

Designs in a coset geometry: Delsarte theory revisited

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Designs in a coset geometry: Delsarte theory revisited

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Delsarte [2] formulated design theory in the framework of commutative association schemes, especially P- and Q-polynomial schemes. It enabled us to interpret the combinatorial aspect of designs in terms of representations of the Bose-Mesner algebras. In this article, we revisit the Delsarte theory, shifting the framework from association schemes to coset geometries. When a group acts transitively on the underlying sets, this attempt broadens the category of designs and makes all the clearer the relation between combinatorial and algebraic structures of designs. As an application, t-transitive sets are constructed from the classical t-designs.

1. Let X, Ω be finite sets and G a finite group acting transitively both on X and Ω . With the action to be from the right, G acts on $X \times \Omega$ by $(x, \alpha)^a = (x^a, \alpha^a)$. Let \mathcal{O} be an orbit of G on $X \times \Omega$. Then \mathcal{O} defines an incidence relation $I = I_{\mathcal{O}}$ between X and Ω :

$$xI\alpha \Leftrightarrow (x, \alpha) \in \mathcal{O}.$$

For $(x_0, \alpha_0) \in \mathcal{O}$, let H, K be the stabilizers of x_0, α_0 in G , respectively. If we identify X, Ω with the cosets $H \backslash G, K \backslash G$, then

$$HaIKb \Leftrightarrow Ha \cap Kb \neq \emptyset.$$

For a subset Y of X and an element α of Ω , let $\lambda(\alpha)$ be the number of elements x in Y that are incident to α :

$$\lambda(\alpha) = \lambda_I(\alpha) = \#\{x \in Y | xI\alpha\}.$$

Y is called an *I-design* if $\lambda(\alpha)$ is a constant λ for all $\alpha \in \Omega$. Y is called a *combinatorial design* or simply a *design* if Y is an *I-design* for each $I = I_{\mathcal{O}}$.

2. Let V be the vector space over \mathbf{C} with X a basis:

$$V = \bigoplus_{x \in X} \mathbf{C}x.$$

V affords the permutation character θ of G on X . For an irreducible character χ of G appearing in θ , let V_χ be the homogeneous component of V corresponding to χ , i.e., the sum of all irreducible G -subspaces of V affording χ . Then V is decomposed into the direct sum of these V_χ :

$$V = \bigoplus_{\chi} V_\chi,$$

where χ runs over the irreducible characters of G appearing in θ . Let χ_0 be the principal character 1_G of G . Then the transitivity of G on X implies

$$V_{\chi_0} = \mathbf{C}\underline{X},$$

where $\underline{X} = \sum_{x \in X} x$. Similarly the other G -module

$$W = \bigoplus_{\alpha \in \Omega} \mathbf{C}\alpha$$

affords the permutation character π of G on Ω , and W is decomposed into the direct sum of homogeneous components W_χ :

$$W = \bigoplus_{\chi} W_\chi,$$

$$W_{\chi_0} = \mathbf{C}\underline{\Omega}.$$

With an incidence relation $I = I_{\mathcal{O}}$, we associate a linear mapping f_I from V to W :

$$f_I(x) = \sum_{xI\alpha} \alpha \text{ for } x \in X.$$

Then for any subset Y of X , it holds that

$$f_I(\underline{Y}) = \sum_{\alpha \in \Omega} \lambda(\alpha)\alpha,$$

where $\underline{Y} = \sum_{x \in Y} x$, and $\lambda(\alpha) = \#\{x \in Y | xI\alpha\}$. So we have:

Lemma *A subset Y of X is an I -design if and only if $f_I(\underline{Y}) \in W_{\chi_0}$.*

3. Let $Hom_G(V, W)$ be the set of linear mappings f from V to W that commute with the action of G : $f(v^a) = f(v)^a$ for $v \in V$, $a \in G$. The mappings f_I form a basis of $Hom_G(V, W)$ as a vector space over \mathbf{C} , where $I = I_{\mathcal{O}}$ and \mathcal{O} ranges over the G -orbits of $X \times \Omega$. We give a brief proof of this fact. For $f \in Hom_G(V, W)$ and $x \in X$, set

$$f(x) = \sum_{\alpha \in \Omega} c_{\alpha x} \alpha \text{ with } c_{\alpha x} \in \mathbf{C}.$$

Then from $f(x) = f(x^a)^{a^{-1}}$, it follows that $c_{\alpha x}$ is a constant $c_{\mathcal{O}}$ on each G -orbit \mathcal{O} . So we have $f = \sum_{\mathcal{O}} c_{\mathcal{O}} f_{I_{\mathcal{O}}}$.

By the above lemma, a subset Y of X is a combinatorial design if and only if $f_I(\underline{Y}) \in W_{\chi_0}$ for all $I = I_{\mathcal{O}}$. Since the mappings f_I form a basis of $Hom_G(V, W)$, Y is a combinatorial design if and only if $f(\underline{Y}) \in W_{\chi_0}$ for all $f \in Hom_G(V, W)$.

By Schur's lemma, $Hom_G(V, W)$ is decomposed into the direct sum of $Hom_G(V_{\chi}, W_{\chi})$:

$$Hom_G(V, W) = \bigoplus_{\chi} Hom_G(V_{\chi}, W_{\chi}),$$

where V_{χ}, W_{χ} run over the homogeneous components of V, W , respectively. We understand that $V_{\chi} = 0$ (resp. $W_{\chi} = 0$) if χ does not appear in the permutation character θ (resp. π) of G on X (resp. Ω). Let $Irr(\theta), Irr(\pi)$ be the set of irreducible characters of G appearing in θ, π , respectively. Then $Hom_G(V_{\chi}, W_{\chi}) = 0$ unless $\chi \in Irr(\theta) \cap Irr(\pi)$.

Let p_{χ} be the projection of V onto the homogeneous component V_{χ} . Then by $Hom_G(V, W) = \bigoplus_{\chi} Hom_G(V_{\chi}, W_{\chi})$, $f(\underline{Y})$ belongs to W_{χ_0} for all $f \in Hom_G(V, W)$ if and only if $p_{\chi}(\underline{Y}) = 0$ for all $\chi \in Irr(\theta) \cap Irr(\pi)$ ($\chi \neq 1_G$). Thus we have:

Theorem *A subset Y of X is a combinatorial design if and only if*

$$p_{\chi}(\underline{Y}) = 0 \text{ for all } \chi \in Irr(\theta) \cap Irr(\pi) \text{ } (\chi \neq 1_G).$$

In view of this theorem, we introduce the notion of a T -design, where T is a set of irreducible characters of G . A subset Y of X is said to be a T -design if

$$p_\chi(\underline{Y}) = 0 \text{ for all } \chi \in T \ (\chi \neq 1_G).$$

Notice that T can be replaced by $T \cap \text{Irr}(\theta)$ or by any T' with $T \cap \text{Irr}(\theta) = T' \cap \text{Irr}(\theta)$, since $p_\chi = 0$ for $\chi \notin \text{Irr}(\theta)$. The theorem above states that Y is a combinatorial design if and only if Y is a T -design for $T = \text{Irr}(\pi)$.

4. The projection p_χ of V onto the homogeneous component V_χ is given by the formula ([3] Theorem 8):

$$p_\chi(v) = \frac{\chi(1)}{|G|} \sum_{a \in G} \chi(a)^* v^a \text{ for } v \in V,$$

where $*$ stands for the complex conjugate. This formula is valid for any irreducible character χ of G , in particular the sum on the right hand side vanishes if χ does not appear in the G -module V . Notice that $\chi(a)^* = \chi(a^{-1})$. So for a subset Y of X and $\underline{Y} = \sum_{x \in Y} x$, we have

$$p_\chi(\underline{Y}) = \frac{\chi(1)}{|G|} \sum_{y \in X} c_y y \text{ with } c_y = \sum_{y^a \in Y} \chi(a).$$

Equip V with a Hermitian form $\langle \rangle$ such that X is an orthonormal basis: $\langle x, y \rangle = \delta_{xy}$ for $x, y \in X$. Notice that the Hermitian form $\langle \rangle$ is G -invariant and so the homogeneous components V_χ are orthogonal each other. It holds that $\langle p_\chi(\underline{Y}), p_\chi(\underline{Y}) \rangle = \langle p_\chi(\underline{Y}), \underline{Y} \rangle = \frac{\chi(1)}{|G|} \sum_{a \in G} \chi(a) |Y^a \cap Y|$. So we have:

Delsarte's Condition For a subset Y of X and an irreducible character χ of G ,

$$\langle p_\chi(\underline{Y}), p_\chi(\underline{Y}) \rangle = \frac{\chi(1)}{|G|} \sum_{a \in G} \chi(a) |Y^a \cap Y| \geq 0.$$

Y is a T -design if and only if the equality holds in Delsarte's condition for $\chi \in T$ ($\chi \neq 1_G$).

Since the Hermitian form $\langle \rangle$ is G -invariant and since p_χ commutes with the action of G , we have

$$\langle p_\chi(x), y \rangle = \langle p_\chi(x'), y' \rangle$$

for $(x, y), (x', y')$ in the same G -orbit of $X \times X$. Fix an element x_0 in X and let H be the stabilizer of x_0 in G . For each G -orbit Λ of $X \times X$, choose $x_\Lambda \in X$ and $t_\Lambda \in G$ such that $(x_0, x_\Lambda) \in \Lambda$ and $x_0^{t_\Lambda} = x_\Lambda$. Then for $(x, y) \in \Lambda$, we have $\langle p_\chi(x), y \rangle = \langle p_\chi(x_0), x_\Lambda \rangle = \frac{\chi(1)}{|G|} \chi(\underline{Ht_\Lambda})^*$, where $\underline{Ht_\Lambda}$ is the sum of elements of the coset Ht_Λ . We understand that χ is extended to the character of the group algebra $\mathbf{C}[G]$. Since $\langle p_\chi(x_0), x_\Lambda \rangle = \langle p_\chi(x_0), x_\Lambda^h \rangle$ for $h \in H$, we have $\chi(\underline{Ht_\Lambda}) = \chi(\underline{Ht_\Lambda h})$. So for $(x, y) \in \Lambda$

$$\langle p_\chi(x), y \rangle = \frac{\chi(1)}{|G||H : H \cap t_\Lambda^{-1} H t_\Lambda|} \chi(\underline{Ht_\Lambda H})^*,$$

where $\underline{Ht_\Lambda H}$ is the sum of elements of the double coset $Ht_\Lambda H$. Thus computing $\langle p_\chi(\underline{Y}), p_\chi(\underline{Y}) \rangle = \langle p_\chi(\underline{Y}), \underline{Y} \rangle$, we have another formulation of Delsarte's condition:

Delsarte's Condition *For a subset Y of X and an irreducible character χ of G ,*

$$\langle p_\chi(\underline{Y}), p_\chi(\underline{Y}) \rangle = \frac{\chi(1)}{|G|} \sum_{\Lambda} \frac{|\Lambda \cap Y \times Y|}{|H : H \cap t_\Lambda^{-1} H t_\Lambda|} \chi(\underline{Ht_\Lambda H}) \geq 0,$$

where Λ runs over the G -orbits of $X \times X$.

Delsarte's condition is the basis on which Delsarte discussed the linear programming bound for the size of a subset in an association scheme [2], but we go no further into this direction in this article.

5. As is seen in Delsarte's condition, what really matters to T -designs is the Hecke algebra $Hom_G(V, V)$, to which the projections p_χ belong. It is well known that the algebra $Hom_G(V, V)$ is semisimple and that as a $Hom_G(V, V)$ -module, $V = \bigoplus_{\chi} V_\chi$ is still the decomposition into the homogeneous components. The space V_χ is irreducible as a $(\mathbf{C}[G] \otimes Hom_G(V, V))$ -module.

Let $\chi_0 = 1_G, \chi_1, \dots, \chi_r$ be the irreducible characters appearing in the permutation character θ of G on X . We shall abbreviate p_{χ_i}, V_{χ_i} to p_i, V_i , respectively. Let \mathcal{A} be the linear subspace of $Hom_G(V, V)$ spanned by the projections p_i :

$$\mathcal{A} = Span\{p_0, p_1, \dots, p_r\}.$$

Then in the Hecke algebra $Hom_G(V, V)$, we have

$$p_i p_j = \delta_{ij} p_i, \quad 1 = p_0 + p_1 + \cdots + p_r,$$

and \mathcal{A} becomes a commutative semisimple algebra of dimension $r + 1$.

Besides the ordinary product, $Hom_G(V, V)$ is endowed with the Schur product \circ , which is also called the Hadamard product:

$$(f \circ g)(x) = \sum_{y \in X} c_{yx} d_{yx} y \quad \text{for } x \in X$$

if $f(x) = \sum_{y \in X} c_{yx} y$, $g(x) = \sum_{y \in X} d_{yx} y$. $Hom_G(V, V)$ is closed with respect to the Schur product, since the coefficients $c_{yx} d_{yx}$ of $f \circ g$ are G -invariant. If we express the elements of $Hom_G(V, V)$ as matrices with respect to the basis X of V , the Schur product is the entrywise product of matrices. The Schur product is defined with respect to the basis X and hence it must carry certain information of X .

Assume that \mathcal{A} is closed with respect to the Schur product, and let \mathcal{A}° be the commutative algebra that the linear space \mathcal{A} gives rise to with respect to the Schur product. Since $p_0(x) = \frac{1}{|X|} \underline{X}$ for $x \in X$, the mapping $|X|p_0$ is the identity of the algebra \mathcal{A}° . Since \mathcal{A}° has no nilpotent elements, \mathcal{A}° is semisimple. Let f_0, f_1, \dots, f_r be the primitive idempotents of \mathcal{A}° :

$$f_i \circ f_j = \delta_{ij} f_i, \quad |X|p_0 = f_0 + f_1 + \cdots + f_r.$$

Let A_i, E_i be the matrices of f_i, p_i with respect to the basis X , respectively. Then the algebra spanned by A_i ($0 \leq i \leq r$) is the Bose-Mesner algebra of a commutative association scheme; A_i, E_i are the adjacency matrices, the primitive idempotents of the Bose-Mesner algebra, respectively ([1] Section 2.2). The proof is rather routine and is left to the reader. We have now reached the place where Delsarte built his design theory.

6. We keep the notations of **5**. Let χ_i^* be the complex conjugate character of χ_i . Then $\chi_i^* = \chi_{\hat{i}}$ for some \hat{i} ($0 \leq \hat{i} \leq r$), and we have an involutive permutation $\hat{\cdot}$ of the indices $0, 1, \dots, r$.

Let $\mathcal{A} = Span\{p_0, p_1, \dots, p_r\}$. Assume that \mathcal{A} is closed with respect to the Schur product:

$$p_i \circ p_j = \frac{1}{|X|} \sum_{k=0}^r q_{ij}^k p_k.$$

The coefficients q_{ij}^k are the Krein parameters and known to be nonnegative real numbers. For a subset Y of X , p_Y denotes the orthogonal projection of V onto the subspace V_Y of V spanned by Y : $V_Y = \bigoplus_{x \in Y} \mathbb{C}x$. Delsarte showed ([2] Theorem 3.15):

Delsarte's Criterion *Let Y be a subset of X and i, j arbitrarily given distinct indices.*

(i) *The subspaces $p_Y(V_i)$, $p_Y(V_j)$ are orthogonal each other if and only if $(p_i \circ p_j)(\underline{Y}) \in V_0$, i.e., $q_{ij}^k p_k(\underline{Y}) = 0$ for all $k \neq 0$.*

(ii) *The mapping $\sqrt{\frac{|X|}{|Y|}} p_Y$ is an isometry of V_i into V_Y if and only if $(p_i \circ p_i)(\underline{Y}) \in V_0$, i.e., $q_{ii}^k p_k(\underline{Y}) = 0$ for all $k \neq 0$.*

A Fisher type inequality is derived from Delsarte's Criterion. For a subset S of the indices $0, 1, \dots, r$, set

$$S \circ \hat{S} = \{k \mid q_{ij}^k \neq 0 \text{ for some } i, j \in S\}.$$

For a T -design Y , choose S such that

$$S \circ \hat{S} \subseteq T \cup \{0\}.$$

Then by Delsarte's Criterion, p_Y maps $\bigoplus_{i \in S} V_i$ into V_Y injectively. Thus we have:

Fisher Type Inequality

$$|Y| \geq \sum_{i \in S} \dim V_i.$$

To find an explicit formula for the Krein parameters q_{ij}^k , we calculate the trace of $p_k(p_i \circ p_j) = \frac{1}{|X|} q_{ij}^k p_k$. Then $\text{Tr } p_k = \dim V_k$ and $\text{Tr } p_k(p_i \circ p_j) = \sum_{x \in X} \langle p_k(p_i \circ p_j)(x), x \rangle = \sum_{x \in X} \langle (p_i \circ p_j)(x), p_k(x) \rangle$. Keeping the notations of 4, we have

$$p_\chi(x) = \frac{\chi(1)}{|G|} \sum_{\Lambda} \frac{1}{|H : H \cap t_{\Lambda}^{-1} H t_{\Lambda}|} \chi(\underline{H t_{\Lambda} H})^* \underline{\Lambda(x)},$$

where Λ runs over the G -orbits of $X \times X$, $\Lambda(x) = \{y \in X \mid (x, y) \in \Lambda\}$ and $\underline{\Lambda(x)}$ is the sum of elements of $\Lambda(x)$. So $\langle (p_i \circ p_j)(x), p_k(x) \rangle$ equals

$$\frac{\chi_i(1)\chi_j(1)\chi_k(1)}{|G|^3} \sum_{\Lambda} \frac{\chi_i(\underline{Ht_{\Lambda}H})^* \chi_j(\underline{Ht_{\Lambda}H})^* \chi_k(\underline{Ht_{\Lambda}H})}{|H : H \cap t_{\Lambda}^{-1}Ht_{\Lambda}|^3} |\Lambda(x)|.$$

Since $|\Lambda(x)| = |H : H \cap t_{\Lambda}^{-1}Ht_{\Lambda}|$ and $\dim V_k = \chi_k(1) \text{mult}(\chi_k)$ with $\text{mult}(\chi_k)$ the multiplicity of χ_k in the permutation character θ of G on X , we have

$$q_{ij}^k = \frac{\chi_i(1)\chi_j(1)|X|^2}{|G|^3 \text{mult}(\chi_k)} \sum_{\Lambda} \frac{\chi_i(\underline{Ht_{\Lambda}H})^* \chi_j(\underline{Ht_{\Lambda}H})^* \chi_k(\underline{Ht_{\Lambda}H})}{|H : H \cap t_{\Lambda}^{-1}Ht_{\Lambda}|^2}.$$

7. There are two important cases which satisfy the condition that the linear subspace $\mathcal{A} = \text{Span}\{p_0, p_1, \dots, p_r\}$ is closed with respect to the Schur product. One is the case in which the permutation character θ of G on X is multiplicity free, i.e., each homogeneous component V_i is irreducible or equivalently the Hecke algebra $\text{Hom}_G(V, V)$ is commutative. The other is the case in which $X = G$ and G acts on X as the regular representation.

Suppose first that the permutation character θ of G on X is multiplicity free. Then $\mathcal{A} = \text{Hom}_G(V, V)$ and hence \mathcal{A} is closed with respect to the Schur product. Notice that $f \in \mathcal{A}$ acts on each homogeneous component V_{χ} as a scalar and so $\omega = \frac{1}{\chi(1)} \text{Tr}|_{V_{\chi}}$ is a linear representation of \mathcal{A} . For more information about the association scheme $\mathcal{A} = \text{Hom}_G(V, V)$, see [2] Section 2.11.

Let us go back to the I -design situation discussed in **1**, **2**, **3**. We have two finite sets X, Ω on which a group G acts transitively. The I -designs are defined with respect to the incidence relation $I = I_{\mathcal{O}}$ associated with an G -orbit \mathcal{O} of $X \times \Omega$. Let $V = \bigoplus_{x \in X} \mathbf{C}x$, $W = \bigoplus_{\alpha \in \Omega} \mathbf{C}\alpha$ be the permutation modules with characters θ, π , respectively. By our assumption, every homogeneous component V_{χ} of V is irreducible. For $f \in \text{Hom}_G(V, W)$, the kernel of f is G -invariant and so is a direct sum of V_{χ} 's. Let $\text{Supp}(f)$ be the rest:

$$\text{Supp}(f) = \{\chi \in \text{Irr}(\theta) \cap \text{Irr}(\pi) \mid f(V_{\chi}) \neq 0\}.$$

Then we have

$$V = \ker(f) \bigoplus_{\chi \in \text{Supp}(f)} V_{\chi}.$$

Hence by the lemma in **2**, we have a stronger version of the theorem in **3**:

Theorem Assume that the permutation character θ of G on X is multiplicity free. Then a subset Y of X is an I -design if and only if

$$p_\chi(\underline{Y}) = 0 \text{ for all } \chi \in \text{Supp}(f_I) \ (\chi \neq 1_G),$$

where $f_I(x) = \sum_{xI\alpha} \alpha$ for $x \in X$.

Notice that $\chi_0 = 1_G$ is contained in $\text{Supp}(f_I)$, since V_{χ_0} is spanned by \underline{X} . Hence we have:

Corollary An I -design is an I' -design if $\text{Supp}(f_I) \supseteq \text{Supp}(f_{I'})$, i.e., $\ker(f_I) \subseteq \ker(f_{I'})$.

8. Let us consider the case in which $X = G$ and G acts on X as the regular representation. In this case, the permutation character θ contains every irreducible character χ of G with multiplicity $\chi(1)$. The projection p_χ is given by

$$p_\chi(x) = \frac{\chi(1)}{|G|} \sum_{y \in G} \chi(y^{-1}x)y \text{ for } x \in G,$$

and so for $x \in G$

$$(p_i \circ p_j)(x) = \frac{\chi_i(1)\chi_j(1)}{|G|^2} \sum_{y \in G} \chi_i(y^{-1}x)\chi_j(y^{-1}x)y.$$

Let us decompose $\chi_i\chi_j$ as a character of the group G :

$$\chi_i\chi_j = \sum_{k=0}^r c_{ij}^k \chi_k.$$

Then

$$(p_i \circ p_j) = \frac{1}{|X|} \sum_{k=0}^r q_{ij}^k p_k$$

with

$$q_{ij}^k = \frac{\chi_i(1)\chi_j(1)}{\chi_k(1)} c_{ij}^k.$$

This is also checked by the formula of q_{ij}^k in **6**, thanks to the orthogonality relation of group characters. Thus the linear subspace $\mathcal{A} = \text{Span}\{p_0, p_1, \dots, p_r\}$

is closed with respect to the Schur product. The association scheme \mathcal{A} is discussed in detail in [2] Section 2.7.

For a subset Y of $X = G$, Delsarte's Condition is

$$\langle p_\chi(\underline{Y}), p_\chi(\underline{Y}) \rangle = \frac{\chi(1)}{|G|} \sum_{x, y \in Y} \chi(y^{-1}x) \geq 0.$$

This can be checked by the second formulation of Delsarte's Condition in 4 or directly by $\langle p_\chi(x), p_\chi(y) \rangle = \langle p_\chi(x), y \rangle = \frac{\chi(1)}{|G|} \chi(y^{-1}x)$. From Delsarte's Condition, we see that if Y is a T -design, then aY and Ya are also T -designs for all $a \in G$.

Given a transitive permutation representation of G on Γ and a T -design Δ in Γ , we can 'lift' Δ to a T -design in $X = G$. Choose a point $\gamma_0 \in \Gamma$. Let H be the stabilizer of γ_0 in G and identify Γ with the cosets $H \backslash G$. With this identification, define a subset Y of $X = G$ to be

$$Y = \sum_{Ht \in \Delta} Ht,$$

where the symbol \sum stands for the direct sum of subsets. It is a straightforward consequence of the first formulation of Delsarte's Condition in 4 that Y becomes a T -design. We shall refer to this T -design Y as the *lifting* of Δ .

Let us consider the combinatorial meaning of I -designs in $X = G$. Besides the regular representation of G on X , we have a set Ω on which G acts transitively. Each pair $(1, \alpha) \in X \times \Omega$ belongs to a unique G -orbit of $X \times \Omega$, which we shall denote by \mathcal{O}_α . Let I_α be the incidence relation associated with \mathcal{O}_α . Then $x I_\alpha \beta$ if and only if $\alpha^x = \beta$. For a subset Y of X , set $\lambda_\alpha(\beta) = \#\{x \in Y \mid x I_\alpha \beta\}$. Then

$$\lambda_\alpha(\beta) = \#\{x \in Y \mid \alpha^x = \beta\}.$$

Thus a subset Y of X is an I_α -design if and only if Y contains exactly $\lambda = \frac{|Y|}{|\Omega|}$ members that send α to β , independent of the choice of $\beta \in \Omega$. A subset Y of X is a combinatorial design if and only if Y contains exactly $\lambda = \frac{|Y|}{|\Omega|}$ members that send α to β , independent of the choice of $\alpha, \beta \in \Omega$; such a subset Y is called a *transitive* set on Ω . A *transitive* set with $\lambda = 1$ is called *regular*.

9. Let us consider the classical t -designs in the framework we have discussed. Let Ω be a finite set and G the symmetric group on Ω . Let $\Omega_{\{k\}}$ be the set of unordered k -sets of Ω and Ω_k that of ordered k -sets of Ω :

$$\Omega_{\{k\}} = \{\{\alpha_1, \alpha_2, \dots, \alpha_k\} \mid \alpha_i \in \Omega, \alpha_i \text{ pairwise distinct}\},$$

$$\Omega_k = \{(\alpha_1, \alpha_2, \dots, \alpha_k) \mid \alpha_i \in \Omega, \alpha_i \text{ pairwise distinct}\}.$$

G acts both on $\Omega_{\{k\}}$, Ω_k , transitively. Let $\pi_{\{k\}}$, π_k be the permutation characters of G on $\Omega_{\{k\}}$, Ω_k , respectively. It is well known that $\pi_{\{k\}}$ is multiplicity free and that for $1 \leq t \leq k \leq \frac{1}{2}|\Omega|$

$$\text{Irr}(\pi_{\{k\}}) \cap \text{Irr}(\pi_t) = \text{Irr}(\pi_{\{t\}}),$$

in particular, $\text{Irr}(\pi_{\{t\}}) \subseteq \text{Irr}(\pi_t)$, where $\text{Irr}(\pi)$ is the set of irreducible characters of G appearing in π .

For $1 \leq t \leq k \leq \frac{1}{2}|\Omega|$, the G -orbits on $\Omega_{\{k\}} \times \Omega_{\{t\}}$ are

$$\mathcal{O}_i = \{(\alpha, \beta) \in \Omega_{\{k\}} \times \Omega_{\{t\}} \mid |\alpha \cap \beta| = t - i\} \quad (0 \leq i \leq t).$$

We shall abbreviate $I_{\mathcal{O}_i}$ to I_i . The incidence relation I_0 is the inclusion relation between the unordered k -sets and t -sets of Ω . The classical t -design is by definition a subset of $\Omega_{\{k\}}$ that is an I_0 -design in our terms. The incidence relation I_0 is a particular one in the sense that the linear mapping f_{I_0} from $V = \bigoplus_{\alpha \in \Omega_{\{k\}}} \mathbf{C}\alpha$ to $W = \bigoplus_{\beta \in \Omega_{\{t\}}} \mathbf{C}\beta$ is surjective. In terms of **7**, $\text{Supp}(f_{I_0}) = \text{Irr}(\pi_{\{t\}})$. By the theorem in **7**, an I_0 -design is a T -design for $T = \text{Irr}(\pi_{\{t\}})$. By the theorem in **3**, an I_0 -design is a combinatorial design, i.e., an I_i -design for all i . Notice that an I_0 -design is also a T -design for $T = \text{Irr}(\pi_t)$, since $\text{Irr}(\pi_{\{k\}}) \cap \text{Irr}(\pi_t) = \text{Irr}(\pi_{\{t\}})$.

Given a classical t -design Δ , the notion of which is defined in the G -set $\Omega_{\{k\}} \times \Omega_{\{t\}}$, we can regard Δ as a T -design with $T = \text{Irr}(\pi_t)$, the notion of which is defined in the G -set $\Omega_{\{k\}}$. As is explained in **8**, the T -design Δ in the G -set $\Omega_{\{k\}}$ can be lifted to a T -design Y in G . Place the T -design Y in the G -set $G \times \Omega_t$ (resp. the G -set $G \times \Omega_{\{t\}}$). Then Y turns out to be a combinatorial design, i.e., a transitive set on Ω_t (resp. $\Omega_{\{t\}}$). A transitive set on Ω_t (resp. $\Omega_{\{t\}}$) is called a t -transitive (resp. t -homogeneous) set. Thus a classical t -design is lifted to a t -transitive set.

From the viewpoint of representation theory, it is clear that classical t -designs, t -transitive sets, t -homogeneous sets are also $(t-1)$ -designs, $(t-1)$ -transitive sets, $(t-1)$ -homogeneous sets, respectively. The problem of extending a classical $(t-1)$ -design to a classical t -design is settled by Teirlinck [4]

by a combinatorial method. It is a problem of some significance how much possible it is to reconstruct the work of Teirlinck in terms of representation theory.

By our construction of t -transitive sets from classical t -designs, there are a large number of nontrivial t -transitive sets for arbitrary t , whereas there are no t -transitive groups for $6 \leq t$ other than the trivial ones, i.e., the symmetric or alternating groups. However, it is yet to be settled whether there exists a sharply t -transitive set for large t , i.e., a t -transitive set Y with $|Y| = |\Omega_t|$.

Designs in coset geometries seem to be particularly interesting when G is a group of Lie type, and X, Ω are $H \backslash G, K \backslash G$ with H, K (maximal) parabolic subgroups of G . In case of $G = GL(n, q)$, we have a q -analogue of the classical t -designs. However, the existence problem of such designs is yet to be settled; q -analogues of the classical t -designs are constructed only for $t \leq 3$ so far. Let G be a group of Lie type, (G_I, G_J) a pair of (maximal) parabolic subgroups of G , W the Weyl group of G , and (W_I, W_J) the pair of the corresponding parabolic subgroups of W . It is an interesting problem whether there exists a correspondence in some sort between the designs in $G_I \backslash G \times G_J \backslash G$ and those in $W_I \backslash W \times W_J \backslash W$.

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