General masters in parallel condensation of eigenvalue problems

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Abstract

In the dynamic analysis of structures using finite element methods very often prohibitively many degrees of freedom are required to model the structure sufficiently accurate. Condensation methods are often used to reduce the number of unknowns to manageable size. Substructuring and choosing the master variables as the degrees of freedom on the interfaces of the substructures yields data structures which are well suited to be implemented on parallel computers. In this paper we discuss the use of additional non nodal masters in substructuring. The data structure is preserved such that the condensed problem can be determined substructurewise.

Keywords: eigenvalue problem, condensation, parallel method

1 Introduction

In the analysis of the dynamic response of a linear structure using finite element methods very often prohibitively many degrees of freedom are needed to model the behaviour of the system sufficiently accurate. In this situation condensation methods are employed to economize the computation of a selected group of eigenvalues and eigenvectors. These methods which were introduced by Guyan [2] and Irons [4] choose from the degrees of freedom a small number of master variables which appear to be representative. In a Gaussian elimination type procedure the rest of the variables (termed slaves) is eliminated leaving a much smaller problem for the master variables only.

Partitioning the structure under consideration into substructures and choosing the degrees of freedom on the interfaces of the substructures as

masters leads to data structures and formulae which are well suited to be implemented on distributed memory MIMD parallel computers. Taking advantage of these properties Rothe and the second author obtained a fully parallel condensation method for generalized eigenvalue problems (cf. [11])

Usually the approximation properties of condensation methods are not very good and only few eigenvalues at the lower end of the spectrum are obtained with sufficient quality. Several attempts have been reported in the literature to improve the accuracy of the eigenvalue and eigenvector approximations.

In a recent paper [7] the authors generalized Guyan's method to enable general degrees of freedom to be master variables. This allows a priori information about the desired eigenvectors to be introduced into the condensation process, and the method can be enhanced considerably. Examples demonstrating this improvement using modal masters are contained in [7], [12] and [14].

In this paper we carry over the parallelization concept from [10] to condensation in the presence of general masters. If the structure under consideration is partitioned into substructures, if all degrees of freedom on the interfaces are chosen as masters, and if we additionally consider general masters the supports of which are contained in a single substructure each then the condensed eigenvalue problem can be determined substructurewise, and therefore in parallel.

The paper is organized as follows: In Section 2 we briefly sketch nodal condensation and its parallelization taking advantage of substructuring. Section 3 reviews the results from [7] demonstrating that non nodal condensation can be performed without explicit access to the slave part of general coordinates. Section 4 discusses the parallelization of non nodal condensation using substructuring. Section 5 demonstrates the gain of accuracy using known exact eigenvectors of a similar problem as general masters.

2 Nodal Condensation

The following matrix eigenvalue problem results from the finite element analysis of a structure undergoing free harmonic oscillations

$$Kx = \lambda Mx. \tag{1}$$

Here the stiffness matrix $K \in \mathbb{R}^{(n,n)}$ and the mass matrix $M \in \mathbb{R}^{(n,n)}$ are real symmetric and positive definite, x is the vector of modal displacements, and $\lambda = \omega^2$ where ω denotes the natural frequencies of the system. To describe the structure accurately enough the mathematical model (1) often requires a very large number n of degrees of freedom, and reduction of the

number of unknowns is required before proceeding to the calculation of the required frequencies and modal shapes.

To reduce the number of the degrees of freedom to manageable size the vector x is partitioned into a set of variables x_s (termed slaves) which are to be eliminated and the remaining variables x_m (termed masters) which are to be retained. After reordering the unknowns and equations system (1) obtains the following block form:

$$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm} & K_{ss} \end{bmatrix} \left\{ \begin{array}{c} x_m \\ x_s \end{array} \right\} = \lambda \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \left\{ \begin{array}{c} x_m \\ x_s \end{array} \right\} \tag{2}$$

Solving the second row of equation (2) for x_s one obtains

$$x_s(\lambda) = -(K_{ss} - \lambda M_{ss})^{-1}(K_{sm} - \lambda M_{sm})x_m =: S(\lambda)x_m. \tag{3}$$

Thus, if $\tilde{x}_m$ is the master portion of an eigenvector $\tilde{x}$ corresponding to the eigenvalue $\tilde{\lambda}$, and if $\tilde{\lambda}$ is not in the spectrum of the slave eigenvalue problem

$$K_{ss}\phi = \lambda M_{ss}\phi \tag{4}$$

then $S(\tilde{\lambda})\tilde{x}_m$ is the slave part of $\tilde{x}$. Hence, if we are interested in eigenvalues close to $\hat{\lambda}$, it is reasonable to project the eigenvalue problem (2) to the linear space

$$\left\{ \begin{bmatrix} I \\ S(\hat{\lambda}) \end{bmatrix} x_m \ : \ x_m \in \mathbb{R}^m \right\},$$

i.e. to consider the projected eigenvalue problem

$$P^t K P x_m = \lambda P^t M P x_m, \quad P := \begin{bmatrix} I \\ S(\hat{\lambda}) \end{bmatrix}. \tag{5}$$

To keep the paper at reasonable length we only consider the statically condensed eigenproblem which was introduced by Guyan [2] and Irons [4] and which is obtained for $\hat{\lambda} = 0$:

$$K_0 x_m = \lambda M_0 x_m \tag{6}$$

where

$$\begin{aligned} K_0 &:= K_{mm} - K_{ms}K_{ss}^{-1}K_{sm}, \\ M_0 &:= M_{mm} - K_{ms}K_{ss}^{-1}M_{sm} - M_{ms}K_{ss}^{-1}K_{sm} + K_{ms}K_{ss}^{-1}M_{ss}K_{ss}^{-1}K_{sm}. \end{aligned}$$

Notice, however, that the method to be developed applies to dynamic ($\hat{\lambda} \neq 0$ fixed), spectral (cf. [8]), and exact condensation ($\hat{\lambda}$ variable) as well.

Usually in the literature approximations to some of the smallest eigenvalues of (1) and to the master portions x_m of the corresponding eigenvectors

are obtained from the statically condensed problem (6), and the slave portions x_s are calculated by equation (3). Observe however, that in many cases only very few eigenmodes are derived from (6) with sufficient accuracy. Several attempts have been made to enhance their quality most of them being very time consuming since an iterative process is involved (cf. [5], [6], [9]). In [3] we improved eigenvalue and eigenvector approximations substantially by condensation-projection. A different approach which takes advantage of the Rayleigh functional of the exactly condensed problem was introduced in [13], [10].

The condensation can be performed completely in parallel if the slave variables can be chosen such that the matrices K_{ss} and M_{ss} are block diagonal (cf. [11]). Suppose that r substructures are considered and that they connect to each other through the master variables on the interfaces only. If the slave variables are numbered appropriately, then the stiffness matrix is given by

$$
K = \begin{bmatrix}
K_{mm} & K_{ms1} & K_{ms2} & \ldots & K_{msr} \\
K_{sm1} & K_{ss1} & O & \ldots & O \\
K_{sm2} & O & K_{ss2} & \ldots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
K_{smr} & O & O & \ldots & K_{ssr}
\end{bmatrix},
$$

and the mass matrix M has the same block form.

Taking advantage of the blockstructure of K and M the reduced matrices K_0 and M_0 can be calculated substructurewise, and hence, completely in parallel. Obviously,

$$
K_0 = K_{mm} - \sum_{j=1}^{r} K_{mmj} := K_{mm} - \sum_{j=1}^{r} K_{msj} K_{ssj}^{-1} K_{smj}
$$

and

$$
M_0 = M_{mm} - \sum_{j=1}^{r} M_{mmj},
$$

where

$$
M_{mmj} := K_{msj} K_{ssj}^{-1} M_{smj} + M_{msj} K_{ssj}^{-1} K_{smj} - K_{msj} K_{ssj}^{-1} M_{ssj} K_{ssj}^{-1} K_{smj}.
$$

3   Non nodal masters in condensation

We already mentioned that static condensation usually allows accurate approximations only at the lower end of the spectrum. The approximation properties can be enhanced considerably if we introduce a priori knowledge about the eigenvectors via general masters. For instance in component

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mode synthesis [1] vibration modes of subsystems are already known, and in eigenvalue reanalysis [15] eigenvectors of similar problems have been obtained in previous calculations. Either of them may serve as general, non nodal masters.

These general masters could be incorporated in the condensation process in the following straightforward manner.

Let $\{z_1, \ldots, z_m\}$ be a set of linearly independent mastervectors and $y_{m+1}, \ldots, y_n$ be a complementary basis of

$$\{z_1, \ldots, z_m\}^\perp := \{y \ : \ y^t V z_j = 0, \ j = 1, \ldots, m\}$$

where $V \in \mathbb{R}^{(n,n)}$ is a positive definite metric matrix.

We put $Z := (z_1, \ldots, z_m) \in \mathbb{R}^{(n,m)}$ and $Y := (y_{m+1}, \ldots, y_n) \in \mathbb{R}^{(n,n-m)}$. Then every vector $x \in \mathbb{R}^n$ can be written as $x = Z x_m + Y x_s$, $x_m \in \mathbb{R}^m$, $x_s \in \mathbb{R}^{n-m}$. Going with this representation into equation (1) and multiplying with the regular matrix $(Z, Y)^t$ from the left one obtains the eigenvalue problem

$$\begin{bmatrix} K_{zz} & K_{zy} \\ K_{yz} & K_{yy} \end{bmatrix} \left\{ \begin{array}{c} x_m \\ x_s \end{array} \right\} = \lambda \begin{bmatrix} M_{zz} & M_{zy} \\ M_{yz} & M_{zz} \end{bmatrix} \left\{ \begin{array}{c} x_m \\ x_s \end{array} \right\} \tag{7}$$

where

$$L_{zz} := Z^t L Z, \ L_{zy} := Z^t L Y, \ L_{yz} := L_{zy}^t, \ L_{yy} := Y^t L Y, \ L \in \{K, M\}. \tag{8}$$

In the special case that $V = I$ is the identity and the columns of Z and Y consist of coordinate vectors corresponding to the master and slave variables, respectively, then equation (7) reproduces the splitting in (2).

The decomposition in (7) could serve as a basis for condensation using general masters. However, there is a strong practical objection to this naive approach: For large systems the small number of general masters $z_1, \ldots, z_m$ will usually be accessible whereas the (large number of) complementary vectors $y_{n-m}, \ldots, y_n$ are definitely not.

In [7] the authors presented a general method to perform condensation using only the master vectors $z_1, \ldots, z_m$. The following theorem contains a method for computing the condensed problems in the presence of general masters without using the complementary basis $y_{n-m}, \ldots, y_n$.

Theorem 1: *Let $Z \in \mathbb{R}^{(n,m)}$ have full rank m, and let $V \in \mathbb{R}^{(n,n)}$ be a symmetric and positive definite metric matrix. Then with $X := VZ$ the statically condensed problem with general masters $z_1, \ldots, z_m$ is given by*

$$P^t K P x_m = \lambda P^t M P x_m$$

where

$$P = K^{-1} X (X^t K^{-1} X)^{-1} X^t Z. \tag{9}$$

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$(X^t K^{-1} X)^{-1} X^t Z \in {\rm I\!R}^{(m,m)}$ is a nonsingular matrix. Hence the column space of P is spanned by the columns of $K^{-1} V Z$ as well, and with $V = M$ static condensation is nothing else but one step of simultaneous inverse iteration with initial vectors $z_1, \ldots, z_m$.

The following alternative characterization of the projection matrix P from [7] is the key to determining the condensed eigenvalue problem substructurewise in the presence of general masters.

Theorem 2: *Let $z_1, \ldots, z_m \in {\rm I\!R}^n$ be linearly independent, and let $V \in {\rm I\!R}^{(n,n)}$ be a symmetric and positive definite metric matrix. Then the statically condensed eigenvalue problem corresponding to problem (7) is given by*

$$P^t K P x_m = \lambda P^t M P x_m \tag{10}$$

where the matrix $P \in {\rm I\!R}^{(n,m)}$ can be calculated from

$$\begin{bmatrix} K & -VZ \\ -Z^t V & O \end{bmatrix} \begin{bmatrix} P \\ S \end{bmatrix} = \begin{bmatrix} O \\ -I_m \end{bmatrix}. \tag{11}$$

Moreover, if $Z^t V Z = I_m$ then the condensed stiffness matrix is given by

$$P^t K P = S. \tag{12}$$

4   General masters and substructuring

We consider the free vibrations of a structure which is decomposed into r substructures. Let the vibration problem be discretized (by finite elements or finite differences) in correspondence to the substructure decomposition, i.e. $k_{ij} = 0$ and $m_{ij} = 0$ whenever i and j denote indices of interior nodes of different substructures. We choose as nodal masters those degrees of freedom which are located on the boundaries of the substructures, and additionally we allow general masters. We assume that the supports of any of the general masters are contained in exactly one substructure each. Here we have in mind nodal interior masters, or modal masters, i.e. eigenvectors of the eigenvalue problem restricted to the substructure under consideration, or restrictions of global approximations of eigenvectors (for instance from reanalysis) to the substructures (cf. Section 5).

In this section we demonstrate how the condensed problem can be computed substructurewise. This is the basis of a fully parallel condensation algorithm in the presence of general masters.

We number the variables in the usual way where the coupling of the boundary masters (i.e. the nodal masters on the boundaries of the substructures) is given by K_{mm} and M_{mm}, and the interaction of the interior degrees of freedom of the j-th substructure and the boundary masters is given by $K_{smj} = K_{msj}^t$ and $M_{smj} = M_{msj}^t$. The general masters corresponding to the j-th substructure are collected in the matrix $Z_j \in \mathbb{R}^{(s_j, m_j)}$ where s_j denotes the number of interior degrees of freedom of the j-th substructure and m_j the number of general masters having their support in the j-th substructure.

Finally, we assume that the metric matrix V is block diagonal

$$V = \text{diag} \{I_m, V_1, \ldots, V_r\}.$$

Then the condensed eigenvalue problem is given by

$$P^t K P \xi = \lambda P^t M P \xi,$$

and by Theorem 2 the matrix

$$P =: (P^{(0)}, P^{(1)}, \ldots, P^{(r)}), \quad P^{(0)} \in \mathbb{R}^{(n,m)}, \quad P^{(j)} \in \mathbb{R}^{(n,m_j)}, \quad j = 1, \ldots, r,$$

can be calculated from the linear system

$$
\begin{bmatrix}
K_{mm} & K_{ms1} & \ldots & K_{msr} & -I_m & O & \ldots & O \\
K_{sm1} & K_{ss1} & \ldots & O & O & -V_1 Z_1 & \ldots & O \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
K_{smr} & O & \ldots & K_{ssr} & O & O & \ldots & -V_r Z_r \\
-I_m & O & \ldots & O & O & O & \ldots & O \\
O & -Z_1^t V_1 & \ldots & O & O & O & \ldots & O \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & -Z_r^t V_r & O & O & \ldots & O
\end{bmatrix}
$$

$$
\cdot
\begin{bmatrix}
& & & \\
P^{(0)} & P^{(1)} & \ldots & P^{(r)} \\
& & & \\
& & & \\
S^{(0)} & S^{(1)} & \ldots & S^{(r)}
\end{bmatrix}
=
\begin{bmatrix}
O & O & \ldots & O \\
O & O & \ldots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & O \\
-I_m & O & \ldots & O \\
O & -I_{m_1} & \ldots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & -I_{m_r}
\end{bmatrix}
$$

where $P^{(0)}$ contains that part of the matrix P which belongs to the boundary masters, and $P^{(j)}$ denotes the part of the masters belonging to the general masters of the j–th substructure.

Partitioning $P^{(0)}$ and $S^{(0)}$ as

$$P^{(0)} := \begin{bmatrix} P_0^{(0)} \\ P_1^{(0)} \\ \vdots \\ P_r^{(0)} \end{bmatrix} \quad \text{and} \quad S^{(0)} := \begin{bmatrix} S_0^{(0)} \\ S_1^{(0)} \\ \vdots \\ S_r^{(0)} \end{bmatrix}$$

the first (block-) column yields the linear system

$$K_{mm} P_0^{(0)} + \sum_{j=1}^{r} K_{msj} P_j^{(0)} - S_0^{(0)} \;=\; O \tag{13}$$

$$K_{smj} P_0^{(0)} + K_{ssj} P_j^{(0)} - V_j Z_j S_j^{(0)} \;=\; O, \; j = 1, \ldots, r \tag{14}$$

$$-P_0^{(0)} \;=\; -I_m \tag{15}$$

$$-Z_j^t V j P_j^{(0)} \;=\; O, \; j = 1, \ldots, r \tag{16}$$

From equation (15) one obtains $P_0^{(0)} = I_m$, and therefore (14) and (16) can be rewritten to r systems of equations

$$\begin{bmatrix} K_{ssj} & -V_j Z_j \\ -Z_j^t V_j & O \end{bmatrix} \begin{bmatrix} P_j^{(0)} \\ S_j^{(0)} \end{bmatrix} = \begin{bmatrix} -K_{smj} \\ O \end{bmatrix}, \; j = 1, \ldots, r \tag{17}$$

which are independent of each other.

Since the matrices K_{ssj} are positive definite, and since the matrices Z_j have full rank, the coefficient matrix of system (17) is nonsingular, and the matrices $P_j^{(0)}$ and $S_j^{(0)}$ can be computed substructurewise.

If the general masters are orthonormal with respect to V, i.e. if $Z_j^t V_j Z_j = I_{m_j}$, $j = 1, \ldots, r$, then the condensed stiffness matrix equals S, and we determine from equation (13) that

$$S_0^{(0)} = K_{mm} + \sum_{j=1}^{r} K_{msj} P_j^{(0)}.$$

For $k = 1, \ldots, r$ the matrices

$$P^{(k)} := \begin{bmatrix} P_0^{(k)} \\ P_1^{(k)} \\ \vdots \\ P_r^{(k)} \end{bmatrix} \quad \text{and} \quad S^{(0)} := \begin{bmatrix} S_0^{(k)} \\ S_1^{(k)} \\ \vdots \\ S_r^{(k)} \end{bmatrix}$$

satisfy the system of equations

$$K_{mm} P_0^{(k)} + \sum_{j=1}^{r} K_{msj} P_j^{(k)} - S_0^{(k)} \;=\; O \tag{18}$$

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$$K_{smj}P_0^{(k)} + K_{ssj}P_j^{(k)} - V_j Z_j S_j^{(k)} = O, \ j = 1, \ldots, r \tag{19}$$

$$-P_0^{(k)} = O \tag{20}$$

$$-Z_j^t V_j P_j^{(k)} = O, \ j = 1, \ldots, r, \ j \neq k \tag{21}$$

$$-Z_k^t V_k P_k^{(k)} = -I_{m_k} \tag{22}$$

From equation (20) one obtains $P_0^{(k)} = O$, and therefore for $j = 1, \ldots, r$, $j \neq k$ eqns. (19) and (21) decouple to r homogeneous linear systems

$$\begin{bmatrix} K_{ssj} & -V_j Z_j \\ -Z_j^t V_j & O \end{bmatrix} \begin{bmatrix} P_j^{(k)} \\ S_j^{(k)} \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix}, \ j = 1, \ldots, r, \ j \neq k, \tag{23}$$

from which we obtain $P_j^{(k)} = O$ and $S_j^{(k)} = O$ for $j = 1, \ldots, r$, $j \neq k$.

For $j = k$ we get

$$\begin{bmatrix} K_{ssk} & -V_k Z_k \\ -Z_k^t V_k & O \end{bmatrix} \begin{bmatrix} P_k^{(k)} \\ S_k^{(k)} \end{bmatrix} = \begin{bmatrix} O \\ -I_{m_k} \end{bmatrix}, \tag{24}$$

from which the matrices $P_k^{(k)}$ and $S_k^{(k)}$, too, can be determined substructurewise. Notice that the coefficient matrices in (17) and (24) are identical. Hence, a decomposition of the matrix in (17) can be reused to solve (24).

The matrix P has the following form:

$$P = \begin{bmatrix} I & O & \ldots & O \\ P_1^{(0)} & P_1^{(1)} & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ P_r^{(0)} & O & \ldots & P_r^{(r)} \end{bmatrix} =: \begin{bmatrix} I & O & \ldots & O \\ P_1 & Q_1 & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ P_r & O & \ldots & Q_r \end{bmatrix}.$$

Thus the reduced mass matrix $M_0 = P^t M P$ is given by

$$\begin{bmatrix} I & \ldots P_j^t \ldots \\ \vdots & \\ O & \mathrm{diag}(Q_j^t) \\ \vdots & \end{bmatrix} \begin{bmatrix} M_{mm} & \ldots M_{msj} \ldots \\ \vdots & \\ M_{smj} & \mathrm{diag}(M_{ssj}) \\ \vdots & \end{bmatrix} \begin{bmatrix} I & \ldots O \ldots \\ \vdots & \\ P_j & \mathrm{diag}(Q_j) \\ \vdots & \end{bmatrix} =$$

$$\begin{bmatrix} M_{mm} + \sum\limits_{j=1}^{r}(M_{msj}P_j + P_j^t M_{smj} + P_j^t M_{ssj}P_j) & \ldots M_{msj}Q_j + P_j^t M_{ssj}Q_j \ldots \\ \vdots & \vdots \\ Q_j^t M_{smj} + Q_j^t M_{ssj}P_j & \mathrm{diag}(Q_j^t M_{ssj}Q_j) \\ \vdots & \vdots \end{bmatrix}$$

If $Z_j^t V_j Z_j = I_{m_j}$, $j = 1, \ldots, r$, then by the last part of Theorem 2 the condensed stiffness matrix satisfies $K_0 := P^t K P = S$. Since K_0 is symmetric

the matrices $S_0^{(k)} = (S_k^{(0)})^t$ need not be computed from equation (18), and

$$K_0 = \begin{bmatrix} S_0^{(0)} & (S_1^{(0)})^t & (S_2^{(0)})^t & \ldots & (S_r^{(0)})^t \\ S_1^{(0)} & S_1^{(1)} & O & \ldots & O \\ S_2^{(0)} & O & S_2^{(2)} & \ldots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_r^{(0)} & O & O & \ldots & S_r^{(r)} \end{bmatrix}.$$

Otherwise $K_0 = P^t K P$ has to be determined in the same way as the condensed mass matrix M_0.

Details of the implementation of the algorithm on a distributed memory transputer system are given in [12].

5 A numerical example

The transversal vibrations of a tapered cantilever beam of length 1 with area of cross section $A_x := A_0(1 - 0.5x)^2$, $0 \leq x \leq 1$, are governed by the eigenvalue problem

$$\begin{aligned} ((1 - 0.5x)^4 y'')'' &= \lambda(1 - 0.5x)^2 y, \ 0 < x < 1, \\ y(0) = y'(0) &= y''(1) = y'''(1) = 0, \end{aligned}$$

where $\lambda = \omega^2 \rho A_0/(E I_0)$, A_0 and I_0 are the area of the cross section and the moment of inertia at $x = 0$, respectively, ρ is the mass per unit volume, E is the modulus of elasticity and ω denotes the natural circular frequencies of the beam.

We discretized the problem by finite elements with cubic hermite splines (beam elements). We divided the beam into 3 substructures of the same length and subdivided each substructure into 20 elements of the same length. Thus, problem (13) has dimension $n = 120$ and is condensed to dimension $m = 6$.

Additionally we introduced as general masters modes of the uniform cantilever beam

$$y^{(4)} = \lambda y, \ 0 < x < 1, \ y(0) = y'(0) = y''(1) = y'''(1) = 0$$

which are assumed to be known from previous calculations. Let v_j be the eigenvector corresponding to the j-smallest eigenvalue of the uniform cantilever beam problem discretized by finite elements with beam elements on a uniform grid with stepsize $1/60$.

Let $w_j := Mv_j$ where M denotes the mass matrix of the discretized tapered beam. We partition these vectors corresponding to the substructuring, i.e.

$$z_{j,1} := w_j(1:38), \ z_{j,2} := w_j(41:78), \ z_{j,3} := w_j(81:118).$$

Table 1 contains the 6 smallest eigenvalues in its first column, the relative errors of the approximate eigenvalues obtained by nodal condensation using 6 boundary masters in its second column, and the relative errors if we add 1, 2 or 3 general masters in each substructure obtained from the eigenmodes of the uniform beam as described above. As metric matrix we chose the identity matrix $V = I$.

j	λ_j	nodal c.	1 general m.	2 general m.	3 general m.
1	$2.139201E01$	$9.89E-04$	$1.23E-07$	$1.60E-11$	$4.63E-14$
2	$3.821092E02$	$1.02E-02$	$4.53E-04$	$3.76E-07$	$5.12E-10$
3	$2.359911E03$	$2.32E-02$	$7.24E-03$	$9.89E-05$	$4.24E-07$
4	$8.429599E03$	$3.46E-01$	$1.23E-02$	$2.54E-03$	$3.14E-05$
5	$2.231745E04$	$8.27E-01$	$5.82E-02$	$1.10E-02$	$8.31E-04$
6	$4.898665E04$	$1.58E+00$	$1.61E-01$	$3.40E-02$	$5.18E-03$

Tab. 1.

For comparison we give in Table 2 the relative errors which are obtained if we add in each substructure 3 uniformly distributed displacements as interior nodal masters and 3 modal masters (i.e. the eigenvectors corresponding to the 3 smallest eigenvalues of the clamped substructures), respectively. In the latter case we chose the mass matrices of the substructures as metric matrices V_j.

j	3 general masters	3 interior nodal m.	3 modal masters
1	$4.63E-14$	$4.11E-06$	$5.67E-07$
2	$5.12E-10$	$6.31E-05$	$2.23E-05$
3	$4.24E-07$	$3.94E-04$	$2.53E-04$
4	$3.14E-05$	$1.47E-03$	$3.31E-04$
5	$8.31E-04$	$4.36E-03$	$9.53E-04$
6	$5.18E-03$	$1.04E-02$	$1.62E-03$

Tab. 2.

Further examples demonstrating the superiority of modal masters to nodal masters for membrane and plate problems are contained in [7], [12] and [14].

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