Available online at www.sciencedirect.com
SCIENCE DIRECT®

Systems & Control Letters 48 (2003) 407–414

SYSTEMS & CONTROL LETTERS

www.elsevier.com/locate/sysconle

Hankel norm approximation for well-posed linear systems

Ruth F. Curtain[a,*], Amol J. Sasane[b]

[a] *Department of Mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen, Netherlands*
[b] *Department of Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands*

Received 17 January 2002; received in revised form 23 September 2002; accepted 10 October 2002

Abstract

The sub-optimal Hankel norm approximation problem is solved for a well-posed linear system with generating operators (A, B, C) and transfer function G satisfying some mild assumptions. In the special case of the sub-optimal Nehari problem, an explicit parameterization of all solutions is obtained in terms of the system parameters A, B, C and $G(0)$.

Keywords: Well-posed infinite-dimensional linear systems; Hankel norm approximation problem; Nehari problem

1. Introduction

The standard way to obtain explicit formulae for solutions to the sub-optimal Hankel norm approximation problem for a rational transfer function G is to use a minimal realization $G(s)=C(sI-A)^{-1}B$ and then obtain formulae in terms of the generators (A, B, C) (see [2]). These formulae typically involve the controllability and observability Gramians L_B, L_C or solutions of various Lyapunov equations. Such an approach has been extended to certain classes of infinite-dimensional linear systems (see [12,13]), but the limiting factor to extending these results to the general case of well-posed linear systems is the difficulty in manipulating with the unbounded operators B and C. For example, in [12,13], explicit solutions to the sub-optimal Hankel norm approximation problem for exponentially stable smooth Pritchard–Salamon systems and exponentially stable analytic systems, respectively, were obtained via the solution to the appropriate J-spectral factorization problem using the smoothing properties of these classes. However, it was not possible to extend this technique to more general well-posed linear systems. In general, it is not clear that the candidate spectral factor is even well-posed (see [14]).

In this paper, we suggest translating the problem to the analogous one for reciprocal systems which we now define. The *reciprocal system* of the well-posed linear system Σ with generating operators A, B, C and transfer function G, such that $0 \in \rho(A)$ (here $\rho(A)$ denotes the resolvent set of A), is the well-posed linear system with the bounded generating operators $A^{-1}, A^{-1}B, -CA^{-1}, G(0)$ and transfer function

$$G_-(s) = G(0) - CA^{-1}(sI - A^{-1})^{-1}A^{-1}B.$$

* Corresponding author. Tel.: +31-50-363-3987; fax: +31-50-363-3800.
E-mail addresses: r.f.curtain@math.rug.nl (R.F. Curtain), a.j.sasane@math.utwente.nl (A.J. Sasane).

0167-6911/03/$ - see front matter
PII: S0167-6911(02)00301-8

For the theory of well-posed linear systems, we refer the reader to [15] or [16,17]. The key to our approach is the relationship between the transfer function G of the original well-posed linear system and the transfer function G_- of its reciprocal system. Note that A^{-1}, $A^{-1}B$, CA^{-1} are all bounded operators and for all nonzero $s \in \rho(A)$ there holds

$$G(s) = G(0) + C[(sI - A)^{-1} + A^{-1}]B \tag{1}$$

$$= G(0) - CA^{-1}\left(\frac{1}{s} - A^{-1}\right)^{-1} A^{-1}B \tag{2}$$

$$= G_-\left(\frac{1}{s}\right). \tag{3}$$

Let $H_\infty(\mathscr{L}(U,Y))$ denote the space of $\mathscr{L}(U,Y)$-valued functions of a complex variable defined in the open right half-plane which are bounded and analytic in the open right half-plane. Then from (3), it is clear that $G \in H_\infty(\mathscr{L}(U,Y))$ iff $G_- \in H_\infty(\mathscr{L}(U,Y))$. In addition, the reciprocal system has the same controllability and observability Gramians, which we prove in the following lemma.

Lemma 1.1. *Let A,B,C be generating operators of a regular linear system with transfer function G. Suppose that $0 \in \rho(A)$ and G_- is the transfer function of its reciprocal system with generating operators A^{-1}, $A^{-1}B$, $-CA^{-1}$ and feedthrough operator $G(0)$. Then the following hold*:

(1) *C is an infinite-time admissible observation operator for A iff $-CA^{-1}$ is an infinite-time admissible observation operator for A^{-1}. If either C or $-CA^{-1}$ is infinite-time admissible* (*for the semigroups generated by A or A^{-1}, respectively*), *then the observability Gramians are identical.*
(2) *B is an infinite-time admissible control operator for A iff $A^{-1}B$ is an infinite-time admissible control operator for A^{-1}. If either B or $A^{-1}B$ is infinite-time admissible* (*for the semigroups generated by A or A^{-1}, respectively*), *then the controllability Gramians are identical.*
(3) *$G \in H_\infty(\mathscr{L}(U,Y))$ iff $G_- \in H_\infty(\mathscr{L}(U,Y))$.*

Proof.

(1) From [7] (see also [6]), we know that C is an infinite-time admissible observation operator iff the Lyapunov equation

$$\langle Az_1, L_C z_2\rangle + \langle L_C z_1, Az_2\rangle = -\langle Cz_1, Cz_2\rangle, \tag{4}$$

for all z_1 and z_2 in $D(A)$, has a nonnegative definite solution $L_C = L_C^* \geqslant 0$. Eq. (4) is clearly equivalent to the Lyapunov equation

$$\langle x_1, L_C A^{-1} x_2\rangle + \langle L_C A^{-1} x_1, x_2\rangle = -\langle CA^{-1}x_1, CA^{-1}x_2\rangle \tag{5}$$

for all x_1 and x_2 in X, which establishes the equivalence. Moreover, the observability Gramians are the smallest positive solution and so the Gramians are identical.
(2) This is dual to part (1) above.
(3) This follows from (3). □

The idea is then to translate the sub-optimal Hankel norm approximation problem for the well-posed linear system with transfer function G into one for the system with transfer function G_-, the latter having *bounded* generating operators. We now elaborate on this. But first we will recall the definition of the (frequency domain) Hankel operator corresponding to a symbol $G \in L_\infty(\mathrm{i}\mathbb{R}, \mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ and the definition of its singular values.

Let $H_2(\mathbb{C}^{\mathtt{k}})$ denote the set of all analytic functions $f:\mathbb{C}_0^+\to\mathbb{C}^{\mathtt{k}}$ such that

$$\|f\|_2 := \sup_{\zeta>0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\|f(\zeta+\mathrm{i}\omega)\|^2\,\mathrm{d}\omega\right)^{1/2} < \infty.$$

For $G\in L_\infty(\mathrm{i}\mathbb{R},\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ we define the *Hankel operator with symbol* G, denoted by H_G, acting from $H_2(\mathbb{C}^{\mathtt{m}})$ to $H_2(\mathbb{C}^{\mathtt{p}})$, as follows:

$$H_G f = \Pi(M_G f_-) \quad \text{for } f\in H_2(\mathbb{C}^{\mathtt{m}}),$$

where M_G is the multiplication map on $L_2(\mathrm{i}\mathbb{R},\mathbb{C}^{\mathtt{m}})$ induced by G, Π is the orthogonal projection operator from $L_2(\mathrm{i}\mathbb{R},\mathbb{C}^{\mathtt{p}})$ onto $H_2(\mathbb{C}^{\mathtt{p}})$ and $f_-(s) := f(-s)$.

Now, we recall the notion of singular values of a bounded linear operator from a Hilbert space $\mathscr{H}_1$ to a Hilbert space $\mathscr{H}_2$. For $\mathtt{k}\in\{1,2,\ldots\}$ the $\mathtt{k}$th *singular value* of an operator $\mathtt{H}\in\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ (denoted by $\sigma_{\mathtt{k}}(\mathtt{H})$) is defined to be the distance with respect to the norm in $\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ of $\mathtt{H}$ from the set of operators in $\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ of rank at most $\mathtt{k}-1$. Thus $\sigma_1(\mathtt{H})=\|\mathtt{H}\|$, and $\sigma_1(\mathtt{H})\geqslant\sigma_2(\mathtt{H})\geqslant\sigma_3(\mathtt{H})\geqslant\cdots\geqslant 0$. If $\mathtt{H}$ is compact, then $\mathtt{H}^*\mathtt{H}$ is compact and nonnegative, and so the spectrum of $\mathtt{H}^*\mathtt{H}$ is a pure point spectrum with countably many nonnegative eigenvalues. The square roots of these eigenvalues are then the singular values of $\mathtt{H}$.

If $G\in L_\infty(\mathrm{i}\mathbb{R},\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$, we sometimes denote the singular values of H_G, $\sigma_{\mathtt{k}}(H_G)$, simply by $\sigma_{\mathtt{k}}(G)$. The $\sigma_{\mathtt{k}}(G)$'s are then referred to as the *Hankel singular values of* G. The following theorem (see for instance [9, Corollary 4.10, p. 46]) gives a necessary and sufficient condition on G for the corresponding Hankel operator H_G to be compact.

Theorem 1.2 (Hartman). *$G\in L_\infty(\mathrm{i}\mathbb{R},\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ determines a compact Hankel operator H_G iff $G_-(\cdot)\in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})+\mathscr{C}_0(\mathrm{i}\mathbb{R},\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$, where $\mathscr{C}_0(\mathrm{i}\mathbb{R},\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ denotes the space of continuous $\mathtt{p}\times\mathtt{m}$ complex matrix-valued functions defined on $\mathrm{i}\mathbb{R}$, with a unique limit at $\pm\mathrm{i}\infty$.*

Let $H_{\infty,\ell}(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ denote the set of all $\mathtt{p}\times\mathtt{m}$ matrix-valued functions K of a complex variable defined in the open right half-plane such that $K=G_{\mathtt{f}}+F$, where F is an element in $H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ and $G_{\mathtt{f}}$ is the transfer function of a finite-dimensional system with order at most ℓ, with all its poles in the open right half-plane. The set $H_{\infty,\ell}(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ is a subset of $L_\infty(\mathrm{i}\mathbb{R},\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$.

Clearly, $K(-\cdot)\in H_{\infty,\ell}(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ iff $K_{\mathtt{r}}(-\cdot)\in H_{\infty,\ell}(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$, where K and $K_{\mathtt{r}}$ are related by

$$K_{\mathtt{r}}(s) = K\left(\frac{1}{s}\right) + G(0) \quad \text{for all } s\in\mathbb{C}_0^+. \tag{6}$$

Now suppose that $G\in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ and $G(\cdot)\in\mathscr{C}_0(\mathrm{i}\mathbb{R},\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$, and define

$$G_{\mathtt{r}}(s) = G\left(\frac{1}{s}\right) - G(0) \quad \text{for all } s\in\mathbb{C}_0^+.$$

Consequently we have $\|G(i\cdot)+K(i\cdot)\|_\infty=\|G_{\mathtt{r}}(i\cdot)+K_{\mathtt{r}}(i\cdot)\|_\infty$. Thus from the following theorem [1] of Adamjan et al. [1], it follows that $\sigma_{\mathtt{k}}(G)=\sigma_{\mathtt{k}}(G_{\mathtt{r}})$ for all $\mathtt{k}\in\mathbb{N}$.

Theorem 1.3. *If $G\in L_\infty(\mathrm{i}\mathbb{R},\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$, then*

$$\inf_{K(-\cdot)\in H_{\infty,\ell}(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})}\|G(i\cdot)+K(i\cdot)\|_\infty = \sigma_{\ell+1}(G).$$

The *sub-optimal Hankel norm approximation problem* is the following:

[1] The discrete-time matrix case was proved in [8] and the continuous-time matrix case was proved in [5].

Let $G(i\cdot) \in L_\infty(\mathbb{R}, \mathbb{C}^{p\times m})$. If $\sigma_{\ell+1} < \sigma < \sigma_\ell$, then find $K(-\cdot) \in H_{\infty,\ell}(\mathbb{C}^{p\times m})$ such that $\|G(i\cdot)+K(i\cdot)\|_\infty \leqslant \sigma$.

First we will solve the sub-optimal Hankel norm approximation problem for well-posed linear systems. The sub-optimal Nehari problem is a special case in which $\ell = 0$ and $\sigma > \sigma_1$. In this special case, using the results from [3], we can give a parameterization of *all* solutions, and we do this in the last section of this paper.

2. The sub-optimal Hankel norm approximation problem

In this section, we will solve the sub-optimal Hankel norm approximation problem for well-posed linear systems, by first translating the problem to its reciprocal system with *bounded* generating operators, albeit a *nonexponentially* stable semigroup. Subsequently we use a result from [11] in order to obtain explicit formulae for solutions to the sub-optimal Hankel norm approximation problem for such systems. We remark that the sub-optimal Nehari problem, which is a special case in which $\ell = 0$, will be treated separately in the next section, and we get stronger results for this special case.

In [11], the following theorem was proved:

Theorem 2.1. *Suppose that the triple (A, B, C) satisfies the following assumptions*:

$\mathfrak{A}1$. *A is the infinitesimal generator of a strongly continuous semigroup on the Hilbert space X, $B \in \mathscr{L}(\mathbb{C}^m, X)$ and $C \in \mathscr{L}(X, \mathbb{C}^p)$.*

$\mathfrak{A}2$. *The impulse response $h(\cdot) = CT(\cdot)B \in L_2([0,\infty), \mathbb{C}^{p\times m})$ is such that $G(\cdot) \in \mathscr{C}_0(\mathrm{i}\mathbb{R}, \mathbb{C}^{p\times m})$, where $G(s) := C(sI - A)^{-1}B$.*

$\mathfrak{A}3$. *For every $\zeta > 0$, $A - \zeta I$ is the infinitesimal generator of an exponentially stable, strongly continuous semigroup.*

Let $\sigma_{\ell+1} < \sigma < \sigma_\ell$ and $Q(-\cdot) \in H_\infty(\mathbb{C}^{p\times m})$ with $\|Q(i\cdot)\|_\infty \leqslant 1$. For every $\zeta > 0$ denote

$$K(s) = R_1(s)R_2(s)^{-1},$$

where

$$\begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix} = \Lambda(s)^{-1} \begin{bmatrix} Q(s-\zeta) \\ I_m \end{bmatrix}$$

and

$$\Lambda(s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \begin{bmatrix} -\dfrac{1}{\sigma^2} C L_B^\zeta \\ \dfrac{1}{\sigma} B^* \end{bmatrix} [N_\sigma^\zeta]^* (sI + A^* - \zeta I)^{-1} [C^* \ \ L_C^\zeta B].$$

In the above, L_B^ζ and L_C^ζ denote the controllability Gramian and the observability Gramian, respectively, of the exponentially stable system $(A - \zeta I, B, C)$ and $N_\sigma^\zeta := [I - (1/\sigma^2) L_B^\zeta L_C^\zeta]^{-1}$.

Then there exists a $\delta > 0$ such that for every $\zeta \in (0, \delta)$, $K(-\cdot) \in H_{\infty,\ell}(\mathbb{C}^{p\times m})$ and $\|G(i\cdot) + K(i\cdot)\|_\infty \leqslant \sigma$.

We remark that in the above Theorem 2.1, from Theorem 1.2 and assumption $\mathfrak{A}2$, it follows that the Hankel operator H_G is compact.

Using Theorem 2.1, we will solve the sub-optimal Hankel norm approximation problem for the well-posed linear system Σ on a Hilbert space X with generating operators A, B, C and transfer function G under the following assumptions:

$\mathfrak{H}1$.	The input and output spaces are finite-dimensional; $U = \mathbb{C}^{\mathtt{m}}$ and $Y = \mathbb{C}^{\mathtt{p}}$.
$\mathfrak{H}2$.	$0 \in \rho(A)$ and $\sigma(A) \cap \mathbb{C}_0^+$ is empty.
$\mathfrak{H}3$.	B is an infinite-time admissible control operator for $\{T(t)\}_{t \geqslant 0}$.
$\mathfrak{H}4$.	C is an infinite-time admissible observation operator for $\{T(t)\}_{t \geqslant 0}$.
$\mathfrak{H}5$.	$G(\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ and $G(\cdot) \in \mathscr{C}_0(\mathrm{i}\mathbb{R}, \mathbb{C}^{\mathtt{p}\times\mathtt{m}})$.

First of all, we will show that if assumptions $\mathfrak{H}1$–$\mathfrak{H}5$ are satisfied by the original well-posed linear system Σ, then assumptions $\mathfrak{A}1$–$\mathfrak{A}3$ are satisfied by the system with the generating operators $A^{-1}, A^{-1}B, -CA^{-1}$. Furthermore, from (3) and Theorem 1.2, the Hankel operator H_G of the original transfer function and the Hankel operator $H_{G_{\mathtt{r}}}$ of the new system are both compact. Also, as discussed in the introduction, it follows from Theorem 1.3 that the Hankel singular values of G and $G_{\mathtt{r}}$ are equal. Applying Theorem 2.1, we give solutions to the sub-optimal Hankel norm approximation problem for the transfer function $G_{\mathtt{r}}$ and hence also to the original transfer function G. But first we will prove the following elementary result.

Lemma 2.2. *If A, B, C and G satisfy $\mathfrak{H}1$–$\mathfrak{H}5$, then the triple $(A^{-1}, A^{-1}B, -CA^{-1})$ satisfies $\mathfrak{A}1$–$\mathfrak{A}3$.*

Proof. $\mathfrak{A}1$ is obvious.

$\mathfrak{A}2$: Owing to the infinite-time input admissibility of $-CA^{-1}$ for the reciprocal system, it is clear that for all $x \in X$, $-CA^{-1}\mathrm{e}^{A^{-1}}x \in L_2([0,\infty), \mathbb{C}^{\mathtt{p}})$. Consequently, $-CA^{-1}\mathrm{e}^{A^{-1}}A^{-1}B \in L_2([0,\infty), \mathbb{C}^{\mathtt{p}\times\mathtt{m}})$. Furthermore, the continuity of G_- on the imaginary axis follows from Eq. (3), and $\mathfrak{H}5$.

$\mathfrak{A}3$: It follows from $\mathfrak{H}2$ and the spectral mapping theorem for the resolvent (see [4, 1.13.(i), p. 243]) that $\sigma(A^{-1}) \cap \mathbb{C}_0^+$ is empty. Furthermore, since A^{-1} is bounded, it satisfies the spectrum determined growth assumption (see for instance [4, Corollary 2.4, p. 252]). Thus it is clear that the growth bound of the semigroup generated by A^{-1} is nonpositive, and so $A^{-1} - \zeta I$ is the infinitesimal generator of an exponentially stable, strongly continuous semigroup. □

We end this section with our main result.

Theorem 2.3. *Suppose that the well-posed linear system Σ satisfies $\mathfrak{H}1$–$\mathfrak{H}5$. Let $\sigma_{\ell+1} < \sigma < \sigma_\ell$ and $Q(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ with $\|Q(i\cdot)\|_\infty \leqslant 1$. For every $\zeta > 0$, denote*

$$K_{\mathtt{r}}(s) = R_1(s)R_2(s)^{-1},$$

where

$$\begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix} = \Lambda(s)^{-1} \begin{bmatrix} Q(s-\zeta) \\ I_{\mathtt{m}} \end{bmatrix}$$

and

$$\Lambda(s) = \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & \sigma I_{\mathtt{m}} \end{bmatrix} + \begin{bmatrix} -\frac{1}{\sigma^2}CA^{-1}L_B^\zeta \\ \frac{1}{\sigma}(A^{-1}B)^* \end{bmatrix} [N_\sigma^\zeta]^*[sI + (A^{-1})^* - \zeta I]^{-1}[(CA^{-1})^* \ \ L_C^\zeta A^{-1}B].$$

Here L_B^ζ and L_C^ζ denote the controllability Gramian and the observability Gramian, respectively, of the exponentially stable system with the generating operators $A^{-1} - \zeta I, A^{-1}B, -CA^{-1}$, and $N_\sigma^\zeta := [I - (1/\sigma^2)L_B^{[\zeta]}L_C^{[\zeta]}]^{-1}$.

Then there exists a $\delta > 0$ *such that for every* $\zeta \in (0,\delta)$, $K_{\mathtt{r}}(-\cdot) \in H_{\infty,\ell}(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$. *Furthermore, defining*

$$K(s) = K_{\mathtt{r}}\left(\frac{1}{s}\right) - G(0), \tag{7}$$

we have $K(-\cdot) \in H_{\infty,\ell}(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ *and* $\|G(i\cdot) + K(i\cdot)\|_\infty \leqslant \sigma$.

Proof. From Lemma 2.2 it follows that the triple $(A^{-1}, A^{-1}B, -CA^{-1})$ satisfies assumptions $\mathfrak{A}1$–$\mathfrak{A}3$. If $G_{\mathtt{r}} = -CA^{-1}(sI - A^{-1})^{-1}A^{-1}B$, we have

$$\sigma_\ell(G_{\mathtt{r}}) = \sigma_\ell(G) > \sigma > \sigma_{\ell+1}(G) = \sigma_{\ell+1}(G_{\mathtt{r}}).$$

Consequently, using Theorem 2.1, we have that $K_{\mathtt{r}}(-\cdot) \in H_{\infty,\ell}(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ and $\|G_{\mathtt{r}}(i\cdot) + K_{\mathtt{r}}(i\cdot)\|_\infty \leqslant \sigma$. Finally it is clear that K defined by (7) is such that $K(-\cdot) \in H_{\infty,\ell}(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ and

$$\|G(i\cdot) + K(i\cdot)\|_\infty = \|G_{\mathtt{r}}(i\cdot) + K_{\mathtt{r}}(i\cdot)\|_\infty \leqslant \sigma. \qquad \square$$

3. The sub-optimal Nehari problem

Finally, in this section we solve the sub-optimal Nehari problem for the well-posed linear system on a Hilbert space X with generating operators A, B, C and transfer function G under the following assumptions:

$\mathfrak{N}1$. The input and output spaces are finite-dimensional; $U = \mathbb{C}^{\mathtt{m}}$ and $Y = \mathbb{C}^{\mathtt{p}}$.
$\mathfrak{N}2$. $0 \in \rho(A)$.
$\mathfrak{N}3$. B is an infinite-time admissible control operator for $\{T(t)\}_{t\geqslant 0}$.
$\mathfrak{N}4$. C is an infinite-time admissible observation operator for $\{T(t)\}_{t\geqslant 0}$.
$\mathfrak{N}5$. $G(\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$.

The sub-optimal Nehari problem is the following:

If $\sigma > \sigma_1$ $(=\|H_G\|$, the norm of Hankel operator $H_G)$, then find $K(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ such that $\|G(i\cdot) + K(i\cdot)\|_\infty \leqslant \sigma$. K is then called a solution of the sub-optimal Nehari problem.

The sub-optimal Nehari problem can be thought of as a special case of the sub-optimal Hankel norm approximation problem with $\ell = 0$. So in principle, the results of the previous section apply to this case. However, we can improve on these results considerably. This is because Theorem 2.1 in the previous section can be now replaced (in the *special* case of the Nehari problem) by a more powerful result from [3], which will enable us to even obtain a *parameterization* of *all* solutions to the sub-optimal Nehari problem. Also, we find that with this alternate approach we can solve the problem under weaker assumptions than those demanded in the previous section (notice the differences in $\mathfrak{N}2$ versus $\mathfrak{H}2$, and $\mathfrak{N}5$ versus $\mathfrak{H}5$). However, the broad approach in both sections remains the same: instead of looking at the original system, we translate the problem to the reciprocal system, solve it, and finally retrieve solutions to the original problem.

Suppose that $\sigma > \sigma_1(G) = \sigma_1(G_{\mathtt{r}})$. If $K(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ satisfies $\|G(i\cdot) + K(i\cdot)\|_\infty < \sigma$, then $K_{\mathtt{r}}(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ satisfies $\|G_{\mathtt{r}}(i\cdot) + K_{\mathtt{r}}(i\cdot)\|_\infty < \sigma$, where

$$K_{\mathtt{r}}(s) := K\left(\frac{1}{s}\right) + G(0).$$

Conversely, if $K_{\mathtt{r}}(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ satisfies $\|G_{\mathtt{r}}(i\cdot) + K_{\mathtt{r}}(i\cdot)\|_\infty < \sigma$, then $K(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ satisfies $\|G(i\cdot) + K(i\cdot)\|_\infty < \sigma$, where

$$K(s) := K_{\mathtt{r}}\left(\frac{1}{s}\right) - G(0).$$

So instead of solving the suboptimal Nehari problem for G, we solve the suboptimal Nehari problem for the reciprocal system with the bounded generating operators A^{-1}, $A^{-1}B$, $-CA^{-1}$ and feedthrough operator 0. This system satisfies all the conditions in the following result from [3], which we now recall.

Theorem 3.1. *Suppose that the triple* (A, B, C) *satisfies the following assumptions*:

$\mathfrak{B}1$. *A is the infinitesimal generator of a strongly continuous semigroup on the Hilbert space X, $B \in \mathscr{L}(\mathbb{C}^{\mathtt{m}}, X)$ and $C \in \mathscr{L}(X, \mathbb{C}^{\mathtt{p}})$.*
$\mathfrak{B}2$. *$G(i\cdot) = C(\cdot iI - A)^{-1}B \in L_\infty(\mathbb{R}, \mathbb{C}^{\mathtt{p}\times\mathtt{m}})$.*
$\mathfrak{B}3$. *B is an infinite-time admissible control operator for $\{T(t)\}_{t\geqslant 0}$.*
$\mathfrak{B}4$. *C is an infinite-time admissible observation operator for $\{T(t)\}_{t\geqslant 0}$.*

Let $\sigma > \sigma_1$ and let Λ be given by

$$\Lambda(s) = \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & \sigma I_{\mathtt{m}} \end{bmatrix} + \begin{bmatrix} -\dfrac{1}{\sigma^2} C L_B \\ \dfrac{1}{\sigma} B^* \end{bmatrix} N_\sigma^* [sI + A^*]^{-1} [C^* \;\; L_C B].$$

where L_B and L_C denote the controllability Gramian and the observability Gramian, respectively, of the system with generating operators A, B, C, and $N_\sigma := [I - (1/\sigma^2) L_B L_C]^{-1}$. Then $K(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ satisfies $\|G(i\cdot) + K(i\cdot)\|_\infty \leqslant \sigma$ iff $K(s) = R_1(s) R_2(s)^{-1}$, where

$$\begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix} = \Lambda(s)^{-1} \begin{bmatrix} Q(s) \\ I_{\mathtt{m}} \end{bmatrix}$$

for some $Q(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ with $\|Q(i\cdot)\|_\infty \leqslant 1$.

Indeed, under assumptions $\mathfrak{N}1$–$\mathfrak{N}5$, the reciprocal system has its transfer function in $H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ and the operators $A^{-1}B$ and $-CA^{-1}$ are infinite-time admissible with the same Gramians L_B and L_C as the original system. So in light of what has been said above, upon applying Theorem 3.1 to the triple $(A^{-1}, A^{-1}B, -CA^{-1})$, we obtain the following result:

Theorem 3.2. *Suppose that the well-posed linear system Σ satisfies $\mathfrak{N}1$–$\mathfrak{N}5$. Let $\sigma > \sigma_1$ and $Q(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ with $\|Q(i\cdot)\|_\infty \leqslant 1$. Let Λ be given by*

$$\Lambda(s) = \begin{bmatrix} I_{\mathtt{p}} & 0 \\ 0 & \sigma I_{\mathtt{m}} \end{bmatrix} + \begin{bmatrix} -\dfrac{1}{\sigma^2} C A^{-1} L_B \\ \dfrac{1}{\sigma} (A^{-1}B)^* \end{bmatrix} N_\sigma^* [sI + (A^{-1})^*]^{-1} [(CA^{-1})^* \;\; L_C A^{-1} B],$$

where L_B and L_C denote the controllability Gramian and the observability Gramian, respectively, of the system Σ, and $N_\sigma := [I - (1/\sigma^2) L_B L_C]^{-1}$. Then $K(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ satisfies $\|G(i\cdot) + K(i\cdot)\|_\infty \leqslant \sigma$ iff

$$K(s) = K_{\mathtt{r}}\left(\frac{1}{s}\right) - G(0),$$

where $K_{\mathtt{r}}(s) = R_1(s) R_2(s)^{-1}$, and

$$\begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix} = \Lambda(s)^{-1} \begin{bmatrix} Q(s) \\ I_{\mathtt{m}} \end{bmatrix},$$

for some $Q(-\cdot) \in H_\infty(\mathbb{C}^{\mathtt{p}\times\mathtt{m}})$ with $\|Q(i\cdot)\|_\infty \leqslant 1$.

While it is tempting to try to write Λ and K in terms of its reciprocal, we know that this will not (in general) be well-defined (see [14]). So we leave the explicit solution as it stands.

It is known (see [10,14]) that a given transfer function $G \in H_\infty(\mathbb{C}^{p\times m})$ always has a realization as a well-posed linear system satisfying assumptions $\mathfrak{N}1$–$\mathfrak{N}5$. Unfortunately, $\mathfrak{N}2$ may not always be satisfied. So an interesting open problem is to find natural sufficient conditions on G for this to hold.

The assumption that $0 \in \rho(A)$ is not central and can be replaced by the assumption that $i\omega \in \rho(A)$ for some real ω. Then one defines the ω-reciprocal system with generating operators A_ω^{-1}, $A_\omega^{-1}B$, $-CA_\omega^{-1}$ and feedthrough operator $G(i\omega)$, where $A_\omega = A - i\omega I$. Its transfer function G_-^ω satisfies

$$G_-^\omega\left(\frac{1}{s}\right) = G(s + i\omega).$$

References

[1] V.M. Adamjan, D.Z. Arov, M.G. Krein, Analytic properties of the Schmidt pairs of a Hankel operator and the generalized Schur–Takagi problem, Math. Sb. (N.S.) 86 (1971) 34–75.

[2] J.A. Ball, A.C.M. Ran, Optimal Hankel norm model reductions and Wiener–Hopf factorization I: the canonical case, SIAM J. Control Optim. 25 (2) (1987) 362–382.

[3] R.F. Curtain, J.C. Oostveen, The Nehari problem for nonexponentially stable systems, Integral Equations Operator Theory 31 (1998) 307–320.

[4] K-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, Berlin, 2000.

[5] K. Glover, All optimal Hankel-norm approximations of linear multivariable systems and their L_∞ error bounds, Int. J. Control 39 (1984) 1115–1193.

[6] P. Grabowski, On the spectral-Lyapunov approach to parametric optimization of distributed parameter systems, IMA J. Math. Control Inform. 7 (1990) 317–338.

[7] S. Hansen, G. Weiss, New results on the operator Carleson measure criterion, IMA J. Math. Control Inform. 14 (1997) 3–32.

[8] S.Y. Kung, D.W. Lin, Optimal Hankel-norm model reductions: multivariable systems, IEEE Trans. Automat. Control 26 (1981) 832–852.

[9] J.R. Partington, An Introduction to Hankel Operators, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 1988.

[10] D. Salamon, Realization theory in Hilbert space, Mathematical Systems Theory 21 (1989) 147–164.

[11] A.J. Sasane, Hankel Norm Approximation for Infinite-dimensional Systems, Lecture Notes in Control and Information Sciences, Vol. 277, Springer, Berlin, 2002.

[12] A.J. Sasane, R.F. Curtain, Sub-optimal Hankel norm approximation problem for the Pritchard–Salamon class of infinite-dimensional systems, Integral Equations Operator Theory 39 (2001) 98–126.

[13] A.J. Sasane, R.F. Curtain, Sub-optimal Hankel norm approximation problem for the analytic class of infinite-dimensional systems, Integral Equations Operator Theory 43 (2002) 356–377.

[14] O.J. Staffans, Admissible factorizations of Hankel operators induced by well-posed linear systems, Systems Control Lett. 37 (1999) 301–307.

[15] O.J. Staffans, Well-posed linear systems I: general theory, in preparation. Draft available on the world wide web at http://www.abo.fi/∼staffans/publ.htm.

[16] G. Weiss, Regular linear systems with feedback, Math. Control Signals Systems 7 (1994) 23–57.

[17] G. Weiss, Transfer functions of regular linear systems, part 1: characterizations of regularity, Trans. Amer. Math. Soc. 342 (1994) 827–854.