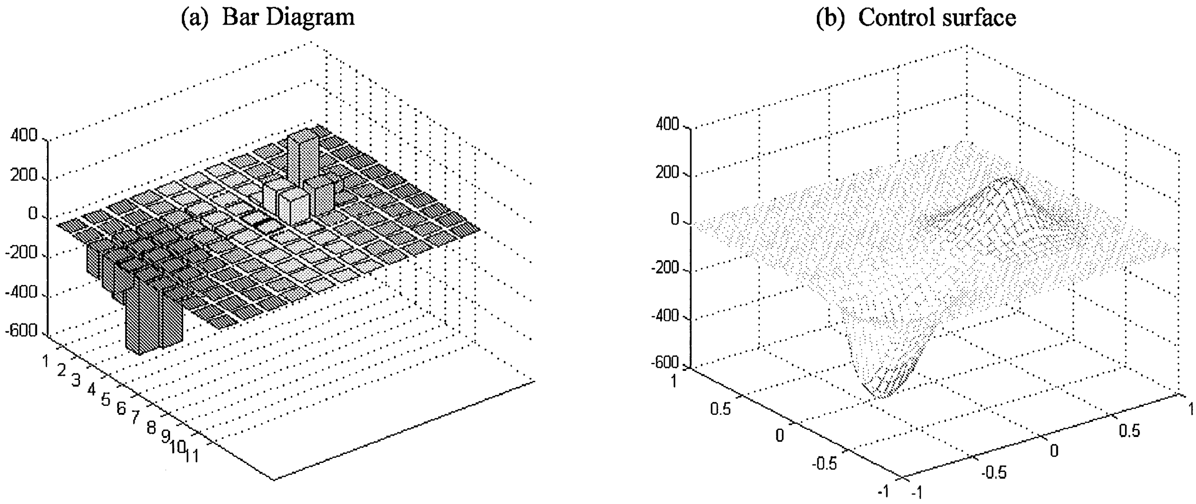

Fig. 7. The 3d bar diagram for the parameters of the second fuzzy controller and control surface.

5. Conclusion

In this paper, we proposed a novel adaptive fuzzy variable structure control via backstepping for a class of SISO nonlinear systems which can solve the traditional model reference adaptive control problem in the presence of system uncertainties. It was rigorously proved that the stability of the overall system is assured and the tracking error can be driven to the designated dead-zone range. Besides, with undesirable chattering from the "hard" high-gain control laws can be avoided due to the adoption of the smooth B-spline-type membership functions. Salient features of the present work includes that the involved rules are locally weighted and the output control is rather smooth.

Appendix A

This appendix presents the derivation of the backstepping-based error model (8). First we realize the filter (7) in the state-space form as follows:

$$\dot{\eta} = A_f \eta + b_f \delta(y) u, \quad \eta \in \Re^{\rho-1}, \tag{45}$$

where

$$A_f = \begin{bmatrix} -\lambda_1 & 1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & -\lambda_{\rho-1} \end{bmatrix}, \qquad b_f = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix},$$

$\eta = [\eta_1, \ldots, \eta_{\rho-1}]$, and $\eta(0) = \eta_0$. Then, augment the error model (6) with this filter (45) as follows:

$$\begin{bmatrix} \dot{e} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_c & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} e \\ \eta \end{bmatrix} + \begin{bmatrix} -b(\alpha) \\ b_f \end{bmatrix} \delta(y)u + \begin{bmatrix} -ay - \Phi(y,\alpha) + b_m r_m \\ 0 \end{bmatrix}$$
$$e_o = c^{\mathrm{T}} e, \tag{46}$$

which is equivalent to the following:

$$\dot{\zeta} = A_c \zeta - ay - \Phi(y,\alpha) + b_m r_m - d(\alpha)\eta_1,$$
$$e_o = c^{\mathrm{T}} \zeta \tag{47}$$

from the I/O point of view, where $d = [d_1, \ldots, d_n]$ is a vector of Hurwitz coefficients of degree one, derived from the following transfer function:

$$W(s)W_f(s) = \frac{b_\rho s^{n-\rho} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_{\rho-1})$$
$$= \frac{d_1 s^{n-1} + \cdots + d_n}{s^n + a_1 s^{n-1} + \cdots + a_n}. \tag{48}$$

Apparently, $d_1 = b_\rho$. After applying the transformation developed by Marino and Tomei [10], we define a new vector $\xi = [\xi_1, \ldots, \xi_{n-1}]^{\mathrm{T}}$ as follows:

$$\xi_1 = \zeta_2 - \frac{d_2(\alpha)}{d_1(\alpha)} e_o,$$
$$\vdots$$
$$\xi_{n-1} = \zeta_n - \frac{d_n(\alpha)}{d_1(\alpha)} e_o, \tag{49}$$

whereby we can obtain a different dynamic model as shown below:

$$\dot{\xi} = \begin{bmatrix} -d_2/d_1 & 1 & 0 & \cdots & 0 \\ -d_3/d_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_{n-1}/d_1 & 0 & 0 & \cdots & 1 \\ -d_n/d_1 & 0 & 0 & \cdots & 0 \end{bmatrix} \xi + \begin{bmatrix} (d_3/d_1) - (d_2^2/d_1^2) \\ (d_4/d_1) - (d_3 d_2/d_1^2) \\ \vdots \\ (d_n/d_1) - (d_{n-1} d_2/d_1^2) \\ -d_n d_2/d_1^2 \end{bmatrix} e_o$$
$$+ \begin{bmatrix} (d_2/d_1)\Phi_1(y,\alpha) - \Phi_2(y,\alpha) \\ (d_3/d_1)\Phi_1(y,\alpha) - \Phi_3(y,\alpha) \\ \vdots \\ (d_{n-1}/d_1)\Phi_1(y,\alpha) - \Phi_{n-1}(y,\alpha) \\ (d_n d_1)\Phi_1(y,\alpha) - \Phi_n(y,\alpha) \end{bmatrix} + \begin{bmatrix} (d_2/d_1)a_1 y_m - a_2 y_m \\ \vdots \\ (d_\rho/d_1)a_1 y_m - (a_\rho y_m - b_{m\rho} r_m) \\ \vdots \\ (d_{n-1}/d_1)a_1 y_m - (a_{n-1} y_m - b_{mn-1} r_m \\ (d_n/d_1)a_1 y_m - (a_n y_m - b_{mn} r_m) \end{bmatrix}$$
$$= \Gamma\xi + \beta e_o + \Psi(y,\alpha) + \Psi_r(y_m, r_m), \tag{50}$$

where Γ is apparently a Hurwitz matrix. From assumption (A4), it can be easily verified that

$$\|\Psi(y,\alpha) - \Psi(y_m,\alpha)\| \leqslant l_\Phi(e_o, y_m, \alpha)|e_o| \quad \text{for some } l_\Phi(e_o, y_m, \alpha) \geqslant 0. \tag{51}$$

On the other hand, from transformation (49) and Eq. (47), the output backstepping-based error model (8) can be easily yielded.

Appendix B

This appendix presents the proof of Propositions 3.1–3.3.

B.1. Proof of Proposition 3.1

First, we define $\tilde{\xi} = \xi - \hat{\xi}$ as the difference between ξ and $\hat{\xi}$, and derive its time derivative as follows:

$$\dot{\tilde{\xi}} = \Gamma\tilde{\xi} + \beta e_o + \Psi(y,\alpha) - \Psi(y_m,\alpha). \tag{52}$$

Then, we consider a Lyapunov function candidate

$$V_1 = \tfrac{1}{2}e_o^2 + \tilde{\xi}^{\mathrm{T}} P\tilde{\xi}, \tag{53}$$

where P is a symmetric positive-definite matrix which satisfies the following:

$$\Gamma^{\mathrm{T}}P + P\Gamma = -Q \leqslant -(q_0 + \rho\varepsilon_2)I < 0, \tag{54}$$

where I is an $(n-1)\times(n-1)$ identity matrix and $\varepsilon_2 > 0$ is a positive constant. By differentiating V_1 along the solution trajectories of (8) and (52), we obtain

$$\begin{aligned}
\dot{V}_1 &= e_o\dot{e}_o + \tilde{\xi}^{\mathrm{T}}P\dot{\tilde{\xi}} + \dot{\tilde{\xi}}^{\mathrm{T}}P\tilde{\xi} \\
&= e_o(\tilde{\xi}_1 + \hat{\xi}_1) + \frac{d_2}{d_1}e_o^2 - e_o[\phi(e_o, y_m, \alpha) + a_1 y_m] - |d_1|k_{11}e_o^2 \\
&\quad -|d_1|k_{12}|e_o| - \tilde{\xi}^{\mathrm{T}}Q\tilde{\xi} + 2\tilde{\xi}^{\mathrm{T}}P[\beta e_o + \Psi(y,\alpha) - \Psi(y_m,\alpha)] \\
&\leqslant |e_o|(\|\hat{\xi}\| + |\phi| + |a_1||y_m| - |d_1|\delta_0 k_{12}) + \left(\left|\frac{d_2}{d_1}\right| - |d_1|k_{11}\right)e_o^2 \\
&\quad + 2\|\tilde{\xi}\||e_o|\left[\tfrac{1}{2} + \|P\|(\|\beta\| + l_\Phi)\right] - q_0\|\tilde{\xi}\|^2 - \rho\varepsilon_2\|\tilde{\xi}\|^2 \\
&\leqslant |e_o|(\|\hat{\xi}\| + |\phi| + |a_1||y_m| - |d_1|k_{12}) + \left(\left|\frac{d_2}{d_1}\right| - |d_1|k_{11}\right)e_o^2 \\
&\quad + \frac{[\frac{1}{2} + \|P\|(\|\beta\| + l_\Phi)]^2}{q_0}e_o^2 - \left[\frac{\frac{1}{2} + \|P\|(\|\beta\| + l_\Phi)}{\sqrt{q_0}}e_o - \sqrt{q_0}\|\tilde{\xi}\|\right]^2 - \rho\varepsilon_2\|\tilde{\xi}\|^2 \\
&\leqslant -\rho\varepsilon_1 e_o^2 - \rho\varepsilon_2\|\tilde{\xi}\|^2.
\end{aligned} \tag{55}$$

This implies boundness of e_o, $\tilde{\xi}$ and the zero convergence of e_o by stability theory of Lyapunov, which completes the proof. □

B.2. Proof of Proposition 3.2

Define a new variable $e_{o\Delta}$ as follows:

$$e_{o\Delta} = \begin{cases} e_o & \text{as } e_o < -\Delta_e \text{ or } e_o > \Delta_e, \\ 0 & \text{otherwise (i.e. } e_o \in [-\Delta_e, \Delta_e]); \end{cases} \tag{56}$$

so that

$$\dot{e}_{o\Delta} = \dot{e}_o \quad \text{for } e_{o\Delta} \neq 0.$$

Consider a Lyapunov function candidate

$$V_\Delta = \tfrac{1}{2}e_{o\Delta}^2 + \tilde{\xi}^{\mathrm{T}} P \tilde{\xi}. \tag{57}$$

Similar to the procedure of proving Proposition 3.1, we can easily derive that

$$\dot{V}_\Delta \leqslant \begin{cases} -\rho\varepsilon_1 e_{o\Delta}^2 - \rho\varepsilon_2 \|\tilde{\xi}\|^2 & \text{for } e_{o\Delta} > 0, \\ -\rho\varepsilon_2\|\tilde{\xi}\|^2 + \left[\dfrac{\frac{1}{2} + \|P\|(\|\beta\| + l_\Phi)}{\sqrt{q_0}} e_o\right]^2 & \text{for } e_{o\Delta} = 0, \end{cases} \tag{58}$$

which, first of all, implies the boundedness of both $e_{o\Delta}$ (hence e_o) and $\tilde{\xi}$. Next, we can conclude that $e_{o\Delta} \to 0$ again by the stability theory of Lyapunov, which implies that $e_o \to [-\Delta_e, \Delta_e]$. □

B.3. Proof of Proposition 3.3

Consider a Lyapunov function candidate

$$V_2 = \tfrac{1}{2}e_{o\Delta}^2 + \tilde{\xi}^{\mathrm{T}} P \tilde{\xi} + \tfrac{1}{2}\tilde{\eta}_1^2 \tag{59}$$

whose time derivative for $e_{o\Delta} > 0$ can be derived as follows:

$$\begin{aligned} \dot{V}_2 &= e_{o\Delta}(\dot{e}_{o_\Delta} - d_1\eta_1^\dagger + d_1\eta_1^\dagger) + \tilde{\xi}^{\mathrm{T}} P \dot{\tilde{\xi}} + \dot{\tilde{\xi}}^{\mathrm{T}} P \tilde{\xi} + \tilde{\eta}_1 \dot{\tilde{\eta}}_1 \\ &\leqslant -\rho\varepsilon_1 e_{o\Delta}^2 - \rho\varepsilon_2\|\tilde{\xi}\|^2 + d_1 e_{o\Delta}\tilde{\eta}_1 + \tilde{\eta}_1\dot{\tilde{\eta}}_1 \\ &\leqslant -\rho\varepsilon_1 e_{o\Delta}^2 - \rho\varepsilon_2\|\tilde{\xi}\|^2 + \tilde{\eta}_1\left[d_1 e_{o\Delta} + \frac{\partial\eta_1^\dagger}{\partial e_o}\dot{e}_o - (\lambda_1 + k_{21})\tilde{\eta}_1 \; - k_{22}\,\mathrm{sgn}(\tilde{\eta}_1)\right] \\ &\leqslant -\rho\varepsilon_1 e_{o\Delta}^2 - \rho\varepsilon_2\|\tilde{\xi}\|^2 - \left(\lambda_1 - d_1\frac{\partial\eta_1^\dagger}{\partial e_o} + k_{21}\right)\tilde{\eta}_1{}^2 + \frac{\partial\eta_1^\dagger}{\partial e_o}\tilde{\eta}_1\tilde{\xi}_1 + \left(d_1 + \frac{d_2}{d_1}\frac{\partial\eta_1^\dagger}{\partial e_o}\right)e_{o\Delta}\tilde{\eta}_1 \\ &\quad + \tilde{\eta}_1\left\{\frac{\partial\eta_1^\dagger}{\partial e_o}\left[\hat{\xi}_1 - \phi - a_1 y_m - d_1\eta_1^\dagger + \frac{d_2}{d_1}(e_o - e_{o\Delta})\right] - k_{22}\,\mathrm{sgn}(\tilde{\eta}_1)\right\} \\ &\leqslant -(\rho - 1)\varepsilon_1 e_{o\Delta}^2 - (\rho - 1)\varepsilon_2\|\tilde{\xi}\|^2 - \lambda_1\tilde{\eta}_1^2 \end{aligned}$$

and

$$\dot{V}_2 \leqslant -(\rho-1)\varepsilon_2\|\tilde{\xi}\|^2 - \lambda_1\tilde{\eta}_1^2 + \left[\frac{\frac{1}{2}+\|P\|(\|\beta\|+l_\Phi)}{\sqrt{q_0}}e_o\right]^2 + \varepsilon_1 e_o^2 \quad \text{for } e_{o\Delta}=0$$

which similarly implies the boundedness of all involved signals and the zero convergence of $e_{o\Delta}$. □

Appendix C

This appendix contains the definition of $k_3(\cdot), \ldots, k_\rho(\ldots)$. First, we have to derive the time derivative of $\widetilde{\eta_j}$ for $j=2,\ldots,\rho-1$ as follows:

$$\begin{aligned}
\dot{\widetilde{\eta_j}} &= \dot{\eta}_j^\dagger - \dot{\eta}_j = \dot{\eta}_j^\dagger + \lambda_j\eta_j - \eta_{j+1} = \dot{\eta}_j^\dagger - \lambda_j\widetilde{\eta_j} - k_{j+1} - k_{\delta j+1}\widetilde{\eta_j} + \widetilde{\eta_{j+1}} \\
&= \frac{\partial\eta_j^\dagger}{\partial e_o}\dot{e}_o + \frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_1}}\dot{\widetilde{\eta_1}} + \cdots + \frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_{j-1}}}\dot{\widetilde{\eta_{j-1}}} - \lambda_j\widetilde{\eta_j} - k_{j+1} - k_{\delta j+1}\widetilde{\eta_j} + \widetilde{\eta_{j+1}} \\
&= \left(\widetilde{\xi_1} + \widehat{\xi_1} + \frac{d_2}{d_1}e_o - \phi - a_1 y_m - d_1\eta_1^\dagger + d_1\widetilde{\eta_1}\right)\left[\frac{\partial\eta_j^\dagger}{\partial e_o} + \left(\sum_{i=1}^{j-1}\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_i}}\frac{\partial\eta_i^\dagger}{\partial e_o}\right) + \left(\sum_{i=2}^{j-1}\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_i}}\frac{\partial\eta_i^\dagger}{\partial\widetilde{\eta_{i-1}}}\frac{\partial\eta_{i-1}^\dagger}{\partial e_o}\right)\right. \\
&\quad \left. + \cdots + \left(\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_{j-1}}}\frac{\partial\eta_{j-1}^\dagger}{\partial\widetilde{\eta_{j-2}}}\cdots\frac{\partial\eta_1^\dagger}{\partial e_o}\right)\right] + (\widetilde{\eta_2} - \lambda_1\widetilde{\eta_1})\left[\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_1}} + \left(\sum_{i=2}^{j-1}\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_i}}\frac{\partial\eta_i^\dagger}{\partial\widetilde{\eta_1}}\right) + \left(\sum_{i=3}^{j-1}\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_i}}\frac{\partial\eta_i^\dagger}{\partial\widetilde{\eta_{i-1}}}\frac{\partial\eta_{i-1}^\dagger}{\partial\widetilde{\eta_1}}\right)\right. \\
&\quad \left. + \cdots + \left(\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_{j-1}}}\frac{\partial\eta_{j-1}^\dagger}{\partial\widetilde{\eta_{j-2}}}\cdots\frac{\partial\eta_1^\dagger}{\widetilde{\eta_1}}\right)\right] + \cdots + (\widetilde{\eta_j} - \lambda_{j-1}\widetilde{\eta_{j-1}})\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_{j-1}}} - \lambda_j\widetilde{\eta_j} - k_{j+1} - k_{\delta j+1}\widetilde{\eta_j} + \widetilde{\eta_{j+1}} \\
&= g_{1_j}(\widehat{\xi_1} - \phi - a_1 y_m - d_1\eta_1^\dagger) + g_{1_j}\widetilde{\xi_1} + \frac{d_2}{d_1}g_{1_j}e_o + g_{2_j}\widetilde{\eta_1} + \cdots + g_{j_j}\widetilde{\eta_{j-1}} + g_{(j+1)_j}\widetilde{\eta_j} \\
&\quad - \lambda_j\widetilde{\eta_j} - k_{j+1} - k_{\delta j+1}\widetilde{\eta_j} + \widetilde{\eta_{j+1}},
\end{aligned} \tag{60}$$

where $\widetilde{\eta_\rho} = \delta(y)u - \eta_\rho^\dagger = 0$ and

$$g_{1_j} = \frac{\partial\eta_j^\dagger}{\partial e_o} + \left(\sum_{i=1}^{j-1}\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_i}}\frac{\partial\eta_i^\dagger}{\partial e_o}\right) + \left(\sum_{i=2}^{j-1}\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_i}}\frac{\partial\eta_i^\dagger}{\partial\widetilde{\eta_{i-1}}}\frac{\partial\eta_{i-1}^\dagger}{\partial e_o}\right) + \cdots + \left(\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_{j-1}}}\frac{\partial\eta_{j-1}^\dagger}{\partial\widetilde{\eta_{j-2}}}\cdots\frac{\partial\eta_1^\dagger}{\partial e_o}\right),$$

$$g_{2_j} = d_1 g_{1_j} - \lambda_1\left[\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_1}} + \left(\sum_{i=2}^{j-1}\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_i}}\frac{\partial\eta_i^\dagger}{\partial\widetilde{\eta_1}}\right) + \left(\sum_{i=3}^{j-1}\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_i}}\frac{\partial\eta_i^\dagger}{\partial\widetilde{\eta_{i-1}}}\frac{\partial\eta_{i-1}^\dagger}{\partial\widetilde{\eta_1}}\right) + \cdots + \left(\frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_{j-1}}}\frac{\partial\eta_{j-1}^\dagger}{\partial\widetilde{\eta_{j-2}}}\cdots\frac{\partial\eta_1^\dagger}{\widetilde{\eta_1}}\right)\right],$$

$$\vdots$$

$$g_{(j+1)_j} = \frac{\partial\eta_j^\dagger}{\partial\widetilde{\eta_{j-1}}}.$$

Then, $k_3(\cdot),\ldots,k_\rho(\cdot)$ are designed as follows:

$$k_{j+1}(t) = \begin{cases} k_{(j+1)_1}(t)\widetilde{\eta_j} + k_{(j+1)_2}(t)\,\mathrm{sgn}(\widetilde{\eta_j}), & \widetilde{\eta_j} \notin [-\Delta_{\eta_j}, \Delta_{\eta_j}], \\ k_{s\,j+1}(t) \quad \text{otherwise}, \end{cases} \tag{61}$$

where $k_{s\,j+1}(\cdot)$ is a smooth function in order to make $\eta_j^\dagger$ globally smooth, and

$$k_{(j+1)_1}(t) \geqslant \frac{((d_2/d_1)g_{1_j})^2}{4\varepsilon_1} + \frac{g_{1_j}^2}{4\varepsilon_2} + \frac{g_{2_j}^2}{4\varepsilon_3} + \cdots + \frac{g_{j-1_j}^2}{4\varepsilon_j} + \frac{(1+g_{j_j})^2}{4\varepsilon_{j+1}} + |g_{(j+1)_j}|,$$

$$k_{(j+1)_2}(t) \geqslant |g_{1_j}| \left(\|\xi\| + |\phi| + |a_1||y_m| + |d_1\eta_1^\dagger| + \left|\frac{d_2}{d_1}\right| \Delta_e \right) + |g_{2_j}|\Delta_{\eta_1} + \cdots + |g_{j_j}|\Delta_{\eta_{j-1}} \tag{62}$$

for $j = 2,\ldots,\rho-1$ and $\varepsilon_3,\ldots,\varepsilon_\rho$ are defined as $(\rho-k)\varepsilon_{k+2} = \lambda_k$ for $k = 1,\ldots,\rho-1$, and $k_{\delta 1},\ldots,k_{\delta\rho-1}$ are defined as follows:

$$k_{\delta 1} \geqslant \frac{\Delta_{\eta_1}}{\Delta_e},$$

$$k_{\delta 2}(t) \geqslant \frac{\Delta_{\eta_2}}{\Delta_{\eta_1}} + \left|\frac{d_2}{d_1}\right| \left|\frac{\partial \eta_1^\dagger}{\partial e_o}\right| \frac{\Delta_e}{\Delta_{\eta_1}},$$

$$\vdots$$

$$k_{\delta j}(t) \geqslant \frac{\Delta_{\eta_j}}{\Delta_{\eta_{j-1}}} + \left|\frac{d_2}{d_1}\right| |g_{1(j-1)}| \frac{\Delta_e}{\Delta_{\eta_{j-1}}} + |g_{2(j-1)}| \frac{\Delta_{\eta_1}}{\Delta_{\eta_{j-1}}} + \cdots + |g_{(j-1)(j-1)}| \frac{\Delta_{\eta_{j-2}}}{\Delta_{\eta_{j-1}}},$$

$$\vdots$$

$$k_{\delta\rho-1}(t) \geqslant \frac{\Delta_{\eta_{\rho-1}}}{\Delta_{\eta_{\rho-2}}} + \left|\frac{d_2}{d_1}\right| |g_{1(\rho-2)}| \frac{\Delta_e}{\Delta_{\eta\rho-2}} + |g_{2(\rho-2)}| \frac{\Delta_{\eta 1}}{\Delta_{\eta\rho-2}} + \cdots + |g_{(\rho-2)(\rho-2)}| \frac{\Delta_{\eta\rho-3}}{\Delta_{\eta\rho-2}}. \tag{63}$$

Appendix D

Proof of Theorem 3.1. Consider a Lyapunov function candidate

$$V_\Delta = \tfrac{1}{2}e_{o\Delta}^2 + \tilde{\xi}^{\mathrm{T}} P \tilde{\xi} + \frac{1}{2}\sum_{j=1}^{\rho-1} \tilde{\eta}_{j\Delta}^2. \tag{64}$$

From the former propositions, the time derivative of V_Δ for $e_{o\Delta} > 0$ can be derived as follows:

$$\begin{aligned}
\dot{V}_\Delta &= e_{o\Delta}(\dot{e}_{o_\Delta} - d_1\eta_1^\dagger + d_1\eta_1^\dagger) + \tilde{\xi}^{\mathrm{T}} P \dot{\tilde{\xi}} + \dot{\tilde{\xi}}^{\mathrm{T}} P \tilde{\xi} + \tilde{\eta}_{1\Delta}(\dot{\tilde{\eta}}_1 - \eta_2^\dagger + \eta_2^\dagger) + \sum_{j=2}^{\rho-1}(\tilde{\eta}_{j\Delta}\dot{\tilde{\eta}}_j) \\
&\leqslant -\rho\varepsilon_1 e_{o\Delta}^2 - \rho\varepsilon_2\|\tilde{\xi}\|^2 - e_{o\Delta}[k_{\delta 1}e_o - d_1(\tilde{\eta}_1 - \tilde{\eta}_{1\Delta})] \\
&\quad + \tilde{\eta}_{1\Delta}\left[d_1 e_{o\Delta} + \frac{\partial\eta_1^\dagger}{\partial e_o}\dot{e}_o - (\lambda_1 + k_{21} + k_{\delta 2})\tilde{\eta}_1 - k_{22}\,\mathrm{sgn}(\tilde{\eta}_1) \right] + \sum_{j=2}^{\rho-1}(\tilde{\eta}_{j\Delta}\dot{\tilde{\eta}}_j) \\
&\leqslant -(\rho-1)\varepsilon_1 e_{o\Delta}^2 - (\rho-1)\varepsilon_2\|\xi\|^2 - \lambda_1\tilde{\eta}_{1_\Delta}^2 - e_{o\Delta}[k_{\delta 1}e_o - d_1(\tilde{\eta}_1 - \tilde{\eta}_{1\Delta})] \\
&\quad - \tilde{\eta}_{1\Delta}[k_{\delta 2}\tilde{\eta}_1 - (\tilde{\eta}_2 - \tilde{\eta}_{2\Delta}) + \tilde{\eta}_{2\Delta}] + \sum_{j=2}^{\rho-1}\left\{ \tilde{\eta}_{j\Delta}\left[g_{1_j}(\hat{\tilde{\xi}}_1 - \phi - a_1 y_m - d_1\eta_1^\dagger) + g_{1_j}\tilde{\xi}_1 \right.\right. \\
&\quad \left.\left. + \frac{d_2}{d_1}g_{1_j}e_o + g_{2_j}\tilde{\eta}_1 + \cdots + (g_{j_j}+1)\,\widetilde{\eta_{j-1}} + g_{(j+1)_j}\tilde{\eta}_j - k_{j+1} - k_{\delta j+1}\tilde{\eta}_j + \widetilde{\eta_{j+1}} \right] \right\}
\end{aligned}$$

$$\leqslant -(\rho-1)\varepsilon_1 e_{o\Delta}^2 - (\rho-1)\varepsilon_2\|\tilde{\xi}\|^2 - (\rho-1)\varepsilon_3\tilde{\eta}_{1\Delta}^2 + \sum_{j=2}^{\rho-1}\left\{\tilde{\eta}_{j\Delta}\left[g_{1j}(\hat{\xi}_1 - \phi - a_1 y_m - d_1\eta_1^{\dagger}) + g_{1j}\tilde{\xi}_1\right.\right.$$

$$\left.\left. + \frac{d_2}{d_1}g_{1j}e_{o\Delta} + g_{2j}\tilde{\eta}_{1\Delta} + \cdots + (g_{jj}+1)\widetilde{\eta_{j-1}}_{\Delta} + g_{(j+1)j}\tilde{\eta}_{j\Delta} - k_{j+1}\right]\right\} - \sum_{j=2}^{\rho-1}[(\rho-j)\varepsilon_{j+2}\tilde{\eta}_{j\Delta}^2]$$

$$-\sum_{j=2}^{\rho-1}\left\{\tilde{\eta}_{j\Delta}\left[k_{\delta j+1}\tilde{\eta}_j - (\widetilde{\eta_{j+1}} - \widetilde{\eta_{j+1}}_{\Delta}) - \frac{d_2}{d_1}g_{1j}(e_o - e_{o\Delta}) - \cdots - g_{jj}(\widetilde{\eta_{j-1}} - \widetilde{\eta_{j-1}}_{\Delta})\right]\right\}$$

$$\leqslant -\varepsilon_1 e_{o\Delta}^2 - \varepsilon_2\|\tilde{\xi}\|^2 - \sum_{j=1}^{\rho-1}\varepsilon_{j+2}\tilde{\eta}_{j\Delta}^2 - \sum_{j=2}^{\rho-1}\left[\left(\sqrt{\varepsilon_1}e_{o\Delta} - \frac{(d_2/d_1)g_{1j}}{2\sqrt{\varepsilon_1}}\tilde{\eta}_{j\Delta}\right)^2 + \left(\sqrt{\varepsilon_2}\|\tilde{\xi}\| - \frac{g_{1j}}{2\sqrt{\varepsilon_2}}\tilde{\eta}_{j\Delta}\right)^2\right.$$

$$+\left(\sqrt{\varepsilon_3}\tilde{\eta}_{1\Delta} - \frac{g_{2j}}{2\sqrt{\varepsilon_3}}\tilde{\eta}_{j\Delta}\right)^2 + \cdots + \left(\sqrt{\varepsilon_j}\widetilde{\eta_{j-2}}_{\Delta} - \frac{(g_{j-1j})}{2\sqrt{\varepsilon_j}}\tilde{\eta}_{j\Delta}\right)^2$$

$$\left. + \left(\sqrt{\varepsilon_{j+1}}\widetilde{\eta_{j-1}}_{\Delta} - \frac{(g_{jj}+1)}{2\sqrt{\varepsilon_{j+1}}}\tilde{\eta}_{j\Delta}\right)^2\right]$$

$$\leqslant -\varepsilon_1 e_{o\Delta}^2 - \varepsilon_2\|\tilde{\xi}\|^2 - \sum_{j=1}^{\rho-1}\varepsilon_{j+2}\tilde{\eta}_{j\Delta}^2, \tag{65}$$

whereas

$$\dot{V}_{\Delta} \leqslant \varepsilon_2\|\tilde{\xi}\|^2 - \sum_{j=1}^{\rho-1}\varepsilon_{j+2}\tilde{\eta}_{j\Delta}^2 + \left[\frac{\frac{1}{2} + \|P\|(\|\beta\| + l_{\Phi})}{\sqrt{q_0}}e_o\right]^2 + (\rho-1)\varepsilon_1 e_o^2 \quad \text{for } e_{o\Delta} = 0.$$

By Lyapunov stability theory, it is clear that when $e_{o\Delta} > 0$, $e_{o\Delta}$ and $\tilde{\eta}_{j\Delta}$ will converge to zero globally and exponentially, which then implies the boundness of the input u. Referring to Eq. (6), we can then conclude the boundness of the state e. □

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