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Node-disjoint paths in incomplete WK-recursive networks

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Abstract

The incomplete WK-recursive networks have been recently proposed to relieve the restriction on the sizes of the WK-recursive networks. In this paper, a maximal set of node-disjoint paths is constructed between arbitrary two nodes of an incomplete WK-recursive network. The effectiveness of the constructed paths is verified by both theoretic analysis and extensive experiments. A tight upper bound on the maximal length is suggested. On the other hand, experimental results show that for arbitrary two nodes, the expected maximal length is not greater than twice their distance and about equal to the diameter. When the two nodes are the farthest pair, the maximal length is not greater than twice the diameter and the expected maximal length is not greater than 1.5 times the diameter. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Graph-theoretic interconnection network; Incomplete WK-recursive network; Node-disjoint paths; WK-recursive network

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1. Introduction

The WK-recursive networks [23] own two attractive topological properties: expansibility and equal degree. A network is *expansible* if no changes to node configuration and link connection are necessary when it is expanded, and of *equal degree* if its nodes have the same degree no matter what its size is. A network with these two properties will gain the advantages of easy implementation and low cost when it is manufactured. A VLSI implementation of a 16-node WK-recursive network has been realized at the Hybrid Computing Research Center [23]. This prototype network was further extended to 64 nodes later [24]. Recently two variants of the WK-recursive networks have been proposed in [5,6].

Although the WK-recursive networks own many favorable properties (see [1,3,4,7–9,11,23,24]), there is a rigorous restriction on their sizes. As will become clear in Section 2, the number of nodes contained in a WK-recursive network must satisfy d^t, where $d > 1$ is the size of the basic building block and $t \geqslant 1$ is the level of expansion. Thus, as $d = 4$, extra $3 \times 4^7 = 49152$ nodes are required to expand from a 7-level WK-recursive network to an 8-level one. Almost all announced networks have suffered from the same restriction. In order to relieve this restriction, some incomplete networks have been proposed recently. Among them, incomplete hypercubes [12], incomplete star networks [14,17], clustered-star graphs [13], incomplete rotator graphs [16], and incomplete WK-recursive networks [19] are some examples. Previously several results on the incomplete WK-recursive networks were obtained; topological properties were investigated in [21], a shortest-path routing algorithm appeared in [22], and a broadcasting algorithm was proposed in [20].

Given a network, it is both theoretically interesting and practically important to find node-disjoint paths (disjoint paths for short) between any two of its nodes. With disjoint paths, transmission rate can be accelerated and transmission reliability can be enhanced. In the past, a maximal set of disjoint paths was constructed for many (complete) networks, e.g., hypercubes [18], star graphs [2], and WK-recursive networks [3]. However, the same problem remained unsolved for all incomplete networks but the clustered-star graphs (see Ref. [10]). In this paper, a maximal set of disjoint paths is constructed for the incomplete WK-recursive networks.

In the next section, the incomplete WK-recursive networks are first reviewed. A prerequisite step for constructing disjoint paths is presented in Section 3. By its aid a maximal set of disjoint paths is constructed in Section 4. Moreover, a tight upper bound on their maximal length is suggested. In Section 5, extensive experiments are further made to verify their effectiveness. Finally, we conclude this paper with some remarks in Section 6.

2. Incomplete WK-recursive networks

The WK-recursive networks can be built incrementally with basic building blocks. Any complete graph can serve as a basic building block. Throughout this paper we use graph and network interchangeably. Let $K(d, t)$ denote a WK-recursive network of level t whose basic building block is a d-node complete graph, where $d > 1$ and $t \geqslant 1$. $K(d,1)$, which is the basic building block, is the d-node complete graph, and $K(d,t)$ for $t \geqslant 2$ is a d-supernode complete graph, where each supernode is a $K(d,t-1)$. Each node of a $K(d,t)$ is assigned with a unique identifier which consists of a d-ary sequence of length t. The following definition is due to Chen and Duh [1].

Definition 2.1. The node set of a $K(d,t)$ is denoted by $\{a_{t-1}a_{t-2}\cdots a_1a_0 \mid a_i \in \{0,1,\ldots,d-1\}$ for $0 \leqslant i \leqslant t-1\}$. Node adjacency is defined as follows: $a_{t-1}a_{t-2}\cdots a_1a_0$ is adjacent to (1) $a_{t-1}a_{t-2}\cdots a_1b$, where $0 \leqslant b \leqslant d-1$ and $b \neq a_0$, and (2) $a_{t-1}a_{t-2}\cdots a_{j+1}a_{j-1}a_j^j$ if $a_j \neq a_{j-1}$ and $a_{j-1} = a_{j-2} = \cdots = a_0$ for some $1 \leqslant j \leqslant t-1$, where a_j^j represents j consecutive a_js. The links of (1) are called *substituting links*, and are labeled 0. The link of (2), if existing, is called *j-flipping link* (or simply *flipping link*), and is labeled j. Besides, if $a_{t-1} = a_{t-2} = \cdots = a_0$, there is a link, called *open link*, incident to node $a_{t-1}a_{t-2}\cdots a_1a_0$. The open link, which is labeled t, is reserved for further expansion.

The structures of $K(4,1)$ and $K(4,3)$ are illustrated in Fig. 1. The links within basic building blocks are substituting links, and those connecting two embedded $K(d,j)$s are j-flipping links. For example, the link between nodes 311 and 133 is a 2-flipping link, and the other links incident to node 311 are all substituting links. The open links are incident to nodes 000, 111, 222, and 333.

Definition 2.2. Define $c_{t-1}c_{t-2}\cdots c_m \cdot K(d,m)$ to be the subgraph of a $K(d,t)$ induced by $\{c_{t-1}c_{t-2}\cdots c_m a_{m-1}\cdots a_1a_0 \mid a_j \in \{0,1,\ldots,d-1\}$ for $0 \leqslant j \leqslant m-1\}$, where $1 \leqslant m \leqslant t-1$ and $c_{t-1}, c_{t-2}, \ldots, c_m$ are all integers from $\{0,1,\ldots,d-1\}$.

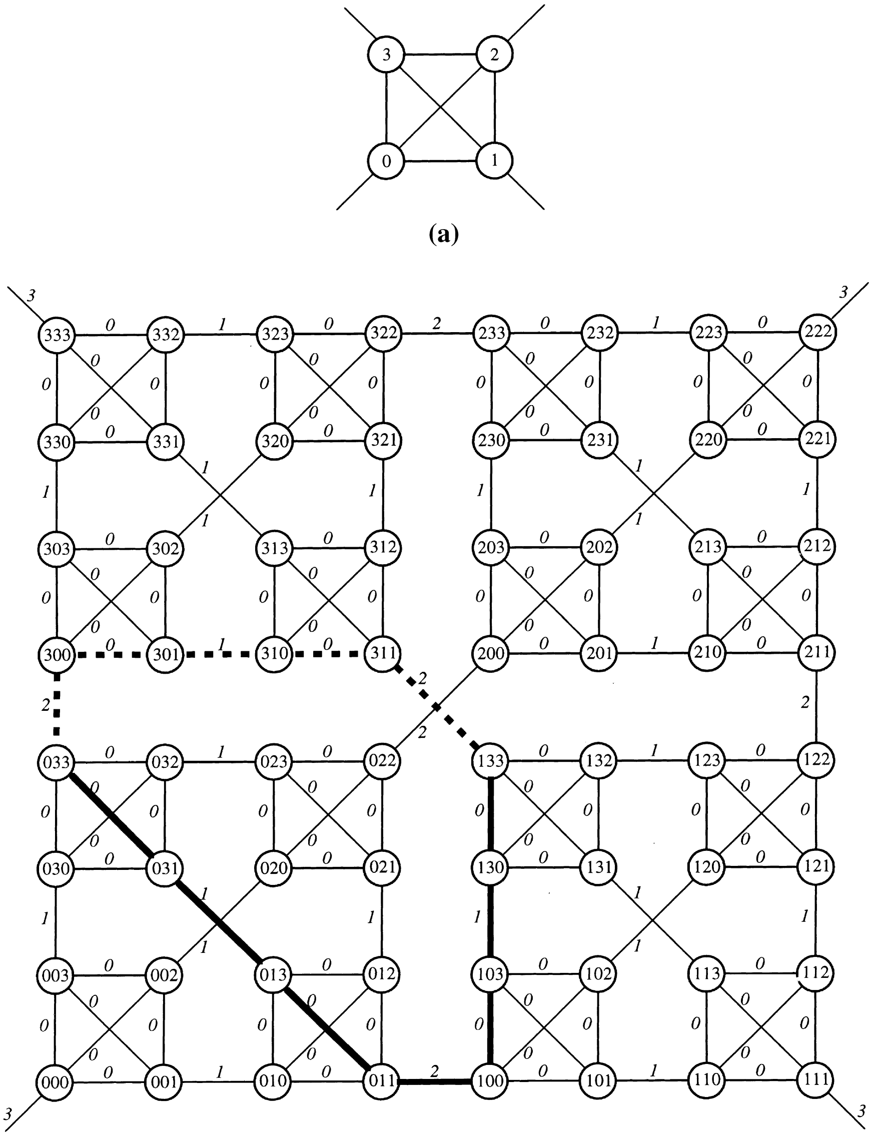
For example, refer to Fig. 1(b), where $31 \cdot K(4,1)$ is the subgraph of $K(4,3)$ induced by {310, 311, 312, 313}.

Definition 2.3. Node $a_{t-1}a_{t-2}\cdots a_1a_0$ is a *k-frontier* if $a_{k-1} = a_{k-2} = \cdots = a_1 = a_0$, where $1 \leqslant k \leqslant t$.

By definition a k-frontier is automatically an l-frontier for $1 \leqslant l \leqslant k-1$. Both end nodes of a k-flipping link are k-frontiers. For $1 \leqslant m \leqslant t-1$, an embedded $K(d,m)$ contains one $(m+1)$-frontier and $d-1$ m-frontiers. These d frontiers are $2^m - 1$ distant from each other.

The incomplete WK-recursive networks, which were originally defined in [19], are induced subgraphs of the WK-recursive networks. If we number the nodes of a $K(d,t)$ according to their lexicographical order, then an N-node incomplete WK-recursive network is the subgraph of the $K(d,t)$ induced by the first N nodes. Throughout this paper we use $IK(d,t)$ to denote an N-node incomplete WK-recursive network, where $d^{t-1} < N < d^t$ and N is a multiple of d.

Associated with an N-node $IK(d,t)$ is a *coefficient vector* $(b_{t-1}, b_{t-2}, \ldots, b_1)$, where $0 \leqslant b_j \leqslant d-1$ for all $1 \leqslant j \leqslant t-1$ and $N = b_{t-1}d^{t-1} + b_{t-2}d^{t-2} + \cdots + b_1 d$. It means that an $IK(d,t)$ with coefficient vector $(b_{t-1}, b_{t-2}, \ldots, b_1)$ contains b_j



(b)

Fig. 1. The structrue of: (a) $K(4,1)$ and (b) $K(4,3)$. This figure also shows a heuristic routing path and the shortest routing path between nodes 033 and 133.

embedded $K(d,j)$s: $b_{t-1}b_{t-2}\cdots b_{j+1}0\cdot K(d,j)$, $b_{t-1}b_{t-2}\cdots b_{j+1}1\cdot K(d,j),\ldots,b_{t-1}b_{t-2}\cdots b_{j+1}(b_j-1)\cdot K(d,j)$. For example, the structure of an $IK(4,3)$ with coefficient vector (3, 2) is shown in Fig. 2. It contains three embedded $K(4,2)$s, i.e., $0\cdot K(4,2)$, $1\cdot K(4,2)$, and $2\cdot K(4,2)$, and two embedded $K(4,1)$s, i.e., $30\cdot K(4,1)$

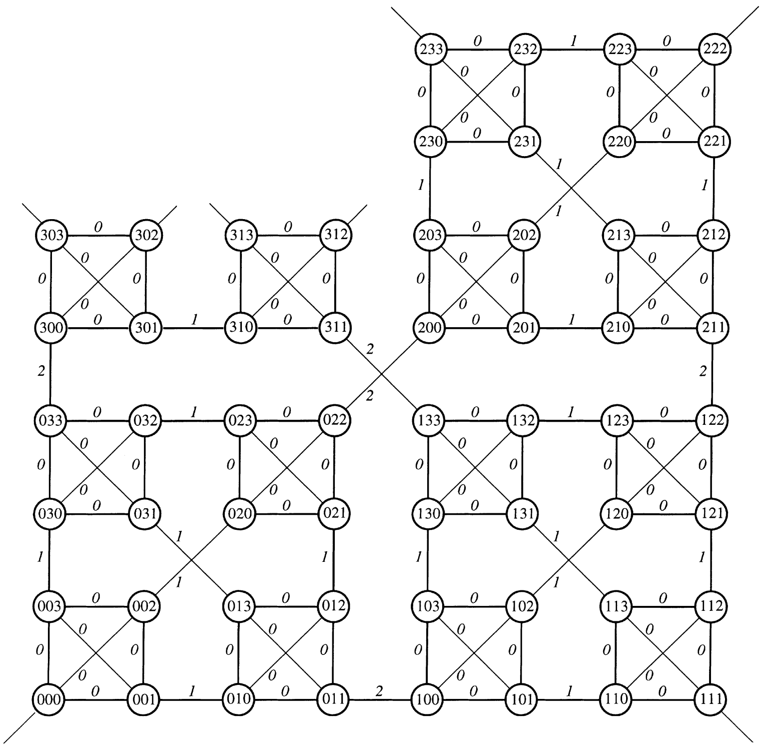


Fig. 2. The structure of $IK(4,3)$ with coefficient vector $(3,2)$.

and $31 \cdot K(4,1)$. In the rest of this paper, coefficient vector $(b_{t-1}, b_{t-2}, \ldots, b_1)$ is written as $(b_{t-1}, b_{t-2}, \ldots, b_i, *)$, provided $b_i \neq 0$ and $b_{i-1} = b_{i-2} = \cdots = b_1 = 0$, where $1 \leqslant i \leqslant t-1$. For example, (2, 0, 4, 0, 0) is written as (2, 0, 4, $*$).

Given an $IK(d,t)$ with coefficient vector $(b_{t-1}, b_{t-2}, \ldots, b_i, *)$, let S_m represent the subgraph induced by the nodes of $b_{t-1}b_{t-2}\cdots b_{m+1}0 \cdot K(d,m)$, $b_{t-1}b_{t-2}\cdots b_{m+1}1 \cdot K(d,m), \ldots, b_{t-1}b_{t-2}\cdots b_{m+1}(b_m - 1) \cdot K(d,m)$, where $i \leqslant m \leqslant t-1$. For example, given an $IK(5,7)$ with coefficient vector (4, 2, 4, 3, 1, 4, $*$), S_6 contains $0 \cdot K(5,6)$, $1 \cdot K(5,6)$, $2 \cdot K(5,6)$, and $3 \cdot K(5,6)$, S_5 contains $40 \cdot K(5,5)$ and $41 \cdot K(5,5)$, and so on. We note that the embedded $K(d,m)$s within S_m join one another through m-flipping links. That is, S_m is a b_m-supernode complete graph with each supernode being a $K(d,m)$. If each S_m is regarded as a stage, then the structure of the $IK(d,t)$ forms a $(t-i)$-stage graph, denoted by $S_{t-1} + S_{t-2} + \cdots + S_i$. Refer to Fig. 3, where three examples are shown. For the sake of simplicity, each embedded $K(d,m)$ within S_m is drawn as a circle, and the one $b_{t-1}b_{t-2}\cdots b_{m+1}j \cdot K(d,m)$ is denoted by C_m^j, where $0 \leqslant j \leqslant b_m - 1$. All the links within S_m are omitted.

There are $\min\{b_m, b_{m-1}\}$ m-flipping links between S_m and S_{m-1} that connect C_m^j and C_{m-1}^j for all $0 \leqslant j \leqslant \min\{b_m, b_{m-1}\} - 1$. Besides, there may exist a u-flipping link

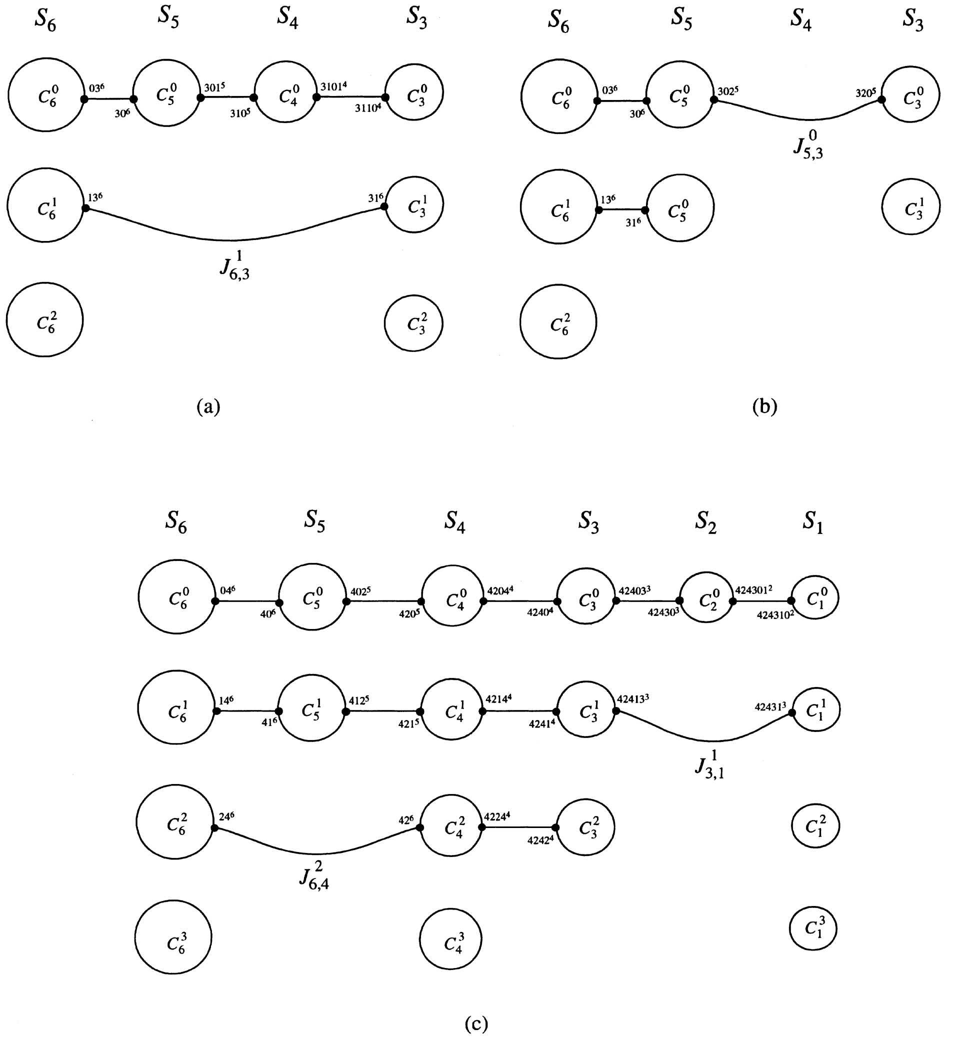


Fig. 3. Multistage graph representation of $IK(6, 10)$ with coefficient vector (4, 5, 5, 3, 5, 3, 1, 1, 4, $*$).

between S_u and S_v, where $i \leqslant v < u \leqslant t-1$ and $u - v > 1$. Such a link, if it exists, is called a *jumping u-flipping link*. A necessary and sufficient condition for the existence of jumping flipping links is presented below.

Theorem 2.1. *Given an* $IK(d, t)$ *with coefficient vector* $(b_{t-1}, b_{t-2}, \ldots, b_i, *)$, *one jumping u-flipping link exists between* S_u *and* S_v *if and only if* $b_u > b_{u-1} = b_{u-2} = \cdots = b_{v+1} < b_v$, *where* $i \leqslant v < u \leqslant t-1$ *and* $u - v > 1$. *Moreover, this jumping flipping link connects* C_u^e *and* C_v^e, *where* $e = b_{u-1} = b_{u-2} = \cdots = b_{v+1}$.

Proof. ($\Leftarrow$) According to the definition of $IK(d,t)$, there are e u-flipping links between S_u and S_{u-1} that connect C_u^j and C_{u-1}^j for all $0 \leqslant j \leqslant b_{u-1} - 1$. Besides, there exists one jumping u-flipping link connecting C_u^e and C_v^e whose two end nodes are $b_{t-1}b_{t-2}\cdots b_{u+1}eb_u^u \in C_u^e$ and $b_{t-1}b_{t-2}\cdots b_{u+1}b_ue^u = b_{t-1}b_{t-2}\cdots b_{u+1}b_uee\cdots ee^{v+1} \in C_v^e$. For $b_{u-1} < j \leqslant b_u - 1$, the jumping flipping link $(b_{t-1}b_{t-2}\cdots b_{u+1}jb_u^u,\ b_{t-1}b_{t-2}\cdots b_{u+1}b_uj^u)$ does not exist because $b_{t-1}b_{t-2}\cdots b_{u+1}b_uj^u$ is not a node of the $IK(d,t)$.

($\Rightarrow$) Without loss of generality, assume the jumping u-flipping link emits from C_u^α to S_v for some $0 \leqslant \alpha \leqslant b_u - 1$. We first show $b_u > b_{u-1}$ by contradiction. Suppose $b_u \leqslant b_{u-1}$. There is a u-flipping link between C_u^j and C_{u-1}^j for all $0 \leqslant j \leqslant b_u - 1$. Regarding $S_{u-1} + S_{u-2} + \cdots + S_i$ as an embedded $IK(d,u)$, there are two u-flipping links between C_u^α and the embedded $IK(d,u)$: one is between C_u^α and C_{u-1}^α and the other is between C_u^α and S_v. This is a contradiction because at most one u-flipping link may exist between any two embedded $K(d,u)$s and an $IK(d,u)$ is a subgraph of a $K(d,u)$. Similarly, $\alpha \geqslant b_{u-1}$ can be proved.

We continue to show that $\alpha = b_{u-1},\ b_{u-1} = b_{u-2} = \cdots = b_{v+1} < b_v$, and the jumping u-flipping link connects C_u^α and C_v^α. According to the definition of u-flipping link, the end node of the jumping u-flipping link in S_u (actually in C_u^α) is $b_{t-1}b_{t-2}\cdots b_{u+1}\alpha b_u^u$. Thus the other end node in S_v is $b_{t-1}b_{t-2}\cdots b_{u+1}b_u\alpha^u$. It is not difficult to see that $b_{t-1}b_{t-2}\cdots b_{u+1}b_u\alpha^u$ does not belong to the $IK(d,t)$ if $\alpha > b_{u-1}$. Consequently, we have $\alpha = b_{u-1}$ and the end node in S_v is $b_{t-1}b_{t-2}\cdots b_{u+1}b_ub_{u-1}^u = b_{t-1}b_{t-2}\cdots b_{u+1}b_ub_{u-1}b_{u-1}\cdots b_{u-1}(b_{u-1})^{v+1}$. The latter further implies that $b_{u-1} = b_{u-2} = \cdots = b_{v+1} < b_v$ and the end node is located in $C_v^\alpha = C_v^{b_{u-1}}$. $\square$

In the rest of this paper we use $J_{u,v}^e$ to denote the jumping u-flipping link that connects C_u^e and C_v^e (refer to Fig. 3 for illustration). Theorem 2.1 provides a fast way to determine all jumping flipping links from the coefficient vector $(b_{t-1}, b_{t-2}, \ldots, b_i, *)$. We only need to examine $(b_{t-1}, b_{t-2}, \ldots, b_i, *)$ from the left to the right so that $J_{u,v}^e$ exists if $b_u > b_{u-1} = b_{u-2} = \cdots = b_{v+1} < b_v$, where $u - v > 1$ and $e = b_{u-1} = b_{u-2} = \cdots = b_{v+1}$.

We note that for $i \leqslant n \leqslant m \leqslant t-1$, $b_m \neq 0$, and $b_n \neq 0$, $S_m + S_{m-1} + \cdots + S_n$ forms an embedded $IK(d, m+1)$ with coefficient vector $(b_m, b_{m-1}, \ldots, b_n, *)$ whose each node has its identifier prefixed with $b_{t-1}b_{t-2}\cdots b_{m+1}$. For example, refer to Fig. 3(c), where $S_3 + S_2 + S_1$ forms an embedded $IK(5,4)$ with coefficient vector (3, 1, 4, $*$) whose each node has its identifier prefixed with 424. Theorem 2.1 can be applied to $S_m + S_{m-1} + \cdots + S_n$ as well.

3. A prerequisite step

Suppose X and Y are arbitrary two nodes of an $IK(d,t)$ with coefficient vector $(b_{t-1}, b_{t-2}, \ldots, b_i, *)$. Without loss of generality, we assume $X \in C_m^\alpha$ and $Y \in C_n^\beta$, where $i \leqslant n \leqslant m \leqslant t-1$, $0 \leqslant \alpha \leqslant b_m - 1$ and $0 \leqslant \beta \leqslant b_n - 1$. In this section, an algorithm that groups stages $S_m, S_{m-1}, \ldots, S_i$ into blocks is proposed. Each block contains one or more consecutive stages, and every two adjacent blocks intersect with one stage. The union of all blocks is the set of all stages. The algorithm will be invoked when we construct disjoint paths between X and Y in Section 4.

With input $(b_m, b_{m-1}, \ldots, b_i, *)$ and α, the algorithm produces a sequence of integers $m_0, m_1, \ldots, m_k$, where $k \geqslant 0$ and $m \geqslant m_0 > m_1 > \cdots > m_k = i$. These integers define $k+1$ blocks, i.e., $S_m + S_{m-1} + \cdots + S_{m_0}, S_{m_0} + S_{m_0-1} + \cdots + S_{m_1}, \cdots, S_{m_{k-1}} + S_{m_{k-1}-1} + \cdots + S_{m_k}$. The algorithm, as shown below, takes $O(m)$ time.

Algorithm (*Stage_Grouping*$((b_m, b_{m-1}, \ldots, b_i, *), \alpha)$. $/* \; 0 \leqslant \alpha \leqslant b_m - 1 \; */$.

(1). Scan $(b_m, b_{m-1}, \ldots, b_i, *)$ from the left to the right and determine in sequence $J^{x_1}_{y_1,z_1}, J^{x_2}_{y_2,z_2}, \ldots, J^{x_c}_{y_c,z_c}$, so that $\alpha > x_1 > x_2 > \cdots > x_c$. That is, $J^{x_1}_{y_1,z_1}$ is the first jumping flipping link encountered in the scanning which has $x_1 < \alpha$. Each $J^e_{u,v}$ between $J^{x_j}_{y_j,z_j}$ and $J^{x_{j+1}}_{y_{j+1},z_{j+1}}$ has $e \geqslant x_j$, where $1 \leqslant j < c$, and each $J^e_{u,v}$ after $J^{x_c}_{y_c,z_c}$ has $e \geqslant x_c$. Let $\boldsymbol{L} = \{x_1, x_2, \ldots, x_c\}$. If no feasible jumping flipping link is found in the scanning, $\boldsymbol{L}$ is empty.

For example, refer to Fig. 3(c) again. If $X \in C^3_6$, then (4, 2, 4, 3, 1, 4, $*$) and 3 are taken as input. Since $J^2_{6,4}$ and $J^1_{3,1}$ are found in the scanning, $\boldsymbol{L} = \{2, 1\}$. Similarly, if $X \in C^2_6$, only $J^1_{3,1}$ is found and thus $\boldsymbol{L} = \{1\}$. On the other hand, if X belong to C^1_6 or C^0_6, $\boldsymbol{L}$ is empty because no feasible jumping flipping link can be found. We also note that $m \geqslant y_1 > z_1 \geqslant y_2 > z_2 \geqslant \cdots \geqslant y_c > z_c \geqslant i$. By the aid of Theorem 2.1, this step can be completed in $O(m)$ time.

(2). Determine $m_0 = \min\{r \mid b_r > \alpha \text{ and } b_j \geqslant \alpha \text{ for all } m \geqslant j > r\}$. If $\boldsymbol{L}$ is not empty, determine $m_1, m_2, \ldots, m_c$ sequentially as follows: $m_1 = \min\{r \mid b_r > x_1 \text{ and } b_j \geqslant x_1 \text{ for all } m_0 > j > r\}$, $m_2 = \min\{r \mid b_r > x_2 \text{ and } b_j \geqslant x_2 \text{ for all } m_1 > j > r\}, \ldots,$ $m_c = \min\{r \mid b_r > x_c \text{ and } b_j \geqslant x_c \text{ for all } m_{c-1} > j > r\}$.

By examining $(b_m, b_{m-1}, \ldots, b_i, *)$ from the left to the right, this step can be completed in $O(m)$ time. For example, refer to Fig. 3(c) again. If $\alpha = 3$, we have $m_0 = 6$, $m_1 = 3$, and $m_2 = 1$. If $\alpha = 2$, we have $m_0 = 3$ and $m_1 = 1$. If $\alpha = 1$ or 0, we have $m_0 = 1$. We note that $m \geqslant m_0 \geqslant y_l > z_1 \geqslant m_1 \geqslant y_2 > z_2 \geqslant \cdots \geqslant m_{c-1} \geqslant y_c > z_c \geqslant m_c \geqslant i$.

(3). Output $m_0, m_1, \ldots, m_k (= i)$, where $k = c$ or $c+1$, according to the following four cases:

Case 1. $\boldsymbol{L}$ is empty and $m_0 = i$. Output m_0.

Case 2. $\boldsymbol{L}$ is empty and $m_0 > i$. Set $m_1 = i$ and output m_0, m_1.

Case 3. $\boldsymbol{L}$ is not empty and $m_c = i$. Output $m_0, m_1, \ldots, m_c$.

Case 4. $\boldsymbol{L}$ is not empty and $m_c > i$. Set $m_{c+1} = i$ and output $m_0, m_1, \ldots, m_c, m_{c+1}$.

Refer to Fig. 3(c) again. The algorithm will output (6, 3, 1), (3, 1), (1), and (1) if $\alpha = 3, 2, 1$, and 0, respectively. The output $(m_0, m_1, \ldots, m_k)$ defines $k+1$ blocks, denoted by $B_0, B_1, \ldots, B_k$, where $B_0 = S_m + S_{m-1} + \cdots + S_{m_0}$ and $B_l = S_{m_{l-1}} + S_{m_{l-1}-1} + \cdots + S_{m_l}$ for all $1 \leqslant l \leqslant k$. Every two adjacent blocks B_{l-1} and B_l contain one common stage $S_{m_{l-1}}$.

Lemma 3.1 [22]. *Let $J^{x_1}_{y_1,z_1}, J^{x_2}_{y_2,z_2}, \ldots, J^{x_c}_{y_c,z_c}$ and $m_0, m_1, \ldots, m_c$ be defined as in Algorithm Stage_Grouping. Then, for all $1 \leqslant j \leqslant c$,*

1. $m_{j-1} \geqslant y_j$;
2. *if* $m_{j-1} > y_j$, *then* $b_{m_{j-1}} > (x_{j-1} \geqslant) b_{m_{j-1}-1} \geqslant \cdots \geqslant b_{y_j}$;
3. $z_j \geqslant m_j$;

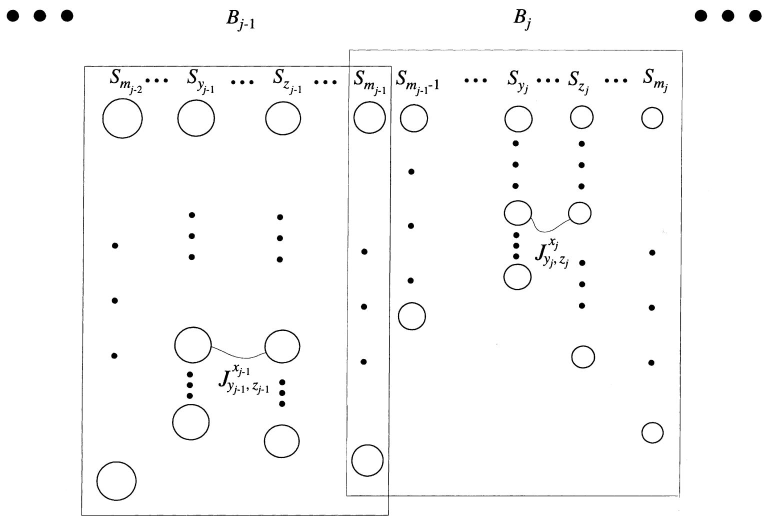


Fig. 4. Illustration of Lemma 3.1.

4. *if* $z_j > m_j$, *then* $b_{z_j} > x_j$, $b_{m_j} > x_j$, *and* $b_q \geqslant x_j$ *for all* $z_j > q > m_j$;
5. $x_j = \min\{b_{m_{j-1}}, b_{m_{j-1}-1}, \ldots, b_{y_j}, \ldots, b_{z_j}, \ldots, b_{m_j}\}$,*where* $x_0 = \alpha$ *is assumed.*

This lemma is illustrated in Fig. 4, where $m_{j-1} > y_j$ and $z_j > m_j$ are assumed. According to Lemma 3.1, $J^{x_1}_{y_1,z_1}, J^{x_2}_{y_2,z_2}, \ldots, J^{x_c}_{y_c,z_c}$ are the leftmost and upmost jumping flipping links in $B_1, B_2, \ldots, B_c$, respectively (the smaller the value x_j is, the upper $J^{x_j}_{y_j,z_j}$ is). That is, for any $J^e_{u,v}$ in B_j we have $u \leqslant y_j$ and $e \geqslant x_j$ (actually $u \leqslant z_j$ if $J^e_{u,v} \neq J^{x_j}_{y_j,z_j}$). We note that B_0 may or may not contain jumping flipping links and B_{c+1}, if it exists, does not contain any jumping flipping link. For B_{c+1} we have $b_{m_c} > (x_c \geqslant) b_{m_c-1} \geqslant \cdots \geqslant b_{m_{c+1}} (m_{c+1} = i)$. We also note that B_0 contains at least one stage, B_j for $0 < j \leqslant c$ contains at least three stages, and B_{c+1}, if it exists, contains at least two stages.

4. Construction of disjoint paths

In this section, disjoint paths are constructed between arbitrary two nodes X and Y of an $IK(d,t)$ with coefficient vector $(b_{t-1}, b_{t-2}, \ldots, b_i, *)$. Without loss of generality, we assume $X \in C^{\alpha}_m$ and $Y \in C^{\beta}_n$, where $i \leqslant n \leqslant m \leqslant t-1$, $0 \leqslant \alpha \leqslant b_m - 1$, and $0 \leqslant \beta \leqslant b_n - 1$. The disjoint paths have maximal length not greater than $2^{m+1} + 2^m - 1$. The construction time is $O(d \times D_m)$, where $D_m = 2^m - 1$ is the diameter of a $K(d,m)$.

First we consider a trivial case of $m = n$ and $\alpha = \beta$. Since X and Y belong to the same embedded $K(d,m)$, $d-1$ disjoint paths between X and Y can be obtained by Duh and Chen's work [3]. These $d-1$ paths are all within $C^{\alpha}_m (= C^{\beta}_n)$, and their maximal length is not greater than $3D_{m-1}+2=2^m+2^{m-1}-1$. It should be mentioned that there may exist one more disjoint path between X and Y that goes outside C_m^α. This path, if it exists, is much longer than those $d-1$ paths within C_m^α. We exclude this path from our discussion.

In the rest of this section, a maximal set of disjoint paths between X and Y is constructed within $S_m+S_{m-1}+\cdots+S_i$ for: (i) $m=n$ and $\alpha\neq\beta$ and (ii) $m>n$. It is still possible that there exists one additional disjoint path going outside $S_m+S_{m-1}+\cdots+S_i$. This path, if it exists, is excluded from our discussion because it is too long as compared with those within $S_m+S_{m-1}+\cdots+S_i$.

To begin, we have to review Vecchia and Sanges' routing algorithm [23] for a $K(d,t)$ because it is necessary to the construction of the disjoint paths. Suppose A and B are arbitrary two nodes of a $K(d,t)$. We define $A=_r B$ if they belong to the same embedded $K(d,r)$, and $A\neq_r B$ otherwise, where $1\leqslant r\leqslant t$. For example, refer to Fig. 1(b), where $033 =_3 133$, but $033 \neq_2 133$. A routing path from A to B within a $K(d,t)$ can be obtained by the following procedure.

1. Determine the level r so that $A=_r B$ but $A\neq_{r-1} B$, where $1\leqslant r\leqslant t$.
2. Determine the flipping link, say (W,Z), so that $A=_{r-1} W$ and $Z=_{r-1} B$.
3. Determine a routing path from A to W and a routing path from Z to B, recursively.

A routing path from A to B is obtained by concatenating the routing path from A to W, the flipping link (W,Z), and the routing path from Z to B. For example, a routing path from node 033 to node 133 within $K(4,3)$ is shown with bold lines in Fig. 1(b). When a message is transmitted from A to B, it is first routed to the nearest $(r-1)$-frontier, say Z, with $Z=_{r-1} B$, then routed to the nearest $(r-2)$-frontier, say Z', with $Z'=_{r-2} B$, and so on. In other words, when the message is going along the routing path, the identifiers of the traversed nodes are gradually equalized with B from the left to the right. For example, let us consider the routing path from node 033 to node 133 that is indicated with bold lines in Fig. 1(b). The left digit is equalized at node 100, the middle digit is equalized at node 130, and finally all the three digits are equalized at the destination node 133. The following observation is immediate.

Observation 4.1. When routing a message according to Vecchia and Sanges' algorithm, the identifiers of the traversed nodes are gradually equalized with B from the left to the right.

Vecchia and Sanges' algorithm, although simple, does not guarantee the shortest path. For example, the shortest path from node 033 to node 133 is shown with dashed lines in Fig. 1(b). Let $p(A,B)$ denote the routing path from node A to node B within a $K(d,t)$ that is produced by Vecchia and Sanges' algorithm. The following two lemmas have been proved in [3].

Lemma 4.1 [3]. *Suppose A and B are arbitrary two nodes of a $K(d,t)$. If $A=_r B$ and either of them is an r-frontier, then $p(A,B)$ is the shortest, where $1\leqslant r\leqslant t$. Moreover, it takes at most $O(D_t)$ time to determine $p(A,B)$, where $D_t=2^t-1$ is the diameter of the $K(d,t)$.*

Lemma 4.2 [3]. *Suppose A is an arbitrary node of a $K(d,t)$, and let $V_{l,0}, V_{l,1}, \ldots, V_{l,d-1}$ be the dl-frontiers of an embedded $K(d,l)$ that contains A, where $1 \leqslant l \leqslant t$. Then, the d paths $p(A,V_{l,0}), p(A,V_{l,1}), \ldots, p(A,V_{l,d-1})$ are mutually disjoint, exclusive of A.*

By Lemma 4.1, $p(A,V_{l,0}), p(A,V_{l,1}), \ldots, p(A,V_{l,d-1})$ are all the shortest and they can be determined in $\mathrm{O}(d \times D_l)$ time. Now we are ready to construct disjoint paths between $X \in C_m^\alpha$ and $Y \in C_n^\beta$. First we consider the situation of $m = n$ and $\alpha \neq \beta$. Within S_m there are $b_m - 1$ disjoint paths between X and Y. Besides, there may exist one more disjoint path that passes through the embedded $IK(d,m)$ formed by $S_{m-1} + S_{m-2} + \cdots + S_i$. These b_m paths are pictorially expressed in Fig. 5, where $\alpha \notin \{0, b_m - 1\}, \beta \notin \{0, b_m - 1\}$, and $\alpha < \beta$ are assumed. In Fig. 5, each thin line represents a flipping (or jumping flipping) link and each bold line represents a subpath obtained by Vecchia and Sanges' algorithm. According to Lemmas 4.1 and 4.2, these b_m paths are mutually disjoint (exclusive of X and Y) and they can be determined in $\mathrm{O}(b_m \times D_m) \leqslant \mathrm{O}(d \times D_m)$ time.

We also note that the path passing the embedded $IK(d,m)$ exists only if the two nodes $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\alpha^m$ and $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^m$ belong to the embedded $IK(d,m)$ (refer to Fig. 5). According to Observation 4.1, $p(b_{t-1}b_{t-2}\cdots b_{m+1}b_m\alpha^m,$ $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^m)$, which is constructed within $b_{t-1}b_{t-2}\cdots b_{m+1}b_m \cdot K(d,m)$, can be expressed as follows:

$$
\begin{aligned}
& b_{t-1}b_{t-2}\cdots b_{m+1}b_m\alpha^m \to \cdots \to b_{t-1}b_{t-2}\cdots b_{m+1}b_m\alpha\beta^{m-1},\\
\to\ & b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta\alpha^{m-1} \to \cdots \to b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta\alpha\beta^{m-2},\\
\to\ & b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^2\alpha^{m-2} \to \cdots \to b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^2\alpha\beta^{m-3},\\
\to\ & \qquad\qquad \vdots\\
\to\ & b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^{m-2}\alpha^2 \to b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^{m-2}\alpha\beta,\\
\to\ & b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^{m-1}\alpha \to b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^m.
\end{aligned}
$$

Each node in the subpath from $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\alpha^m$ to $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\alpha\beta^{m-1}$ has leading digits $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\alpha$, each node in the subpath from $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta\alpha^{m-1}$ to $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta\alpha\beta^{m-2}$ has leading digits $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta\alpha$, and so on. That is, every node in $p(b_{t-1}b_{t-2}\cdots b_{m+1}b_m\alpha^m,$ $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^m)$ precedes $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^m$ lexicographically if $\alpha < \beta$, and precedes $b_{t-1}b_{t-2}\cdots b_{m+1}b_m\alpha^m$ lexicographically if $\alpha > \beta$. Hence, $p(b_{t-1}b_{t-2}\cdots b_{m+1}b_m\alpha^m,\ b_{t-1}b_{t-2}\cdots b_{m+1}b_m\beta^m)$ is entirely contained in the embedded $IK(d,m)$, provided the two end nodes are contained in the embedded $IK(d,m)$.

Then we consider the situation of $m > n$. With input $(b_m, b_{m-1}, \ldots, b_i, *)$ and α, Algorithm *Stage_Grouping* is first executed to produce $m_0, m_1, \ldots, m_k$ so that $B_0 = S_m + S_{m-1} + \cdots + S_{m_0}$, $B_1 = S_{m_0} + S_{m_0-1} + \cdots + S_{m_1}, \ldots, B_k = S_{m_{k-1}} + S_{m_{k-1}-1}$

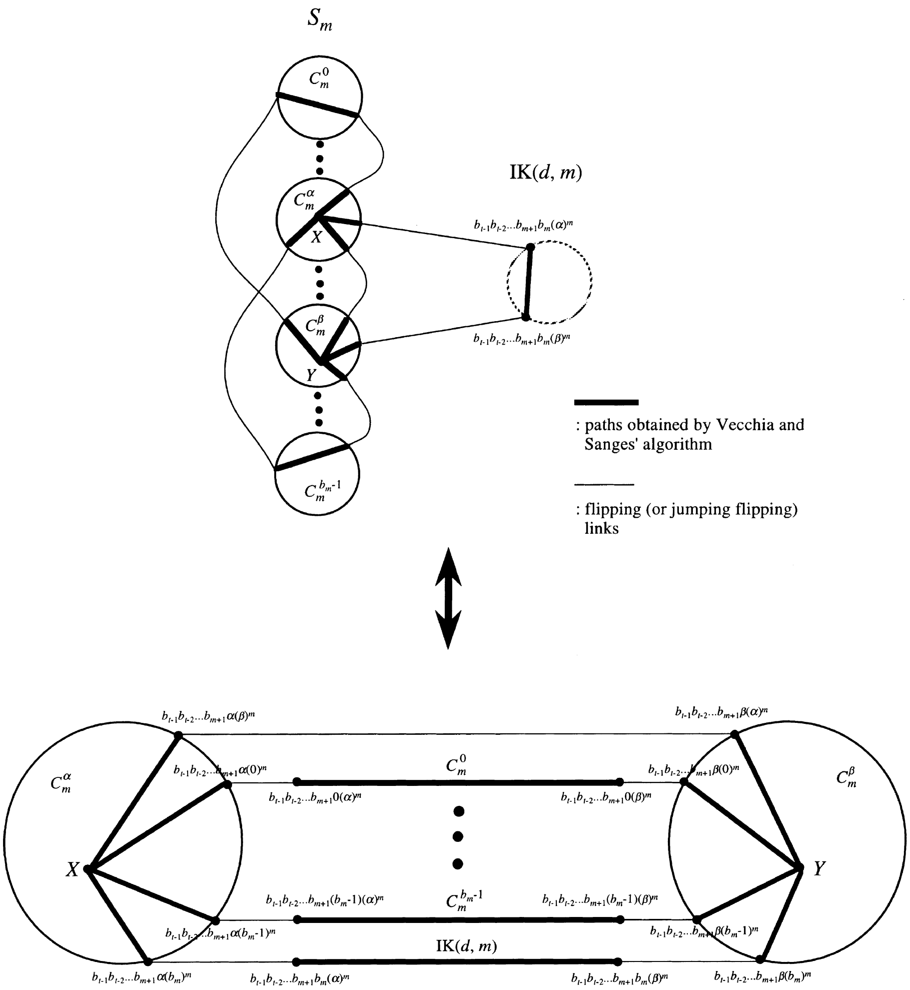


Fig. 5. b_m Disjoint paths between X and Y, where $\alpha \notin \{0, b_m - 1\}$, $\beta \notin \{0, b_m - 1\}$, and $\alpha < \beta$ are assumed.

$+ \cdots + S_{m_k}$, where $k \geqslant 0$ and $m_k = i$. Suppose $Y \in C_n^\beta$ belongs to B_l, where $0 \leqslant l \leqslant k$. If $Y \in S_{m_l}$, we consider $Y \in B_l$ but $\notin B_{l+1}$. Two cases are discussed below.

Case 1 ($l=0$). We have $m > n \geqslant m_0$. Let $\kappa = \min\{b_m, b_{m-1}, \ldots, b_n\}$ if $\min\{b_m, b_{m-1}, \ldots, b_n\} = b_m$ or b_n and $\kappa = \min\{b_m, b_{m-1}, \ldots, b_n\} + 1$ else. There are κ disjoint paths between X and Y, denoted by $P_0, P_1, \ldots, P_{\kappa-1}$, where P_r passes $C_m^\alpha, C_m^r, C_{m-1}^r, \ldots, C_n^r, C_n^\beta$ in sequence for all $0 \leqslant r \leqslant \kappa - 1$ ($C_m^\alpha = C_m^r$ as $\alpha = r$, and $C_n^r = C_n^\beta$ as $r = \beta$). Besides, there is an additional disjoint path, denoted by P_κ, if $\kappa = b_n < b_m$ and there is a jumping flipping link, say $J_{u,v}^{b_n}$, under S_n, where $m > u > n > v$. It is still possible that P_κ contains other jumping flipping links

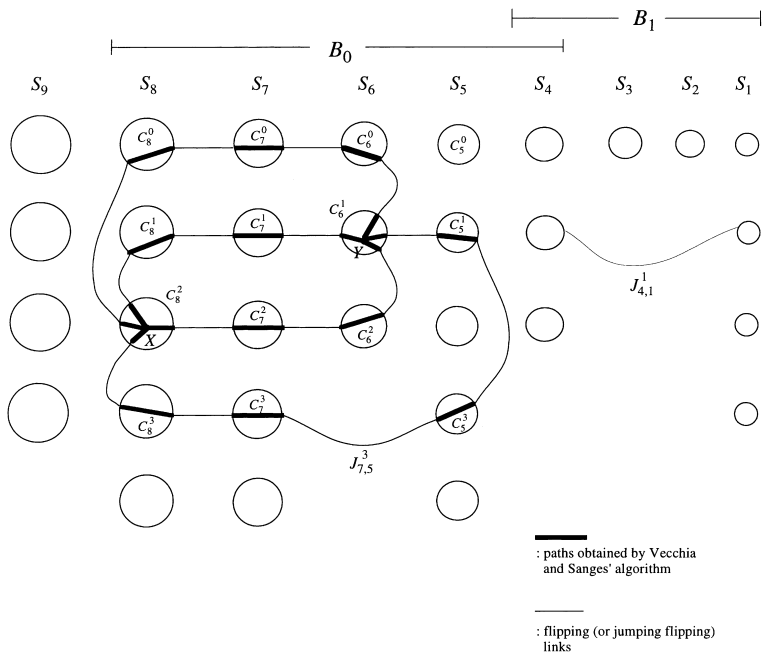


Fig. 6. Disjoint paths between X and Y for Case 1.

between S_m and S_u. Without loss of generality, we assume P_κ contains only $J_{u,v}^{b_n}$, and so P_κ passes $C_m^\alpha, C_m^\kappa, C_{m-1}^\kappa, \ldots, C_u^\kappa, C_v^\kappa, C_v^\beta, C_{v+1}^\beta, \ldots, C_n^\beta$, in sequence. According to Lemmas 4.1 and 4.2, these P_κ paths are mutually disjoint (exclusive of X and Y) and they can be determined in $\mathrm{O}(\kappa \times D_m) \leqslant \mathrm{O}(d \times D_m)$ time.

For example, let us consider $X \in C_6^2$ and $Y \in C_5^1$ in an $IK(5,7)$ with coefficient vector (4, 2, 4, 3, 1, 4, $*$) (refer to Fig. 6). The execution of Algorithm *Stage_Grouping* produces $B_0 = S_6 + S_5 + S_4 + S_3$ and $B_1 = S_3 + S_2 + S_1$. We have $\kappa = \min\{4, 2\} = 2$. Since $\kappa = b_5 < b_6$ and one jumping flipping link goes under S_5, there are $\kappa + 1 = 3$ disjoint paths between X and Y. These three paths are shown in Fig. 6, where each thin line represents a flipping (or jumping flipping) link and each bold line represents a subpath obtained by Vecchia and Sanges' algorithm.

Case 2 ($0 < l \leqslant k$). We have $m_{l-1} > n \geqslant m_l$. Let $(\alpha >)x_1 > x_2 > \cdots > x_c$ be defined as in Algorithm *Stage_Grouping*, where $c = k$ or $k - 1$. Recall the discussion in the last paragraph of Section 3. There is at least one jumping flipping link in B_l unless $l = c + 1$. We first assume $l < c + 1$ and let $J_{y_l,z_l}^{x_l}$ represent the leftmost and upmost jumping flipping link in B_l. Three subcases have to be discussed below.

Subcase 1 ($m_{l-1} > n \geqslant y_l$). There are b_n disjoint paths between X and Y, denoted by $P_0, P_1, \ldots, P_{b_n-1}$, where P_r passes $C_m^\alpha, C_m^r, C_{m-1}^r, \ldots, C_n^r, C_n^\beta$ in sequence for all

$0 \leqslant r \leqslant b_n - 2$ and P_{b_n-1} passes $C_m^{\alpha}, C_{m-1}^{\alpha}, \ldots, C_{m_0}^{\alpha}$, $C_{m_0}^{x_1}, C_{m_0-1}^{x_1}, \ldots, C_{m_1}^{x_1}, C_{m_1}^{x_2}, C_{m_1-1}^{x_2}, \ldots,$ $C_{m_2}^{x_2}, C_{m_2}^{x_3}, \ldots, C_{m_{l-1}}^{x_{l-1}}, C_{m_{l-1}}^{b_n-1}, C_{m_{l-1}-1}^{b_n-1}, \ldots, C_n^{b_n-1}, C_n^{\beta}$ in sequence.

For example, refer to Fig. 7, where an $IK(5,7)$ with coefficient vector (4, 2, 4, 3, 1, 4, $*$) is shown. Suppose $X \in C_6^3$ and $Y \in C_3^1$. The execution of Algorithm *Stage_Grouping* produces $B_0 = S_6, B_1 = S_6 + S_5 + S_4 + S_3$, and $B_2 = S_3 + S_2 + S_1$. There are $b_3 = 3$ disjoint paths between X and Y as shown in Fig. 7(a).

Subcase 2 ($y_l > n \geqslant m_l$ and $b_n = x_l$). There are $b_n + 1$ disjoint paths between X and Y. Since $b_n = x_l = \min\{b_{m_{l-1}}, b_{m_{l-1}-1}, \ldots, b_{m_l}\}$ (by Lemma 3.1), $b_{m_{l-1}} > x_{l-1} > x_l$, and $b_{m_l} > x_l$, there exists a jumping flipping link under S_n, say $J_{u,v}^{b_n}$, where $m_{l-1} \geqslant u > v \geqslant m_l$. We use $P_0, P_1, \ldots, P_{b_n}$ to denote the $b_n + 1$ disjoint paths, where P_r passes $C_m^{\alpha}, C_m^{r}, C_{m-1}^{r}, \ldots, C_n^{r}, C_n^{\beta}$ in sequence for all $0 \leqslant r \leqslant b_n - 1$ and P_{b_n} passes $C_m^{\alpha}, C_{m-1}^{\alpha}, \ldots,$ $C_{m_0}^{\alpha}$, $C_{m_0}^{x_1}, C_{m_0-1}^{x_1}, \ldots, C_{m_1}^{x_1}$, $C_{m_1}^{x_2}, C_{m_1-1}^{x_2}, \ldots, C_{m_2}^{x_2}$, $C_{m_2}^{x_3}, \ldots, C_{m_{l-1}}^{x_{l-1}}$, $C_{m_{l-1}}^{x_l}$, $C_{m_{l-1}-1}^{x_l}$, $C_{m_{l-1}-2}^{x_l}, \ldots, C_u^{x_l}$, $C_v^{x_l}$, $C_v^{\beta}, C_{v+1}^{\beta}, \ldots, C_n^{\beta}$ in sequence.

For example, suppose $X \in C_6^3$ and $Y \in C_2^0$ belong to the same $IK(5,7)$. There are $b_2 + 1 = 2$ disjoint paths between X and Y as shown in Fig. 7(b).

Subcase 3 ($y_l > n \geqslant m_l$ and $b_n > x_l$). Actually we have $z_l \geqslant n \geqslant m_l$ and $b_n > x_l$ because $b_{y_l-1} = b_{y_l-2} = \cdots = b_{z_l+1} = x_l$ can be assured by Theorem 2.1. There are $x_l + 1$ disjoint paths between X and Y, denoted by $P_0, P_1, \ldots, P_{x_l}$, where P_r passes

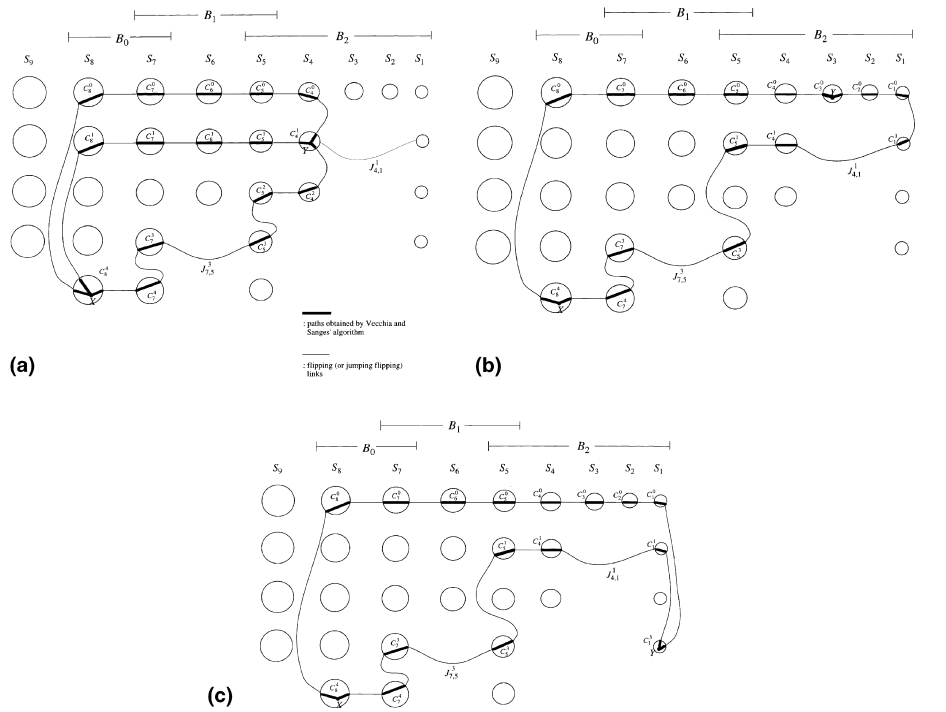


Fig. 7. Disjoint paths between X and Y for Case 2. (a) $m_{l-1} > n \geqslant y_l$; (b) $y_l > n \geqslant m_l$ and $b_n = x_l - 1$; (c) $y_l > n \geqslant m_l$ and $b_n \geqslant x_l$.

$C_m^\alpha, C_m^r, C_{m-1}^r, \ldots, C_n^r, C_n^\beta$ in sequence for all $0 \leqslant r \leqslant x_l - 1$ and P_{x_l} passes $C_m^\alpha, C_{m-1}^\alpha, \ldots, C_{m_0}{}^\alpha,\ C_{m_0}^{x_1}, C_{m_0-1}^{x_1}, \ldots, C_{m_1}^{x_1},\ C_{m_1}^{x_2}, C_{m_1-1}{}^{x_2}, \ldots, C_{m_2}^{x_2},\ C_{m_2}^{x_3}, \ldots, C_{m_{l-1}}^{x_{l-1}},\ C_{m_{l-1}}^{x_l}, C_{m_{l-1}-1}^{x_l}, C_{m_{l-1}-2}^{x_l}, \ldots, C_{y_l}^{x_l},\ C_{z_l}^{x_l}, \ldots, C_n^{x_l}, C_n^\beta$ in sequence.

For example, suppose $X \in C_6^3$ and $Y \in C_1^2$ belong to same $IK(5,7)$. There are $x_l + 1 = 2$ disjoint paths between X and Y as shown in Fig. 7(c).

On the other hand, if $l = c + 1$, we have $b_{m_c} > (x_c \geqslant) b_{m_{c-1}} \geqslant \cdots \geqslant b_n \geqslant \cdots \geqslant b_{m_{c+1}} (m_{c+1} = i)$. There are b_n disjoint paths between X and Y whose construction is similar to Subcase 1.

According to Lemmas 4.1 and 4.2, the paths obtained for Case 2 are mutually disjoint (exclusive of X and Y), and they can be determined in $\mathrm{O}(\max\{b_n + 1, x_l + 1\} \times D_m) \leqslant \mathrm{O}(d \times D_m)$ time.

The following theorem holds as a consequence of our discussion above.

Theorem 4.1. *Suppose $X \in C_m^\alpha$ and $Y \in C_n^\beta$ belong to an $IK(d,t)$ with coefficient vector $(b_{t-1}, b_{t-2}, \ldots, b_i, *)$, where $i \leqslant n \leqslant m \leqslant t-1,\ 0 \leqslant \alpha \leqslant b_m - 1$, and $0 \leqslant \beta \leqslant b_n - 1$. Then the disjoint paths between X and Y can be determined in $\mathrm{O}(d \times D_m)$ time.*

Let $len(X,Y)$ be the maximal length of the disjoint paths between X and Y. In the following we show that $len(X,Y)$ has an upper bound of $2^{m+1} + 2^m - 1$.

Theorem 4.2. *Suppose $X \in C_m^\alpha$ and $Y \in C_n^\beta$ belong to an $IK(d,t)$ with coefficient vector $(b_{t-1}, b_{t-2}, \ldots, b_i, *)$, where $i \leqslant n \leqslant m \leqslant t-1,\ 0 \leqslant \alpha \leqslant b_m - 1$, and $0 \leqslant \beta \leqslant b_n - 1$. Then $len(X,Y) \leqslant 2^{m+1} + 2^m - 1$.*

Proof. If $m = n$, there are at most b_m disjoint paths between X and Y. By the aid of Lemma 4.1, the path passing the embedded $IK(d,m)$ has length at most $D_m + 1 + D_m + 1 + D_m = 2^{m+1} + 2^m - 1$, and the others each have length at most $D_m + 1 + D_m + 1 + D_m = 2^{m+1} + 2^m - 1$. Hence, $len(X,Y) \leqslant 2^{m+1} + 2^m - 1$. In the rest of the proof, we assume $m > n$ and $Y \in B_l$ for some $0 \leqslant l \leqslant k$.

If $l = 0$, there are at most $\kappa + 1$ disjoint paths $P_0, P_1, \ldots, P_\kappa$ between X and Y. The path P_κ, if it exists, has length at most $D_m + 1 + D_m + 1 + D_{m-1} + 1 + \cdots + D_{u+1} + 1 + D_u + 1 + D_v + 1 + D_v + 1 + D_{v+1} + 1 + \cdots + D_n < 2^{m+1} + 2^m - 1$, and the others each have length at most $D_m + 1 + D_m + 1 + D_{m-1} + 1 + D_{m-2} + 1 + \cdots + D_{n+1} + 1 + D_n + 1 + D_n = 2^{m+1} + 2^m - 1$. Hence, $len(X,Y) \leqslant 2^{m+1} + 2^m - 1$.

If $0 < l \leqslant k$, there are b_n or $b_n + 1$ or $x_l + 1$ disjoint paths between X and Y, according to three subcases. Similarly, $len(X,Y) \leqslant 2^{m+1} + 2^m - 1$. □

5. Experiments and results

In this section the effectiveness of the disjoint paths is verified by extensive experiments. The following two algorithms were implemented for the need of our experiments.

- Su, Chen, and Duh's algorithm that computes the diameter of an $IK(d,t)$ [21].
- Su, Chen, and Duh's algorithm that computes the shortest path between arbitrary two nodes of an $IK(d,t)$ [22].

Remarks. Adopting the prune-and-search technique [15], the algorithm of Su et al. [21] can compute the diameter of an $IK(d,t)$ and the farthest pair of nodes in $O(t)$ time. Although the diameter of an $IK(d,t)$ can be computed by Su, Chen, and Duh's algorithm, no formula is available for computing it.

First we compare $len(X,Y)$ with $dis(X,Y)$, where $dis(X,Y)$ is the distance between X and Y. Fig. 8 shows the average ratios of $len(X,Y)$ to $dis(X,Y)$ for $IK(d,t)$s with $4 \leqslant d \leqslant 6$ and $2 \leqslant t \leqslant 10$. The values of $dis(X,Y)$ were computed by the algorithm of Su et al. [22]. For each pair of d and t, 10^6 random instances were run and their average ratio was computed. The averages got stable after running as many as 10^6 instances. A randomly generated coefficient vector combined with two nodes, also randomly generated, of an $IK(d,t)$ forms an instance. Experimental results showed that $len(X,Y)$ is not greater than twice $dis(X,Y)$ in average.

Then we compare $len(X,Y)$ with the diameter. Fig. 9 shows the average ratios of $len(X,Y)$ to the diameter for $IK(d,t)$s with $4 \leqslant d \leqslant 6$ and $2 \leqslant t \leqslant 10$. The diameters were computed by the algorithm of Su et al. [21]. Like Fig. 8, the average ratio for 10^6 random instances was taken for each pair of d and t. Experimental results showed that $len(X,Y)$ is smaller than the diameter in average both as $d=4$ and $t \geqslant 2$ and as $d=5$ and $t \geqslant 5$. Besides, $len(X,Y)$ tends to the diameter as $d=6$ and t increases.

When X and Y were selected to be the farthest pair of nodes (i.e., $dis(X,Y)$ is equal to the diameter of the $IK(d,t)$), the average ratios of $len(X,Y)$ to the diameter were shown in Fig. 10. The farthest pair of nodes can be determined by the algorithm of Su et al. [21]. The average ratios each were obtained by running 10^5 random instances because the averages got stable after running as many as 10^5 instances.

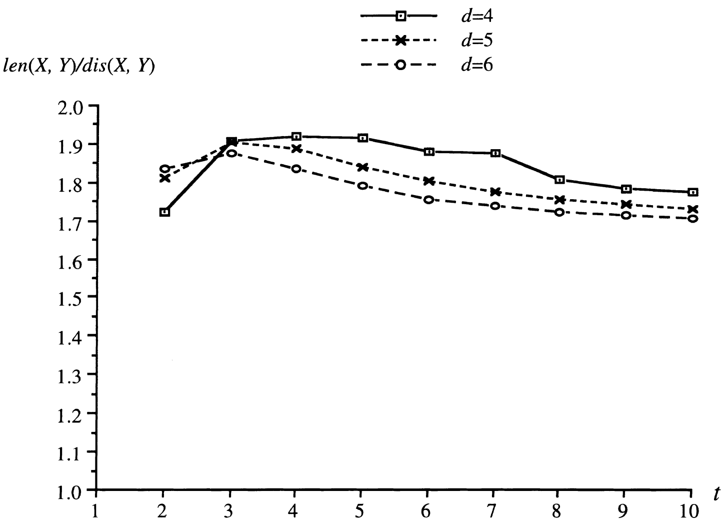


Fig. 8. Average ratios of $len(X,Y)$ to $dis(X,Y)$ for $IK(d,t)$.

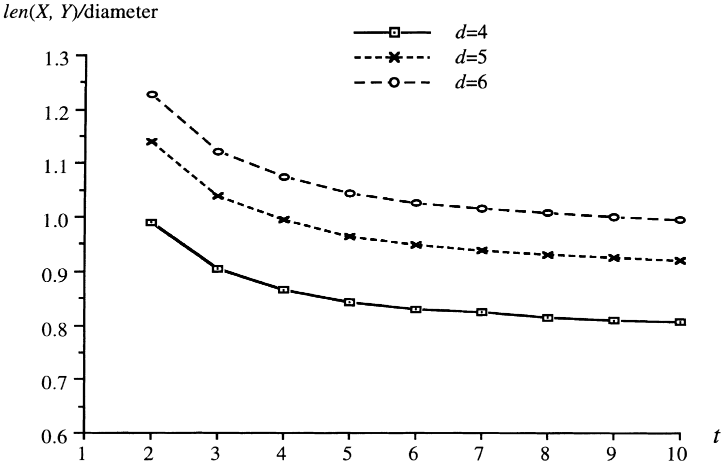


Fig. 9. Average ratios of $len(X, Y)$ to diameter for $IK(d, t)$.

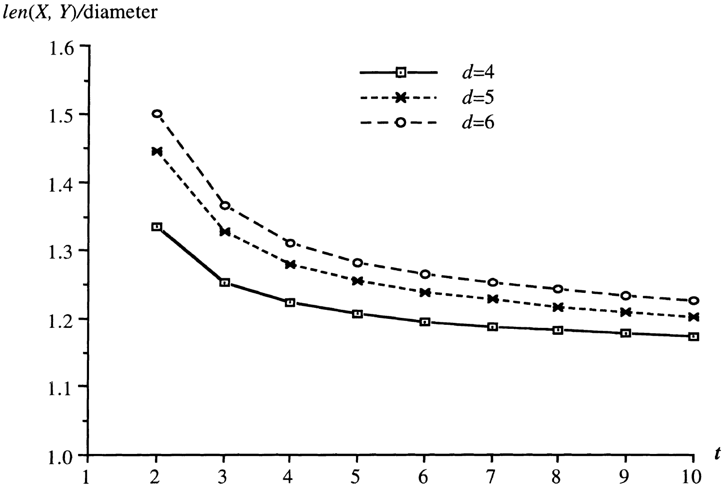


Fig. 10. Average ratios of $len(X, Y)$ to $dis(X, Y)$ for $IK(d, t)$, where X and Y are the farthest pair nodes.

Experimental results showed that most of the averages fall in the range of 1.2–1.3. The maximal average ratio does not exceed 1.5.

It is worth mentioning that for the experiments of Fig. 10, no instance has $len(X, Y)$ exceeding twice the diameter. The distributions of the 10^5 ratios obtained for an $IK(4, 8)$, an $IK(5, 8)$, and an $IK(6, 8)$ were shown in Figs. 11(a)–(c), respectively. For example, for the $IK(4, 8)$ there are 54,566 ratios (about 54%) fall in the range of 1.0–1.1 and there are 4044 ratios (about 4%) fall in the range of 1.1–1.2.

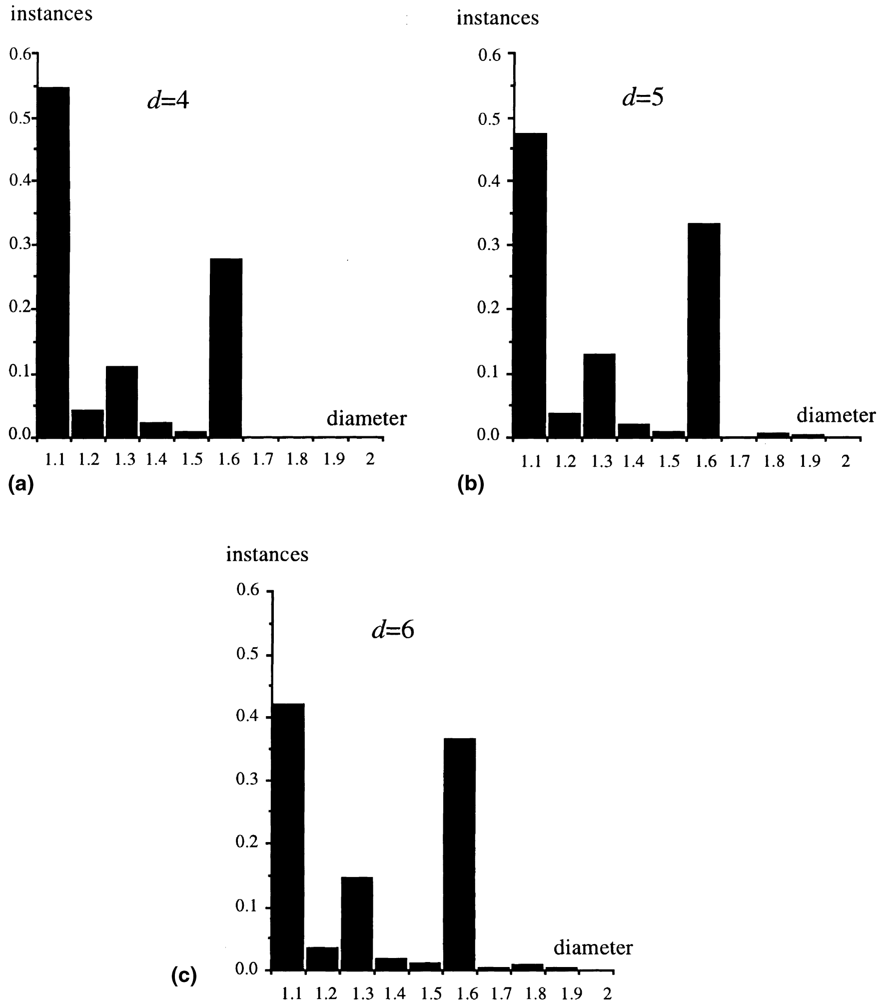


Fig. 11. Distribution of ratios over 10^5 instances for $t = 8$. (a) $d = 4$; (b) $d = 5$; (c) $d = 6$.

6. Concluding remarks

In this paper, we have constructed a maximal set of disjoint paths between X and Y, which are arbitrary two nodes of an $IK(d, t)$. The construction time is bounded by $O(d \times D_t)$. We have shown that the disjoint paths have maximal length not greater than $2^{m+1} + 2^m - 1$, where $X \in C_m^\alpha$, $Y \in C_n^\beta$, and $i \leqslant n \leqslant m \leqslant t - 1$ are assumed. The effectiveness of the disjoint paths was further verified by extensive experiments. Experimental results showed that the disjoint paths have expected maximal length not greater than twice their distance and about equal to the diameter. Besides, when X and Y are the farthest pair, the disjoint paths have maximal length not greater than twice the diameter and expected maximal length not greater than 1.5 times the diameter.

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