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A note on conic fitting by the gradient weighted least-squares estimation: refined eigenvector solution

G.Y. Wang [a], Z. Houkes [b,*], B. Zheng [a], X. Li [a]

[a] *Department of Electrical Engineering, Ocean University of Qingdao, Yushan Road 5, Qingdao, 266003 China*

[b] *Measurement and Instrumentation Laboratory, Faculty of Electrical Engineering, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands*

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Abstract

The gradient weighted least-squares criterion is a popular criterion for conic fitting. When the non-linear minimisation problem is solved using the eigenvector method, the minimum is not reached and the resulting solution is an approximation. In this paper, we refine the existing eigenvector method so that the minimisation of the non-linear problem becomes exactly. Consequently we apply the refined algorithm to the re-normalisation approach, by which the new iterative scheme yields to bias-corrected solution but based on the exact minimiser of the cost function. Experimental results show the improved performance of the proposed algorithm.

Keywords: Conic fitting; Gradient weighted least-squares; Eigenvector method; Bias-corrected

1. Introduction

Conic (or quadric) fitting is frequently used as a model-based approach to parameter estimation problems involved in many applications of computer vision. Among the criteria applied for fitting algebraic modelling functions, the gradient weighted least-squares criterion is widely used, where the cost function is defined by an approximated measure of the geometric distance of a set of measured points to a model, i.e. a conic (or quadric), providing the fitting curve (Bolle and Vemuri, 1991; Taubin, 1991; Zhang, 1997).

As one of the major themes of pattern recognition, conic fitting has been studied in varieties. Recently, similar work has also been published by Chojnacki et al. (2000). They have discussed this classical parameter estimation problem along with new insights by considering the covariance matrices characterising the uncertainty of the measurements. They proposed a new iterative scheme to obtain the minimiser as its theoretical limits to the cost function with maximum likelihood formulation. But the biasedness of the minimiser is not considered in their approach. Also, a notable outcome using the heterocedastic regression algorithm, was given by Matei and Meer (2000) as well

* Corresponding author. Tel.: +31-53-489-2790; fax: +31-53-489-1067.

E-mail address: z.houkes@el.utwente.nl (Z. Houkes).

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as by Leedan and Meer (2000). They used the criterion of maximum likelihood estimation and proposed a general computational method with EIV model to solve the estimation problem with bilinear constraint. They also proved that the gradient weighted least-squares criterion is just an approximated case in their proposed EIV model by neglecting higher order moments of the heterocedastic covariance matrix. For the case of ellipse-specific fitting, which is a special case of conic fitting problem, Fitzgibbon et al. (1999) presented a direct computational method to obtain an initial estimate, holding the advantage of one-step computation and being convergence guaranteed. However, as a trade-off between the optimisation of criterion and the simplicity of computation, the gradient weighted least-squares estimation is a very popular criterion for a wide-range of applications in conic fitting, with the very attractive advantage that the eigenvector solution assures the global minimisation in case of convergence.

Given an algebraic function in the implicit form $f(\boldsymbol{x}) = 0$, with $\boldsymbol{x} = (x, y)^{\mathrm{T}}$, the curve of the function is defined by $C = \{\boldsymbol{x} | f(\boldsymbol{x}) = 0\}$, i.e., the set of zeros of f. Supposing $\{\boldsymbol{x}_i\} \forall i \in \{1, \ldots, n\}$ to be the data points observed, the squared distance of a point $\boldsymbol{x}_i$ to the curve can be approximated by (Bolle and Vemuri, 1991; Keren et al., 1994)

$$\mathrm{dist}(\boldsymbol{x}_i, C)^2 \approx \frac{f^2(\boldsymbol{x}_i)}{\|\nabla f(\boldsymbol{x}_i)\|^2} \tag{1}$$

The gradient weighted least-squares criterion for fitting the function to the data set $\{\boldsymbol{x}_i\}$ minimises the mean-square distance, denoted by the cost function

$$\Theta = \frac{1}{n} \sum_{i=1}^{n} \mathrm{dist}(\boldsymbol{x}_i, C)^2 = \sum_{i=1}^{n} w_i f^2(\boldsymbol{x}_i) \tag{2}$$

where $w_i = 1/\|\nabla f(\boldsymbol{x}_i)\|^2$ is the "gradient weight".

In general, minimisation of (2) is a non-linear problem. Well established non-linear least-squares estimation techniques can be applied. In the case of conic fitting (more generally, in the case that f is a polynomial function), the eigenvector method can be employed to solve the minimisation problem (Kanatani, 1994, 1996; Taubin, 1991; Zhang, 1997). To cope with the dependency of the weights $\{w_i\}$ on the parameters, a re-weighing procedure is applied to update the weights $\{w_i\}$.

Although there is no guarantee for convergence, the eigenvector method is very tractable to search for the global minimum. It has been utilised to get approximate estimates for polynomial fitting (Taubin et al., 1994). However, as we will see in the next section, the solution of the existing eigenvector method does not yield an *exact* minimisation of the non-linear cost function of (2) because the dependency of the "weights" $\{w_i\}$ on the parameters is neglected in the formulation of the minimisation problem. To obtain the exact solution, further non-linear numerical techniques are still required.

In this paper, we present a refined version of the eigenvector method for the solution of the gradient weighted least-squares conic fitting problem. The dependency of the "weight" on the parameters is explicitly represented in the initial formulation of the minimisation problem, so convergence of the re-weighing procedure yields to an exact minimiser of the cost function defined by (2), without further processing by other non-linear numerical techniques. Therefore, accuracy of estimates can be guaranteed when the refined version of the eigenvector method is used, while the merit of searching the global minimum makes the computation robust and tractable.

It has been proved that the gradient weighted least-squares solution of conic fitting is statistically biased (Kanatani, 1994, 1996). To correct the bias of the estimates, Kanatani (1994, 1996) proposed the *re-normalisation* method. The re-normalisation method is emphasised in our concerns because of its property of unbiasedness. This property opens the possibility to model the uncertainties in parameter estimates with an explicit probabilistic representation, which can be used to optimise the model-based recognition scheme (Wang, 2000). It should be noted that the existing computational algorithm of the re-normalisation method (Kanatani, 1996) is based on the inexact minimising of the cost function (2) by eigenvector computation, which iterative scheme would degrades the reliability of the final estimate (Leedan and Meer, 2000).

To refine the re-normalisation method, we utilise the refined version of the eigenvector method and incorporate it with the bias-corrected re-normalisation algorithm. Consequently the reliability of the estimate is improved while the optimality of the unbiased estimate is preserved.

The paper is organised as follows. Section 2 introduces the well-known eigenvector method for the gradient weighted least-squares conic fitting problem and the principle of re-normalisation. In Section 3, we formulate the non-linear minimisation problem with an exact representation and the refined algorithm of the eigenvector solution is given. Based on the new eigenvector solution, a refined iterative scheme of the re-normalisation is described in Section 4. In Section 5, we present some experimental results of the proposed approach and compare these results obtained with the existing approach. Finally a summary is given in Section 6.

2. The well-known eigenvector method for conic fitting and the re-normalisation algorithm

When we suppose the conic function to be given in implicit form as described by

$$f(x,y) = ax^2 + by^2 + cxy + dx + ey + k = 0$$

and when we define a vector $\boldsymbol{p} = (a,b,c,d,e,k)^{\mathrm{T}}$, then the cost function described by (2) can be expressed as

$$\Theta = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{N}\boldsymbol{p} \quad \text{with } \boldsymbol{N} = \sum_{i=1}^{n} w_i \boldsymbol{N}_i,$$

$$\boldsymbol{N}_i = \boldsymbol{m}_i\boldsymbol{m}_i^{\mathrm{T}} \quad \text{and } \boldsymbol{m}_i = \left(x_i^2, y_i^2, x_iy_i, x_i, y_i, 1\right)^{\mathrm{T}}$$

Because any non-zero scalar multiplying $\boldsymbol{p}$ results in the same expression, constraints on $\boldsymbol{p}$ must be set. Although there are alternative selections of the normalisation of $\boldsymbol{p}$, we set $\|\boldsymbol{p}\|_2 = 1$, which is tractable to approach the eigenvector principle without loss of generality. Under such constraint, the algebraic parameters of the conic function is estimated by minimising the cost function defined in (2), i.e.,

$$\boldsymbol{p} = \arg\min_{p\in P}\Theta \tag{3}$$

where $P = \{\boldsymbol{p} \mid \|\boldsymbol{p}\|_2 = 1\}$ denotes a manifold of the parameters.

To solve the constraint minimisation problem of (3), the *Lagrange Multiplier* method can be applied. Defining an augmented cost function as

$$\Theta' = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{N}\boldsymbol{p} - L(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{p} - 1) \tag{4}$$

where L is the so-called Lagrange Multiplier, then solutions of $\boldsymbol{p}$ and L are derived by setting

$$\frac{\partial\Theta'}{\partial\boldsymbol{p}} = 0 \quad \text{and} \quad \frac{\partial\Theta'}{\partial L} = 0$$

If we neglect the dependence of $\{w_i\}$ on $\boldsymbol{p}$ in the derivatives, it is obtained

$$\begin{cases} \boldsymbol{N}\boldsymbol{p} = L\boldsymbol{p} \\ \boldsymbol{p}^{\mathrm{T}}\boldsymbol{p} = 1 \end{cases} \tag{5}$$

Thus the estimate of $\boldsymbol{p}$ is just the eigenvector of $\boldsymbol{N}$ associated to the smallest eigenvalue. The initial estimate is given by setting $w_i = 1 \forall i$, and the weights are updated using the current eigenvector solution of $\boldsymbol{p}$ through iterative computations, i.e., the re-weighing procedure. If this process converges, such an eigenvector solution is usually used as an approximate estimate for the solution of the gradient weighted least-squares problem.

The gradient weighted least-squares criterion is based on the minimum mean squared geometric distances. It has been proved that such a criterion is statistically biased and the re-normalisation method was proposed to remove (although not completely) the bias (Kanatani, 1996).

It can be proved that, due to the perturbation of the set of data points $\{\boldsymbol{x}_i\}\forall i \in \{1,\ldots,n\}$ by noise, the expectation of the perturbation $\Delta\boldsymbol{N}$ of the matrix $\boldsymbol{N}$ is non-zero. It can be formulated as (Zhang, 1997)

$$E[\Delta\boldsymbol{N}] = c\sum_{i=1}^{n} w_i\boldsymbol{B}_i$$

where c is the variance of the noise and $\boldsymbol{B}_i$ is a matrix derived from the Taylor expansion of $\boldsymbol{N}_i$. Because the perturbation of the eigenvector is linear in the perturbation of the matrix $\boldsymbol{N}$, the eigenvector of $\boldsymbol{N}$ is thus biased. The re-normalisation approach compensates the bias by replacing the matrix $\boldsymbol{N}$ in (5) by $\boldsymbol{N}' = \boldsymbol{N} - E[\Delta\boldsymbol{N}]$, so the bias of the eigenvector solution is corrected because the $E[\Delta\boldsymbol{N}'] = \boldsymbol{0}$. Because the variance of the noise is unknown in practice, its variance c is estimated from an imposed constraint $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{N}'\boldsymbol{p} = 0$ through the iterative computations, while the weights $\{w_i\}\forall i \in \{1, \ldots, n\}$ are updated by the current eigenvector solution $\boldsymbol{p}$. The re-normalisation estimator is optimal under the small noise level assumption (Kanatani, 1996).

The re-normalisation process was established on the eigenvector solution represented by (5). However, the formulation of (5) is not an exact representation of the non-linear minimisation problem formulated by (3), because the dependency of the weights $\{w_i\}$ on $\boldsymbol{p}$ has been neglected in the derivation. Although the re-weighing procedure has been applied to update the weights $\{w_i\}$, the converged $\boldsymbol{p}$ is only an approximated solution with respect to the non-linear minimisation problem. Due to such insufficiency in the iterative minimising process, the iterative scheme of the re-normalisation would become unstable when the noise level increases.

Instead of further refinement using other non-linear minimisation techniques, we reformulate the estimation problem expressed by (3) with an exact representation that can also be approached as an eigenvector problem. Applying the perturbation theory, a new computing algorithm is proposed to improve the solution of (5), as well as the reliability of the re-normalisation method. This will be described in Section 3.

3. The exact eigenvector representation

Starting with the defined cost function of (4), the minimisation problem is resolved by setting

$$\frac{\partial \Theta'}{\partial \boldsymbol{p}} = 0 \quad \text{and} \quad \frac{\partial \Theta'}{\partial L} = 0$$

To get an exact solution of the minimisation problem of (3), the implicit dependence of $\{w_i\}$ on $\boldsymbol{p}$ has to be encoded in formulating the derivatives. Thus we have

$$\begin{cases} \boldsymbol{N}\boldsymbol{p} + \frac{1}{2}\sum_{i=1}^{n}\left[(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{N}_i\boldsymbol{p})\left(\frac{\partial w_i}{\partial \boldsymbol{p}}\right)\right] = L\boldsymbol{p} \\ \boldsymbol{p}^{\mathrm{T}}\boldsymbol{p} = 1 \end{cases} \tag{6}$$

If the second term in the LHS of (6), which is associated with the dependency of the weights $\{w_i\}$ on $\boldsymbol{p}$, is ignored, the expression (6) degenerates to (5).

In case of noise absence or low-level noise perturbation in data set, the quantities of $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{N}_i\boldsymbol{p}$ multiplied to $(\partial w_i/\partial \boldsymbol{p})$ are either zero or small enough so that the difference between solutions of (5) and (6) can be ignored. But in case of a significant noise perturbation, the solution of (5) becomes unreliable.

First, we reformulate the second term of the LHS of (6) in the form

$$\frac{1}{2}\sum_{i=1}^{n}\left[(\boldsymbol{p}^{\mathrm{T}}\boldsymbol{N}_i\boldsymbol{p})\left(\frac{\partial w_i}{\partial \boldsymbol{p}}\right)\right] = \frac{1}{2}\sum_{i=1}^{n}\left[\left(\frac{\partial w_i}{\partial \boldsymbol{p}}\right)\boldsymbol{p}^{\mathrm{T}}\boldsymbol{N}_i\right]\boldsymbol{p} \tag{7}$$

Supposing the dimensionality of $\boldsymbol{p}$ is m, denoting the $m \times m$ matrix

$$\boldsymbol{E} = \frac{1}{2}\sum_{i=1}^{n}\left[\left(\frac{\partial w_i}{\partial \boldsymbol{p}}\right)\boldsymbol{p}^{\mathrm{T}}\boldsymbol{N}_i\right] \tag{8}$$

then (6) can be expressed in the form

$$(\boldsymbol{N} + \boldsymbol{E})\boldsymbol{p} = L\boldsymbol{p} \tag{9}$$

Defining a matrix

$$\boldsymbol{N}_e = \boldsymbol{N} + \boldsymbol{E} \tag{10}$$

the solution of $\boldsymbol{p}$ becomes just the eigenvector of the new matrix $\boldsymbol{N}_e$.

Since the matrix $\boldsymbol{E}$ is a refined item, it is small compared with $\boldsymbol{N}$, it can be treated as a perturbation of $\boldsymbol{N}$. According to perturbation theory, the perturbation of an eigenvector is linear in the perturbation of matrix. Denoting $\boldsymbol{p}$ and $\boldsymbol{p}_e$ as the eigenvector of $\boldsymbol{N}$ and $\boldsymbol{N}_e$ respectively, it is shown (Stewart, 1973) that

$$\Delta\boldsymbol{p} = \boldsymbol{p}_e - \boldsymbol{p} = \boldsymbol{U}\left(\lambda\boldsymbol{I} - \boldsymbol{U}^{\mathrm{T}}\boldsymbol{N}U\right)^{-1}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{E}\boldsymbol{p} \tag{11}$$

Here, $\boldsymbol{U}$ is an $m \times (m-1)$ matrix whose columns are the other $m-1$ eigenvectors of $\boldsymbol{N}$ except the one indicated by $\boldsymbol{p}$. The eigenvalue associated with $\boldsymbol{p}$ is indicated by λ.

Therefore, the eigenvector of $\boldsymbol{N}_e$ can be computed as $\boldsymbol{p}_e = \boldsymbol{p} + \Delta\boldsymbol{p}$ at each iteration step. Briefly, to obtain an accurate solution from expression (6), the eigenvector solution by (5) should be corrected with $\Delta\boldsymbol{p}$ during the re-weighing process. The solution of (6) is then obtained when both $\boldsymbol{p}$ and $\boldsymbol{p}_e$ converge.

4. The refined iterative scheme for the re-normalisation method

Since the re-normalisation process is initialised from minimising the cost function (2), the refined eigenvector method should consequently be applied to the re-normalisation process.

In the sense of the bias correction, the matrix $\boldsymbol{N}$ in expression of (6) should be corrected using

$$\boldsymbol{N}' = \boldsymbol{N} - E\{\Delta\boldsymbol{N}\} = \boldsymbol{N} - c\sum_{i=1}^{n}\boldsymbol{B}_i \tag{12}$$

Replacing the matrix $\boldsymbol{N}$ in (6) by $\boldsymbol{N}'$ obtained from (12), the estimation problem becomes

$$\begin{cases} \boldsymbol{N}'\boldsymbol{p} + \frac{1}{2}\sum_{i=1}^{n}\left([\boldsymbol{p}^{\mathrm{T}}(\boldsymbol{N}_i - c\boldsymbol{B}_i)\boldsymbol{p}]\left(\frac{\partial w_i}{\partial \boldsymbol{p}}\right)\right) = L\boldsymbol{p} \\ \boldsymbol{p}^{\mathrm{T}}\boldsymbol{p} = 1 \end{cases} \tag{13}$$

In the same way we have from (7) to (10)

$$\sum_{i=1}^{n}\left[\boldsymbol{p}^{\mathrm{T}}(N_i - c\boldsymbol{B}_i)\boldsymbol{p}\left(\frac{\partial w_i}{\partial \boldsymbol{p}}\right)\right] = \sum_{i=1}^{n}\left[\left(\frac{\partial w_i}{\partial \boldsymbol{p}}\right)\boldsymbol{p}^{\mathrm{T}}(\boldsymbol{N}_i - c\boldsymbol{B}_i)\right]\boldsymbol{p} \tag{14}$$

Defining

$$\boldsymbol{E}' = \frac{1}{2}\sum_{i=1}^{n}\left[\left(\frac{\partial w_i}{\partial \boldsymbol{p}}\right)\boldsymbol{p}^{\mathrm{T}}(\boldsymbol{N}_i - c\boldsymbol{B}_i)\right] \tag{15}$$

we have instead of (9)

$$(\boldsymbol{N}' + \boldsymbol{E}')\boldsymbol{p} = L\boldsymbol{p} \tag{16}$$

Therefore, at each iteration step, $\boldsymbol{p}$ is still obtained as an eigenvector, but of a new matrix $\boldsymbol{N}'_e$, which is defined as

$$\boldsymbol{N}'_e = \boldsymbol{N}' + \boldsymbol{E}'. \tag{17}$$

Denoting $\boldsymbol{p}'$ and $\boldsymbol{p}'_e$ as the eigenvectors of $\boldsymbol{N}'$ and $\boldsymbol{N}'_e$ respectively, applying the perturbation theory expressed by (11), the refined bias-corrected estimate $\boldsymbol{p}'_e$ can be computed from $\boldsymbol{p}'_e = \boldsymbol{p}' + \Delta\boldsymbol{p}'$, where $\Delta\boldsymbol{p}'$ is the perturbation of $\boldsymbol{p}'$ with respect to the perturbation matrix $\boldsymbol{E}'$.

In summary, the procedure of the refined re-normalisation process is as described below. Given a set of points $\{\boldsymbol{x}_i\}\forall i \in \{1,\ldots,n\}$,

1. Let $w_i = 1\forall i \in \{1,\ldots,n\}$ and $c = 0$.
2. Using the data to compute $\boldsymbol{N}$, we compute the eigenvectors of
   $$\boldsymbol{N}' = \boldsymbol{N} - c\sum_{i=1}^{n} w_i\boldsymbol{B}_i,$$
   which we denoted as the set $\{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_6\}$. It is supposed that $\boldsymbol{p}_1$ is the solution associated with the smallest eigenvalue $\lambda_{\min}$.
3. Correct $\boldsymbol{p}_1$ to be
   $$\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{U}(\lambda_{\min}\boldsymbol{I} - \boldsymbol{U}^{\mathrm{T}}\boldsymbol{N}'\boldsymbol{U})^{-1}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{E}'\boldsymbol{p}_1,$$
   where $\boldsymbol{U} = [\boldsymbol{p}_2,\ldots,\boldsymbol{p}_6]$, $\boldsymbol{I}$ the unity matrix and
   $$\boldsymbol{E}' = \frac{1}{2}\sum_{i=1}^{n}\left[\left(\frac{\partial w_i}{\partial \boldsymbol{p}_1}\right)\boldsymbol{p}_1^{\mathrm{T}}(\boldsymbol{N}_i - c\boldsymbol{B}_i)\right]$$
4. According to Zhang (1997), c is updated using
   $$c = c + \frac{\lambda_{\min}}{\boldsymbol{p}^{\mathrm{T}}\left(\sum_{i=1}^{n} w_i\boldsymbol{B}_i\right)\boldsymbol{p}}$$
   Update $\{w_i\}$ using the new corrected estimate $\boldsymbol{p}$, so to update $\boldsymbol{N}'$ with new $\{w_i\}$ and c.
5. If the updates of $\boldsymbol{p}_1$, $\boldsymbol{p}$ and c have converged, return the latest $\boldsymbol{p}$ as the final solution. Otherwise go to step 2 for iteration.

5. Experimental results

Experimental results verify the reliability of the refined algorithm when applying both the gradient weighted least-squares eigen-computation and the re-normalisation method.

Figs. 1 and 2 show the results of gradient weighted least-squares fitting using synthetic data. The data points were sampled from a segment of synthetic ellipses. Gaussian noise was added to the x and y co-ordinates of each data point. There are 80 points used for fitting. Given a set of points, both the known eigenvector method based on (5) (without ambiguity, we denote it as the old version), and the proposed new version based on (6), were used to give a comparison. The solid lines are

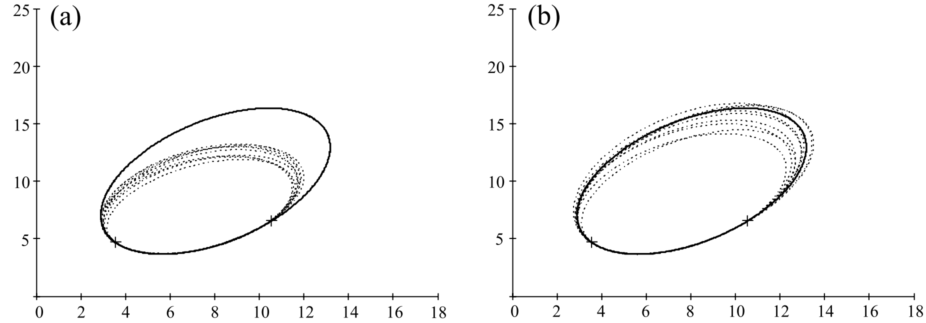

Fig. 1. Results of eigenvector solutions for the gradient weighted least-squares fitting. The noise level was $\sigma = 0.08$. (a) shows the result of using the old version and (b) shows the results of using the new version. Data points were sampled from the lower segments within the range marked by "+"s.

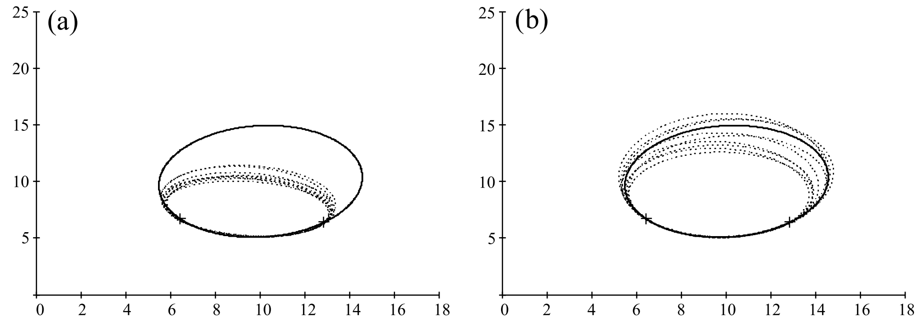

Fig. 2. Results of the gradient weighted least-squares fitting for a different prototype. The noise level is $\sigma = 0.15$.

the noise-free prototypes of the ellipse. Results were obtained for different noise levels and with different prototypes. Plots indicated as (a) show the results using the old version and plots indicated as (b) show the results using the new version.

Figs. 3–5 show the results of the refined version of the re-normalisation approach compared with the old version. In these figures (a) shows the results of the old version and (b) shows the results of new version.

Figs. 6–8 show the histograms of the errors of the estimates using the old and the new versions. By applying the transformation of rotation and translation to the algebraic coefficients, we calcu-

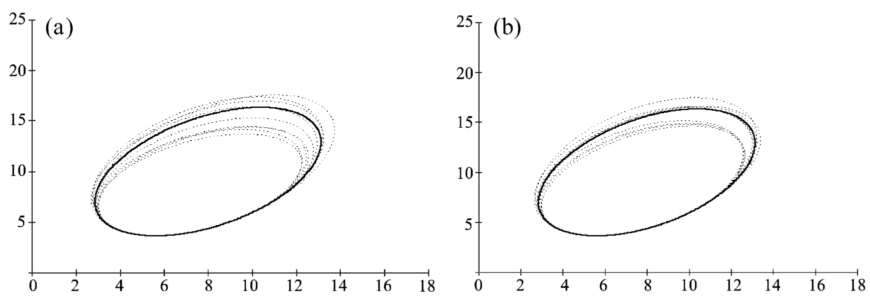

Fig. 3. Results of using the re-normalisation approach. The data points were sampled from the same range of the prototype as used in Fig. 1. The noise level was $\sigma = 0.08$. The estimates by the old and new versions are shown in (a) and (b), respectively.

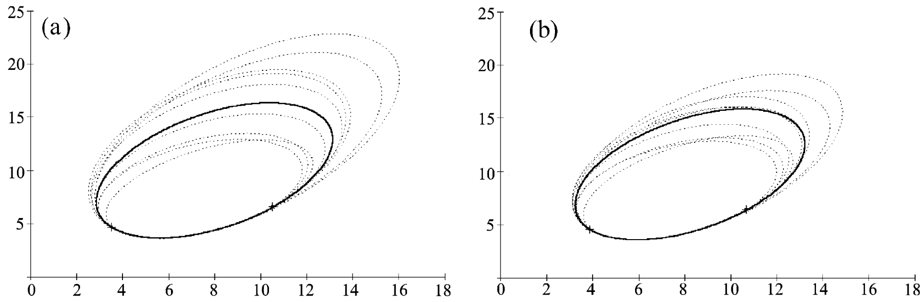

Fig. 4. Results at noise level of $\sigma = 0.15$. The prototype was the same as the one of Fig. 3.

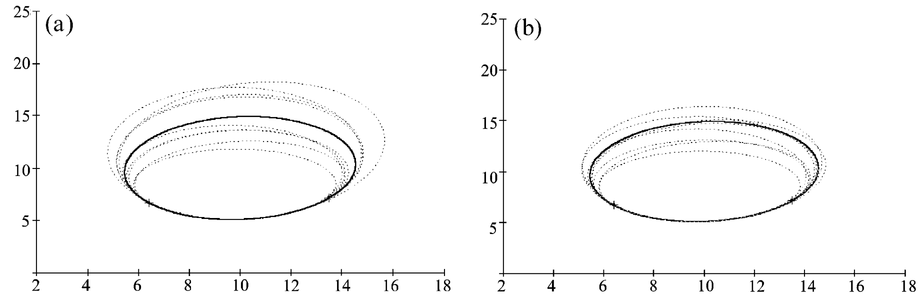

Fig. 5. Results of using the re-normalisation approach for a different prototype. The noise level was $\sigma = 0.15$.

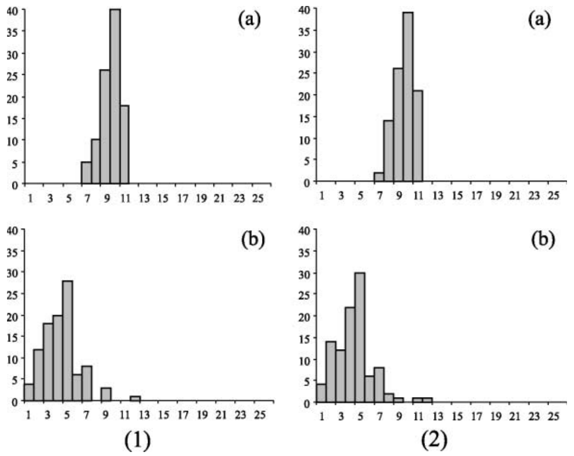

Fig. 6. Histogram of errors by the gradient weighted LSE using the old version (a) and the new version (b). The results for shape estimates and the position estimates of the ellipse are shown in (1) and (2) respectively. The noise level was $\sigma = 0.15$.

lated the two major lengths of the ellipse (denoted as r_1 and r_2) and its central position (denoted as t_x and t_y). Defining a vector $v_s = [r_1, r_2]$, the Euclidean distance between the true and the estimated v_s was used as a measure of the shape deviation. The same way, the error of a vector composed of t_x and t_y was also calculated to indicate the deviation of position estimate. Fig. 6 shows a comparison of the results using old and new versions of the gradient weighted LSE. Figs. 7 and 8 show the comparisons of the results using old and new versions of the re-normalisation approach at different noise level. In these figures, the number of trials is 100. The histograms of the errors of shape estimates and the position estimates were shown in (1) and (2) respectively, in which (a) shows the result of the old version and (b) shows the result of new version. The vertical axis indicates the number of estimates corresponding to the error level quantified at the horizontal axis. The error level (the Euclidean distance) was quantified by an interval of 0.25. Fig. 9 shows the results obtained with real image data. Fig. 9(a) shows the grey-level image of a part of a cup and the detected curved edge (noise was added to the original grey-level image). Data points were sampled from a portion

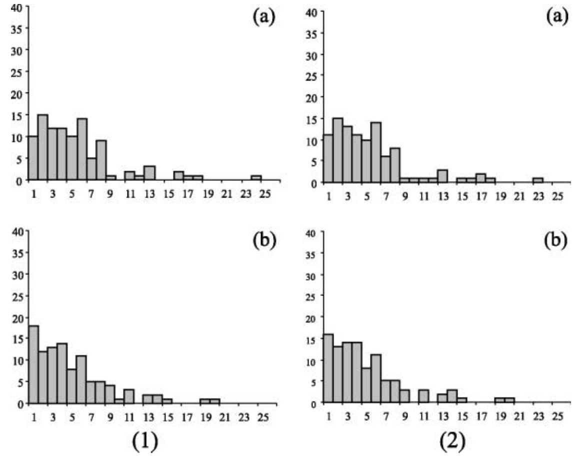

Fig. 7. Histogram of errors by the re-normalisation approach using the old version (a) and the new version (b). The results for shape estimates and the position estimates of the ellipse are shown in (1) and (2) respectively. The noise level was $\sigma = 0.08$.

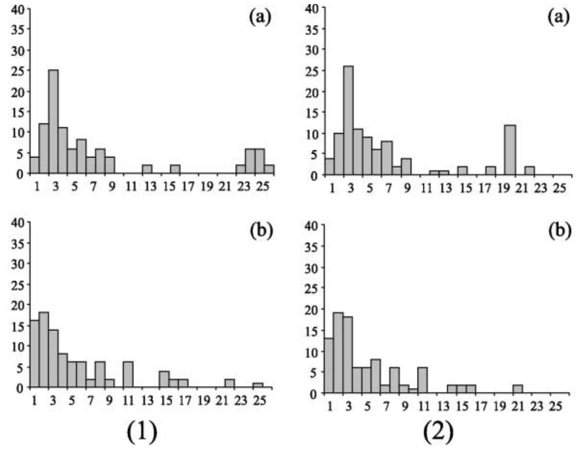

Fig. 8. Histograms of the errors using the old version (a) of the re-normalisation approach and the new version (b). The errors in the estimated shape and the position parameters of the ellipse are shown in (1) and (2) respectively. The noise level $\sigma = 0.15$.

of the detected edge at upper side for conic fitting (the number of fitted points is 150). Fig. 9(b) shows the comparison of the eigenvector solutions by both old and new versions of the gradient weighted least squares fitting. Fig. 9(c) shows the results of the old version of the re-normalisation method and the refined version.

In the above experiments, only a few iterations (3)–(10) were required for convergence of both the old and the refined versions.

From the results we can see that the fittings in the range occupied by sampled points were good and almost indistinguishable for both the old and the new methods, but the difference in global description was obvious. From the Figs. 1 and 2 it appears, just as we expected, because of the inexact eigenvector solution of (5), that the solutions of the old version have large errors, although the reweighing process converged. In contrast with these results, the refined algorithm used for eigenvector solution, which is derived from an exact formulation of (6), gives rise to much better results.

In our experiments, the results of the old and the new versions of the re-normalisation method are almost indistinguishable at small noise level. But at higher noise level, the new version yielded less errors in average and showed more stable than the old version. For all trials during experiments at the noise level of $\sigma = 0.15$ (the number of trials is 100), there were cases (8%) that failed to convergence in using the old version of the re-normalisation approach, while the new version converged

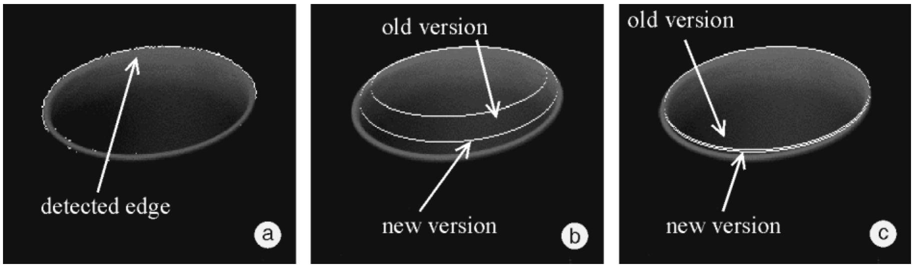

Fig. 9. Results of fitting real image data: (a) shows the greylevel image of part of a cup and the detected edge (the white trace) after adding noise; (b) show the results of the gradient weighted least-squares fitting using the old and new versions of the eigenvector solution and (c) show the results of applying the re-normalisation method, both of the old and the new versions were used.

for all cases. Such an improvement of the new version is due to the more reliable modelling of the eigenvector solution in the refined iterative scheme when applying the re-normalisation.

6. Conclusions

Conics belong to the most fundamental features, which are frequently encountered in images. The reliable estimation of their parameters is critical for further image analysis. In this paper, we presented a refined algorithm of eigenvector solution to conic fitting based on the gradient weighted least-squares criterion. By reformulating the minimisation problem with a refined eigenvector representation, the non-linear minimisation problem can be approached using the refined eigenvector method towards an exact minimum. While taking advantage of searching the global minimum by eigenvector computing, the accuracy of solution is guaranteed without employing other well-established non-linear minimising techniques. Applying the refined algorithm to the bias-corrected renormalisation process, the reliability of the estimator is improved, by which the new iterative scheme yields to bias-corrected solution but based on the exact minimiser of the cost function. Furthermore, the proposed algorithm can be directly applied to approach higher order polynomial fitting in a wide range of applications. Due to the restricted subject of this paper, a general comparison of our method and other methods for conic fitting is beyond the scope of this paper.

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