# Classes of Lattices induced by Chip Firing 

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#### Abstract

In this paper we study three classes of models widely used in physics, computer science and social science: the Chip Firing Game, the Abelian Sandpile Model and the Chip Firing Game on a mutating graph. We study the set of configurations reachable from a given initial configuration, called the configuration space of a model, and try to determine the main properties of such sets. We study the order induced over the configurations by the evolution rule. This makes it possible to compare the power of expression of these models. It is known that the configuration spaces we obtain are lattices, a special kind of partially ordered set. Although the Chip Firing Game on a mutating graph is a generalization of the usual Chip Firing Game, we prove that these models generate exactly the same configuration spaces. We also prove that the class of lattices induced by the Abelian Sandpile Model is strictly included in the class of lattices induced by the Chip Firing Game, but contains the class of distributive lattices, a very well known class.


## Introduction

The Chip Firing Game (CFG), the Abelian Sandpile Model (ASM), and the Chip Firing Game on a mutating graph (which we will call Mutating Chip Firing Game or MCFG) are closely related models studied in physics [BTW87, DRSV95], computer science [GMP98, BLS91, BL92] and social science [Big97, Big99, Heu99]. These three models are variations of the following game: given a graph and a distribution of chips on its vertices (called configuration), one may select a vertex that contains at least as many chips as its outdegree, and move one chip from this vertex along each of its outgoing edges, to the vertex at the other extremity. Many questions have naturally arisen and given matter for research about this game: given a graph and an initial configuration, does the game stop after some time, or can it be played forever [BLS91, BL92, Eri96] ?
determine the differences between the classes of configuration spaces induced by each model, i.e. given the configuration space of a game, we will try to decide if it is isomorphic to the configuration space of a game of another type. For instance, given the configuration space of a CFG, does there exist a MCFG or an ASM such that its configuration space is isomorphic to it?

In Section 1, we give the definitions and known results used in this paper, as well as the definitions of the models. In Section 2, we show that CFGs and MCFGs induce exactly the same class of configuration spaces, and in Section 3, we will compare the configuration spaces of CFGs and ASMs. Since an ASM is a special CFG, we will study under which condition a CFG can be transformed into an ASM. We will give a sufficient (but not necessary) condition of such a transformation, and we will give an example of a CFG that cannot be transformed into an ASM.

## 1 Definitions and known results

We will need some definitions from order and lattice theory to describe the configuration spaces of the models. We give them first, after what we give the definition of each model, as well as the previously known results about them.

### 1.1 Posets and lattices

A partially ordered set (or poset) is a set equipped with an order relation $\leq$ (i.e. a transitive, reflexive and antisymmetric relation). If $x$ and $y$ are two elements of a poset, we say that $x$ is covered by $y$ (or $y$ covers $x$ ), and write $x \prec y$ (or $y \succ x$ ) if $x<y$ and $x \leq z<y$ implies $z=x$. We then say that $x$ is a lower cover of $y$ (or that $y$ is an upper cover of $x$ ). The interval $[x, y]$ is the set $\{z, x \leq z \leq y\}$. To represent a poset $P$ we will use its Hasse diagram, defined as follows :

- each element $x$ of $P$ is represented by a point $p_{x}$ of the plane,
- if $x<y$, then $p_{x}$ is lower than $p_{y}$, and
- $p_{x}$ and $p_{y}$ are joined by a line if and only if $x \prec y$.

Two posets $P$ and $P^{\prime}$ are isomorphic if there exists a bijection $\varphi: P \longrightarrow P^{\prime}$ satisfying: for all $x, y \in P, x \leq y \Longleftrightarrow \varphi(x) \leq \varphi(y)$.


Figure 1: From left to right: a poset, a lattice, a distributive lattice and a hypercube of dimension 3
hypercube. Lower locally distributive (LLD) lattices are defined dually. All ULD lattices are ranked, i.e. all the paths in the covering relation from the minimal to the maximal element have the same length. A lattice $L$ is distributive if it satisfies one of the two following relations of distributivity (that are equivalent and imply each other):

$$
\begin{aligned}
& \text { for all } x, y, z \in L, x \vee(y \wedge z)=(x \vee y) \wedge(x \vee z) \\
& \text { for all } x, y, z \in L, x \wedge(y \vee z)=(x \wedge y) \vee(x \wedge z)
\end{aligned}
$$

A distributive lattice is a lattice that is at the same time upper and lower locally distributive, i.e. if the interval between any element and, on the one hand the join of its upper covers, and on the other hand the meet of all its lower covers both are hypercubes. In the sequel we will only be concerned with distributive and ULD lattices. Figure 1 shows examples of posets and different types of lattices.

For a more complete introduction to posets and lattices, see for instance [DP90].

### 1.2 The different kinds of Chip Firing Games

In this section we give the definitions of the Abelian Sandpile Model, the Chip Firing Game and the Mutating Chip Firing Game. We begin by presenting the features shared by the three models. Then we detail what is specific about each of them. We will see that some models are generalizations of others. Finally, we give some results about the CFG which will be useful in this paper.

### 1.2.1 Definitions

Each model is defined over a graph $G=(V, E)$, called the support graph of

initial configuration
Figure 2: An initial configuration of a CFG and its configuration space. Each edge ( $\sigma, \sigma^{\prime}$ ) is labelled by the name of the vertex fired to reach $\sigma^{\prime}$ from $\sigma$.
vertex. We call this process firing $v$. If $\sigma$ is the configuration we start from, and $\sigma^{\prime}$ is obtained from $\sigma$ by firing $v$, we write $\sigma \xrightarrow{v} \sigma^{\prime}$, and we call $\sigma$ a predecessor of $\sigma^{\prime}$.

Note 1.1 We consider that the firing rule cannot be applied to a vertex with no outgoing edges because firing it does not change the configuration of the game and is therefore of no interest to us.

The Chip Firing Game (CFG) [BL92] is defined over a directed graph. We give an example of a CFG together with its configuration space in Figure 2. The Abelian Sandpile Model (ASM) [BTW87] is defined over an undirected graph, with a distinguished vertex called the sink and denoted by $\perp$. The sink can never be fired. Finally the Chip Firing Game on a mutating graph (MCFG) [Eri96] is played on a directed graph that changes during the game in the following way: after the firing of a vertex, its outgoing edges are removed and new ones are added in a pre-determined way (the vertex set $V$ remains the same). A position of a MCFG consists in:

- a directed graph on $V$,
- on each node $v \in V$ a nonnegative number of chips,
- for each node $v \in V$ an infinite sequence $M_{v}$ of multisets of nodes in $V$, which is the mutation sequence of $v$.

Figure 3: An example of a firing sequence in a MCFG, where the beginning of the mutation sequences of the vertices are : $M_{a}=(\{c\}, \ldots), M_{b}=(\{a\}, \ldots), M_{c}=$ $(\{a, b\}, \ldots)$

These three games are strongly convergent games [Eri93], which implies that, given an initial configuration, either a given game can be played forever or it reaches a unique fixed point (where no firing is possible), called the final configuration, which does not depend on the order in which the vertices are fired. We will only consider convergent games, i.e. games that reach a fixed point. We call execution of a game any sequence of firing that, starting from the initial configuration, reaches the final configuration.

We call configuration space of a game $C$, and we denote it by $L(C)$, the set of all the configurations reachable from the initial configuration, ordered by the reflexive and transitive closure of the predecessor relation. It is known [BLS91, LP01, Eri96] that when a game is convergent, its configuration space is a ULD lattice. This is a very strong property, because ULD lattices are very structured sets. For instance, an immediate consequence of this is the fact that all the firing sequences, in any convergent ASM, CFG or MCFG, from the initial configuration to the final configuration, have the same length. We will say that two convergent games $C$ and $C^{\prime}$ are equivalent if $L(C)$ is isomorphic to $L\left(C^{\prime}\right)$.

We denote by $L(C F G), L(A S M)$ and $L(M C F G)$ the classes of lattices that are the configurations spaces of convergent CFGs, ASMs and MCFGs respectively. It is possible to guarantee that a CFG is convergent by the presence in the support graph of a sink (a vertex with no outgoing edges) reachable from all vertices [LP01]. This also holds for MCFGs. In the ASM, the presence of the distinguished vertex which can never be fired (called the sink because it has the same function as in the directed case) guarantees that the game is always convergent.

These three models are very close to one another, and it is easy to see that some of these are generalizations of others (the class of lattices they induce are included in one another): since undirected graphs are particular directed graphs, one can consider an ASM as a particular CFG, where each undirected edge $\{u, v\}$ with $u, v \neq \perp$ is replaced by two opposite directed edges $(u, v)$ and $(v, u)$, and where each undirected edge $\{v, \perp\}$ is replaced only by the directed edoe $(v, \mid)$ Therefore we obtain that $I(A S M) \subset I(C F G)$ Likewise a CFG

Thanks to this result, all the CFGs considered in the sequel will be simple, and such that their support graph has exactly one sink (denoted by $\perp$ ) and such that all vertices except $\perp$ are fired during an execution. This is always possible because, if this is not the case:

- either there is no sink and all vertices are fired during an execution; then we can add an isolated vertex to the graph, which becomes the sink,
- or there exists a vertex $v$ that is never fired during any execution but is not a sink; then we can remove its outgoing edges without changing the configuration space of the CFG, and $v$ becomes the sink,
- or there is more than one sink; then we can merge them into a single vertex without changing the configuration space.

In the sequel, we will restrict ourselves to CFGs for which the support graph has no loops (i.e. no ( $v, v$ ) edges). This is always possible because, for a simple CFG $C$, if there is a loop on a vertex $v$ ( $v$ cannot be the sink), we can replace it by an edge $(v, \perp)$, and the resulting CFG is equivalent to $C$, since $C$ is simple.

Given a vertex $v$, we denote by $d^{-}(v)$ its indegree, by $d^{+}(v)$ its outdegree, by $d_{\perp}(v)$ the number of edges from $v$ to $\perp$, and we define $d(v)=d^{+}(v)-d_{\perp}(v)$. Given two vertices $u$ and $v$, we denote by $d(u, v)$ the number of edges from $u$ to $v$. Given a CFG and a vertex $v$ of its support graph, the initial number of chips in $v$ is denoted by $\sigma_{0}(v)$, and the number of chips in $v$ in the final configuration by $\sigma_{f}(v)$ (since all the CFGs considered are simple, we have $\sigma_{f}(v)=\sigma_{0}(v)+$ $\left.d^{-}(v)-d^{+}(v)\right)$. The number of chips needed to fire $v$ is the difference between $\sigma_{0}(v)$ and $d^{+}(v)$, i.e. 0 if $\sigma_{0}(v) \geq d^{+}(v)$, and $d^{+}(v)-\sigma_{0}(v)$ otherwise.

The configuration spaces of simple CFGs can be described more easily than in the general case. Indeed we have the following results:

Lemma 1.4 [MPV01, LP01] In a simple CFG, if, starting from the same configuration, two sequences of firing lead to the same configuration, then the vertices fired in each sequence are the same.

This allows us to define the shot-set $s(\sigma)$ of a configuration $\sigma$ as the set of vertices fired to reach $\sigma$ from the initial configuration. Given a CFG with support graph $(V, E)$, we say that a subset $X \subseteq V$ is a valid shot-set if there exists a configuration $\sigma$ reachable from the initial configuration such that $s(\sigma)=X$. A
the shot-sets of the minimal configurations in which $c$ can be fired are $\{a\}$ and $\{b\}$, and we say that the first times at which $c$ can be fired are $a$ and $b$. The minimal configuration in which $a$ and $b$ can be fired is the minimal point of the lattice, and its shot-set is the empty set. We say that the first time at which $a$ and $b$ can be fired is the beginning of any execution. The knowledge, for all vertices, of the first times at which they can be fired, is a characterization of the configuration space of a CFG, as it is stated in the following proposition:

Proposition 1.5 Let $C$ be a simple $C F G$ and let $L=L(C)$. Let $\left\{v_{1}, \ldots, v_{n}, \perp\right\}$ be the set of vertices of $C$. For each $i, \in\{1, \ldots, n\}$, let $X_{i}$ be the set representing the first times at which $v_{i}$ can be fired (i.e. $X_{i}$ is the set of the shot-sets of the minimal configurations at which $v_{i}$ can be fired). Then $L$ is completely determined by $\left\{v_{1}, \ldots, v_{n}\right\}$ and $\left\{X_{1}, \ldots, X_{n}\right\}$.

Proof: We will construct $L$ from $\left\{v_{1}, \ldots, v_{n}\right\}$ and $\left\{X_{1}, \ldots, X_{n}\right\}$. Notice that, for each $i=0, \ldots, n$, and for each $x \in X_{i}$, the set $\left\{v_{i}\right\} \cup x$ is a valid shot-set of $C$ : by definition, $x$ is the shot-set of a configuration in which $v_{i}$ can be fired. We claim that all subsets $Y$ of $\left\{v_{1}, \ldots, v_{n}\right\}$ satisfying the following condition:

$$
\text { for all } v_{i} \in Y \text {, there exists } Z \in X_{i} \text {, such that } Z \subseteq Y
$$

are valid shot-sets of elements of $L$. Let $Y$ be such a set. $Y$ can be decomposed into a union of sets $Y_{1}, \ldots, Y_{k}$ such that, for each $j=1, \ldots, k$, there exists $i$ such that $Y_{j}$ is the union of $\left\{v_{i}\right\}$ and of an element $x$ of $X_{i}$ (notice that the sets $Y_{1}, \ldots, Y_{k}$ are not a partition of $Y$, some sets may overlap). Therefore, each of the sets $Y_{j}$ is a valid shot-set of $C$. Since the shot-sets are stable by union, $Y$ is itself a valid shot-set of $C$. On the other hand, every shot-set of $C$ satisfies the condition above.

Therefore the set of subsets of $\left\{v_{1}, \ldots, v_{n}\right\}$ that satisfy this condition is the set of shot-sets of $C$, which, ordered by inclusion, is isomorphic to $L$.

This result will be useful in the sequel when we have to guarantee that a modification made to a game does not change its configuration space. We now have all the definitions and tools needed to compare the classes of lattices induced by the models we have introduced.

## Theorem 2.1 Any convergent $M C F G$ is equivalent to a simple $M C F G$.

Proof: Let $C$ be a MCFG with vertex set $V$ and initial configuration $\sigma$. Suppose $C$ is not simple: there exists a vertex $a$ which is fired twice or more during any execution of $C$. The idea of the proof is to split $a$ into two vertices $a_{0}$ and $a_{1}$ which will be fired alternatively (the first firing of $a$ in $C$ corresponds to a firing of $a_{0}$ in $C^{\prime}$, the second to a firing of $a_{1}$, and so on), so that each of them is fired strictly less often than $a$ during an execution. Therefore, by iterating this process, one can transform $C$ into a simple MCFG.

Let $G$ be the initial support graph of $C$. If for each vertex $v$, the sequence of mutation is $M_{v}=\left(S_{v}^{1}, S_{v}^{2}, S_{v}^{3}, \ldots\right)$, we define $S_{v}^{0}$ to be the set of outgoing edges of $v$ in $G$. Therefore from now on, a MCFG will be defined by a set of vertices, an initial configuration, and for each vertex $v$ an inifinite sequence of edge sets $M_{v}=\left(S_{v}^{0}, S_{v}^{1}, S_{v}^{2}, \ldots\right)$. At any time in the game, if a vertex $v$ with mutating sequence ( $S_{v}^{0}, S_{v}^{1}, S_{v}^{2}, \ldots$ ) has been fired $i$ times, then the current outgoing edges of $v$ are those given by $S_{v}^{i}$. We then denote by $l_{i}(v)$ the number of loops on $v$ (i.e. the number of occurrences of $v$ in $S_{v}^{i}$ ), by $d_{i}{ }^{>}(v)$ the number of edges going out of $v$ which are not loops (i.e. $d_{i}{ }^{>}(v)=d_{i}^{+}(v)-l_{i}(v)$ ), and we define $d_{i}<(v)$ dually: $\left.d_{i}<(v)=d_{i}{ }^{-}(v)-l_{i}(v)\right)$. Finally we define $N$ to be twice the number of chips present in the game.

Let us define the MCFG $C^{\prime}$ in the following way:

- the vertex set of $C^{\prime}$ is $V^{\prime}=V \backslash\{a\} \cup\left\{a_{0}, a_{1}\right\}$, with $a_{0} \notin V$ and $a_{1} \notin V$.
- for each $v \neq a \in V$, and for each $i \in \mathbb{N}$ we define $S_{v}^{i}$, the $i$-th set in the mutation sequence of $v$, in the following way:
- for each occurrence of a vertex $v^{\prime} \neq a$ in $S_{v}^{i}$, there are two occurrences of $v^{\prime}$ in $S_{v}^{i}$,
- for each occurrence of $a$ in $S_{v}^{i}$, there is one occurrence of $a_{0}$ and one occurrence of $a_{1}$ in $S_{v}^{i}$.
- for each $i \in \mathbb{N}$, we define the sets $S_{a_{0}}^{i}$ and $S_{a_{1}}^{i}$, the $i$-th sets in the mutating sequences of $a_{0}$ and $a_{1}$, in the following way:
- for each occurrence of a vertex $w \neq a$ in $S_{a}^{2 i}$ (resp. $S_{a}^{2 i+1}$ ), there are two occurrences of $w$ in $S_{a_{0}}^{i}$ (resp. $S_{a_{1}}^{i}$ ).
- for each occurrence of $a$ in $S^{2 i}$ (resp. $S^{2 i+1}$ ). there is one occurrence

$a$ in $C$ after $2 j$ firings


Figure 4: The outgoing edges of the vertices in the MCFG constructed during the simplification of a MCFG.

An example of the construction of the graph is given in Figure 4.
We call the double of $C$ the CFG obtained from $C$ by multiplying by two the initial configuration of all vertices, as well as the number of edges between each pair of vertices. This CFG obviously is equivalent to $C$, and in the sequel we will show that $C^{\prime}$ is equivalent to the double of $C$. Notice that, except the two vertices $a_{0}$ and $a_{1}, C^{\prime}$ is the same as the double of $C$. So if the firings of $a_{0}$ and $a_{1}$ take place in turn as stated before, then we can easily show that $L\left(C^{\prime}\right)$ is isomorphic to $L(C)$.

We first prove by induction that any firing sequence $s$ of $C$, leading to a configuration $\sigma$, corresponds to a valid firing sequence $s^{\prime}$ in $C^{\prime}$ (by replacing the occurrences of $a$ by occurrences of $a_{0}$ and $a_{1}$ in turn), leading to a configuration $\sigma^{\prime}$, satisfying Proposition 1 and Proposition 2 or $2^{\prime}$.

$$
\begin{gather*}
\text { for all } v \neq a_{0}, a_{1}, \sigma^{\prime}(v)=2 \sigma(v)  \tag{1}\\
\sigma^{\prime}\left(a_{0}\right)=N+\sigma(a), \text { and } \sigma^{\prime}\left(a_{1}\right)=\sigma(a)  \tag{2}\\
\sigma^{\prime}\left(a_{0}\right)=\sigma(a), \text { and } \sigma^{\prime}\left(a_{1}\right)=N+\sigma(a)
\end{gather*}
$$

The initial configuration of $C$ and $C^{\prime}$ satisfy Propositions 1 and 2. So does any configuration reached by a firing sequence that does not contain $a$, because, except $a_{0}$ and $a_{1}, C^{\prime}$ is the same as the double of $C$, and because for each $v \neq a_{0}, a_{1}$, there is the same number of edges from $v$ to $a_{0}$ than from $v$ to $a_{1}$.
or $2^{\prime}$. Notice that, for each configuration $\sigma$ of $C$ corresponding to a configuration $\sigma^{\prime}$ in $C^{\prime}$, the vertices that can be fired in both configurations are the same. Therefore, there is no other valid firing sequence in $C^{\prime}$ than those corresponding to firing sequences of $C$. Therefore $L\left(C^{\prime}\right)$ is isomorphic to $L(C)$.

For simple MCFGs, the shot-set of a configuration is defined in the same way as for CFGs. Likewise, the configuration space of any convergent MCFG is isomorphic to the set of its shot-sets ordered by inclusion. We now show that any MCFG is equivalent to a CFG.

Theorem 2.2 Any convergent $M C F G$ is equivalent to a $C F G$.
Proof: Let $C$ be a convergent MCFG. From Theorem 2.1 we can suppose that $C$ is simple. Let $G=(V, E)$ and $\sigma$ be the initial support graph and configuration of $C$. To prove our result, we will use $C$ to construct an equivalent (simple) CFG. Given a vertex $v$, we define the initial degree of $v$ to be its degree in $G$. When $v$ is fired (if fired at all), a mutation occurs and its outdegree changes. Since $C$ is simple, $v$ will be fired only once before the end of an execution, so its outdegree changes only once. We define $v$ 's final degree to be its degree in the final configuration (in the case where $v$ is not fired during an execution, the initial and final degrees are the same).

In the final configuration, $v$ contains less chips than its final outdegree (otherwise $v$ could be fired and we would not be in the final configuration). Two cases can occur: either $v$ contains also less chips than its initial outdegree, or not. In the first case, if we suppress the mutation of $v$ (the outgoing edges of $v$ are the same before and after the mutation), $v$ still cannot be fired in the final configuration. If this is the case for all vertices, we can suppress all mutations to obtain a simple classical CFG.

In the other case, there exists a vertex $w$ which contains at the end of an execution more chips than its initial outdegree (which means that the outdegree of the vertex has been increased by the mutation). We will modify $C$ to prevent this from happening. We proceed in the following way: let $d^{+}(w)$ be the initial outdegree of $w, \sigma_{f}(w)$ be the number of chips in $w$ in the final configuration, and $n$ be the number of chips needed to fire $w($ if $\sigma(v) \geq d$, then $n=0$ ). We add $\sigma_{f}(w)+1-d^{+}(w)$ edges from $w$ to the sink, and we set the initial configuration of $w$ to $\sigma_{f}(w)+1-n$ chips. Figure 5 shows an example of this construction. After this medification $w$ still needs $n$ chins to he fired therefore the firct times

Figure 5: An example of a simple MCFG transformed into a CFG. On the left, the initial and final configurations of the MCFG. Notice that the lower vertex contains in the final configuration more chips than its initial outdegree. On the right, the initial and final configuration of the corresponding CFG.
the Abelian Sandpile Model, which is a CFG on an undirected graph. Finding out if a model defined as a Mutating Abelian Sandpile Model generates the same lattices as the usual ASM would help understand these differences better.

## 3 Comparison of CFGs and ASMs

In this section, we compare the classes of configuration spaces induced by CFGs and ASMs. Since we know from Section 1.2.1 that any ASM is equivalent to a CFG, we try to determine at which conditions a CFG is equivalent to an ASM. We will show that this is always the case when the CFG is simple and its support graph has no cycle, by giving an algorithm to transform such a CFG into an ASM. Since we know that any distributive lattice is the configuration space of a simple CFG with no cycle [MPV01], we obtain as a corollary that the class of lattices induced by ASM contains the distributive lattices. However, we will prove that not all CFGs are equivalent to an ASM. Since examples of lattices in $L(A S M)$ that are not distributive appear often in the study of the model, we obtain the result that $L(A S M)$ is strictly between the distributive lattices and the lattices induced by CFG, which is surprising because these two classes are very close to one another.

We will try to transform CFGs into ASMs using local transformations on the support graph of a CFG that do not change its configuration space. Using a combination of these transformations, we will see that it is possible in some cases to obtain a CFG such that its support graph contains one edge $(u, v)$ for each edge ( $v, u$ ), i.e. the graph can be viewed as undirected, so that we have an ASM. We begin by giving the two basic transformations we will use. They do not change the vertex set of a CFG, only its edge set and its initial configuration. Thanks to Proposition 1.5, we will prove that these modifications preserve the configuration space by checking that the first times each vertex can be fired are not changed, and that the CFG remains simple (indeed, in some cases, a

- multiplying by $n$ the initial configuration of $v$,
- adding $(n-1) \cdot d^{+}(v)$ edges from $v$ to the sink,
- for each immediate predecessor $u$ of $v$, adding $(n-1) \cdot d(u, v)$ edges $(u, v)$, and adding as much chips to the initial configuration of $u$.

The next two lemmas state that these modifications do not change the configuration space of the CFG to which they are applied. The first result being obvious, its proof is ommitted.

Lemma 3.1 The CFG obtained by applying the Grounding Modification to a $C F G C$ is equivalent to $C$.

Lemma 3.2 The CFG obtained by applying the Multiplying Modification to a $C F G C$ is equivalent to $C$.

Proof: Let $C^{\prime}$ be the CFG obtained from $C$ by applying the Multiplying Modification to a vertex $v$ with a factor $n . L\left(C^{\prime}\right)$ is isomorphic to $L(C)$ since:

- The outdegree and the initial configuration of $v$ have been multiplied by $n$, thus $v$ needs $n$ times more chips to be fired in $C^{\prime}$ than in $C$. Since, for each immediate predecessor $u$ of $v$ the number of edges from $u$ to $v$ has also been multiplied by $n$, the first times at which $v$ can be fired do not change. We can easily see that $v$ can be fired only once in $C^{\prime}$, since:

$$
\begin{aligned}
\sigma^{\prime}(v)+\sum d^{\prime}(u, v) & =n \sigma(v)+n \sum d(u, v) \\
& <n d^{+}(v) \\
& <d^{\prime+}(v)
\end{aligned}
$$

- the modifications made to the predecessors of $v$ (the modification of the initial configuration and the edges added towards $v$ ) are similar to applying the Grounding Modification to these vertices (with $v$ playing the role of the sink).

```
Algorithm 1 Transformation of a simple CFG with no cycle into an
equivalent ASM
Input: A simple CFG \(C\) with support graph \(G=(V, E)\) and initial
        configuration \(\sigma_{0}\), such that \(G\) has no cycle
    Output: An ASM equivalent to \(C\)
    Compute \(L(C)\);
    For each \(v \in V \backslash\{\perp\}, D[v] \leftarrow \max _{\sigma \in L(C)}\left(d^{+}(v)-\sigma(v)\right)\);
    Mark \(\perp\);
    \(L \leftarrow\) empty-list;
    while there are unmarked vertices do
        Choose an unmarked vertex \(v\) with all successors marked;
        Mark \(v\);
        \(L \leftarrow L, v ;\)
    Step 1: while \(L\) is not empty do
        \(v \leftarrow\) head (L)
        if \(D[v] \leq d(v)\) then
            Apply the Multiplying Modification to \(v\) with factor
            \(\lceil(d(v)+1) / D[v]\rceil\);
            For each edge \((u, v)\) added by the Multiplying Modification, add
            1 to \(D[u]\);
        \(L \leftarrow \operatorname{tail}(L) ;\)
    Step 2: for each \(v \in V \backslash\{\perp\}\) do
    if \(\sigma_{0}(v)+d^{-}(v)+d(v) \geq 2 d^{+}(v)\) then
    Apply the Grounding Modification to \(v\) with factor
                \(\left\lceil\left(d(v)+d^{-}(v)+1\right) /\left(2 \cdot d^{+}(v)-\sigma_{0}(v)\right)\right\rceil ;\)
Step 3: for each \(v \in V \backslash\{\perp\}\) do
    for each edge \((u, v)\) in \(E\) do
            Add one chip to the initial configuration of \(v ;\)
            Add one edge \((v, u)\) to \(E\);
```

this reversal of edges. We first show that, at each step of the algorithm, the current CFG is equivalent to $C$.

Step 1 In this step the indegree of a vertex is increased so that it exceeds by a certain factor its outdegree. This is possible because the graph has no cycle, and the graph has only one sink. In this step, the only modification applied is the Multiplying Modification, therefore by Lemma 3.2, at each step the current CFG is equivalent to $C$. At the end of this step, we have, for each vertex $v$ and for all configuration $\sigma$ :

$$
d^{+}(v)-\sigma(v)>d(v) .
$$

Step 2 The only modification applied in this step is the Grounding Modification, therefore, by Lemma 3.1, at each step the current CFG is equivalent to $C$. At the end of this step we have, for each vertex $v$ :

$$
\sigma_{0}(v)+d^{-}(v)+d(v)<2 d^{+}(v) .
$$

Step 3 In this step an edge $(u, v)$ is added for each pre-existing edge $(v, u)$. The resulting CFG is equivalent to $C$ because:

- a chip is added to the initial configuration of $u$ for each edge $(u, v)$ added. This is similar to applying the Grounding Modification to $u$, with $v$ playing the role of the sink. Figure $6(1-3)$ illustrates why this is necessary.
- After Step 2 we know that $\sigma_{0}(v)+d^{-}(v)+d(v)<2 d^{+}(v)$. This means that, after we have added $d(v, u)$ edges from $u$ to $v$, we still have $\sigma_{0}(v)+d^{-}(v)<$ $2 d^{+}(v)$, which means that $v$ can be fired only once in the resulting CFG. Figure 6 (4) illustrates this.
- After Step 1 we have, for all configuration $\sigma: d^{+}(v)-\sigma(v)>d(v)$. Therefore adding edges from $u$ to $v$ does not change the first times at which $v$ can be fired: there is no configuration $\sigma$ in which $v$ cannot be fired, and in which adding edges from $u$ to $v$ allows $v$ to be fired. Figures 7 and 8 illustrate this.

Corollary 3.4 Let $C$ be a simple CFG with support graph $G=(V, E)$ such that $G$ has no cycle. Then $C$ is equivalent to an $A S M$.

(4)

Figure 6: (2): Simply adding an edge $(u, v)$ to the CFG (1) changes the first times at which $u$ can be fired: $u$ can not be fired at all. (3): therefore, we add a chip to the initial configuration of $u$, and $u$ can be fired after $v$, as in (1). (4): In (3) $v$ can be fired more than once. Therefore, we apply the Grounding Modification to $v$ (i.e. we add a chip to the initial configuration of $v$ and we add one edge $(v, \perp)$ ) before adding the edge $(u, v)$. The resulting CFG is equivalent to (1).



Figure 8: If we want to add one edge $(u, v)$ to the CFG of Figure 7, we have first to add one chip to the initial configuration of $u$, and to apply the Grounding Modification to $v$, as seen before. We thus obtain the CFG shown on the left. In this CFG, it is possible to fire $v$ after $b$ and $u$, and this adds the dotted lines to the configuration space. Therefore, we must apply the Multiplying Modification to $v$ to multiply its indegree by two, before we add the edge $(u, v)$. The resulting CFG (on the right) is equivalent to the original CFG (see Figure 7).

Corollary 3.6 Any distributive lattice is the configuration space of an ASM.
The distributive lattices are not the only ones that can be obtained both by CFG and ASM. Indeed, most of the simple CFGs we have considered during our studies have no cycle, and are therefore equivalent to an ASM. Moreover, we give in Figure 9 an example of a CFG which has a cycle in its support graph, but is equivalent to an ASM.

However, we show that not all CFGs are equivalent to ASMs.
Theorem 3.7 $L(A S M) \varsubsetneqq L(C F G)$.
Proof : We know that $L(A S M) \subseteq L(C F G)$. We will show that the lattice $L$ of Figure 10, which is the configuration space of the CFG of Figure 11, cannot be the configuration space of any ASM (i.e. a CFG such that there exists an edge $(u, v)$ for each edge $(v, u))$. This will show that the inclusion is strict. The proof is in two steps: first we show that $L$ can be obtained only by a simple CFG (i.e. a CFG such that its configuration space is $L$ must have six different vertices, each fired once during an execution, and a sink); we have to show that $L$ can only be obtained by a simple CFG, otherwise we must prove also that $L$ cannot be the configuration space of an ASM with five vertices or less. Then we prove some inequalities on the number of edges between two pairs of vertices, from which we will show that there exist two vertices $a$ and $b$ such that the number


Figure 9: A CFG with a cycle in its support graph that is equivalent to an ASM. We present here the CFG, an equivalent ASM, and their configuration space



Figure 11: A CFG the configuration space of which is the lattice of Figure 10. Notice the cycle in the support graph.








 $\square$

Figure 12: The possible vertex sets of a CFG that generates the lattice of Figure 10
$c$, and the third one, that we denote by $a^{\prime}$, is therefore a vertex different from $b$ and $c . a^{\prime}$ is either a fourth vertex (that can be fired immediatly after $a$ ), or $a$ itself (which would mean that $a$ can be fired twice in a row). The same argument holds for $b$ and $c$. So the vertex set of a CFG $C$ such that $L=L(C)$ is something in-between the two vertex sets of Figure 12.

We will now prove that $a^{\prime}, b^{\prime}$ and $c^{\prime}$ are different from $a, b$ and $c$, and that they are all distinct from each other. First we prove that $a^{\prime}$ is different from $a$ : the configuration $b b^{\prime}$ has three immediate successors, therefore there are three different vertices that can be fired in configuration $b b^{\prime}$. Two of them are $a$ and $c$, so the third one (denoted by $x$, and the firing of which leads to the configuration labelled by $b b^{\prime} x$ ) is either $a^{\prime}$ (in which case $a^{\prime}$ is distinct from $a$ ) or $c^{\prime}$ (in which case $c^{\prime}$ is distinct form $c$ ). We observe that the join of $a a^{\prime}$ and $b b^{\prime}$ (labelled by $\left.a a^{\prime} b b^{\prime}\right)$ is greater than $b b^{\prime} x$, which means that $\left\{b b^{\prime} x\right\} \subseteq\left\{a a^{\prime} b b^{\prime}\right\}$. From this we
$d\left(b, a^{\prime}\right)+d\left(b^{\prime}, a^{\prime}\right)$. Suppose our claim is not true, i.e. $d\left(c^{\prime}, a^{\prime}\right) \geq d\left(b^{\prime}, a^{\prime}\right)$. Then we have $n \leq d\left(c^{\prime}, a^{\prime}\right)+d\left(b^{\prime}, a^{\prime}\right)$, and therefore $n \leq d\left(c^{\prime}, a^{\prime}\right)+d\left(b^{\prime}, a^{\prime}\right)+d\left(c, a^{\prime}\right)$. This means that $a^{\prime}$ can be fired after $c, c^{\prime}$ and $b^{\prime}$. But at point $b^{\prime} c c^{\prime}$, only two vertices can be fired, and they are $a$ and $b^{\prime}$. Therefore we obtain a contradiction, and we must have $d\left(b^{\prime}, a^{\prime}\right)>d\left(c^{\prime}, a^{\prime}\right)$. By similar arguments we can establish that $d\left(c^{\prime}, b^{\prime}\right)>d\left(a^{\prime}, b^{\prime}\right)$ and $d\left(a^{\prime}, c^{\prime}\right)>d\left(b^{\prime}, c^{\prime}\right)$. Now if $L$ was the configuration space of an ASM, we would have $d\left(a^{\prime}, b^{\prime}\right)=d\left(b^{\prime}, a^{\prime}\right), d\left(a^{\prime}, c^{\prime}\right)=d\left(c^{\prime}, a^{\prime}\right)$ and $d\left(b^{\prime}, c^{\prime}\right)=d\left(c^{\prime}, b^{\prime}\right)$, which is in contradiction with the inequalities established above. We conclude that $L$ cannot be the configuration space of an ASM.

By combining Theorem 3.7 and Corollary 3.6, we obtain that $L(A S M)$ is situated strictly between the class of distributive lattices and $L(C F G)$. This shows the complexity of the problems raised by the Chip Firing Game and the Abelian Sandpile Model in lattice theory: $L(C F G)$ and $L(A S M)$ are both between the distributive and ULD lattices, while there is no previously known lattices class satisfying this condition.

## Conclusion and perspectives

In this paper, we have studied from the lattice point of view the Chip Firing Game and two closely related models: the Abelian Sandpile Model and the Mutating Chip Firing Game. It was already known that all these models generate lattices, and that these lattices are in the class of Upper Locally Distributive (ULD) lattices. Our goal was to characterize the classes of lattices $L(C F G)$, $L(A S M)$ and $L(M C F G)$ induced by each model (i.e. determine, given a ULD lattice, by which model(s) it can be obtained). We have first shown that the mutating CFG and the CFG generate exactly the same set of lattices, by giving a way to transform a MCFG into a classical CFG. This means that, although the MCFG is a generalization of the CFG, it brings no more information in lattice terms.

We know that every ASM is equivalent to a CFG, and we have given an example of a CFG that is not equivalent to any ASM. This implies that $L(A S M) \nsubseteq$ $L(C F G)$. Then we have given a sufficient but not necessary condition at which a CFG can be transformed into an ASM, from which it can be shown that the class $D$ of distributive lattices is included in $L(A S M)$. The class of lattices induced by CFG being somewhere between the distributive and the ULD lattices, we obtain that the class of lattices induced by ASM is a new class between the
condition we have given at which a CFG is equivalent to an ASM is that the support graph must contain no directed cycle. This is a strong restriction, but it nonetheless implies that $L(A S M)$ is a very significant part of $L(C F G)$ (it contains more than the distributive lattices class, which is a large part of the ULD class).

The fact that three important models, used in various domains like physics, computer science and social science, all induce strongly structured sets precisely situated between two classical types of lattices, shows the importance of the use of order theory in the context of dynamical models studies. In our case it is even more interesting to notice that the models introduce new classes of lattices which one may study from the order theoretical point of view.

We have seen that the difference between these models does not lie in the fact that the graph is directed or not, but on the existence or not of directed cycles in the graph. Simple CFGs with no cycle in their support graph are very strongly linked with the class $\mathcal{C N}$ of lattices obtained by a sequence of duplications of convex sets. It would be a natural sequel of this work to find out in all lattices of $\mathcal{C N}$ induced by CFG can be obtained by ASM. However, if this is true this does not give the result that $L(A S M)=L(C F G) \cap \mathcal{C N}$, because we have examples of lattices that are in $L(A S M)$, but not in $\mathcal{C N}$.

Finally, lattices also appear in some other kinds of discrete dynamical models defined in the context of tiling theory: for some classes of tiling problems, one can define a local rearrangement of tiles, called flip, which transforms a tiling of a given region into another tiling of the same region. In some cases (mainly tilings with dominoes or with three lozenges [Rem99, BL01]), it has been proved that the flip relation gives the distributive lattice structure to the set of all possible tilings of a given region. A work similar to the one done in this paper, comparing the classes of lattices induced by each tiling problem, may lead to interesting result, both in tiling theory and in the study of discrete dynamical models.

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