# A DUALITY PRINCIPLE FOR HOMOGENEOUS VECTORFIELDS WITH APPLICATIONS

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Abstract. We introduce a duality principle for homogeneous vectorfields. As an application of this duality principle, stability and boundedness results for negative order homogeneous differential equations are obtained, starting from known results for positive order homogeneous differential equations.

**Keywords:** stability, time-varying, homogeneity, geometry, boundedness.

## 1. Introduction

Homogeneous vectorfields are vectorfields possessing a symmetry with respect to a family of dilations. They play a prominent role in various aspects of nonlinear control theory. See, for example, (Hermes, 1991; Kawski, 1990; M'Closkey and Murray, 1993; Morin *et al.*, 1997) for some applications in feedback control.

Recently, interesting results have been obtained for the particular class of *positive* order homogeneous differential equations: Peuteman and Aeyels (1999) have proven that a time-varying positive order homogeneous differential equation is asymptotically stable if the associated averaged differential equation is asymptotically stable; and Peuteman *et al.* (1999) have proven that a time-varying positive order homogeneous differential equation is bounded if each associated time-frozen differential equation is bounded. These results allow to reduce a stability and boundedness analysis of a time-varying differential equation to an analysis of time-invariant differential equations, possibly resulting in an important simplification. The proofs of these results exploit the inherently slow character of solutions of positive order homogeneous differential equations near the origin, and the inherently fast character far away from the origin.

In the present paper, we prove the dual results for *negative* order homogeneous differential equations: a time-varying negative order homogeneous differential equation is asymptotically stable if each time-frozen differential equation is asymptotically stable; it is bounded if the averaged is bounded. We provide an indirect proof for these results, obtaining them as corollaries of the above-mentioned results for positive order homogeneous differential equations: we first introduce a duality principle for homogeneous vectorfields; this duality then provides an elegant way to pass from the results for positive order homogeneous differential equations to the dual results for negative order homogeneous differential equations.

We end this introduction with a well-known example from the literature. Consider the differential equation

$$\dot{x} = x^2,$$

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which is a classic example illustrating the concept of finite escape time. In order to determine its nonzero solutions, it is convenient to introduce the new variable $y = \frac{1}{x}$ yielding

$$\dot{y} = -1.$$

Notice that this change of variable has transformed a homogeneous differential equation of order 1 into a homogeneous differential equation of order $-1$ with respect to the standard dilation[1]. The duality transformation to be introduced in the present paper may be seen as a generalization of this particular transformation, casted in a geometric framework.

This paper is organized as follows. Having introduced the preliminaries in Section 2, we define in Section 3 a duality transformation between positive and negative order homogeneous vectorfields. We start Section 4 with recalling a stability and boundedness result for positive order homogeneous differential equations. Then, as an application of the duality concept, we obtain the dual results for negative order homogeneous differential equations. These results are illustrated on a particular example in Section 5. Section 6 presents a generalization of the duality transformation and Section 7 concludes the paper.

## 2. Preliminaries

2.1. **Time-varying vectorfields and differential equations.** This paper is concerned with differential equations on $\mathbb{R}^n_0 := \mathbb{R}^n \setminus \{0\}$. Notice that we exclude the origin from the state space. Although not standard, this seems to be natural within the study of homogeneous vectorfields; it allows to avoid complications that would otherwise arise from singularities at the origin. We emphasize that results obtained in this framework may be reinterpreted in terms of differential equations on the complete state-space $\mathbb{R}^n$, as is illustrated in Section 5.

We take $\mathbb{R}^n_0$ with the relative topology induced by the usual topology on $\mathbb{R}^n$. Let $\mathcal{F}(\mathbb{R}^n_0)$ (resp. $\mathcal{X}(\mathbb{R}^n_0)$) be the set of all real-valued functions $f : \mathbb{R}\times\mathbb{R}^n_0 \to \mathbb{R} : (t,x) \mapsto f(t,x)$ (resp. the set of all vector-valued maps $X : \mathbb{R} \times \mathbb{R}^n_0 \to \mathbb{R}^n : (t,x) \mapsto X(t,x)$) that are

1. continuous in $(t,x)$,
2. locally Lipschitz in $x$ uniformly with respect to $t \in \mathbb{R}$,
3. bounded in $t$ uniformly with respect to $x$ in compact subsets of $\mathbb{R}^n_0$.

An element $X$ of $\mathcal{X}(\mathbb{R}^n_0)$ is called a *time-varying vectorfield*. Clearly $\mathcal{X}(\mathbb{R}^n_0)$ is closed under summation and multiplication with functions $f \in \mathcal{F}(\mathbb{R}^n_0)$.

Associated to a time-varying vectorfield $X \in \mathcal{X}(\mathbb{R}^n_0)$ is a differential equation

$$\dot{x} = X(t,x)$$

on $\mathbb{R}^n_0$ that has the existence and uniqueness property of trajectories. Its trajectory passing through state $x_0$ at time $t_0$ evaluated at time $t$ is denoted by $x(t,t_0,x_0)$. The map $(t,t_0,x_0) \mapsto x(t,t_0,x_0)$ is called the *flow* of this differential equation.

**Remark 1.** In general the flow $x$ need not be forward complete: trajectories may escape to infinity or approach the origin in finite time. Similarly, the flow $x$ may not be backward complete.

[1] We adhere to the convention according to which a linear vectorfield is homogeneous of order zero with respect to the standard dilation.

2.2. **Push-forward.** Let $\phi : \mathbb{R}^n_0 \to \mathbb{R}^n_0$ be a smooth[2] diffeomorphism. We are interested in "pushing forward" functions $f \in \mathcal{F}(\mathbb{R}^n_0)$ and vectorfields $X \in \mathcal{X}(\mathbb{R}^n_0)$ by this diffeomorphism.

The *push-forward* $\phi_* f$ of $f$ by $\phi$ is defined as

$$\phi_* f : \mathbb{R} \times \mathbb{R}^n_0 \to \mathbb{R} : (t, x) \mapsto (\phi_* f)(t, x) = f(t, \phi^{-1}(x)). \tag{1}$$

It is easy to see that $\phi_* f$ is again in $\mathcal{F}(\mathbb{R}^n_0)$.

For vectorfields, we first introduce the *tangent map* $T_x\phi$ of $\phi$ at $x \in \mathbb{R}^n_0$:

$$T_x\phi : \mathbb{R}^n \to \mathbb{R}^n : v \mapsto (T_x\phi)(v) = D\phi(x) \cdot v$$

where $D\phi(x)$ is the Jacobian of $\phi$ at $x$ and where $\cdot$ indicates the matrix product. The *push-forward* $\phi_* X$ of $X$ by $\phi$ is then defined as

$$\phi_* X : \mathbb{R} \times \mathbb{R}^n_0 \to \mathbb{R}^n : (t, x) \mapsto (\phi_* X)(t, x) = (T_{\phi^{-1}(x)}\phi)(X(t, \phi^{-1}(x))). \tag{2}$$

It is easy to see that $\phi_* X$ is again in $\mathcal{X}(\mathbb{R}^n_0)$.

2.3. **Homogeneity.** We give a geometric definition of homogeneity, in the spirit of (Rosier, 1993; Kawski, 1995; M'Closkey, 1997).

Given $r \in (\mathbb{R}_{>0})^n$, we introduce the 1-parameter family of dilations $\delta_\lambda$ $(\lambda > 0)$

$$\delta_\lambda : \mathbb{R}^n_0 \to \mathbb{R}^n_0 : x \mapsto \delta_\lambda(x) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n). \tag{3}$$

The orbits $\mathcal{O}_x = \{\delta_\lambda(x) : \lambda > 0\}$ of this 1-parameter family of dilations are called *homogeneous rays.*

A time-varying vectorfield $X \in \mathcal{X}(\mathbb{R}^n_0)$ is *homogeneous of order* $\tau \in \mathbb{R}$ if

$$(\delta_\lambda)_* X = \lambda^{-\tau} X \qquad \forall \lambda. \tag{4}$$

A *homogeneous norm* is a smooth function $\rho : \mathbb{R}^n_0 \to \mathbb{R}_{>0}$ that satisfies

$$(\delta_\lambda)_* \rho = \lambda^{-1} \rho \qquad \forall \lambda. \tag{5}$$

**Remark 2.** Equation (4) is equivalent to

$$X(t, \delta_\lambda x) = \lambda^\tau \delta_\lambda X(t, x) \qquad \forall \lambda, t, x.$$

Equation (5) is equivalent to

$$\rho(\delta_\lambda x) = \lambda \rho(x) \qquad \forall \lambda, x.$$

We thus recover the characterization of homogeneity as in (e.g. Peuteman *et al.*, 1999; Moreau and Aeyels, n.d.).

2.4. **Stability and boundedness.**

**Definition 1** (local uniform asymptotic stability)**.** A differential equation $\dot{x} = X(t, x)$ on $\mathbb{R}^n_0$ with flow $x(t, t_0, x_0)$ is *locally uniformly asymptotically stable* if

S-1 $\forall c_2 > 0$, $\exists c_1 > 0$ such that $\forall t_0$, $\forall x_0$, if $\rho(x_0) < c_1$ then $\rho(x(t, t_0, x_0)) < c_2$ $\forall t \geq t_0$ in the domain of $x(\cdot, t_0, x_0)$[3],

S-2 $\exists c_1 > 0$ such that $\forall c_2 > 0$, $\exists T \geq 0$ such that $\forall t_0$, $\forall x_0$, if $\rho(x_0) < c_1$ then $\rho(x(t, t_0, x_0)) < c_2$ $\forall t \geq t_0 + T$ in the domain of $x(\cdot, t_0, x_0)$.

**Definition 2** (boundedness)**.** A differential equation $\dot{x} = X(t, x)$ on $\mathbb{R}^n_0$ with flow $x(t, t_0, x_0)$ is *bounded* if

B-1 $\forall c_1 > 0$, $\exists c_2 > 0$ such that $\forall t_0$, $\forall x_0$, if $\rho(x_0) \leq c_1$ then $\rho(x(t, t_0, x_0)) \leq c_2$ $\forall t \geq t_0$ in the domain of $x(\cdot, t_0, x_0)$,

[2] That is, of class $C^\infty$.

[3] We include the restriction "in the domain of $x(\cdot, t_0, x_0)$" because $x(\cdot, t_0, x_0)$ may approach the origin in finite time and thus may not be forward complete. Similar remarks hold for conditions S-2, B-1 and B-2 introduced further in the paper.

B-2 $\exists c_2 > 0$ such that $\forall c_1 > 0$, $\exists T \geq 0$ such that $\forall t_0$, $\forall x_0$, if $\rho(x_0) \leq c_1$ then $\rho(x(t, t_0, x_0)) \leq c_2$ $\forall t \geq t_0 + T$ in the domain of $x(\cdot, t_0, x_0)$.

**Remark 3.** The notions of local uniform asymptotic stability and boundedness are defined in terms of a homogeneous norm $\rho$. This is merely a matter of convenience: replacing the homogeneous norm $\rho$ by the Euclidean norm would result in equivalent definitions.

**Remark 4.** The notion of local uniform asymptotic stability introduced in Definition 1 differs from the classical notion of local uniform asymptotic stability of an equilibrium point at the origin, in that the origin itself is excluded from the state-space.

**Remark 5.** Condition B-1 (resp. B-2) of Definition 2 corresponds to the notion of *uniform boundedness* (resp. *uniform ultimate boundedness*) from (Yoshizawa, 1966; Peuteman *et al.*, 1999) for differential equation on $\mathbb{R}^n$.

## 3. Duality transformation

Associated to a 1-parameter family of dilations $\delta_\lambda$ and a homogeneous norm $\rho$, we introduce the smooth map

$$S : \mathbb{R}^n_0 \to \mathbb{R}^n_0 : x \mapsto S(x) = \delta_{\rho(x)^{-2}}(x). \tag{6}$$

Clearly $S$ leaves homogeneous rays invariant, and on homogeneous rays, $S$ is completely characterized by

$$\rho(S(x)) = \rho(x)^{-1}. \tag{7}$$

Indeed,

$$\begin{aligned} \rho(S(x)) &= \rho(\delta_{\rho(x)^{-2}}(x)) && \text{(definition of } S\text{)} \\ &= \rho(x)^{-2}\rho(x) && \text{(homogeneity of } \rho\text{)} \\ &= \rho(x)^{-1}. \end{aligned}$$

The map $S$ may be interpreted as a reflection with respect to the homogeneous unit ball $\{\rho(x) = 1\}$ along homogeneous rays.

It follows that $S \circ S$ is the identity map on $\mathbb{R}^n_0$, and thus, in particular, $S$ is a smooth diffeomorphism. Furthermore

**Lemma 1.** *$S$ and the 1-parameter family of dilations $\delta_\lambda$ are related by*

$$\delta_\lambda \circ S = S \circ \delta_{\lambda^{-1}} \qquad \forall \lambda. \tag{8}$$

*Proof.* Both the left and the right hand side of (8) leave homogeneous rays invariant, and on homogeneous rays we have:

$$\begin{aligned} \rho((\delta_\lambda \circ S)(x)) &= \lambda\rho(S(x)) && \text{(homogeneity of } \rho\text{)} \\ &= \lambda\rho(x)^{-1} && \text{(equation (7))} \\ &= (\lambda^{-1}\rho(x))^{-1} \\ &= \rho(\delta_{\lambda^{-1}}x)^{-1} && \text{(homogeneity of } \rho\text{)} \\ &= \rho((S \circ \delta_{\lambda^{-1}})(x)) && \text{(equation (7)).} \end{aligned}$$

□

We now introduce three $\mathcal{F}(\mathbb{R}_0^n)$-linear maps from $\mathcal{X}(\mathbb{R}_0^n)$ onto $\mathcal{X}(\mathbb{R}_0^n)$: $X \mapsto X^S$, $X \mapsto X^T$ and $X \mapsto X^D$ defined by

$$X^S = S_* X, \tag{9}$$
$$X^T(t,x) = -X(-t,x), \tag{10}$$
$$X^D = (X^S)^T. \tag{11}$$

The two operations $S$ and $T$ commute and we may thus write

$$X^D = (X^S)^T = (X^T)^S = X^{ST}.$$

Furthermore

$$X^{SS} = X^{TT} = X^{DD} = X. \tag{12}$$

The differential equation $\dot{x} = X^D(x)$ is called the *dual* differential equation associated to $\dot{x} = X(x)$.

The trajectories of $X$, $X^S$, $X^T$ and $X^D$ are related:

**Lemma 2.** *The following four statements are equivalent:*

*(i)* $t \mapsto \xi(t)$ *is a solution of* $\dot{x} = X(t,x)$.
*(ii)* $t \mapsto (S \circ \xi)(t)$ *is a solution of* $\dot{x} = X^S(t,x)$.
*(iii)* $t \mapsto \xi(-t)$ *is a solution of* $\dot{x} = X^T(t,x)$.
*(iv)* $t \mapsto (S \circ \xi)(-t)$ *is a solution of* $\dot{x} = X^D(t,x)$.

*Proof.* "(i)⇒(ii)" is a property of the push-forward. "(i)⇒(iii)" follows from direct verification. These two properties together imply "(i)⇒(iv)" since $X^D = X^{ST}$. The inverse implications follow from $X^{SS} = X^{TT} = X^{DD} = X$. □

The flows of $X$ and $X^D$, respectively denoted by $x$ and $x^D$, are thus related by:

$$x(t_1, t_0, x_0) = x_1 \;\Leftrightarrow\; x^D(-t_0, -t_1, S(x_1)) = S(x_0). \tag{13}$$

**Duality Theorem 1** (stability and boundedness)**.** *The following two statements are equivalent:*

*(i)* $\dot{x} = X(t,x)$ *is locally uniformly asymptotically stable.*
*(ii)* $\dot{x} = X^D(t,x)$ *is bounded.*

Schematically Theorem 1 may be represented as follows:

$$\boxed{\text{local uniform asymptotic stability} \;\overset{D}{\longleftrightarrow}\; \text{boundedness}}$$

We emphasize that, although the duality transformation is based on a family of dilations and a homogeneous norm, the Duality Theorem 1 applies to general differential equations on $\mathbb{R}_0^n$, not necessarily homogeneous.

*Proof.* Assume that $\dot{x} = X^D(t,x)$ does not satisfy condition B-1 of Definition 2; that is,

$$\exists c_1 > 0 \text{ such that } \forall c_2 > 0, \quad \exists t_0, t_1, x_0, x_1 \text{ with}$$
$$t_1 \geq t_0, \quad \rho(x_0) \leq c_1, \quad \rho(x_1) > c_2 \text{ and } x^D(t_1, t_0, x_0) = x_1. \tag{14}$$

With the notation

$$\begin{aligned} c_1 &= 1/\bar{c}_2, & t_0 &= -\bar{t}_1, & x_0 &= S(\bar{x}_1), \\ c_2 &= 1/\bar{c}_1, & t_1 &= -\bar{t}_0, & x_1 &= S(\bar{x}_0), \end{aligned}$$

expression (14) may be rewritten as

$$\begin{aligned}&\exists \overline{c}_2 > 0 \text{ such that } \forall \overline{c}_1 > 0, \quad \exists \overline{t}_0, \overline{t}_1, \overline{x}_0, \overline{x}_1 \text{ with}\\ &\qquad -\overline{t}_0 \geq -\overline{t}_1, \quad \rho(S(\overline{x}_1)) \leq 1/\overline{c}_2, \quad \rho(S(\overline{x}_0)) > 1/\overline{c}_1\\ &\qquad\qquad \text{and } x^D(-\overline{t}_0, -\overline{t}_1, S(\overline{x}_1)) = S(\overline{x}_0).\end{aligned} \tag{15}$$

Since

$$\begin{aligned} -\overline{t}_0 \geq -\overline{t}_1 &\Leftrightarrow \overline{t}_1 \geq \overline{t}_0 \\ \rho(S(\overline{x}_1)) \leq 1/\overline{c}_2 &\Leftrightarrow \rho(\overline{x}_1) \geq \overline{c}_2 && \text{(equation (7))}\\ \rho(S(\overline{x}_0)) > 1/\overline{c}_1 &\Leftrightarrow \rho(\overline{x}_0) < \overline{c}_1 && \text{(equation (7))}\\ x^D(-\overline{t}_0, -\overline{t}_1, S(\overline{x}_1)) = S(\overline{x}_0) &\Leftrightarrow x(\overline{t}_1, \overline{t}_0, \overline{x}_0) = \overline{x}_1 && \text{(equation (13))}\end{aligned}$$

expression (15) is equivalent to

$$\begin{aligned}&\exists \overline{c}_2 > 0 \text{ such that } \forall \overline{c}_1 > 0, \quad \exists \overline{t}_0, \overline{t}_1, \overline{x}_0, \overline{x}_1 \text{ with}\\ &\qquad \overline{t}_1 \geq \overline{t}_0, \quad \rho(\overline{x}_1) \geq \overline{c}_2, \quad \rho(\overline{x}_0) < \overline{c}_1 \text{ and } x(\overline{t}_1, \overline{t}_0, \overline{x}_0) = \overline{x}_1;\end{aligned} \tag{16}$$

which is equivalent to: $\dot{x} = X(t,x)$ does not satisfy condition S-1 of Definition 1.

We have thus proven that $\dot{x} = X(t,x)$ satisfies condition S-1 iff $\dot{x} = X^D(t,x)$ satisfies condition B-1. *Mutatis mutandis*, it may be proven that $\dot{x} = X(t,x)$ satisfies condition S-2 iff $\dot{x} = X^D(t,x)$ satisfies condition B-2. This is left to the reader. □

**Duality Theorem 2** (homogeneity)**.** *The following two statements are equivalent:*

*(i)* $X$ *is homogeneous of order* $\tau$.
*(ii)* $X^D$ *is homogeneous of order* $-\tau$.

Schematically Theorem 2 may be represented as follows:

$$\boxed{\text{homogeneity of order } \tau \overset{D}{\longleftrightarrow} \text{homogeneity of order } -\tau}$$

*Proof.* We first prove "(i)⇒(ii)". Clearly $X^D$ is homogeneous of order $-\tau$ iff $X^S$ is homogeneous of order $-\tau$. It therefore suffices to prove that $(\delta_\lambda)_*(X^S) = \lambda^\tau X^S$. We have:

$$\begin{aligned}(\delta_\lambda)_*(X^S) &= (\delta_\lambda)_* S_* X && \text{(definition of } X^S\text{)}\\ &= (\delta_\lambda \circ S)_* X && \text{(chain rule for push-forward)}\\ &= (S \circ \delta_{\lambda^{-1}})_* X && \text{(Lemma 1)}\\ &= S_* (\delta_{\lambda^{-1}})_* X && \text{(chain rule for push-forward)}\\ &= S_* \lambda^\tau X && \text{(homogeneity of } X\text{)}\\ &= \lambda^\tau S_* X && \text{(linearity of push-forward)}\\ &= \lambda^\tau X^S && \text{(definition of } X^S\text{)},\end{aligned}$$

which thus proves "(i)⇒(ii)". The inverse implication "(ii)⇒(i)" follows from $X^{DD} = X$. □

**Remark 6.** The proof of Theorem 2 is simple and straightforward, because of the geometric characterization of homogeneity and duality via push-forward. This motivates the geometric approach taken in the present paper.

## 4. Local uniform asymptotic stability and boundedness of time-varying homogeneous differential equations

We first recall a stability and boundedness result for *positive* order homogeneous differential equations. Based on the duality introduced above, we then derive the dual result for *negative* order homogeneous differential equations.

Consider a time-varying vectorfield $X \in \mathcal{X}(\mathbb{R}_0^n)$ that satisfies the following additional hypotheses:

H-1 $X$ is periodic in $t$ with period $T > 0$ independent of $x$,
H-2 $X$ is continuously differentiable in $(t, x)$.

Various time-invariant vectorfields may be associated to $X$: we introduce the time-averaged vectorfield $X_{\text{av}}$

$$X_{\text{av}} : \mathbb{R}_0^n \to \mathbb{R}^n : x \mapsto X_{\text{av}}(x) = \frac{1}{T} \int_0^T X(t, x)\, dt, \tag{17}$$

and a collection of time-frozen vectorfields $X_\sigma$ $(\sigma \in \mathbb{R})$

$$X_\sigma : \mathbb{R}_0^n \to \mathbb{R}^n : x \mapsto X_\sigma(x) = X(\sigma, x). \tag{18}$$

In general, a stability or a boundedness analysis of the time-varying differential equation

$$\dot{x} = X(t, x) \tag{19}$$

is highly nontrivial. Reducing the problem to the study of time-invariant differential equations

$$\dot{x} = X_{\text{av}}(x) \tag{20}$$

or

$$\dot{x} = X_\sigma(x) \tag{21}$$

may constitute an important simplification.

This has actually been proven to be possible for positive order homogeneous differential equations:

**Theorem 3.** *Consider $X \in \mathcal{X}(\mathbb{R}_0^n)$ satisfying the hypotheses* H-1 *and* H-2. *Assume that $X$ is homogeneous of order $\tau > 0$. Then*

1. *the time-varying differential equation* (19) *is locally uniformly asymptotically stable if the time-averaged differential equation* (20) *is locally uniformly asymptotically stable,*
2. *the time-varying differential equation* (19) *is bounded if each time-frozen differential equation* (21) *is bounded.*

Theorem 3 is essentially a paraphrased version of (Peuteman and Aeyels, 1999, Theorem 1) and (Peuteman *et al.*, 1999, Theorem 3). See also (Moreau and Aeyels, n.d., Theorem 3). The proof of this result is based on the inherently slow character of solutions of positive order homogeneous differential equations near the origin, and the inherently fast character far away from the origin.

We now illustrate the strength of the duality principle introduced in this paper. Starting from the result for positive order homogeneous differential equations Theorem 3, a straightforward application of the Duality Theorems 1 and 2 immediately yields the dual result for negative order homogeneous differential equations:

**Corollary 1.** *Consider $X \in \mathcal{X}(\mathbb{R}_0^n)$ satisfying the hypotheses* H-1 *and* H-2. *Assume that $X$ is homogeneous of order $\tau < 0$. Then*

1. *the time-varying differential equation* (19) *is locally uniformly asymptotically stable if each time-frozen differential equation* (21) *is locally uniformly asymptotically stable.*

2. *the time-varying differential equation* (19) *is bounded if the time-averaged differential equation* (20) *is bounded.*

*Proof.* We first prove part 1. Assume that each time-frozen differential equation

$$\dot{x} = X_\sigma(x)$$

is locally uniformly asymptotically stable. Then, by Theorem 1 each differential equation

$$\dot{x} = (X_\sigma)^D(x)$$

is bounded. From (9), (10) and (11) it is immediate to see that $(X_\sigma)^D = (X^D)_{-\sigma}$. Hence each differential equation

$$\dot{x} = (X^D)_{-\sigma}(x)$$

is bounded. But $X^D$ is homogeneous of order $-\tau > 0$ by Theorem 2, and thus

$$\dot{x} = X^D(t, x)$$

is also bounded by Theorem 3. Finally, by Theorem 1, we conclude that

$$\dot{x} = X(t, x)$$

is locally uniformly asymptotically stable. The proof of part 2 is completely similar, and therefore left to the reader. □

## 5. Example

We study the stability and boundedness properties of the time-varying differential equation

$$\dot{x} = ||x||^p A(t)x, \quad x \in \mathbb{R}^2_0 \tag{22}$$

with $p$ a real parameter and where $A(t)$ is given by

$$A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t \cos t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}. \tag{23}$$

First of all, we make the following observations:

O-1 If $p = 0$, then differential equation (22) is the restriction to $\mathbb{R}^2_0$ of the linear differential equation

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^2.$$

This linear differential equation, and hence also (22), has been shown to be unstable (Khalil, 1996, Example 3.22).

O-2 For each $t$ the matrix $A(t)$ has the characteristic polynomial

$$s^2 + \frac{1}{2}s + \frac{1}{2},$$

and thus $A(t)$ is Hurwitz for each $t$. Hence, each time-frozen differential equation

$$\dot{x} = ||x||^p A(\sigma)x, \quad x \in \mathbb{R}^2_0$$

is locally uniformly asymptotically stable and bounded.

O-3 The matrix $A(t)$ is a periodic function of $t$ with corresponding averaged matrix

$$A_{\text{av}} = \begin{bmatrix} -\frac{1}{4} & 1 \\ -1 & -\frac{1}{4} \end{bmatrix}.$$

The characteristic polynomial of $A_{\text{av}}$ is

$$s^2 + \frac{1}{2}s + \frac{17}{16},$$

and thus $A_{\mathrm{av}}$ is Hurwitz. Hence, the averaged differential equation

$$\dot{x} = ||x||^p A_{\mathrm{av}} x, \quad x \in \mathbb{R}^2_0$$

is locally uniformly asymptotically stable and bounded.

Although (22) has been shown to be unstable for $p = 0$, we will now see that (22) is both locally uniformly asymptotically stable and bounded for *all* nonzero values of $p$. The analysis is based on Theorem 3 and Corollary 1.

5.1. **Case 1:** $p > 0$**.** In this case, the right hand side of (22) is homogeneous of order $p > 0$ with respect to the standard dilation. We conclude from observations O-2 and O-3 and from Theorem 3 that (22) is both locally uniformly asymptotically stable and bounded.

We now interpret this result in terms of differential equations on the complete state-space $\mathbb{R}^2$. Notice that for $p > 0$, $||x||^p A(t)x$ is defined in $x = 0$. We may thus consider the differential equation

$$\dot{x} = ||x||^p A(t)x, \quad x \in \mathbb{R}^2. \tag{24}$$

Compared with (22), differential equation (24) has only one extra solution: the null-solution. This follows from standard uniqueness results for ordinary differential equations. We may thus conclude that for $p > 0$, the origin of (24) is a locally uniformly asymptotically stable equilibrium point and the solutions of (24) are uniformly bounded and uniformly ultimately bounded (see Remark 5).

5.2. **Case 2:** $p < 0$**.** In this case, the right hand side of (22) is homogeneous of order $p < 0$ with respect to the standard dilation. We conclude from observations O-2 and O-3 and from Corollary 1 that (22) is both locally uniformly asymptotically stable and bounded.

We now interpret this result in terms of differential equations on the complete state-space $\mathbb{R}^2$. Notice that for $-1 < p < 0$, $\lim_{x\to 0} ||x||^p A(t)x$ exists and equals 0. We may then consider the differential equation

$$\dot{x} = \begin{cases} ||x||^p A(t)x & \text{if } x \in \mathbb{R}^2_0 \\ 0 & \text{if } x = 0 \end{cases}. \tag{25}$$

The right hand side of (25) is continuous but not locally Lipschitz at the origin. Hence (25) is not guaranteed to have the uniqueness property of solutions at the origin. However, differential equation (25) cannot have solutions that leave the origin in forward time, since this would contradict the stability properties of (22). The unique forward solution of (25) starting in the origin is the null-solution[4]. We may thus conclude that, for $-1 < p < 0$, the origin of (25) is a locally uniformly asymptotically stable equilibrium point and the solutions of (25) are uniformly bounded and uniformly ultimately bounded (see Remark 5).

## 6. Generalization

In Section 3 we have developed a duality theory based on the smooth map

$$S : \mathbb{R}^n_0 \to \mathbb{R}^n_0 : x \mapsto S(x) = \delta_{\rho(x)^{-2}}(x).$$

Here we consider more generally

$$S_\mu : \mathbb{R}^n_0 \to \mathbb{R}^n_0 : x \mapsto S_\mu(x) = \delta_{\rho(x)^{\mu-1}}(x) \tag{26}$$

with $\mu$ a real parameter. As will become apparent from the forthcoming analysis, the case $\mu < 0$ is very similar to the case described before in Section 3 (duality),

[4]Compared with (22), differential equation (25) has an extra equilibrium solution at the origin, and in addition, every solution of (22) that approaches the origin in finite time corresponds to a solution of (25) that reaches the origin in finite time and stays there afterwards.

whereas $\mu > 0$ gives rise to fundamentally different results. As an application of the results to follow, we mention that a stability or boundedness analysis of a nonzero order homogeneous differential equation may be transformed into a stability or boundedness analysis of a homogeneous differential equation of arbitrary *prescribed* nonzero order with respect to the same dilation. In other words, with regard to stability and boundedness results, the class of homogeneous differential equations of order, let's say, one is representative for the class of *all nonzero order* homogeneous differential equations. See Theorems 4 and 5.

Details of the analysis are left to the reader. $S_\mu$ leaves homogeneous rays invariant, and on homogeneous rays, $S_\mu$ is characterized by

$$\rho(S(x)) = \rho(x)^\mu. \tag{27}$$

It follows readily that

1. $$S_{\mu_1} \circ S_{\mu_2} = S_{\mu_2} \circ S_{\mu_1} = S_{\mu_1 \mu_2} \tag{28}$$
2. For $\mu \neq 0$, $S_\mu \circ S_{1/\mu} = S_{1/\mu} \circ S_\mu = S_1$ is the identity map on $\mathbb{R}^n_0$, and thus in particular $S_\mu$ is a smooth diffeomorphism for $\mu \neq 0$.
3. $S_0$ corresponds to the projection on the homogeneous unit ball $\{\rho(x) = 1\}$ along homogeneous rays.

Assume from now on that $\mu \neq 0$. We introduce $\mathcal{F}(\mathbb{R}^n_0)$-linear maps from $\mathcal{X}(\mathbb{R}^n_0)$ onto $\mathcal{X}(\mathbb{R}^n_0)$: $X \mapsto X^{D_\mu}$ defined by

$$X^{D_\mu} = \begin{cases} (S_\mu)_* X & \text{if } \mu > 0 \\ ((S_\mu)_* X)^T & \text{if } \mu < 0 \end{cases} \tag{29}$$

where the operator $T$ is defined by (10). Clearly

$$(X^{D_{\mu_1}})^{D_{\mu_2}} = (X^{D_{\mu_2}})^{D_{\mu_1}} = X^{D_{\mu_1 \mu_2}} \tag{30}$$

and thus

$$(X^{D_\mu})^{D_{1/\mu}} = (X^{D_{1/\mu}})^{D_\mu} = X. \tag{31}$$

**Theorem 4** (stability and boundedness)**.** *If $\mu > 0$ the following two statements are equivalent:*

*(i) $\dot{x} = X(t,x)$ is locally uniformly asymptotically stable (resp. bounded).*
*(ii) $\dot{x} = X^{D_\mu}(t,x)$ is locally uniformly asymptotically stable (resp. bounded).*

*If $\mu < 0$ the following two statements are equivalent:*

*(i) $\dot{x} = X(t,x)$ is locally uniformly asymptotically stable (resp. bounded).*
*(ii) $\dot{x} = X^{D_\mu}(t,x)$ is bounded (resp. locally uniformly asymptotically stable).*

**Theorem 5** (homogeneity)**.** *Let $\mu \neq 0$. The following two statements are equivalent:*

*(i) $X$ is homogeneous of order $\tau$.*
*(ii) $X^{D_\mu}$ is homogeneous of order $\frac{\tau}{\mu}$.*

## 7. Conclusion

We have introduced a duality transformation between positive and negative order homogeneous vectorfields. We have then illustrated the strength of this duality concept: starting from a known stability and boundedness result for positive order homogeneous differential equations, we were able to obtain, without much effort, the dual result for negative order homogeneous differential equations.

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