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An approximation algorithm for the maximization version of the two level uncapacitated facility location problem

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Abstract

We consider the maximization version of the two level uncapacitated facility location problem, in the following formulation:

$$\max_{S_1\times S_2\subseteq F\times E} C(S_1,S_2)=\max_{S_1\times S_2\subseteq F\times E}\sum_{k\in D}\max_{(i,j)\in S_1\times S_2} c_{ijk}-\sum_{i\in S_1} f_i-\sum_{j\in S_2} e_j,$$

where F, E are finite sets and c_{ijk}, f_i, $e_j \geqslant 0$ are real numbers. Denote by C^* the optimal value of the problem and by $C_R=\sum_{k\in D}\min_{(i,j)\in F\times E} c_{ijk}-\sum_{i\in F} f_i-\sum_{j\in E} e_j$. We present a polynomial time algorithm based on randomized rounding that finds a solution (S_1,S_2) such that

$$C(S_1,S_2)-C_R\geqslant 0.47(C^*-C_R).$$

Keywords: Facility location; Approximation algorithms; Randomized algorithms

1. Introduction

The two level uncapacitated facility location problem (two level MAX UFLP), in the maximization version, can be described as follows. There are two types of potential facility locations: the hub facilities, denoted by F and the transit facilities, denoted by E. Building (opening) the facility $i\in F$ or $j\in E$ has an associated nonnegative cost f_i, respectively e_j. There is also a set of clients, D, who should be assigned to open pairs of facilities from $F\times E$. If a client $k\in D$ is assigned to the pair (i,j), a profit c_{ijk} is obtained. The problem is to decide simultaneously which facilities from F and which from E to open (at least one from each set) and how to assign the clients to the open facilities, such that the total profit is maximized.

Formally, the problem can be stated as

$$\max_{S_1\times S_2\subseteq F\times E} C(S_1,S_2)=\max_{S_1\times S_2\subseteq F\times E}\sum_{k\in D}\max_{(i,j)\in S_1\times S_2} c_{ijk}-\sum_{i\in S_1} f_i-\sum_{j\in S_2} e_j. \tag{1}$$

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Denote by $C_R = \sum_{k\in D} \min_{(i,j)\in F\times E} c_{ijk} - \sum_{i\in F} f_i - \sum_{j\in E} e_j$ and by C^* the optimal value of the problem. Clearly, $C^* \geqslant C(S_1,S_2) \geqslant C_R$, for each $S_1 \times S_2 \subseteq F \times E$. Note that the objective function $C(S_1,S_2)$ can take both positive and negative values and so there is a difficulty in the definition of measure of relative deviation for approximate solutions to (1). To overcome this, we will consider the problem with the shifted objective function $C(S_1,S_2) - C_R$, which takes only positive values (for a discussion on the idea of shifted objective functions see Cornuejols et al. [5]).

Throughout this paper, a *ρ-approximation algorithm* for a maximization (minimization) problem with positive objective function is a polynomial time algorithm that always finds a feasible solution with objective function value at least (at most) ρ times the optimum. The value ρ is called the *performance guarantee* of the algorithm.

If the set E is a singleton, one obtains the one level uncapacitated facility location problem, in the maximization version (MAX UFLP). It can be easily proven that MAX UFLP is NP-hard, by reduction from the node cover problem (see [6]). Recently, Ageev and Sviridenko [3] showed that the MAX UFLP in the shifted form admits no polynomial time approximation scheme. Cornuejols et al. [5] proved that for MAX UFLP a simple greedy algorithm finds a solution $S \subseteq F$ such that

$$C(S) - C_R \geqslant \left(1 - \frac{1}{e}\right)(C^* - C_R).$$

The $1 - 1/e$ factor was improved to 0.828 by Ageev and Sviridenko [3]. Their algorithm has two steps: in the first one they reduce the one level uncapacitated facility location problem to a special case of the maximum satisfiability problem for which they develop in the second step an 0.828-approximation algorithm. The technique used is that of randomized rounding, proposed in the MAX SAT context by Goemans and Williamson [7].

Being a generalization of the one level uncapacitated facility location problem, the two level MAX UFLP is NP-hard as well. Only a few algorithms for the two level MAX UFLP have been developed (see Aardal et al. [2] for a survey). The techniques which have been used are branch-and-bound, Lagrangean relaxation, cutting planes.

More studied in the last years was the minimization version of the problem, in which one has to select in each level the facilities to be opened and to assign every demand point to a path along open facilities such that the total cost (the cost of opening the facilities and the assignment cost) is minimized. For this problem, Shmoys et al. [10] developed a 3.16-approximation algorithm based on the method of filtering and rounding, proposed by Lin and Vitter [8,9]. Using dependent randomized rounding, Aardal et al. improved the performance guarantee to 3. These algorithms rely heavily on the assumption that the transportation costs verify the triangle inequality.

In this paper, we describe a polynomial time approximation algorithm for the shifted two level MAX UFLP problem based on the technique of independently randomized rounding. We prove that the algorithm delivers a solution $S_1 \times S_2 \subseteq F \times E$ such that

$$C(S_1,S_2) - C_R \geqslant 0.47(C^* - C_R).$$

2. An integer formulation of the two level MAX UFLP

In this section, we present an integer formulation of the two level MAX UFLP and give a new interpretation of its objective function based on an idea used by Ageev and Sviridenko [3] for the reduction between MAX UFLP and a special case of MAX SAT.

To derive an integer programming formulation of the two level MAX UFL problem, we introduce the $0-1$ variables y_i $(i \in F)$ and z_j $(j \in E)$ to indicate whether $i \in F$, respectively, $j \in E$ is open and the $0-1$ variables x_{ijk} $(i \in F, j \in E, k \in D)$ to indicate whether demand point k is served by the pair (i,j).

We will call a pair $(i, j) \in F \times E$ open if both i and j are open. We let

$$c(x) = \sum_{k \in D, i \in F, j \in E} c_{ijk} x_{ijk},$$

$$f(y) = \sum_{i \in F} f_i y_i,$$

and

$$e(z) = \sum_{j \in E} e_j z_j.$$

The two level MAX UFL problem (1) is now equivalent to

$$
(P_{\text{int}}) \quad
\begin{array}{lll}
\max & c(x) - f(y) - e(z) & \\
\text{s.t.} & \sum_{i \in F, j \in E} x_{ijk} = 1 & \text{for each } k \in D, \qquad (2) \\
& x_{ijk} \;\; y_i & \text{for each } i \in F,\ j \in E,\ k \in D, \qquad (3) \\
& x_{ijk} \;\; z_j & \text{for each } i \in F,\ j \in E,\ k \in D, \qquad (4) \\
& x_{ijk} \in \{0, 1\} & \text{for each } i \in F,\ j \in E,\ k \in D, \\
& y_i \in \{0, 1\} & \text{for each } i \in F, \\
& z_j \in \{0, 1\} & \text{for each } j \in E.
\end{array}
$$

Constraints (2) ensure that each $k \in D$ is assigned to only one pair of facilities and constraints (3) and (4) ensure that only open pairs are used.

We consider the LP relaxation of (P_{int}) with all variables taking values in $[0, 1]$. Denote the LP-relaxation with (P_{LP}). Let (x, y, z) be a feasible solution to (P_{LP}). Let $|F| = m$ and $|E| = n$.

For each $k \in D$, we order the $p = mn$ pairs (i, j) such that

$$c_{i_1(k) j_1(k) k} \geqslant c_{i_2(k) j_2(k) k} \geqslant \ldots c_{i_p(k) j_p(k) k}.$$

The idea behind the new interpretation of the objective function is that k will obtain the profit $c_{i_s(k) j_s(k) k} - c_{i_{s+1}(k) j_{s+1}(k) k}$, where $1 \leqslant s \leqslant p - 1$ only if one of the pairs $(i_1, j_1), \ldots, (i_s, j_s)$ is open.

For every $s \in \{1, \ldots, p\}$ define the sets I_{sk} as being the set of the s most profitable pairs for k

$$I_{sk} = \{(i_1(k), j_1(k)), \ldots, (i_s(k), j_s(k))\}.$$

For each set I_{sk} let the variable t_{sk} indicate the fraction in which k is assigned to pairs in I_{sk}. In other words,

$$t_{sk} = \sum_{(i,j) \in I_{sk}} x_{ijk}.$$

Further, associate to each set a number w_{sk} defined by

$$w_{sk} = c_{i_s(k) j_s(k) k} - c_{i_{s+1}(k) j_{s+1}(k) k} \quad \text{for } s \;\; p - 1 \tag{5}$$

and

$$w_{pk} = 0. \tag{6}$$

Using (2), (5) and (6), the objective function value of (P_{LP}) corresponding to (x, y, z) can be rewritten as

$$\begin{aligned} c(x) - f(y) - e(z) &= \sum_{k\in D}\sum_{s=1}^{p} w_{sk}t_{sk} + \min_{(i,j)\in F\times E} c_{ijk} \\ &\quad + \sum_{i\in F} f_i(1-y_i) + \sum_{j\in E} e_j(1-z_j) - \sum_{i\in F} f_i - \sum_{j\in E} e_j \\ &= \sum_{k\in D}\sum_{s=1}^{p} w_{sk}t_{sk} + \sum_{i\in F} f_i(1-y_i) + \sum_{j\in E} e_j(1-z_j) + C_R. \end{aligned}$$

Hence,

$$c(x) - f(y) - e(z) - C_R = \sum_{k\in D}\sum_{s=1}^{p} w_{sk}t_{sk} + \sum_{i\in F} f_i(1-y_i) + \sum_{j\in E} e_j(1-z_j). \tag{7}$$

3. Algorithm and its analysis

Let $(\tilde{x}, \tilde{y}, \tilde{z})$ be an optimal solution of the LP relaxation, and let $\widetilde{C_{\mathrm{LP}}}$ be its optimal value, i.e. $\widetilde{C_{\mathrm{LP}}} = c(\tilde{x}) - f(\tilde{y}) - e(\tilde{z})$.

In the following we will merely use the expression of $\widetilde{C_{\mathrm{LP}}} - C_R$, derived from (7)

$$\widetilde{C_{\mathrm{LP}}} - C_R = \sum_{k\in D}\sum_{s=1}^{p} w_{sk}\widetilde{t_{sk}} + \sum_{i\in F} f_i(1-\widetilde{y_i}) + \sum_{j\in E} e_j(1-\widetilde{z_j}),$$

where $\widetilde{t_{sk}} = \sum_{(i,j)\in I_{sk}} \widetilde{x_{ijk}}$.

Clearly, $\widetilde{C_{\mathrm{LP}}} - C_R \geqslant C^* - C_R$.

Let $\lambda \in [0, 1]$. The algorithm independently sets each y_i to 1 with probability $p_i = (1-\lambda) + \lambda\widetilde{y_i}$ and to 0 with probability $1 - p_i = \lambda(1-\widetilde{y_i})$. Similarly, each z_j will take value 1 with probability $q_j = (1-\lambda) + \lambda\widetilde{z_j}$ and value 0 with probability $1 - q_j = \lambda(1-\widetilde{z_j})$. Further, for each $k \in D$ set $x_{ijk} = 1$ for the pair (i, j) with the biggest profit c_{ijk} among the pairs for which both y_i and z_j were previously set to 1.

In other words, the algorithm independently opens each facility $i \in F$ with probability p_i and each facility $j \in E$ with probability q_j and then assigns each demand point to the most profitable open pair. In this way we obtain for every value of λ in $[0, 1]$ a feasible solution of the integer program.

Denote with $C(\lambda)$ the expected value of the algorithm. To analyze the performance of the algorithm we compare $C(\lambda) - C_R$ with $\widetilde{C_{\mathrm{LP}}} - C_R$.

Theorem 1. *The expected value of the algorithm satisfies*

$$C(\lambda) - C_R \geqslant \rho(\lambda)(\widetilde{C_{\mathrm{LP}}} - C_R),$$

where

$$\rho(\lambda) = \min\left\{\lambda, 4\lambda(1-\lambda), 2(1-\lambda)^2 - (1-\lambda)^4, \min_{r\geqslant 1}(1-\lambda)\left[1 - \lambda^r\left(1-\frac{1}{r}\right)^r\right]\right\}.$$

In particular, for $\lambda = 0.47$ we get

$$C(\lambda) - C_R \geqslant 0.47(C^* - C_R).$$

Proof. For each $k \in D$ and $s = 1, \ldots, p$, denote by t_{sk} the random variable that takes value 1 if k is assigned to a pair $(i, j) \in I_{sk}$ (and 0 otherwise).

From (7) and from the linearity of the expectation it follows:

$$C(\lambda) - C_R = \sum_{k \in D} \sum_{s=1}^{p} w_{sk} \operatorname{Prob}(t_{sk} = 1) + \sum_{i \in F} f_i (1 - \operatorname{Prob}(y_i = 1)) + \sum_{j \in E} e_j (1 - \operatorname{Prob}(z_j = 1))$$

$$= \sum_{k \in D} \sum_{s=1}^{p} w_{sk} \operatorname{Prob}(t_{sk} = 1) + \lambda \sum_{i \in F} f_i (1 - \widetilde{y_i}) + \lambda \sum_{j \in E} e_j (1 - \widetilde{z_j}).$$

To calculate the probabilities that t_{sk} take value 1 we distinguish four cases, depending on the structure of the set I_{sk}. The main idea is that the events of choosing the value 0 or 1 for y_i's and z_j's are independent.

Case 1: $I_{sk} = \{(i, j)\}$. In this case we have

$$\operatorname{Prob}(t_{sk} = 1) = \operatorname{Prob}((y_i = 1) \wedge (z_j = 1)) = \operatorname{Prob}(y_i = 1)\operatorname{Prob}(z_j = 1)$$

$$= [(1 - \lambda) + \lambda \widetilde{y_i}][(1 - \lambda) + \lambda \widetilde{z_j}].$$

Using the inequality

$$a + b \geqslant 2\sqrt{ab} \quad \text{for } a, b \geqslant 0,$$

and (3) and (4) we can obtain the following lower bound:

$$\operatorname{Prob}(t_{sk} = 1) \geqslant 4(1 - \lambda)\lambda \sqrt{\widetilde{y_i}\widetilde{z_j}} \geqslant 4(1 - \lambda)\lambda \widetilde{x_{ijk}}.$$

Hence,

$$\operatorname{Prob}(t_{sk} = 1) \geqslant 4(1 - \lambda)\lambda \widetilde{t_{sk}}.$$

Case 2: $I_{sk} = \{(i, j_1), \ldots, (i, j_r)\},\ r \geqslant 2$. We have

$$\operatorname{Prob}(t_{sk} = 1) = \operatorname{Prob}(y_i = 1)\operatorname{Prob}((z_{j_1} = 1) \vee \cdots \vee (z_{j_r} = 1))$$

$$= \operatorname{Prob}(y_i = 1)(1 - \operatorname{Prob}((z_{j_1} = 0) \wedge \cdots \wedge (z_{j_r} = 0))$$

$$= \operatorname{Prob}(y_i = 1) \left(1 - \prod_{q=1}^{r} \operatorname{Prob}(z_{j_q} = 0) \right)$$

$$= [(1 - \lambda) + \lambda \widetilde{y_i}] \left[1 - \lambda^r \prod_{q=1}^{r} (1 - \widetilde{z_{j_q}}) \right].$$

The arithmetic/geometric mean inequality, applied to $1 - \widetilde{z_{j_q}},\ q = \overline{1, r}$ gives

$$\prod_{q=1}^{r} (1 - \widetilde{z_{j_q}}) \left(1 - \frac{\sum_{q=1}^{r} \widetilde{z_{j_q}}}{r} \right)^r.$$

Hence, we obtain the following lower bound for $\operatorname{Prob}(t_{sk} = 1)$:

$$\operatorname{Prob}(t_{sk} = 1) \geqslant [(1 - \lambda) + \lambda \widetilde{y_i}] \left[1 - \lambda^r \left(1 - \frac{\sum_{q=1}^{r} \widetilde{z_{j_q}}}{r} \right)^r \right].$$

The function $f:[0,1]\to\mathscr{R}$ defined by

$$f(x)=1-a\left(1-\frac{x}{r}\right)^r,\quad \text{where } a\in[0,1],$$

is concave. Observing that any $x\in[0,1]$ can be written as a convex combination between 0 and 1, the concavity of f implies

$$f(x)=f(1*x+0*(1-x))\geqslant x*f(1)+(1-x)*f(0).$$

From

$$f(0)=1-a\geqslant 0,$$

$$f(1)=1-a\left(1-\frac{1}{r}\right)^r,$$

it follows that

$$f(x)\geqslant\left[1-a\left(1-\frac{1}{r}\right)^r\right]x.$$

Substituting in this inequality $a=\lambda^r$, $x=\sum_{q=1}^r\widetilde{z_{j_q}}$ and taking into account that by (4) we have $\sum_{q=1}^r\widetilde{z_{j_q}}\geqslant\widetilde{t_{sk}}$, we obtain

$$1-\lambda^r\left(1-\frac{\sum_{q=1}^r\widetilde{z_{j_q}}}{r}\right)^r\geqslant\left[1-\lambda^r\left(1-\frac{1}{r}\right)^r\right]\sum_{q=1}^r\widetilde{z_{j_q}}\geqslant\left[1-\lambda^r\left(1-\frac{1}{r}\right)^r\right]\widetilde{t_{sk}}.$$

Thus, for this case the lower bound for the probability of t_{sk} being 1 is

$$\mathrm{Prob}(t_{sk}=1)\geqslant[(1-\lambda)+\lambda\widetilde{y_i}]\left[1-\lambda^r\left(1-\frac{1}{r}\right)^r\right]\widetilde{t_{sk}}\geqslant(1-\lambda)\left[1-\lambda^r\left(1-\frac{1}{r}\right)^r\right]\widetilde{t_{sk}}.$$

Case 3: $I_l=\{(i_1,j),\dots,(i_r,j)\}, r\geqslant 2$. In a similar way as in the previous case it can be proven that

$$\mathrm{Prob}(t_{sk}=1)\geqslant(1-\lambda)\left[1-\lambda^r\left(1-\frac{1}{r}\right)^r\right]\widetilde{t_{sk}}.$$

Case 4: $I_{sk}\supseteq\{(i_1,j_1),(i_2,j_2)\}$ with $i_1\neq i_2$ and $j_1\neq j_2$. In this case, the event that the pair (i_1,j_1) is open is independent of the event that the pair (i_2,j_2) is open and consequently,

$$\begin{aligned}\mathrm{Prob}(t_{sk}=1)&\geqslant\mathrm{Prob}[(y_{i_1}=1\wedge z_{j_1}=1)\vee(y_{i_2}=1\wedge z_{j_2}=1)]\\&=\mathrm{Prob}(y_{i_1}=1\wedge z_{j_1}=1)+\mathrm{Prob}(y_{i_2}=1\wedge z_{j_2}=1)\\&\quad-\mathrm{Prob}(y_{i_1}=1\wedge z_{j_1}=1)\,\mathrm{Prob}(y_{i_2}=1\wedge z_{j_2}=1)\\&=p_{i_1}q_{j_1}+p_{i_2}q_{j_2}-p_{i_1}q_{j_1}p_{i_2}q_{j_2}\\&\geqslant 2\sqrt{p_{i_1}q_{j_1}p_{i_2}q_{j_2}}-p_{i_1}q_{j_1}p_{i_2}q_{j_2}.\end{aligned}$$

By the definition of p_i and q_j, $p_i\geqslant 1-\lambda$ and $q_j\geqslant 1-\lambda$ for each i and j. Hence,

$$p_{i_1}q_{j_1}p_{i_2}q_{j_2}\geqslant(1-\lambda)^4.$$

The function $f:\mathscr{R}_+\to\mathscr{R}_+$ defined by $f(x)=2\sqrt{x}-x$ is increasing on $[0,1]$, which together with the inequality above implies that

$$\mathrm{Prob}(t_{sk}=1)\geqslant 2\sqrt{(1-\lambda)^4}-(1-\lambda)^4\geqslant[2(1-\lambda)^2-(1-\lambda)^4]\widetilde{t_{sk}}.$$

From cases 1–4 it follows that for each $\lambda \in [0,1]$,

$$C(\lambda) \geqslant \rho(\lambda)\widetilde{C_{\mathrm{LP}}}.$$

Using the fact that $(1-1/r)^r\,\mathrm{e}^{-1}$, for every $r \geqslant 1$, we obtain that 0.47 is the maximum value of $\rho(\lambda)$ for $\lambda \in [0,1]$ and is attained for $\lambda = 0.47$. Hence, the expected value of the algorithm is at least $0.47\widetilde{C_{\mathrm{LP}}}$, which is at least 0.47 the optimum. □

Remark. The randomized algorithm presented above can be derandomized using the method of conditional expectations [4]. The result is a deterministic algorithm which finds in polynomial time a solution (S_1, S_2) such that

$$C(S_1,S_2) - C_R \geqslant 0.47(C^* - C_R).$$

4. Discussion

We have presented a 0.47-approximation algorithm for the two level MAX UFLP in the shifted form. For the analysis of our algorithm the assumption that there are only two levels of facilities was essential. A natural question is whether the algorithm generalizes to the case when the facilities are located on k levels, with $k \geqslant 2$. In this case one should open facilities in each level and assign each demand point to a path along open facilities such that the total profit is maximized. The problem that occurs is that even if we open the facilities independently, the events corresponding to paths being opened become dependent. As a consequence, for $k > 2$ the analysis of the algorithm is much more difficult.

It remains an open question whether there exists an approximation algorithm with a performance guarantee independent of the number of levels, as is the case with the minimization version of the same problem (see [1,10]).

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