

ON THE EULERIAN POLYNOMIALS OF TYPE D

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ABSTRACT. We define and study sub-Eulerian polynomials of type D , which count the elements of D_n with respect to the number of descents in a refined sense. The recurrence relations and exponential generating functions of the sub-Eulerian polynomials are determined, by which the solution to a problem of Brenti, concerning the recurrence relation for the Eulerian polynomials of type D , is also obtained.

1. INTRODUCTION

One of the classical polynomials of combinatorial significance is the Eulerian polynomial, which enumerates elements of the symmetric group with respect to the number of descents, and whose properties are well studied (see, e.g., [3]).

The Eulerian polynomials and their q -generalizations have also been defined for other Coxeter families (see, e.g., [1] for q -Eulerian polynomials which interpolate between the type A and B , and between type A and D , cases). For instance, the basic properties of the Eulerian polynomials of type B , analogous to their type A counterparts, are known.

On the other hand, the type D theory is not as well developed as the type B one does. Only the type D generating functions and Worpitzky identity are known; the type D recurrence relation is still missing. Finding such a type D recurrence happens to be a problem not as simple as in the type A and B cases. The difficulty lies in that, by imitating the derivations of the corresponding type A and B recurrences, the number of descents changes in a peculiar way so that some refined Eulerian polynomials are needed in order to capture these changes. This leads to the introduction of the sub-Eulerian polynomials, which is the idea central to the present work.

The organization of this paper is as follows. In the next section, we collect the notations that will be used in the rest of this work. In section 3, we introduce the sub-Eulerian polynomials, which enumerate the elements of the Coxeter group of type D in a refined sense. In particular, we obtain a recurrence relation involving the type D Eulerian and sub-Eulerian polynomials. In section 4, we compute the exponential generating functions of the sub-Eulerian polynomials. The generating functions computed enable us to refine the recurrence relation obtained in section 3. In the final section, we determine the partial differential equation of minimal order which the exponential generating function of type D Eulerian polynomials satisfy, and by which a recurrence involving the type D Eulerian polynomials and its derivatives only is derived.

2. NOTATIONS

We collect some definitions, and notations that will be used in the rest of this paper. Let S be a finite set. Denote by $\#S$ the cardinality of S . Denote by \mathfrak{S}_n the symmetric group of degree n , B_n the hyperoctahedral group of rank n , and D_n the Coxeter group of type D of rank n . Let $\pi = \pi_1\pi_2\cdots\pi_n \in D_n$, where $\pi(i) = \pi_i$, for $i = 1, 2, \dots, n$. We say that $i \in [0, n-1]$ is a D -descent of π if $\pi_i > \pi_{i+1}$, where $\pi_0 = -\pi_1$. We shall drop the type designation of descents in the sequel unless circumstances demand the contrary. Denote by $\text{Des}(\pi)$ the descent set of π , and $d(\pi) = \#\text{Des}(\pi)$ the number of descents of π . Denote by $D_{n,k}$ the Eulerian number of type D , which is defined as the number of elements of D_n with k descents. Define the Eulerian polynomial $D_n(t)$ of type D by

$$D_n(t) = \sum_{\pi \in D_n} t^{d(\pi)} = \sum_{k=0}^n D_{n,k} t^k.$$

3. SUB-EULERIAN POLYNOMIALS

The goal of the present work is to study the recurrence relation satisfied by $D_{n,k}$. Towards this end, we have the symmetric group \mathfrak{S}_n , and the hyperoctahedral group B_n as our guiding examples. Recall that \mathfrak{S}_n (resp., B_n) can be constructed by inserting n (resp., $\pm n$) to the elements of \mathfrak{S}_{n-1} (resp., B_{n-1}). By studying how the descent number changes, recurrence relations for the Eulerian numbers of type A and B are obtained.

In the case of D_n , a similar construction is also possible. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{n-1} \in B_{n-1}$. Denote by $\hat{\sigma}$ the element $\bar{\sigma}_1\sigma_2\cdots\sigma_{n-1}$ of B_{n-1} . It is clear that the map $B_{n-1} \rightarrow B_{n-1}$, $\sigma \rightarrow \hat{\sigma}$, is a bijection sending D_{n-1} onto $B_{n-1} \setminus D_{n-1}$. To each element σ of D_{n-1} , we can associate to σ the pair of elements $\{\sigma, \hat{\sigma}\}$ of B_{n-1} . This can also be realized as the orbit of σ under the right action of the subgroup $\langle s_0 \rangle$ of B_{n-1} generated by $s_0 = (1 \ \bar{1})$ (in cycle notation). (It is unfortunate that this right action of $\langle s_0 \rangle$ does not endow the collection of orbits with a group structure because $\langle s_0 \rangle$ is not normal.) With this association in place, D_n is constructible from D_{n-1} by inserting n to σ and $-n$ to $\hat{\sigma}$.

For $n \geq 2$, denote by ${}^1D_{n,k}$ (resp., ${}^{0,1}D_{n,k}$, ${}^{\geq 2}D_{n,k}$, and ${}^{0,\geq 2}D_{n,k}$) the collection of elements of D_n of k D -descents, with descent at position 1 but not at 0 (resp., with descents at both positions 0 and 1, with no descent at positions 0 and 1, and with descent at position 0 but not at 1). Denote by ${}^1\bar{D}_{n,k}$ (resp., ${}^{0,1}\bar{D}_{n,k}$, ${}^{\geq 2}\bar{D}_{n,k}$, and ${}^{0,\geq 2}\bar{D}_{n,k}$) the collection of elements of $B_n \setminus D_n$ of k D -descents, with descent at position 1 but not at 0 (resp., with descents at both positions 0 and 1, with no descent at positions 0 and 1, and with descent at position 0 but not at 1).

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in D_n$ be of k descents. It is of interest to see how the pair $\{\sigma, \hat{\sigma}\}$ distributes amongst ${}^1D_{n,k}$, ${}^1\bar{D}_{n,k}$, etc.

LEMMA 3.1. *We have*

$$\begin{aligned} \sigma \in {}^1D_{n,k} &\iff \hat{\sigma} \in {}^{0,\geq 2}\bar{D}_{n,k}, \\ \sigma \in {}^{0,1}D_{n,k} &\iff \hat{\sigma} \in {}^{0,1}\bar{D}_{n,k}, \\ \sigma \in {}^{\geq 2}D_{n,k} &\iff \hat{\sigma} \in {}^{\geq 2}\bar{D}_{n,k}, \\ \sigma \in {}^{0,\geq 2}D_{n,k} &\iff \hat{\sigma} \in {}^1\bar{D}_{n,k}. \end{aligned}$$

Proof. The map $\sigma \rightarrow \hat{\sigma}$ does not alter $\sigma_2 \cdots \sigma_n$ so that descents beyond the position 1 are unaffected. It remains to figure out how descents at positions 0 and 1 change. But the following calculations

$$\begin{aligned} \sigma \in {}^1D_{n,k} &\iff \left. \begin{array}{l} \sigma_1 + \sigma_2 > 0 \\ \sigma_1 > \sigma_2 \end{array} \right\} &\iff \left. \begin{array}{l} \bar{\sigma}_1 < \sigma_2 \\ 0 > \bar{\sigma}_1 + \sigma_2 \end{array} \right\} &\iff \hat{\sigma} \in {}^{0,\geq 2}\bar{D}_{n,k}, \\ \sigma \in {}^{0,1}D_{n,k} &\iff \left. \begin{array}{l} \sigma_1 + \sigma_2 < 0 \\ \sigma_1 > \sigma_2 \end{array} \right\} &\iff \left. \begin{array}{l} \bar{\sigma}_1 > \sigma_2 \\ 0 > \bar{\sigma}_1 + \sigma_2 \end{array} \right\} &\iff \hat{\sigma} \in {}^{0,1}\bar{D}_{n,k}, \\ \sigma \in {}^{\geq 2}D_{n,k} &\iff \left. \begin{array}{l} \sigma_1 + \sigma_2 > 0 \\ \sigma_1 < \sigma_2 \end{array} \right\} &\iff \left. \begin{array}{l} \bar{\sigma}_1 < \sigma_2 \\ 0 < \bar{\sigma}_1 + \sigma_2 \end{array} \right\} &\iff \hat{\sigma} \in {}^{\geq 2}\bar{D}_{n,k}, \\ \sigma \in {}^{0,\geq 2}D_{n,k} &\iff \left. \begin{array}{l} \sigma_1 + \sigma_2 < 0 \\ \sigma_1 < \sigma_2 \end{array} \right\} &\iff \left. \begin{array}{l} \bar{\sigma}_1 > \sigma_2 \\ 0 < \bar{\sigma}_1 + \sigma_2 \end{array} \right\} &\iff \hat{\sigma} \in {}^1\bar{D}_{n,k} \end{aligned}$$

then show that either the descents at positions 0 and 1 are preserved, or the descent at position 0 of σ is mapped to the descent at position 1 of $\hat{\sigma}$, and conversely. In any case, the number of descents are preserved, and the leading descent structure are given as in the lemma, concluding the proof. \square

We are now ready to look at the insertion process. Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_{n-1} \in {}^*D_{n-1,k}$, where ${}^*D_{n-1,k}$ is any of the subcollections of elements of D_{n-1} of k descents, defined above. The letters $\pm n$ can be inserted at any of the n positions, numbered consecutively from 0 to $n-1$, and the insertion proceeds as follows.

At position 0:

if $\sigma \in {}^1D_{n-1,k}$ ($\hat{\sigma} \in {}^{0,\geq 2}D_{n-1,k}$), then

$$(1) \quad n\sigma_1\sigma_2 \cdots \sigma_{n-1} \in {}^1D_{n,k+1}, \quad \bar{n}\bar{\sigma}_1\sigma_2 \cdots \sigma_{n-1} \in {}^{0,\geq 2}D_{n,k};$$

if $\sigma \in {}^{0,1}D_{n-1,k}$ ($\hat{\sigma} \in {}^{0,1}D_{n-1,k}$), then

$$(2) \quad n\sigma_1\sigma_2 \cdots \sigma_{n-1} \in {}^1D_{n,k}, \quad \bar{n}\bar{\sigma}_1\sigma_2 \cdots \sigma_{n-1} \in {}^{0,\geq 2}D_{n,k};$$

if $\sigma \in {}^{\geq 2}D_{n-1,k}$ ($\hat{\sigma} \in {}^{\geq 2}D_{n-1,k}$), then

$$(3) \quad n\sigma_1\sigma_2 \cdots \sigma_{n-1} \in {}^1D_{n,k+1}, \quad \bar{n}\bar{\sigma}_1\sigma_2 \cdots \sigma_{n-1} \in {}^{0,\geq 2}D_{n,k+1};$$

if $\sigma \in {}^{0,\geq 2}D_{n-1,k}$ ($\hat{\sigma} \in {}^1D_{n-1,k}$), then

$$(4) \quad n\sigma_1\sigma_2 \cdots \sigma_{n-1} \in {}^1D_{n,k}, \quad \bar{n}\bar{\sigma}_1\sigma_2 \cdots \sigma_{n-1} \in {}^{0,\geq 2}D_{n,k+1}.$$

At position 1:

if $\sigma \in {}^1D_{n-1,k}$ ($\hat{\sigma} \in {}^{0,\geq 2}D_{n-1,k}$), then

$$(5) \quad \sigma_1 n \sigma_2 \cdots \sigma_{n-1} \in {}^{\geq 2}D_{n,k}, \quad \bar{\sigma}_1 \bar{n} \sigma_2 \cdots \sigma_{n-1} \in {}^{0,1}D_{n,k+1};$$

if $\sigma \in {}^{0,1}D_{n-1,k}$ ($\hat{\sigma} \in {}^{0,1}D_{n-1,k}$), then

$$(6) \quad \sigma_1 n \sigma_2 \cdots \sigma_{n-1} \in {}^{\geq 2}D_{n,k-1}, \quad \bar{\sigma}_1 \bar{n} \sigma_2 \cdots \sigma_{n-1} \in {}^{0,1}D_{n,k};$$

if $\sigma \in {}^{\geq 2}D_{n-1,k}$ ($\hat{\sigma} \in {}^{\geq 2}D_{n-1,k}$), then

$$(7) \quad \sigma_1 n \sigma_2 \cdots \sigma_{n-1} \in {}^{\geq 2}D_{n,k+1}, \quad \bar{\sigma}_1 \bar{n} \sigma_2 \cdots \sigma_{n-1} \in {}^{0,1}D_{n,k+2};$$

if $\sigma \in {}^{0,\geq 2}D_{n-1,k}$ ($\hat{\sigma} \in {}^1D_{n-1,k}$), then

$$(8) \quad \sigma_1 n \sigma_2 \cdots \sigma_{n-1} \in {}^{\geq 2}D_{n,k}, \quad \bar{\sigma}_1 \bar{n} \sigma_2 \cdots \sigma_{n-1} \in {}^{0,1}D_{n,k+1}.$$

At position i ($2 \leq i \leq n-2$):

if $\sigma \in {}^*D_{n-1,k}$, then

$$(9) \quad \sigma_1 \sigma_2 \cdots \sigma_i n \sigma_{i+1} \cdots \sigma_{n-1} \in \begin{cases} {}^*D_{n,k} & \text{if } i \in \text{Des}(\sigma), \\ {}^*D_{n,k+1} & \text{if } i \notin \text{Des}(\sigma); \end{cases}$$

if $\hat{\sigma} \in {}^*D_{n-1,k}$, then

$$(10) \quad \bar{\sigma}_1 \sigma_2 \cdots \sigma_i \bar{n} \sigma_{i+1} \cdots \sigma_{n-1} \in \begin{cases} {}^*D_{n,k} & \text{if } i \in \text{Des}(\sigma), \\ {}^*D_{n,k+1} & \text{if } i \notin \text{Des}(\sigma); \end{cases}$$

At position $n-1$:

$$(11) \quad \sigma_1 \sigma_2 \cdots \sigma_{n-1} n \in {}^*D_{n,k};$$

$$(12) \quad \bar{\sigma}_1 \sigma_2 \cdots \sigma_{n-1} \bar{n} \in {}^*D_{n,k+1}.$$

Define now the *sub-Eulerian numbers* $D_{n,k}^1$, $D_{n,k}^{0,1}$, $D_{n,k}^{\geq 2}$, $D_{n,k}^{0,\geq 2}$ by

$$D_{n,k}^1 = \#^1 D_{n,k},$$

$$D_{n,k}^{0,1} = \#^{0,1} D_{n,k},$$

$$D_{n,k}^{\geq 2} = \#^{\geq 2} D_{n,k},$$

$$D_{n,k}^{0,\geq 2} = \#^{0,\geq 2} D_{n,k}.$$

The sub-Eulerian numbers defined are obviously refinements of the Eulerian numbers $D_{n,k}$, and they offer a more accurate description of the descent distribution of elements of D_n .

Since the map $\sigma \rightarrow \hat{\sigma}$ is an injection, Lemma 3.1 then says that

$$\#^1 \bar{D}_{n,k} = D_{n,k}^{0,\geq 2},$$

$$\#^{0,1} \bar{D}_{n,k} = D_{n,k}^{0,1},$$

$$\#^{\geq 2} \bar{D}_{n,k} = D_{n,k}^{\geq 2},$$

$$\#^{0,\geq 2} \bar{D}_{n,k} = D_{n,k}^1.$$

Note that, in the insertion process above, the descent numbers can go up by 2 or go down by 1, which never occurs in the case of \mathfrak{S}_n or B_n . However, the introduction of sub-Eulerian numbers helps capture these changes.

PROPOSITION 3.2. For $n \geq 3$, the sub-Eulerian numbers satisfy the following recurrence relations

- (i) $D_{n,k}^1 = (2k-1)D_{n-1,k}^1 + 2(n-k)D_{n-1,k-1}^1 + D_{n-1,k}^{0,1} + D_{n-1,k-1}^{\geq 2} + D_{n-1,k}^{0,\geq 2}$,
- (ii) $D_{n,k}^{0,1} = 2(k-1)D_{n-1,k}^{0,1} + (2n-2k+1)D_{n-1,k-1}^{0,1} + D_{n-1,k-1}^1 + D_{n-1,k-2}^{\geq 2} + D_{n-1,k-1}^{0,\geq 2}$,
- (iii) $D_{n,k}^{\geq 2} = (2k+1)D_{n-1,k}^{\geq 2} + 2(n-k-1)D_{n-1,k-1}^{\geq 2} + D_{n-1,k}^1 + D_{n-1,k+1}^{0,1} + D_{n-1,k}^{0,\geq 2}$,
- (iv) $D_{n,k}^{0,\geq 2} = 2kD_{n-1,k}^{0,\geq 2} + (2n-2k-1)D_{n-1,k-1}^{0,\geq 2} + D_{n-1,k-1}^1 + D_{n-1,k}^{0,1} + D_{n-1,k-1}^{\geq 2}$.

Proof. We only prove (i); the remaining assertions follow from similar considerations. Elements of ${}^1D_{n,k}$ can be obtained by inserting n

to $\sigma \in {}^1D_{n-1,k-1}$	as in (1)	with multiplicity	$D_{n-1,k-1}^1$,
$\sigma \in {}^{0,1}D_{n-1,k}$	(2)		$D_{n-1,k}^{0,1}$
$\sigma \in {}^{\geq 2}D_{n-1,k-1}$	(3)		$D_{n-1,k-1}^{\geq 2}$
$\sigma \in {}^{0,\geq 2}D_{n-1,k}$	(4)		$D_{n-1,k}^{0,\geq 2}$
$\sigma \in {}^1D_{n-1,k}$	(9)		$(k-1)D_{n-1,k}^1$
$\sigma \in {}^1D_{n-1,k}$	(9)		$[(n-2)-(k-1)]D_{n-1,k-1}^1$
$\sigma \in {}^1D_{n-1,k}$	(11)		$D_{n-1,k}^1$

and by inserting $-n$

to $\hat{\sigma} \in {}^1\bar{D}_{n-1,k}$	as in (10)	with multiplicity	$(k-1)D_{n-1,k}^1$,
$\hat{\sigma} \in {}^1\bar{D}_{n-1,k-1}$	(10)		$[(n-2)-(k-1)]D_{n-1,k-1}^1$,
$\hat{\sigma} \in {}^1\bar{D}_{n-1,k-1}$	(12)		$D_{n-1,k-1}^1$.

Summing the multiplicities, we finally have

$$D_{n,k}^1 = (2k-1)D_{n-1,k}^1 + 2(n-k)D_{n-1,k-1}^1 + D_{n-1,k}^{0,1} + D_{n-1,k-1}^{\geq 2} + D_{n-1,k}^{0,\geq 2},$$

as desired. □

It is convenient to record the sub-Eulerian numbers by generating functions. For $n \geq 2$, define the *sub-Eulerian polynomials* $D_n^1(t)$, $D_n^{0,1}(t)$, $D_n^{\geq 2}(t)$, and $D_n^{0,\geq 2}(t)$ by

$$\begin{aligned} D_n^1(t) &= \sum_{k \geq 0} D_{n,k}^1 t^k, \\ D_n^{0,1}(t) &= \sum_{k \geq 0} D_{n,k}^{0,1} t^k, \\ D_n^{\geq 2}(t) &= \sum_{k \geq 0} D_{n,k}^{\geq 2} t^k, \\ D_n^{0,\geq 2}(t) &= \sum_{k \geq 0} D_{n,k}^{0,\geq 2} t^k. \end{aligned}$$

The first few values of $D_n^1(t)$, etc., are given in Tables 1–4.

It is clear that the Eulerian polynomial $D_n(t)$, $n \geq 2$, satisfies

$$(13) \quad D_n(t) = D_n^1(t) + D_n^{0,1}(t) + D_n^{\geq 2}(t) + D_n^{0,\geq 2}(t).$$

The Eulerian polynomials of type A and B satisfy certain difference-differential equations. The same is true of the sub-Eulerian polynomials.

PROPOSITION 3.3. *For $n \geq 3$, the sub-Eulerian polynomials satisfy the following difference-differential equations:*

- (i) $D_n^1(t) = [2(n-1)t-1]D_{n-1}^1(t) + 2t(1-t)(D_{n-1}^1)'(t) + D_{n-1}^{0,1}(t) + tD_{n-1}^{\geq 2}(t) + D_{n-1}^{0,\geq 2}(t),$
- (ii) $D_n^{0,1}(t) = [(2n-1)t-2]D_{n-1}^{0,1}(t) + 2t(1-t)(D_{n-1}^{0,1})'(t) + tD_{n-1}^1(t) + t^2D_{n-1}^{\geq 2}(t) + tD_{n-1}^{0,\geq 2}(t),$
- (iii) $D_n^{\geq 2}(t) = [2(n-2)t+1]D_{n-1}^{\geq 2}(t) + 2t(1-t)(D_{n-1}^{\geq 2})'(t) + D_{n-1}^1(t) + t^{-1}D_{n-1}^{0,1}(t) + D_{n-1}^{0,\geq 2}(t),$
- (iv) $D_n^{0,\geq 2}(t) = (2n-3)tD_{n-1}^{0,\geq 2}(t) + 2t(1-t)(D_{n-1}^{0,\geq 2})'(t) + tD_{n-1}^1(t) + D_{n-1}^{0,1}(t) + tD_{n-1}^{\geq 2}(t).$

Proof. We only prove (i); the remaining assertions follow from similar reasoning. Multiplying Proposition 3.2 (i) by t^k and summing over k yields

$$\begin{aligned}
 \sum_{k=1}^n D_{n,k}^1 t^k &= \sum_{k=1}^n (2k-1)D_{n-1,k}^1 t^k + \sum_{k=1}^n 2(n-k)D_{n-1,k-1}^1 t^k + \sum_{k=1}^n D_{n-1,k}^{0,1} t^k \\
 &\quad + \sum_{k=1}^n D_{n-1,k-1}^{\geq 2} t^k + \sum_{k=1}^n D_{n-1,k}^{0,\geq 2} t^k \\
 &= I + II + III + IV + V.
 \end{aligned}
 \tag{14}$$

The left hand side of (14) is equal to $D_n^1(t)$ because $D_{n,0}^1 = 0$, while terms on the right hand side are equal respectively to

$$\begin{aligned}
 I &= 2t \sum_{k=1}^n k D_{n-1,k}^1 t^{k-1} - \sum_{k=1}^n D_{n-1,k}^1 t^k = 2t(D_{n-1}^1)'(t) - D_{n-1}^1(t), \\
 II &= \sum_{k=0}^{n-1} 2(n-k-1)D_{n-1,k}^1 t^{k+1} = 2(n-1)t \sum_{k=0}^{n-1} D_{n-1,k}^1 t^k - 2t^2 \sum_{k=0}^{n-1} k D_{n-1,k}^1 t^{k-1} \\
 &= 2(n-1)tD_{n-1}^1(t) - 2t^2(D_{n-1}^1)'(t), \\
 III &= \sum_{k=0}^{n-1} D_{n-1,k}^{0,1} t^k = D_{n-1}^{0,1}(t), \\
 IV &= \sum_{k=0}^{n-1} D_{n-1,k}^{\geq 2} t^{k+1} = tD_{n-1}^{\geq 2}(t), \\
 V &= D_{n-1}^{0,\geq 2}(t),
 \end{aligned}$$

because $D_{n,0}^1 = D_{n-1,n}^1 = D_{n,0}^{0,1} = D_{n-1,n}^{0,1} = D_{n-1,0}^{0,\geq 2} = D_{n-1,n}^{0,\geq 2} = 0$. Hence, (i) follows. \square

In view of the decomposition property (13) of the sub-Eulerian polynomials, the following is the immediate consequence of Proposition 3.3.

COROLLARY 3.4. *We have*

$$\begin{aligned}
 D_n(t) &= 2ntD_{n-1}(t) + 2t(1-t)D_{n-1}'(t) + (t^{-1}-t)D_{n-1}^{0,1}(t) + (1-t)^2D_{n-1}^{\geq 2}(t) \\
 &\quad + 2(1-t)D_{n-1}^{0,\geq 2}(t).
 \end{aligned}$$

Proof. Summing Proposition 3.3 (i)–(iv) and using (13). □

The above recurrence relation will be revisited later when further properties of the sub-Eulerian polynomials are found.

n	$D_n^1(t)$
2	t
3	$3t + 3t^2$
4	$7t + 34t^2 + 7t^3$
5	$15t + 225t^2 + 225t^3 + 15t^4$
6	$31t + 1196t^2 + 3306t^3 + 1196t^4 + 31t^5$

TABLE 1. The sub-Eulerian polynomial $D_n^1(t)$ for $n = 2, \dots, 6$.

n	$D_n^{0,1}(t)$
2	t^2
3	$5t^2 + t^3$
4	$17t^2 + 30t^3 + t^4$
5	$49t^2 + 303t^3 + 127t^4 + t^5$
6	$129t^2 + 2132t^3 + 3030t^4 + 468t^5 + t^6$

TABLE 2. The sub-Eulerian polynomial $D_n^{0,1}(t)$ for $n = 2, \dots, 6$.

n	$D_n^{\geq 2}(t)$
2	1
3	$1 + 5t$
4	$1 + 30t + 17t^2$
5	$1 + 127t + 303t^2 + 49t^3$
6	$1 + 468t + 3030t^2 + 2132t^3 + 129t^4$

TABLE 3. The sub-Eulerian polynomial $D_n^{\geq 2}(t)$ for $n = 2, \dots, 6$.

n	$D_n^{0,\geq 2}(t)$
2	t
3	$3t + 3t^2$
4	$7t + 34t^2 + 7t^3$
5	$15t + 225t^2 + 225t^3 + 15t^4$
6	$31t + 1196t^2 + 3306t^3 + 1196t^4 + 31t^5$

TABLE 4. The sub-Eulerian polynomial $D_n^{0,\geq 2}(t)$ for $n = 2, \dots, 6$.

4. GENERATING FUNCTIONS

Proposition 3.3 offers an efficient way of computing $D_n^1(t)$, etc. This section discusses a convenient way of recording them, e.g., by their exponential generating functions. Define

the exponential generating functions for the sub-Eulerian polynomials by

$$\begin{aligned}
 D^1(x, t) &= \sum_{n \geq 2} D_n^1(t) \frac{x^n}{n!}, \\
 D^{0,1}(x, t) &= \sum_{n \geq 2} D_n^{0,1}(t) \frac{x^n}{n!}, \\
 D^{\geq 2}(x, t) &= \sum_{n \geq 2} D_n^{\geq 2}(t) \frac{x^n}{n!}, \\
 D^{0, \geq 2}(x, t) &= \sum_{n \geq 2} D_n^{0, \geq 2}(t) \frac{x^n}{n!}.
 \end{aligned}
 \tag{15}$$

Here, we are interested in obtaining closed form expressions for the right hand sides of (15).

Denote by $\mathfrak{S}_n(t)$ and $B_n(t)$ the n -th Eulerian polynomials of type A and B for \mathfrak{S}_n and B_n , respectively. In the case of $\mathfrak{S}(x, t) = \sum_{n \geq 0} \mathfrak{S}_n(t) x^n / n!$, one determines $f_{n,k}$ for which $\mathfrak{S}_n(t) / (1-t)^{n+1} = \sum_{k \geq 0} f_{n,k} t^k$, then multiply by $x^n / n!$, sum over n , and replace x by $x(1-t)$. This same procedure is complicated for $D^1(x, t)$, etc., because they are coupled via Proposition 3.3 (i)–(iv). Also, it is not clear at present what are the right denominators $Q(t)$ for the rational generating function $D_n^*(t) / Q(t)$.

An alternative, which we believe to be new, way of computing them is as follows. Let us show how it works in the case of $\mathfrak{S}(x, t)$. By taking partial derivatives of $\mathfrak{S}(x, t)$ with respect to x and t , and making use of the difference-differential equation which $\mathfrak{S}_n(t)$ satisfy, namely,

$$\mathfrak{S}_n(t) = [(n-1)t + 1] \mathfrak{S}_{n-1}(t) + t(1-t) \mathfrak{S}'_{n-1}(t),
 \tag{16}$$

we obtain that $\mathfrak{S} = \mathfrak{S}(x, t)$ satisfies the following first order linear partial differential equation

$$t(t-1) \mathfrak{S}_t + (1-xt) \mathfrak{S}_x = \mathfrak{S},
 \tag{17}$$

which, together with the initial condition $\mathfrak{S}(0, t) = 1$, uniquely determine \mathfrak{S} , that is,

$$\mathfrak{S}(x, t) = \frac{(1-t)e^{x(1-t)}}{1-te^{x(1-t)}}.
 \tag{18}$$

For completeness, we record the PDE which $B(x, t) = \sum_{n \geq 0} B_n(t) x^n / n!$ satisfies, namely

$$2t(t-1)B_t + (1-2xt)B_x = (1+t)B.
 \tag{19}$$

The concerned initial condition is $B(0, t) = 1$, and the solution to (19) is of course

$$B(x, t) = \frac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}.
 \tag{20}$$

We shall not give the details of the proof of (17) and (19) but encourage interested readers to work them out by imitating the proof of Proposition 4.1 below.

PROPOSITION 4.1. *We have*

$$\text{(i) } 2t(t-1)D_t^1 + (1-2xt)D_x^1 = -D^1 + D^{0,1} + tD^{\geq 2} + D^{0, \geq 2} + xt;$$

- (ii) $2t(t-1)D_t^{0,1} + (1-2xt)D_x^{0,1} = tD^1 + (t-2)D^{0,1} + t^2D^{\geq 2} + tD^{0,\geq 2} + xt^2;$
- (iii) $2t(t-1)D_t^{\geq 2} + (1-2xt)D_x^{\geq 2} = D^1 + t^{-1}D^{0,1} + (1-2t)D^{\geq 2} + D^{0,\geq 2} + x;$
- (iv) $2t(t-1)D_t^{0,\geq 2} + (1-2xt)D_x^{0,\geq 2} = tD^1 + D^{0,1} + tD^{\geq 2} - tD^{0,\geq 2} + xt.$

Proof. We only prove (iii); the remaining assertions follow from similar reasoning. We have

$$\begin{aligned}
2t(1-t)D_t^{\geq 2} &= \sum_{n \geq 2} 2t(1-t)(D_n^{\geq 2})'(t) \frac{x^n}{n!} \\
&= \sum_{n \geq 2} \{D_{n+1}^{\geq 2}(t) - [2nt - (2t-1)]D_n^{\geq 2}(t) - D_n^{0,\geq 2}(t) - D_n^1(t) \\
&\quad - t^{-1}D_n^{0,1}(t)\} \frac{x^n}{n!} \\
&= \sum_{n \geq 2} D_{n+1}^{\geq 2}(t) \frac{x^n}{n!} - 2xt \sum_{n \geq 2} D_n^{\geq 2}(t) \frac{x^{n-1}}{(n-1)!} \\
&\quad + (2t-1) \sum_{n \geq 2} D_n^{n \geq 2}(t) \frac{x^n}{n!} - \sum_{n \geq 2} D_n^{0,\geq 2}(t) \frac{x^n}{n!} \\
&\quad - \sum_{n \geq 2} D_n^1(t) \frac{x^n}{n!} - t^{-1} \sum_{n \geq 2} D_n^{0,1}(t) \frac{x^n}{n!} \\
&= D_x^{\geq 2} - D_2^{\geq 2}(t)x - 2xtD_x^{\geq 2} + (2t-1)D^{\geq 2} - D^{0,\geq 2} - D^1 \\
&\quad - t^{-1}D^{0,1},
\end{aligned}$$

from which (iii) follows, because $D_2^{\geq 2}(t) = 1$. □

The above set of PDEs are readily solved by the method of characteristics [2]. The idea is to solve for characteristic $x = x(t; A)$, which satisfies $dx/dt = (1-2xt)/2t(t-1)$ and is parametrized by A , where A is the constant of integration; for x and t related by $x = x(t; A)$, the partial derivatives become total derivatives with respect to t , and the PDEs are thus reduced to ODEs and the solutions of which solve the given PDEs. (Here, the constants of integration of the latter ODEs are arbitrary functions of A , and whose forms can be determined by the initial conditions.)

THEOREM 4.2. *We have*

- (i) $D^{0,\geq 2}(x, t) = \frac{t(e^{x(1-t)} - 1)^2}{2(1-t)(1-te^{2x(1-t)})} = D^1(x, t),$
- (ii) $D^{\geq 2}(x, t) = \frac{-1 - x(1-t) + e^{x(1-t)}}{(1-t)(1-te^{2x(1-t)})},$
- (iii) $D^{0,1}(x, t) = \frac{t^2 e^{x(1-t)} - t^2[1 - x(1-t)]e^{2x(1-t)}}{(1-t)(1-te^{2x(1-t)})}.$

Proof. Let $x = x(t)$ be such that $dx/dt = (1-2xt)/2t(t-1)$. The latter linear first order ODE is readily solved, e.g., $t^{1/2}e^{x(1-t)} = A$, where A is a constant. On $x = x(t)$, the left hand sides of Proposition 4.1 become total derivatives with respect to t . In particular,

Proposition 4.1 (iv) is then

$$(21) \quad 2t(t-1) \frac{dD^{0,\geq 2}}{dt} = tD^1 + D^{0,1} + tD^{\geq 2} - tD^{0,\geq 2}.$$

Differentiating (21) with respect to t , using $dx/dt = (1-2xt)/2t(t-1)$, multiplying by $2t(t-1)$, followed by substituting the t -derivatives on the right by the corresponding right hand side in Proposition 4.1, we have that $D^{0,\geq 2}$ satisfies the following ODE,

$$(22) \quad 4t(t-1)^2 \frac{d^2 D^{0,\geq 2}}{dt^2} + 2(t-1)(3t-1) \frac{dD^{0,\geq 2}}{dt} - 4D^{0,\geq 2} = 1,$$

whose solution (obtained by *Mathematica*) is

$$(23) \quad D^{0,\geq 2}(x, t) = \frac{1-t+4(1+t)C_1(A) - 8it^{1/2}C_2(A)}{4(t-1)}.$$

Here, C_1 , and C_2 are arbitrary functions of A . The initial condition $D^{0,\geq 2}(0, t) = 0$ implies that

$$\frac{1-t+4(1+t)C_1(t^{1/2}) - 8it^{1/2}C_2(t^{1/2})}{4(t-1)} = 0$$

so that

$$C_2(t) = \frac{1-t^2+4(1+t^2)C_1(t)}{8it}.$$

Reporting this back in (23) and replacing A by $t^{1/2}e^{x(1-t)}$ yields that

$$(24) \quad D^{0,\geq 2}(x, t) = \frac{(e^{x(1-t)} - 1)[te^{x(1-t)} + 1 + 4(1-te^{x(1-t)})C_1(t^{1/2}e^{x(1-t)})]}{4(t-1)e^{x(1-t)}}.$$

The initial condition $D_x^{0,\geq 2}(0, t) = 0$ then yields that

$$0 = \frac{(1-t)(t+1+4(1-t)C_1(t^{1/2}))}{4(t-1)},$$

from which we deduce that

$$C_1(t) = \frac{t^2+1}{4(t^2-1)}.$$

Substituting $C_1(t)$ back in (24) and after some algebra, the first equality in (i) follows. To prove the second equality in (i), we differentiate

$$(25) \quad 2t(t-1) \frac{dD^1}{dt} = -D^1 + D^{0,1} + tD^{\geq 2} + D^{0,\geq 2} + xt$$

with respect to t along the characteristic $t^{1/2}e^{x(1-t)} = A$, multiply the resulting equation by $2t(t-1)$, and then substitute the t -derivatives by the corresponding right hand sides in Proposition 4.1, the end result being

$$(26) \quad 4t(t-1)^2 \frac{d^2 D^1}{dt^2} + 2(t-1)(3t-1) \frac{dD^1}{dt} - 4D^1 = 1.$$

Note that (26) is the same as (22). Since the initial conditions $D^1(0, t) = D_x^1(0, t) = 0$ are also the same as those for $D^{0,\geq 2}$, we conclude that the second equality in (i) holds.

We have, from Proposition 4.1 (iii) and (iv), that

$$t(t-1)\frac{dD^{\geq 2}}{dt} + tD^{\geq 2} = (t-1)\frac{dD^{0,\geq 2}}{dt} + D^{0,\geq 2},$$

which can be written simply as

$$(27) \quad t\frac{d}{dt}((t-1)D^{\geq 2}) = \frac{d}{dt}((t-1)D^{0,\geq 2}).$$

The right hand side of (27) is readily computed, i.e.,

$$\begin{aligned} \frac{d}{dt}((t-1)D^{0,\geq 2}) &= \frac{d}{dt} \left(\frac{t(e^{x(1-t)} - 1)^2}{2(te^{2x(1-t)} - 1)} \right) = \frac{d}{dt} \left(\frac{A^2 - 2t^{1/2}A + t}{2(A^2 - 1)} \right) \\ &= \frac{-t^{-1/2}A + 1}{2(A^2 - 1)}, \end{aligned}$$

so that (27) now reads

$$t\frac{d}{dt}((t-1)D^{\geq 2}) = \frac{-t^{-1/2}A + 1}{2(A^2 - 1)}.$$

Straightforward integration yields that

$$\begin{aligned} D^{\geq 2} &= \frac{2t^{-1/2}A + \ln t}{2(1-t)(1-A^2)} + \frac{f(A)}{t-1} \\ &= \frac{2e^{x(1-t)} + \ln t}{2(1-t)(1-te^{2x(1-t)})} + \frac{f(t^{1/2}e^{x(1-t)})}{t-1}. \end{aligned}$$

The condition $D^{\geq 2}(0, t) = 0$ then forces

$$f(t) = \frac{1 + \ln t}{1 - t^2}.$$

With this f , and after some algebra, (ii) follows.

We have, from Proposition 4.1 (i) and (ii), that

$$(28) \quad t(t-1)\frac{dD^{0,1}}{dt} + D^{0,1} = t^2(t-1)\frac{dD^1}{dt} + tD^1,$$

By (i),

$$D^1 = \frac{t(e^{x(1-t)} - 1)^2}{2(t-1)(te^{2x(1-t)} - 1)} = \frac{A^2 - 2t^{1/2}A + t}{2(t-1)(A^2 - 1)}.$$

The right hand side of (28) is easily computed, i.e.,

$$t^2(t-1)\frac{dD^1}{dt} + tD^1 = \frac{At^{3/2} - A^2t}{2(t-1)(A^2 - 1)},$$

so that (28) can be rewritten as

$$\frac{dD^{0,1}}{dt} + \frac{D^{0,1}}{t(t-1)} = \frac{At^{1/2} - A^2}{2(t-1)(A^2 - 1)}.$$

By the standard procedure for solving first order linear ODEs, we have that

$$(29) \quad \begin{aligned} D^{0,1} &= \frac{t}{t-1} \left\{ \frac{2At^{1/2} - A^2 \ln t}{2(A^2 - 1)} + g(A) \right\} \\ &= \frac{2t^2 e^{x(1-t)} - t^2 e^{2x(1-t)} \ln t}{2(1-t)(1-te^{2x(1-t)})} + \frac{tg(t^{1/2} e^{x(1-t)})}{t-1}, \end{aligned}$$

where g is an arbitrary function. The condition $D^{0,1}(0, t) = 0$ then implies that

$$g(t) = \frac{t^2 \ln t - t^2}{t^2 - 1}.$$

Reporting g back in (29), and after some algebra, (iii) follows. \square

Some properties of $D_n^1(t)$, etc., are readily deduced from Theorem 4.2.

COROLLARY 4.3. For $n \geq 2$,

- (i) $D_n^1(t) = D_n^{0, \geq 2}(t)$,
- (ii) $D_n^{0,1}(t) = t^n D_n^{\geq 2}(t^{-1})$,
- (iii) $D_n^{\geq 2}(t) = t^n D_n^{0,1}(t^{-1})$,
- (iv) $D_n^{0, \geq 2}(t) = t^n D_n^{0, \geq 2}(t^{-1})$,
- (v) $D_n^1(t) = t^n D_n^1(t^{-1})$,

Proof. Since $D^1(x, t) = D^{0, \geq 2}(x, t)$, equating the coefficients of x^n of $D^1(x, t)$ and $D^{0, \geq 2}(x, t)$, (i) follows. Since

$$\begin{aligned} D^{\geq 2}(xt, t^{-1}) &= \frac{-1 - xt(1 - t^{-1}) + e^{x(1-t^{-1})}}{(1 - t^{-1})(1 - t^{-1}e^{2xt(1-t^{-1})})} \\ &= \frac{t^2 e^{2x(1-t)} [-1 + x(1-t) + e^{-x(1-t)}]}{(1-t)(1 - te^{2x(1-t)})} \\ &= \frac{t^2 e^{x(1-t)} - t^2 [1 - x(1-t)] e^{2x(1-t)}}{(1-t)(1 - te^{2x(1-t)})} = D^{0,1}(x, t), \end{aligned}$$

equating the coefficients of x^n on both sides, (ii) follows. To prove (iii), we replace t by t^{-1} , and x by xt in $D^{\geq 2}(xt, t^{-1}) = D^{0,1}(x, t)$, the end result being

$$D^{\geq 2}(x, t) = D^{\geq 2}((xt)(t^{-1}), (t^{-1})^{-1}) = D^{0,1}(xt, t^{-1}),$$

from which (iii) follows. By a calculation similar to that in (ii), we have that $D^{0, \geq 2}(xt, t^{-1}) = D^{0, \geq 2}(x, t)$ from which (iv) follows. (v) follows from (i) and (iv). \square

REMARK 4.4. Corollary 4.3 (ii)–(v) state that $D_n^1(t)$, etc., have certain reflectional symmetry. For example, the polynomial $t^n D_n^{\geq 2}(t^{-1})$ is obtained by reflecting the coefficients of $D_n^{\geq 2}(t)$ about $x^{n/2}$. Thus, Corollary 4.3 (ii)–(iii) (resp., (iv)–(v)) say that $D_n^{0,1}(t)$ (resp., $D_n^1(t)$) are obtained from $D_n^{\geq 2}(t)$ (resp., $D_n^{0, \geq 2}(t)$) by reflection, and vice versa. Corollary 4.3 (i) further says that $D_n^1(t)$ and $D_n^{0, \geq 2}(t)$ are themselves reflectionally symmetric. However, $D_n^{0,1}(t)$ and $D_n^{\geq 2}(t)$ do not enjoy this property.

Corollary 4.3 (i) enables a further simplification of the recurrence relation for $D_n(t)$ derived in the previous section.

COROLLARY 4.5. *We have*

$$(30) \quad D_n(t) = [(2n-1)t+1]D_{n-1}(t) + 2t(1-t)D'_{n-1}(t) \\ + (1-t)[t^{-1}D_{n-1}^{0,1}(t) - tD_{n-1}^{\geq 2}(t)].$$

Proof. By Corollary 4.3 (i) and (13), the last three terms on the right hand side of Corollary 3.4 can be written as

$$\begin{aligned} & (t^{-1}-t)D_{n-1}^{0,1}(t) + (1-t)^2D_{n-1}^{\geq 2}(t) + 2(1-t)D_{n-1}^{0,\geq 2}(t) \\ &= (1-t)[(1+t^{-1})D_{n-1}^{0,1}(t) + (1-t)D_{n-1}^{\geq 2}(t) + D_{n-1}^1(t) + D_{n-1}^{0,\geq 2}(t)] \\ &= (1-t)D_{n-1}(t) + (1-t)[t^{-1}D_{n-1}^{0,1}(t) - tD_{n-1}^{\geq 2}(t)]. \end{aligned}$$

Aftering regrouping terms, the corollary follows. \square

REMARK 4.6. The first line of (30) is precisely the recurrence relation for the Eulerian polynomial $B_n(t)$ of type B .

Denote by $D(x, t)$ the exponential generating function for the Eulerian polynomial $D_n(t)$ of type D , i.e.,

$$(31) \quad D(x, t) = \sum_{n \geq 0} D_n(t) \frac{x^n}{n!},$$

where $D_0(t) = 1$. Since $D_1(t) = 1$, it is clear that

$$(32) \quad D(x, t) = 1 + x + D^1(x, t) + D^{0,1}(x, t) + D^{\geq 2}(x, t) + D^{0,\geq 2}(x, t).$$

Computing $D(x, t)$ by (32) and Theorem 4.2, we obtain the following result of Brenti [1].

COROLLARY 4.7. *We have*

$$(33) \quad D(x, t) = \frac{(1-t)e^{x(1-t)} - xt(1-t)e^{2x(1-t)}}{1 - te^{2x(1-t)}}.$$

We have now the exponential generating functions for the sub-Eulerian polynomials at our disposal. By reversing the procedure described in the paragraph following the definitions of $D^1(x, t)$, etc., we have the following results.

COROLLARY 4.8. *For $n \geq 2$,*

$$\begin{aligned} \text{(i)} \quad & \frac{D_n^{0,\geq 2}(t)}{t(1-t)^{n-1}} = \sum_{k \geq 0} \{2^{n-1}[(k+1)^n + k^n] - (2k+1)^n\} t^k = \frac{D_n^1(t)}{t(1-t)^{n-1}}, \\ \text{(ii)} \quad & \frac{D_n^{\geq 2}(t)}{(1-t)^{n-1}} = \sum_{k \geq 0} [(2k+1)^n - (n+1)(2k)^n] t^k, \\ \text{(iii)} \quad & \frac{D_n^{0,1}(t)}{t^2(1-t)^{n-1}} = \sum_{k \geq 0} [(2k+1)^n - 2^n(k+1)^n + n2^{n-1}(k+1)^{n-1}] t^k. \end{aligned}$$

Proof. We only prove the first equality in (i); the remaining assertions follow from similar reasoning.

$$\begin{aligned}
\sum_{n \geq 2} \frac{D_n^{0, \geq 2}(t) x^n}{t(1-t)^{n-1} n!} &= \frac{1-t}{t} D^{0, \geq 2}\left(\frac{x}{1-t}, t\right) \\
&= \frac{(e^x - 1)^2}{2(1 - te^{2x})} \\
&= \frac{(e^{2x} - 2e^x + 1)}{2} \sum_{k \geq 0} t^k e^{2kx} \\
&= \sum_{k \geq 0} t^k \frac{(e^{2(k+1)x} - 2e^{(2k+1)x} + e^{2kx})}{2} \\
&= \sum_{k \geq 0} t^k \sum_{n \geq 0} [2^{n-1}(k+1)^n - (2k+1)^n + 2^{n-1}k^n] \frac{x^n}{n!}.
\end{aligned}$$

Equating the coefficient of x^n on both sides, the result follows. \square

5. A RECURRENCE RELATION

In the previous section, we computed the PDEs which the generating functions of the sub-Eulerian polynomials satisfy. By solving these PDEs, we obtained the generating functions. It is important to note that the PDEs encode the recurrence relations for the sub-Eulerian polynomials. This procedure for computing generating functions can be reversed, i.e., starting from the generating function, we determine a PDE of minimal order which the generating function satisfies; the desired recurrence relation then follows upon equating coefficients.

We shall apply this reversed procedure to $D(x, t)$ to obtain a recurrence relation for $D_n(t)$. Proposition 4.1 and (32) suggest the partial differential operator $2t(t-1)\partial_t + (1-2xt)\partial_x$ for $D(x, t)$. The proofs of the following lemma are easy, albeit tedious, exercises of calculus which we omit.

LEMMA 5.1. *We have*

$$\begin{aligned}
\text{(i)} \quad [2t(t-1)\partial_t + (1-2xt)\partial_x]D(x, t) &= \frac{(1-t)[(1+t)e^{x(1-t)} - te^{2x(1-t)}]}{1 - te^{2x(1-t)}}; \\
\text{(ii)} \quad [2t(t-1)\partial_t + (1-2xt)\partial_x]^2 D(x, t) &= \frac{(1-t)[(1+3t^2)e^{x(1-t)} - 2t^2e^{2x(1-t)}]}{1 - te^{2x(1-t)}}; \\
\text{(iii)} \quad [2t(t-1)\partial_t + (1-2xt)\partial_x]^2 &= 4t^2(1-t)^2\partial_{tt} - 4t(1-t)(1-2xt)\partial_{xt} + (1-2xt)^2\partial_{xx} - \\
&\quad 2t(1-2x)\partial_x + 4t(1-t)(1-2t)\partial_t.
\end{aligned}$$

PROPOSITION 5.2. *The function $D = D(x, t)$ satisfies the following PDE:*

$$\begin{aligned}
&4t^2(1-t)^2[x(1+t)-1]D_{tt} - 4t(1-t)(1-2xt)[x(1+t)-1]D_{xt} + \\
&(1-2xt)^2[x(1+t)-1]D_{xx} + 2t(1-t)^2[-2+(3+t)x]D_t + \\
&[4t-x(1+3t)^2+2x^2t(3+2t+3t^2)]D_x + (1-t)^2D = 0.
\end{aligned}$$

Proof. Use Lemma 5.1(i)–(ii) to form the expression

$$(34) \quad \begin{aligned} & \alpha[2t(t-1)\partial_t + (1-2xt)\partial_x]^2 D(x, t) + \beta[2t(t-1)\partial_t + (1-2xt)\partial_x] D(x, t) \\ &= \frac{(1-t)\{\alpha(1+t) + \beta(1+3t^2)\}e^{x(1-t)} + (-\alpha t - 2\beta t^2)e^{2x(1-t)}}{1 - te^{2x(1-t)}} \end{aligned}$$

where $\alpha = \alpha(x, t)$, and $\beta = \beta(x, t)$ are functions to be determined. Now assume that the right side of (34) is equal to

$$(35) \quad \frac{(1-t)[\gamma e^{x(1-t)} - \gamma xte^{2x(1-t)}]}{1 - te^{2x(1-t)}} = \gamma D(x, t),$$

where $\gamma = \gamma(x, t)$ is a function to be determined. Equating the coefficients of $e^{x(1-t)}$ and $e^{2x(1-t)}$ in the numerators of (34) and (35), we have that

$$\alpha(1+t) + \beta(1+3t^2) = \gamma, \quad \text{and} \quad -\alpha t - 2\beta t^2 = -\gamma x t,$$

whose formal solutions are $\alpha = \gamma(1-t)^{-2}[x(1+3t^3) - 2t]$, and $\beta = -\gamma(1-t)^{-2}[x(1+t) - 1]$. Setting $\gamma = (1-t)^2$, we get $\alpha = [x(1+3t^3) - 2t]$ and $\beta = -[x(1+t) - 1]$. Reporting these choices of α , β , and γ in (34), and using Lemma 5.1(iii), the result follows. \square

It is trivial to see that

$$\begin{aligned} D &= \sum_{n \geq 0} D_n(t) \frac{x^n}{n!}, & D_t &= \sum_{n \geq 0} D'_n(t) \frac{x^n}{n!}, & D_x &= \sum_{n \geq 0} D_{n+1}(t) \frac{x^n}{n!}, \\ D_{tt} &= \sum_{n \geq 0} D''_n(t) \frac{x^n}{n!}, & D_{xt} &= \sum_{n \geq 0} D'_{n+1}(t) \frac{x^n}{n!}, & D_{xx} &= \sum_{n \geq 0} D_{n+2}(t) \frac{x^n}{n!}. \end{aligned}$$

We are now ready to state the main result of this section.

THEOREM 5.3. *For $n \geq 1$, the Eulerian polynomials $D_n(t)$ of type D satisfy*

$$(36) \quad \begin{aligned} D_{n+2}(t) &= [n(1+5t) + 4t]D_{n+1}(t) \\ &\quad + 4t(1-t)D'_{n+1}(t) \\ &\quad + [(1-t)^2 - n(1+3t)^2 - 4n(n-1)t(1+2t)]D_n(t) \\ &\quad - [4nt(1-t)(1+3t) + 4t(1-t)^2]D'_n(t) \\ &\quad - 4t^2(1-t)^2D''_n(t) \\ &\quad + [2n(n-1)t(3+2t+3t^2) - 4n(n-1)(n-2)t^2(1+t)]D_{n-1}(t) \\ &\quad + [2nt(1-t)^2(3+t) + 8n(n-1)t^2(1-t)(1+t)]D'_{n-1}(t) \\ &\quad + 4nt^2(1-t)^2(1+t)D''_{n-1}(t). \end{aligned}$$

Proof. Denote by I, II, \dots, VI the successive terms of the PDE as in the previous proposition. Then

$$\begin{aligned}
I &= [-4t^2(1-t)^2 + 4t^2(1-t)^2(1+t)x] \sum_{n \geq 0} D_n''(t) \frac{x^n}{n!}, \\
II &= [4t(1-t) - 4t(1-t)(1+3t)x + 8t^2(1-t)(1+t)x^2] \sum_{n \geq 0} D_{n+1}'(t) \frac{x^n}{n!}, \\
III &= [-1 + (1+5t)x - 4t(1+2t)x^2 + 4t^2(1+t)x^3] \sum_{n \geq 0} D_{n+2}(t) \frac{x^n}{n!}, \\
IV &= [-4t(1-t)^2 + 2t(1-t)^2(3+t)x] \sum_{n \geq 0} D_n'(t) \frac{x^n}{n!}, \\
V &= [4t - (1+3t)^2x + 2t(3+2t+3t^2)x^2] \sum_{n \geq 0} D_{n+1}(t) \frac{x^n}{n!}, \\
VI &= (1-t)^2 \sum_{n \geq 0} D_n(t) \frac{x^n}{n!},
\end{aligned}$$

so that

$$\begin{aligned}
0 = [x^n](I + \dots + VI) &= -\frac{4t^2(1-t)^2 D_n''(t)}{n!} + \frac{4t^2(1-t)^2(1+t) D_{n-1}''(t)}{(n-1)!} \\
&+ \frac{4t(1-t) D_{n+1}'(t)}{n!} - \frac{4t(1-t)(1+3t) D_n'(t)}{(n-1)!} \\
&+ \frac{8t^2(1-t)(1+t) D_{n-1}'(t)}{(n-2)!} - \frac{D_{n+2}(t)}{n!} + \frac{(1+5t) D_{n+1}(t)}{(n-1)!} \\
&- \frac{4t(1+2t) D_n(t)}{(n-2)!} + \frac{4t^2(1+t) D_{n-1}(t)}{(n-3)!} \\
&- \frac{4t(1-t)^2 D_n'(t)}{n!} + \frac{2t(1-t)^2(3+t) D_{n-1}'(t)}{(n-1)!} \\
&+ \frac{4t D_{n+1}(t)}{n!} - \frac{(1+3t)^2 D_n(t)}{(n-1)!} \\
&+ \frac{2t(3+2t+3t^2) D_{n-1}(t)}{(n-2)!} + \frac{(1-t)^2 D_n(t)}{n!},
\end{aligned}$$

where $[x^n](I + \dots + VI)$ denotes the coefficient of x^n in $I + \dots + VI$. Multiplying through by $n!$ and collecting terms, the theorem follows. \square

The recurrence relation (36) involves $D_n(t)$ and its derivatives only. By equating coefficients of t^k , we have the following recurrence relation for $D_{n,k}$.

COROLLARY 5.4.

$$\begin{aligned}
D_{n+2,k} = & (n+4k)D_{n+1,k} \\
& + (5n-4k+8)D_{n+1,k-1} \\
& + [(1-n) - 4(1+n) - 4k(k-1)]D_{n,k} \\
& + [-2(1+n+2n^2) + 8(1-n)(k-1) + 8(k-1)(k-2)]D_{n,k-1} \\
& + [(1-n-8n^2) - 4(1-3n)(k-2) - 4(k-2)(k-3)]D_{n,k-2} \\
& + [6nk + 4nk(k-1)]D_{n-1,k} \\
& + [6n(n-1) + 2n(4n-9)(k-1) - 4n(k-1)(k-2)]D_{n-1,k-1} \\
& + [4n(n-1)^2 + 2n(k-2) - 4n(k-2)(k-3)]D_{n-1,k-2} \\
& + [2n(1-3n+2n^2) + 2n(5-4n)(k-3) + 4n(k-3)(k-4)]D_{n-1,k-3}.
\end{aligned}$$

Theorem 5.3 (or equivalently Corollary 5.4) constitutes the solution to a problem concerning the recurrence relations for the Eulerian polynomials $D_n(t)$ (or the Eulerian number $D_{n,k}$), posed by Brenti [1]. Note that some of the coefficients of $D_{n,k}$ are not linear in n as well as in k . A direct combinatorial proof of Corollary 5.4 is desired, however.

A closely related problem, also posed by Brenti, is whether the Eulerian polynomials $D_n(t)$ have real zeros only. That $\mathfrak{S}_n(t)$ and $B_n(t)$ having only real zeros follow easily from their recurrence relations. Given the complicated nature of the recurrence relation for $D_n(t)$, the corresponding assertion for $D_n(t)$ need not follow in a similar manner.

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