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Hardness of approximation of the discrete time-cost tradeoff problem

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Abstract

We consider the discrete version of the well-known time-cost tradeoff problem for project networks, which has been extensively studied in the project management literature. We prove a strong in-approximability result with respect to polynomial time bicriteria approximation algorithms for this problem.

Keywords: Project management; Decision CPM networks; Bicriteria optimization; Precedence constraints; Approximation algorithm

1. Introduction

The discrete time-cost tradeoff problem is a well-known problem from the project management literature; see e.g. De et al. [3] and Robinson [5]. The instances of this problem are the so-called *projects* that consist of a finite set $\mathscr{A} = \{A_1, \ldots, A_n\}$ of activities together with a partial order $\prec$ on $\mathscr{A}$. Every activity A_j may be executed according to $a(j)$ different alternatives, where the ith alternative $(1 \leqslant i \leqslant a(j))$ takes $d(j,i)$ time and costs an amount $c(j,i)$ of money. Without loss of generality we assume that all values $d(j,i)$ and $c(j,i)$ are non-negative integers. The activities in $\mathscr{A}$ have to be executed in accordance with the precedence constraints; if $A_i \prec A_j$ then activity A_j may not be started before activity A_i has been completed. All activities are available for processing at time zero.

A *realization* $\vec{r} = (r_1, \ldots, r_n)$ of such a project is an assignment of alternatives r_j with $1 \leqslant r_j \leqslant a(j)$ to activities A_j. The *cost* $c(\vec{r})$ of realization $\vec{r}$ equals $\sum_{j=1}^{n} c(j, r_j)$, i.e., the total amount of money spent on $\vec{r}$. The *duration* $d(\vec{r})$ of realization $\vec{r}$ is the finish time of the earliest start schedule which gives duration $d(j, r_j)$ to activity A_j and which starts each activity at the earliest possible point in time while obeying the precedence constraints. In other words, the duration equals the length of the longest chain in the partial order where the length of a chain is the sum of the durations of the activities in the chain.

Ideally, we would like to minimize both duration and cost of a given project. Unfortunately, there is a tradeoff between duration and cost: Short realizations

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are usually expensive, and cheap realizations usually take a long time. By fixing either cost or duration, we get two related optimization problems with the objective to minimize the other parameter: In the Budget Problem, we are given a non-negative budget C and the goal is to find a shortest realization $\vec{r}$ that satisfies $c(\vec{r}) \leqslant C$. In the Deadline Problem, we are given a fixed non-negative deadline D and the goal is to find a cheapest realization $\vec{r}$ that satisfies $d(\vec{r}) \leqslant D$. Without loss of generality we may assume in these two problems that $C \leqslant \sum_{j,i} c(j,i)$ and $D \leqslant \sum_{j,i} d(j,i)$; consequently, the size of C and D is bounded by the input size of the project.

Definition 1. Let $\gamma \geqslant 1$ and $\delta \geqslant 1$ be two real numbers. A polynomial time (γ,δ)-approximation algorithm for the discrete time-cost tradeoff problem takes as input a project together with two non-negative integers C and D. If the project has a realization with cost at most C and duration at most D, then the output must be a realization with cost at most γC and duration at most δD. If the project has no such realization, then the output can be any realization. The running time of the (γ,δ)-approximation algorithm is polynomially bounded in the size of the project.

The concept of a polynomial time (γ,δ)-approximation algorithm captures all kinds of approximability definitions around the discrete time-cost tradeoff problem: By combining a polynomial time $(1,\delta)$-approximation algorithm with bisection search, we get a polynomial time approximation algorithm for the Budget Problem that produces realizations whose duration is at most δ above the optimal duration; this corresponds to a 'classical' performance guarantee of δ. Moreover, any polynomial time approximation algorithm with performance guarantee δ for the Budget Problem is a polynomial time $(1,\delta)$-approximation algorithm for the discrete time-cost tradeoff problem. Similarly, a polynomial time $(\gamma,1)$-approximation algorithm for the discrete time-cost tradeoff problem corresponds to a polynomial time approximation algorithm with performance guarantee γ for the Deadline Problem.

De et al. [4] prove that deciding whether a given project possesses a realization with duration at most 2 and with cost bounded by C (where C is part of the input) is a strongly $\mathcal{NP}$-hard problem. Hence, the Budget Problem and the Deadline Problem both are strongly $\mathcal{NP}$-hard problems. Moreover, unless $\mathcal{P}=\mathcal{NP}$ the Budget Problem does not have a polynomial time approximation algorithm with performance guarantee strictly better than $\frac{3}{2}$; such an algorithm could distinguish in polynomial time between realizations of duration 2 and duration 3. In other words, unless $\mathcal{P}=\mathcal{NP}$ there does not exist a polynomial time $(1,\frac{3}{2}-\varepsilon)$-approximation algorithm with $\varepsilon>0$ for the discrete time-cost tradeoff problem. Skutella [6] derives a polynomial time $(2,2)$-approximation algorithm for the discrete time-cost tradeoff problem by rounding the solutions of a linear programming relaxation.

Result of this note. We show that there exists a real number $\varepsilon>0$ with the following property: Unless $\mathcal{P}=\mathcal{NP}$, there does not exist a polynomial time $(1+\varepsilon,\frac{5}{4}-\beta)$-approximation algorithm for the discrete time-cost tradeoff problem with $\beta>0$. As one consequence, the Deadline Problem cannot possess a polynomial time approximation scheme unless $\mathcal{P}=\mathcal{NP}$. As another consequence, there exists some $\gamma>1$ such that the discrete time-cost tradeoff problem does not have a polynomial time (γ,γ)-approximation algorithm unless $\mathcal{P}=\mathcal{NP}$.

2. The in-approximability result

The proof will be done via an approximation preserving reduction from the vertex cover problem in cubic connected graphs, VC3 for short: An instance of VC3 consists of a connected cubic graph $G=(V,E)$, where cubic means that all vertices are of degree three. The goal is to find a minimum cardinality vertex cover W for G, where a vertex cover is a subset $W \subseteq V$ that intersects every edge in E. Alimonti and Kann [1] proved that problem VC3 is APX-hard. This implies that VC3 cannot have a polynomial time approximation scheme unless $\mathcal{P}=\mathcal{NP}$. In other words, there is some small $\varepsilon>0$ such that the existence of a polynomial time approximation algorithm with performance guarantee $1+\varepsilon$ would imply $\mathcal{P}=\mathcal{NP}$.

For an arbitrary instance $G=(V,E)$ of problem VC3, we will now define a corresponding project P_G for the discrete time-cost tradeoff problem. The well-known theorem of Brooks [2] yields that every connected cubic graph G either is three-colorable, or is the complete graph on four vertices. Moreover, the proof of this theorem gives a polynomial time procedure for finding such a three-coloring in case it exists. Without loss of generality we may assume that G has at least five vertices and hence is three-colorable; let X, Y, and Z denote the corresponding partition of V into three color classes. The project P_G is defined as follows.

- For every $v \in X \cup Y \cup Z$, there is a corresponding activity $A(v)$ in P_G. Activity $A(v)$ may either be executed at cost 0 and duration 2, or it may be executed at cost 1 and duration 0. For $x \in X$ and $y \in Y$, $A(x) \prec A(y)$ if and only if x and y are connected by an edge in E. For $y \in Y$ and $z \in Z$, $A(y) \prec A(z)$ if and only if y and z are connected by an edge in E.
- For every vertex $y \in Y$, there are two corresponding dummy activities $A^-(y)$ and $A^+(y)$. The only way of executing $A^-(y)$ and $A^+(y)$ is at cost 0 and duration 1. Moreover, $A^-(y) \prec A(y)$ and $A(y) \prec A^+(y)$.
- For every edge $e \in E$ that connects a vertex $x \in X$ to a vertex $z \in Z$, there is a corresponding activity $A(x,z)$. The only way of executing $A(x,z)$ is at cost 0 and duration 1. Moreover, $A(x) \prec A(x,z)$ and $A(x,z) \prec A(z)$.

This completes the description of the project P_G. Note that P_G is the so-called *layered* ordered set: The activities $A(x)$ with $x \in X$ and $A^-(y)$ for $y \in Y$ are in the first layer; these are the activities without predecessors. The activities $A(y)$ with $y \in Y$ and $A(x,z)$ for $x \in X$, $z \in Z$, $[x,z] \in E$ are in the second layer; these are all the direct successors of activities in the first layer. Finally, the activities $A(z)$ with $z \in Z$ and $A^+(y)$ for $y \in Y$ are in the third layer; these are all the direct successors of activities in the second layer, and moreover they are the activities without successors.

Lemma 2. *If the graph $G = (V,E)$ has a vertex cover W of cardinality k, then there exists a realization of the project P_G with cost k and duration at most* 4.

Proof. Consider the realization $\vec{r}$ that executes all activities $A(v)$ with $v \in W$ at cost 1 and duration 0, and that executes all remaining activities at cost 0. The duration of $\vec{r}$ is determined by a chain of three activities that crosses all three layers. There are only five possible forms for such a chain:

(i) $A(x) \prec A(x,z) \prec A(z)$. Since $[x,z] \in E$, the vertex cover W contains at least one of x and z, say x. Then in $\vec{r}$ activity $A(x)$ has duration 0, $A(x,z)$ has duration 1, and $A(z)$ has duration at most 2.

(ii) $A(x) \prec A(y) \prec A(z)$. Since $[x,y] \in E$, at least one of x and y is in W. Then the duration of the corresponding activity is 0 whereas the durations of the other two activities are at most 2.

(iii) $A(x) \prec A(y) \prec A^+(y)$. This can be handled similarly as case (ii).

(iv) $A^-(y) \prec A(y) \prec A(z)$. This can be handled symmetrically to case (iii).

(iv) $A^-(y) \prec A(y) \prec A^+(y)$. The durations of $A^-(y)$ and $A^+(y)$ are 1, the duration of $A(y)$ is at most 2.

Hence, indeed $d(\vec{r}) \leqslant 4$ and $c(\vec{r}) = k$ hold as desired. □

Lemma 3. *If the project P_G has a realization $\vec{r}$ of cost k and duration at most* 4, *then there exists a vertex cover of cardinality k for $G = (V,E)$.*

Proof. Consider the set $W \subseteq V$ that contains all $v \in V$ for which realization $\vec{r}$ executes $A(v)$ at duration 0 and at cost 1. Clearly, W has cardinality k. We claim that W is a vertex cover. There are only three possibilities for edges in E: (i) An edge $[x,y]$ must be intersected by W, since otherwise the chain $A(x) \prec A(y) \prec A^+(y)$ makes $d(\vec{r}) \geqslant 5$. (ii) Similarly, an edge $[y,z]$ must be intersected by W, since otherwise the chain $A^-(y) \prec A(y) \prec A(z)$ makes $d(\vec{r}) \geqslant 5$. (iii) An edge $[x,z]$ must be intersected by W, since otherwise the chain $A(x) \prec A(x,z) \prec A(z)$ makes $d(\vec{r}) \geqslant 5$. □

Lemma 4. *Let $\alpha > 0$ and $\beta > 0$ be two real numbers. If there exists a polynomial time $(1 + \alpha, \frac{5}{4} - \beta)$-approximation algorithm for the discrete time-cost tradeoff problem, then there exists a polynomial time approximation algorithm with performance guarantee $1 + \alpha$ for the vertex cover problem in cubic connected graphs.*

Proof. Consider an arbitrary instance $G=(V,E)$ of VC3. Compute the project P_G in polynomial time, as we described above. For every $C=0,1,\ldots,|V|$, apply the polynomial time $(1+\alpha,\frac{5}{4}-\beta)$-approximation algorithm to project P_G with cost bound C and with duration bound $D=4$. Let C^* denote the smallest cost value, for which the algorithm outputs a realization $\vec{r}^*$ of cost at most $(1+\alpha)C^*$ and of duration at most $(\frac{5}{4}-\beta)4$.

First observe that the optimal vertex cover for G has cardinality at least C^*: Otherwise, by Lemma 2 the project P_G had a realization with cost at most C^*-1 and duration at most 4, which contradicts the definition of C^*. Next observe that $d(\vec{r}^*)$ is an integer that is less or equal to $(\frac{5}{4}-\beta)\cdot 4<5$, and hence $d(\vec{r}^*)\leqslant 4$. Therefore, we may use Lemma 3 to get a vertex cover W for G of cardinality at most $(1+\alpha)C^*$. $\square$

Alimonti and Kann [1] proved that there exists an $\varepsilon>0$ such that there is no polynomial time approximation algorithm with performance guarantee $1+\varepsilon$ for VC3 unless $\mathscr{P}=\mathscr{NP}$. Combining this result with the statement of Lemma 4 yields our main result.

Theorem 5. *There exists an $\varepsilon>0$ such that for all $\beta>0$, the existence of a polynomial time $(1+\varepsilon,\frac{5}{4}-\beta)$-approximation algorithm for the discrete time-cost tradeoff problem would imply $\mathscr{P}=\mathscr{NP}$.*

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