ELSEVIER

Systems & Control Letters 43 (2001) 101–109

SYSTEMS & CONTROL LETTERS

www.elsevier.com/locate/sysconle

Exact observability of diagonal systems with a finite-dimensional output operator

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Received 30 September 1999; received in revised form 12 August 2000; accepted 6 December 2000

Abstract

In this paper equivalent conditions for exact observability of diagonal systems with a finite-dimensional output operator are given. One of these equivalent conditions is the conjecture of Russell and Weiss (SIAM J. Control Opt. 32(1) (1994) 1–23). The other conditions are in terms of the eigenvalues and the Lyapunov solutions of finite-dimensional subsystems.

Keywords: Infinite-dimensional systems; Unbounded observation operator; Exact observability; Hautus test; Lyapunov equation

1. Introduction

This paper proves the Hautus test for a class of infinite-dimensional systems. For linear finite-dimensional systems the Hautus test is a well-known test for observability. The Hautus test that we investigate was introduced by Russell and Weiss [18]. We study this test to characterize the exact observability of diagonal systems with a finite-dimensional output space. This class might seem restrictive, but it is fairly general nevertheless, because many semigroups considers in the literature have a Riesz basis of eigenvectors, which makes them isomorphic to diagonal semigroups, and because a technically motivated system has a finite-dimensional output space. The latter correspondence to the fact that one can only measure finitely many signals.

The notion of exact observability (and its dual notion of exact controllability) has generated a huge literature (see e.g., [1,4,5,8,9,11–14,17,18]). Here one can find different approaches for proving exact observability. We will use the fact that a (diagonal) system is exactly observable if and only if a certain set of exponentials form a Riesz basis. The complete characterization for a set of exponentials to form a Riesz basis is given by Nikol'skiĭ and Pavlov [16], and Ivanov (see [1]). These results play an essential role in our proof. We begin with the system definition.

On the Hilbert space Z we consider the following system:

$$\dot{z}(t) = \mathscr{A}z(t), \quad y(t) = \mathscr{C}z(t), \quad z(0) = z_0, \tag{1}$$

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0167-6911/01/$ - see front matter
PII: S0167-6911(00)00117-1

where we assume that

(i) $\mathscr{A}$ is a *diagonal operator*, i.e., $\mathscr{A}z = \sum_{n=1}^{\infty} \lambda_n \langle z, \phi_n \rangle \phi_n$, with $\mathrm{Re}(\lambda_n) < 0$ and with $\{\phi_n\}$ an orthonormal basis of the Hilbert space Z.
(ii) $\mathscr{C}$ is a bounded linear operator from the domain of $\mathscr{A}, D(\mathscr{A})$, to $\mathbb{C}^k$.
(iii) $\mathscr{C}$ is an (*infinite-time*) *admissible output operator*, i.e., there exists a $\kappa > 0$ such that

$$\int_0^{\infty} \|\mathscr{C}T(t)z_0\|^2 \, \mathrm{d}t \leqslant \kappa \|z_0\|^2 \tag{2}$$

for all $z_0 \in D(\mathscr{A})$. Here $T(t)$ is the C_0-semigroup generated by $\mathscr{A}$.

A system (1) that satisfies the above conditions will be denoted by $\Sigma(\mathscr{A}, \mathscr{C})$.

The admissibility of $\mathscr{C}$, Eq. (2), implies that we can extend the mapping $z_0 \to \mathscr{C}T(\cdot)z_0$ to a bounded linear mapping from Z to $L_2((0,\infty);\mathbb{C}^k)$. We denote this mapping by $\mathscr{O}$. Thus, we have that for any initial condition z_0 the solution of Eq. (1) is given by

$$z(t) = T(t)z_0, \quad y(\cdot) = \mathscr{O}z_0.$$

Furthermore, the output is an element of $L_2((0,\infty);\mathbb{C}^k)$.

We define exact observability similar as for bounded output operators $\mathscr{C}$ (see [2, Definition 4.1.12]).

Definition 1. We say the system $\Sigma(\mathscr{A}, \mathscr{C})$ is *exactly observable* (*in infinite time*) if there exists a $\delta > 0$ such that for all $z_0 \in Z$ we have that

$$\|\mathscr{O}z_0\|_{L_2((0,\infty);\mathbb{C}^k)} \geqslant \delta \|z_0\|.$$

In [4], it is shown that exact observability is equivalent to the unique solvability of the following Lyapunov equation by a coercive L:

$$\langle \mathscr{A}z_1, Lz_2 \rangle + \langle Lz_1, \mathscr{A}z_2 \rangle = -\langle \mathscr{C}z_1, \mathscr{C}z_2 \rangle, \quad z_1, z_2 \in D(\mathscr{A}). \tag{3}$$

Using the Lyapunov equation, Russell and Weiss [18] showed that a condition, which corresponds to the Hautus test in the finite-dimensional situation, is necessary for exact observability. Moreover, they proved that for some classes of exponentially stable systems this infinite-dimensional Hautus test is even an equivalent condition and they conjectured that this holds in general.

In this paper, we show that their infinite-dimensional Hautus test is sufficient for our class of systems, i.e., the class of diagonal systems with a finite-dimensional output space. For this class, we do not need the assumption that the system is exponentially stable. Our systems satisfy the weaker condition of strong stability, i.e., $\lim_{t\to\infty} T(t)z_0 = 0$.

The infinite-dimensional Hautus test of Russell and Weiss [18] is one of the four equivalent conditions for exact observability presented in this article. The second condition is in terms of the solution of the Lyapunov equations for $(k+1)$-dimensional subsystems. The third condition is stated in terms of the eigenvalues and finite collections of the vectors $\{C_n e^{\lambda_n \cdot}\}$, whereas the last equivalent condition states that the vectors $\{C_n e^{\lambda_n \cdot}\}$ form a Reisz basis in the closure of its span. This last equivalent condition can also be found in [1, Theorem III.3.3].

2. Main result

Consider the system $\Sigma(\mathscr{A}, \mathscr{C})$ as introduced in the previous section. In the sequel, we use the following notation: We define $\mathscr{C}_n := \mathscr{C}\phi_n$, $\mathbb{C}_- := \{s \in \mathbb{C} \mid \mathrm{Re}\, s < 0\}$, and $\underline{p} := \{1, \ldots, p\}$ for $p \in \mathbb{N}$. The *angle between two vectors* $z_1, z_2 \in Z \backslash \{0\}$ is given by

$$\angle(z_1, z_2) := \arccos\left(\frac{\langle z_1, z_2 \rangle}{\|z_1\| \, \|z_2\|} \right),$$

and it can be calculated using the formula

$$2\sin\left(\frac{\angle(z_1,z_2)}{2}\right) = \left\| \frac{z_1}{\|z_1\|} - \frac{z_2}{\|z_2\|} \right\|.$$

Moreover, the *angle between two subspaces V_1 and V_2 of Z* is defined by

$$\angle(V_1,V_2) := \inf_{v_1\in V_1\setminus\{0\}, v_2\in V_2\setminus\{0\}} \angle(v_1,v_2).$$

Further, a sequence $\{\mu_n\}_n \subset \mathbb{C}_-$ is called a *Carleson set* if

$$\inf_n \prod_{i,i\neq n} \left|\frac{\mu_i-\mu_n}{\mu_i+\bar{\mu}_n}\right| > 0.$$

Further information on Carleson sets can be found in [1,3,15]. However, note that Garnett calls a Carleson set an interpolating sequence [3, Chapter 7] and Nikol'skiĭ says that such a sequence satisfies the Carleson condition (C) [15, Lecture XI].

For the system $\Sigma(\mathscr{A},\mathscr{C})$ we obtain the following four equivalent conditions for exact observability.

Theorem 2. *Consider the diagonal system with finite-dimensional output space $\Sigma(\mathscr{A},\mathscr{C})$. The following conditions are equivalent*:

(i) *The system $\Sigma(\mathscr{A},\mathscr{C})$ is exactly observable.*

(ii) *There exists an $m_1>0$ such that for all $z_0\in D(\mathscr{A})$ and for all $s\in\mathbb{C}_-$*

$$\|(sI-\mathscr{A})z_0\|^2 + |\mathrm{Re}(s)|\|\mathscr{C}z_0\|^2 \geqslant m_1 \mathrm{Re}(s)^2\|z_0\|^2. \tag{4}$$

(iii) *There exists an $m_2>0$ such that for every $T(t)$-invariant subspace $V\subset Z$ with* $\dim V = k+1$ *the solution $L_V\in\mathscr{L}(V)$ of the Lyapunov equation*

$$\mathscr{A}_V^* L_V + L_V\mathscr{A}_V = -\mathscr{C}_V^*\mathscr{C}_V,$$

where $\mathscr{A}_V := \mathscr{A}|_V$ and $\mathscr{C}_V := \mathscr{C}|_V$, satisfies $\langle v, L_V v\rangle \geqslant m_2\|v\|^2$ for every $v\in V$, i.e., L_V is uniformly coercive.

(iv) *$\{\lambda_n\}_{n\in\mathbb{N}}$ can be decomposed in k Carleson sets,*

$$\inf_{n\in\mathbb{N}} \frac{\|\mathscr{C}_n\|^2}{|\mathrm{Re}(\lambda_n)|} > 0, \tag{5}$$

and there exists an $r>0$ such that

$$\inf_{m\in\mathbb{N}} \min_{\lambda_n\in\Lambda_m(r)} \angle\left(\mathscr{C}_n \mathrm{e}^{\lambda_n t}, \operatorname*{span}_{\substack{\lambda_j\in\Lambda_m(r)\\ j\neq n}} \{\mathscr{C}_j \mathrm{e}^{\lambda_j t}\}\right) > 0, \tag{6}$$

where

$$\Lambda_m(r) := \left\{\lambda_n, \left|\frac{\lambda_n-\lambda_m}{\lambda_n+\bar{\lambda}_m}\right| < r\right\}.$$

(v) *The set $\{\mathscr{C}_n \mathrm{e}^{\lambda_n \cdot},\ n\in\mathbb{N}\}$ is a Riesz basis in the closure of its span in $L_2((0,\infty);\mathbb{C}^k)$.*

The statement in item (ii) is the infinite-dimensional Hautus test as introduced by Russell and Weiss [18]. They conjectured that this condition would be sufficient for exact observability for any exponentially stable system. Here we prove this conjecture for our class of diagonal systems. Note that our systems are in general not exponentially stable.

Statements (5) and (6) in item (iv) can be reformulated into conditions on finite-dimensional Lyapunov equations. By the first condition in (iv) there exists an $r>0$ such that for all n we have that $\#\Lambda_n(r)\leqslant k$.

Let $\tilde{V}_n$ be the linear space spanned by those ϕ_j for which the corresponding $\lambda_j \in \Lambda_n(r)$. Since $\#\Lambda_n(r) \leqslant k$, we know that the dimension of this space is at most k. Let L_n denote the solution of the Lyapunov equation corresponding to the system restricted to $\tilde{V}_n$. Then one can show that (5) and (6) are equivalent to saying that the L_n's are uniformly coercive.

Careful reading of the proof of implication item (ii)–(iii) shows that in item (ii) it is sufficient if (4) holds only on the eigenvalues of $\mathscr{A}$, i.e., for $s \in \sigma(\mathscr{A})$.

If the rank of the output operator $\mathscr{C}$ is less than k, then we can find an admissible output operator $\tilde{\mathscr{C}}$ from $D(\mathscr{A})$ to $\mathbb{C}^l$, where l is the rank of $\mathscr{C}$ such that $\mathscr{C}^*\mathscr{C} = \tilde{\mathscr{C}}^*\tilde{\mathscr{C}}$. In this situation, in every statement of Theorem 2, k can be replaced by l.

The result for $k = 1$ can be found in [10]. For this special situation, statement (6) in item (iv) follows from (5).

For the proof of Theorem 2, we need the following results on the solution of the Lyapunov equation.

Lemma 3. *Consider the infinite-dimensional system*

$$\dot{z}(t) = \mathscr{A}_0 z(t), \quad y(t) = \mathscr{C} z(t),$$

where $\mathscr{A}_0$ is the infinitesimal generator of a strongly stable C_0-semigroup $T(t)$, i.e., $\lim_{t\to\infty} T(t)z_0 = 0\ \forall z_0 \in Z$, on the Hilbert space Z, and $\mathscr{C}$ is a bounded linear operator from the domain of $\mathscr{A}_0$ to a Hilbert space Y.

(i) *The output operator $\mathscr{C}$ is admissible if and only if there exists a bounded solution of the Lyapunov equation*

$$\langle \mathscr{A}_0 z_1, L z_2\rangle + \langle L z_1, \mathscr{A}_0 z_2\rangle = -\langle \mathscr{C} z_1, \mathscr{C} z_2\rangle, \quad z_1, z_2 \in D(\mathscr{A}_0). \tag{7}$$

(ii) *If $\mathscr{C}$ is admissible, then the bounded solution to the Lyapunov equation* (7) *is unique, and it is non-negative.*

(iii) *Assume that $\mathscr{C}$ is admissible, and let V be a closed $T(t)$-invariant subspace contained in Z. We denote by A_V and $\mathscr{C}_V$ the restriction of $\mathscr{A}_0$ and $\mathscr{C}$ to V, respectively. Then $\mathscr{C}_V$ is an admissible output operator for A_V and the solution of the corresponding Lyapunov equation is given by*

$$L_V = P_V L \mathscr{I}_V,$$

where P_V is the orthogonal projection of Z onto V and $\mathscr{I}_V$ the embedding map of V into Z.

Proof. Part (i) and (ii) can be found in [7]. Note that admissibility of the output operator $\mathscr{C}$ for $\mathscr{A}_0$ is equivalent to admissibility of the input operator $\mathscr{C}_0^*$ for $\mathscr{A}_0^*$. Part (iii) is easy to see, and it is left to the reader. □

3. On Lyapunov equations for finite-dimensional diagonal systems

In this section, we shall prove some useful results for finite-dimensional systems of the form

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t),$$

where A is a diagonal matrix with its eigenvalues, λ_i, in the open left-half plane. By C_i we denote the ith column of C. Since e^{At} is asymptotically stable, we have that the following Lyapunov equation has a unique non-negative solution:

$$A^*L + LA = -C^*C. \tag{8}$$

Lemma 4. *Suppose that $A \in \mathbb{C}^{p\times p}$ satisfies the assumptions made above. Furthermore, let the rank of C be less than p, and let the solution L of the Lyapunov equation* (8) *satisfy $\tilde{m}\|x\|^2 \leqslant x^*Lx \leqslant M\|x\|^2$. Then*

(i) *For each* $s \in \mathbb{C}_-$ *there exists an* $i \in \underline{p}$ *such that* $|(s-\lambda_i)/\mathrm{Re}(s)| \geqslant \tilde{m}/M$.
(ii) *For every* $i \in \underline{p}$ *we have that* $\angle(C_i e^{\lambda_i t}, \mathrm{span}_{j\neq i}\{C_j e^{\lambda_j t}\}) > 2\arcsin(\sqrt{\tilde{m}/4M})$.

Proof. (i) Suppose that there exists an $s_0 \in \mathbb{C}_-$ such that $|(s_0-\lambda_i)/\mathrm{Re}(s_0)| < \tilde{m}/M$ for every $i \in \underline{p}$. We write the Lyapunov equation (8) as

$$(\bar{s}_0 - A^*)L + L(s_0 - A) = 2\mathrm{Re}(s_0)L + C^*C. \tag{9}$$

Since the rank of C is less than p, there exists an $x \in \mathbb{C}^p$ such that $\|x\| = 1$ and $Cx = 0$. Pre-multiplying (9) with x^* and post-multiplying it with x, gives

$$x^*(\bar{s}_0 - A^*)Lx + x^*L(s_0 - A)x = 2\mathrm{Re}(s_0)x^*Lx. \tag{10}$$

Thus, we have

$$x^*Lx = \frac{x^*(\bar{s}_0 - A^*)Lx}{2\mathrm{Re}(s_0)} + \frac{x^*L(s_0 - A)x}{2\mathrm{Re}(s_0)} \leqslant \left\| \frac{(s_0 I - A)}{\mathrm{Re}(s_0)} x \right\| \|Lx\|$$
$$\leqslant \sup_{i=1,\ldots,p} \left| \frac{s_0 - \lambda_i}{\mathrm{Re}(s_0)} \right| \|x\| \cdot M \cdot \|x\| < \tilde{m}\|x\|^2.$$

This is in contradiction with $x^*Lx \geqslant \tilde{m}\|x\|^2$, and thus (i) is shown.

(ii) Without loss of generality, we may take $i = 1$. We choose $\alpha_2, \ldots, \alpha_p \in \mathbb{C}$ such that $\sum_{j=2}^{p} \alpha_j C_j e^{\lambda_j t}$ has norm one in L_2. Let d_1 denote the L_2-norm of $C_1 e^{\lambda_1 t}$, i.e. $d_1 = \|C_1\|/\sqrt{-2\mathrm{Re}(\lambda_1)}$. Using (8) it is easy to see that $e_1^* L e_1 = (\|C_1\|^2)/(-2\mathrm{Re}(\lambda_1))$. Thus, we get $d_1^{-2} \geqslant 1/M$. Next we define $v^{\mathrm{T}} = (-\alpha_2, \ldots, -\alpha_p)$. Finally, the calculation

$$\left\| \frac{C_1 e^{\lambda_1 t}}{d_1} - \sum_{j=2}^{p} \alpha_j C_j e^{\lambda_j t} \right\|_{L_2}^2 = \left\| C e^{At} \begin{pmatrix} d_1^{-1} \\ v \end{pmatrix} \right\|_{L_2}^2$$
$$= (d_1^{-1}, v^*) L \begin{pmatrix} d_1^{-1} \\ v \end{pmatrix} \geqslant \tilde{m}[d_1^{-2} + \|v\|^2] \geqslant \frac{\tilde{m}}{M}$$

shows

$$2\sin\left(\frac{1}{2}\angle\left(C_i e^{\lambda_i t}, \underset{j\neq i}{\mathrm{span}}\{C_j e^{\lambda_j t}\}\right)\right) \geqslant \sqrt{\frac{\tilde{m}}{M}}. \qquad \square$$

For the following result we introduce a subset of the class of systems studied in this section.

Definition 5. We define $S(p, m, M)$, $p \in \mathbb{N}$, $m, M > 0$, to be the set that contains all pairs $(A, C) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{l\times p}$, $l \in \mathbb{N}$, such that A is diagonal and asymptotically stable, and

(i) The estimate (4) is satisfied, i.e.,

$$\|(sI - A)x\|^2 + |\mathrm{Re}(s)|\|Cx\|^2 \geqslant m|\mathrm{Re}(s)|^2\|x\|^2, \tag{11}$$

for all $s \in \mathbb{C}_-$ and $x \in \mathbb{C}^p$.
(ii) The solution L of the Lyapunov equation (8) satisfies $\|L\| \leqslant M$.

Without proof we state the following properties of $S(p, m, M)$:

Lemma 6. *Let* (A, C) *be an element of* $S(p, m, M)$. *Then*

(i) *For every* $\alpha \in (0, \infty)$ *we have that* $(\alpha A, \sqrt{\alpha}C) \in S(p, m, M)$.
(ii) *For every* ω *on the imaginary axis, we have that* $(A + \omega I, C) \in S(p, m, M)$.

Furthermore, the solution L *of* (8) *for* (A, C) *is the same as that for* $(\alpha A, \sqrt{\alpha}C)$ *and* $(A + \omega I, C)$.

By definition of $S(p,m,M)$ we have that the solution of the Lyapunov equations are uniformly bounded by M. The following proposition shows that they are also uniformly coercive. Although this is only a result on matrices, we were only able to prove it for diagonal A's. The general case is still open.

Proposition 7. *There exists a* $\kappa=\kappa(p,m,M)>0$ *such that for all pairs* $(A,C)\in S(p,m,M)$ *we have that*

$$x^*Lx \geqslant \kappa\|x\|^2 \quad \text{for all } x\in\mathbb{C}^p, \tag{12}$$

where L is the solution of (8).

Proof. We prove this result via induction over p.

Case $p=1$: In the scalar case it is very easy to see that the solution of (8) is given by

$$L=\frac{|C|^2}{-2\mathrm{Re}(A)}.$$

If we choose $s=A$ in (11), then we obtain

$$\frac{|C|^2}{-\mathrm{Re}(A)}\geqslant m.$$

This shows the result for the scalar case.

Case $p>1$: Let us assume that the proposition holds for every set $S(l,m,M)$ with $l<p$. If (12) would not hold, then there exist sequences $(A_n,C_n)\in S(p,m,M)$, and L_n, where L_n is the Lyapunov solution for (A_n,C_n), such that $\lim_n L_n =: L_\infty$ and $\ker L_\infty \neq \{0\}$.

From part (ii) in Lemma 6 we see that without loss of generality we may assume that the eigenvalues $\lambda_{i,n}$, $i\in \underline{p}$ of A_n satisfy $\mathrm{Im}(\lambda_{1,n})=\cdots=\mathrm{Im}(\lambda_{p,n})=0$ or there exist $i,j\in\underline{p}$ such that $\mathrm{Im}(\lambda_{i,n})<0<\mathrm{Im}(\lambda_{j,n})$.

From part (i) in Lemma 6 we see that without loss of generality we may assume that $\|A_n\|=1$. Since every bounded sequence has a converging subsequence, we have that there exists a converging subsequence of $\{A_n\}$. We denote the limit of this subsequence by A_∞. By the special form of A_n we will see that the A_∞ can only have the following form, $A_\infty=-I$, or $A_\infty\neq\alpha I$ for all $\alpha\in\mathbb{C}$. Assume the contrary, then A_∞ would be equal to αI for some $\alpha\in\mathbb{C}\setminus\{-1\}$. Since $\|A_n\|=1$, we see that α must have norm one. Assume that the imaginary part of α is positive, then there would exist an $N\in\mathbb{N}$ such that the eigenvalues of A_N are in the upper half plane, which is in contradiction with the fact that for any $n\in\mathbb{N}$ there is an eigenvalue $\lambda_{i,n}$ of A_n with $\mathrm{Im}(\lambda_{i,n})\leqslant 0$.

Using the fact that $\|L_n\|\leqslant M$ and $\|A_n\|=1$, we obtain from (8) that

$$\|C_nx\|^2\leqslant 2M1\|x\|^2,$$

and thus $\{C_n\}$ is a bounded sequence. Hence, there exist subsequences, and a matrix C_∞ such that

$$A_{n_l}\to A_\infty \quad\text{and}\quad C_{n_l}\to C_\infty \text{ as } l\to\infty. \tag{13}$$

We denote these subsequences again by $\{A_n\}$ and $\{C_n\}$.

Eqs. (8) and (13) imply

$$A_\infty^*L_\infty + L_\infty A_\infty = -C_\infty^*C_\infty. \tag{14}$$

This equation can be written as

$$\langle A_\infty v, L_\infty v\rangle + \langle L_\infty v, A_\infty v\rangle = -\|C_\infty v\|^2.$$

Taking v to be an element of the kernel of L_∞, then the above equality gives that v also lies in the kernel of C_∞. Furthermore, multiplying Eq. (14) from the right by this v gives that $L_\infty A_\infty v=0$. Thus, we see that the kernel of L_∞ is an A_∞-invariant subspace which is contained in the kernel of C_∞. Thus there exists an $\mu\in\bar{\mathbb{C}}_-$ and a non-zero vector $v\in\ker L_\infty\cap\ker C_\infty$ such that $A_\infty v=\mu v$.

Assume that $A_\infty=-I$. Taking $x=v$ and $s=-1(=\mu)$ in (11), we get

$$\|(-I-A_n)v\|^2+\|C_nv\|^2\geqslant m\|v\|^2$$

which is equivalent to

$$\|(A_\infty - A_n)v\|^2 + \|C_n v\|^2 \geqslant m\|v\|^2.$$

Taking the limit for n going to infinity, we have

$$\|C_\infty v\|^2 \geqslant m\|v\|^2.$$

However, since $v \in \ker C_\infty$ we see that this gives a contradiction.

Assume next that $A_\infty \neq \alpha I$ for all $\alpha \in \mathbb{C}$. We define $V_\mu = \ker(A_\infty - \mu I)$.

Since $A_\infty \neq \alpha I$, we have that the dimension of V_μ is less than p. Furthermore, since A_∞ is diagonal, we have that

$$V_\mu = \operatorname*{span}_{j\in\mathbb{J}}\{e_j\},$$

where e_j is the jth standard basis vector of $\mathbb{C}^p$ and $\mathbb{J} \subset \underline{p}$. Let P_μ be the orthogonal projection from $\mathbb{C}^p$ onto V_μ. Since A_n is diagonal, we can restrict these matrices to V_μ and obtain diagonal matrices on a lower-dimensional space. Denote the restriction of A_n by $\tilde{A}_n$, and the restriction of C_n by $\tilde{C}_n$. By Lemma 3 we have that

$$\tilde{L}_n = P_\mu L_n|_{V_\mu}$$

is the solution of the Lyapunov equation associated with $\tilde{A}_n$ and $\tilde{C}_n$. Furthermore, it is easy to see that $\tilde{A}_n, \tilde{C}_n$ satisfies (11), and that $\|\tilde{L}_n\| \leqslant M$. Thus we have that

$$(\tilde{A}_n, \tilde{C}_n) \in S(\dim(V_\mu), m, M). \tag{15}$$

Now we have that

$$\lim_{n\to\infty} \tilde{L}_n v = \lim_{n\to\infty} P_\mu L_n v = P_\mu L_\infty v = 0.$$

However, since we assumed that the result holds for the dimension of the state space less than p, we have a contradiction. □

4. Proof of Theorem 2

We shall prove this theorem via (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (i).

However, first we see that admissibility implies that

$$\begin{aligned} -\frac{\|\mathscr{C}_n\|^2}{2\mathrm{Re}\,\lambda_n} &= \int_0^\infty \|\mathscr{C}_n \mathrm{e}^{\lambda_n t}\|^2\,\mathrm{d}t = \|\mathscr{C}T(t)\phi_n\|^2_{L_2((0,\infty);\mathbb{C}^k)} \\ &= \|\mathscr{O}\phi_n\|^2 \leqslant \|\mathscr{O}\|^2\|\phi_n\|^2 = \|\mathscr{O}\|^2. \end{aligned} \tag{16}$$

Proof of (i) ⇒ (ii). This can be found in Russell and Weiss [18].

Proof of (ii) ⇒ (iii). Let V be a $T(t)$-invariant subspace of dimension $k+1$. By [2, Lemma 2.5.8], we have that V is spanned by $k+1$ eigenfunctions ϕ_i. This implies that $\mathscr{A}_V$ is a diagonal matrix with all its eigenvalues in $\mathbb{C}_-$. Thus, there exists a unique solution, L_V of the Lyapunov equation corresponding to $\mathscr{A}_V$ and $\mathscr{C}_V$. By Lemma 3, we have that L_V is bounded from above by the norm of L. Furthermore, from (4), it follows easily that $(\mathscr{A}_V, \mathscr{C}_V)$ satisfy

$$\|(sI - \mathscr{A}_V)z_0\|^2 + |\mathrm{Re}(s)|\|\mathscr{C}_V z_0\|^2 \geqslant m_1\,\mathrm{Re}(s)^2\|z_0\|^2$$

for $z_0 \in V \cong \mathbb{C}^{k+1}$. Combining these results, we see that $(\mathscr{A}_V, \mathscr{C}_V) \in S(k+1, m_1, \|L\|)$ (see Definition 5). By Proposition 7, we conclude that

$$x^* L_V x \geqslant m_2\|x\|^2$$

for some m_2 which can only depend on $k+1$, m_1, and $\|L\|$. Hence we have proved that (iii) holds.

Proof of (iii) ⇒ (iv). For $\mathbb{J}$ with $\#\mathbb{J}=k+1$ we have that $V=\operatorname{span}_{j\in\mathbb{J}}\phi_j$ is a $T(t)$-invariant subspace, $A_V:=A|_V$ is diagonal and $\{\lambda_j\}_{j\in\mathbb{J}}$ is the set of eigenvalues of A_V. By part (iii), we have $\langle\phi_j, L_V\phi_j\rangle \geqslant m_2$, $j\in\mathbb{J}$. It is easy to see, that the diagonal elements of L_V are given by $-\|\mathscr{C}_n\|^2/2\operatorname{Re}\lambda_n$. Thus,

$$-\frac{\|\mathscr{C}_n\|^2}{2\operatorname{Re}\lambda_n}=\langle\phi_n, L_V\phi_n\rangle \geqslant m_2,$$

which shows that (5) holds.

By part (iii) and the first part of Lemma 4 we have that for every $s\in\mathbb{C}_-$ there are at most k eigenvalues $\lambda_{n_1},\dots,\lambda_{n_k}$ of $\mathscr{A}$ with

$$\left|\frac{s-\lambda_{n_j}}{\operatorname{Re}(s)}\right| < \frac{m_2}{\|L\|}, \quad j=1,\dots,k. \tag{17}$$

Thus, we can split the sequence $\{\lambda_n\}_n$ into k subsequences $\{\lambda_n^1\}_n,\dots,\{\lambda_n^k\}_n$ such that

$$\inf_{n\neq l}\left|\frac{\lambda_n^j-\lambda_l^j}{\operatorname{Re}\lambda_n^j}\right| > 0, \quad j=1,\dots,k. \tag{18}$$

From Garnett [3, Theorem 1.1, p. 287], we have that $\{\lambda_n^j\}_n$, $j=1,\dots,k$, is a Carleson set if and only if (18) holds and there exists a constant $\alpha>0$ such that

$$\sum_{-\lambda_n^j\in Q(h,w)} -\operatorname{Re}(\lambda_n^j) \leqslant \alpha h,$$

where $Q(h,w):=\{s=x+\mathrm{i}y\in\mathbb{C}\mid 0\leqslant x\leqslant h,\ w\leqslant y\leqslant w+h\}$. In order to show that $\{\lambda_n\}_{n\in\mathbb{N}}$ consists of k Carleson sets, using (5), it is thus sufficient to prove that there is a constant $\tilde{\alpha}>0$ such that

$$\sum_{-\lambda_n\in Q(h,w)} \|\mathscr{C}_n\|^2 \leqslant \tilde{\alpha}h, \quad h>0,\ \omega\in\mathbb{R}. \tag{19}$$

However, (19) is implied by the Carleson measure criterion of Weiss [19] (see also Hansen and Weiss [6, Remark 2.4]) for the infinite-time admissibility of $\mathscr{C}$. Thus $\{\lambda_n\}_{n\in\mathbb{N}}$ consists of k Carleson sets.

It now remains to show that (6) holds. If we can show that there exists an $r>0$ such that $\#\Lambda_m(r)\leqslant k+1$ holds, then (6) follows directly from Lemma 4 part (ii). Assume that this is not the case. This would mean that for every $n\in\mathbb{N}$ there is an $m_n\in\mathbb{N}$ such that $\#\Lambda_{m_n}(1/n)>k+1$, and thus

$$\left|\frac{2\operatorname{Re}\lambda_{m_n}}{\lambda_{m_n}-\lambda_j}\right| = \left|1+\frac{\bar{\lambda}_{m_n}+\lambda_j}{\lambda_{m_n}-\lambda_j}\right| > n-1 \quad \text{for } \lambda_j\in\Lambda_{m_n}.$$

However, this is in contradiction with (17), and thus part (iv) holds.

Proof of (iv) ⇒ (v). In [1, Theorem II.2.12, p. 69], it is shown that part (iv) implies that the set $\{\|\mathscr{C}_n e^{\lambda_n\cdot}\|^{-1}\mathscr{C}_n e^{\lambda_n\cdot},\ n\in\mathbb{N}\}$ is a Riesz basis in the closure of its span in $L^2((0,\infty);\mathbb{C}^k)$. Finally, (5) and (16) show that part (v) holds.

Proof of (v) ⇒ (i). Take $z_0\in\operatorname{span}_{n=1,\dots,N}\{\phi_n\}$ for $n\in\mathbb{N}$. Then we see that $\mathscr{C}T(t)z_0=\sum_{n=1}^N\mathscr{C}\phi_n\langle z_0,\phi_n\rangle e^{\lambda_n t}$. Thus

$$\begin{aligned}\|\mathscr{O}z_0\|^2_{L_2((0,\infty);\mathbb{C}^k)} &= \|\mathscr{C}T(t)z_0\|^2_{L_2((0,\infty);\mathbb{C}^k)}\\ &= \left\|\sum_{n=1}^N \mathscr{C}_n\langle z_0,\phi_n\rangle e^{\lambda_n t}\right\|^2_{L_2((0,\infty);\mathscr{C}^k)} \geqslant m_4\sum_{n=1}^N|\langle z_0,\phi_n\rangle|^2.\end{aligned}$$

Here, we used that $\{\mathscr{C}_n e^{\lambda_n\cdot}\}$ is Riesz basis. The m_4 is independent of N. Thus, for a dense subset of Z we have that

$$\|\mathscr{O}z_0\|^2_{L_2((0,\infty);\mathbb{C}^k)} \geqslant m_4\|z_0\|^2$$

and this proves exact observability. □

Acknowledgements

The authors would like to thank George Weiss for suggesting the equivalence of (iii) and (iv) in Theorem 2. Financial support to the authors from the British Council and the NWO (UK–Dutch joint scientific research programme: JRP536) is gratefully acknowledged.

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