Computational Ideal Theory in Finitely Generated Extension Rings

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One of the most general extensions of Buchberger's theory of Gröbner bases is the concept of graded structures due to Robbiano and Mora. But in order to obtain algorithmic solutions for the computation of Gröbner bases it needs additional computability assumptions. In this paper we introduce natural graded structures of finitely generated extension rings and present subclasses of such structures which allow uniform algorithmic solutions of the basic problems in the associated graded ring and, hence, of the computation of Gröbner bases with respect to the graded structure. Among the considered rings there are many of the known generalizations. But, in addition, a wide class of rings appears first time in the context of algorithmic Gröbner basis computations. Finally, we discuss which conditions could be changed in order to find further effective Gröbner structures and it will turn out that the most interesting constructive instances of graded structures are covered by our results.

Key words: ideal membership problem, effective graded structure, Gröbner basis, Buchberger's algorithm

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1 Introduction

In various types of rings such fundamental ideal theoretical problems as the decision of ideal membership and the computation of syzygy modules could be solved in an algorithmic way using the so-called Gröbner bases. Among the more complex applications there are the computation of ideal operations, e.g. intersection or quotient, and the computation of related objects, e.g. Hilbert functions. So, the algorithmic computation of and division modulo Gröbner bases can be considered as the fundamental problems of computational ideal theory. During the last more than three decades Buchberger's algorithm became a central tool in constructive commutative algebra and algebraic geometry (cf.

[1, 8, 9, 11]) and motivated by the achievements in polynomial rings many efforts have been spent in generalizations to other types of rings.

The concept of graded structures due to Robbiano [26] and Mora [22] provides an excellent frame for investigating Gröbner bases in very general situations. What remains to do in a concrete application is to verify a series of computability conditions which have to be fulfilled in order to obtain not only existential statements on Gröbner bases but also constructive results such as decidability of the ideal membership problem or the computability of finite generating sets of syzygy modules. The bottleneck of this approach is the verification and algorithmic solution of properties and problems in the associated graded ring (conditions *iii)-v)* in Definitions 2 and 3). A first approach to illustrate the boundaries of constructivity in the frame of graded structures was presented in [4]. Starting from a graded structure $\mathfrak{R} = (R, \Gamma, \varphi, G, \mathrm{in})$ sufficient conditions ensuring that $\mathfrak{R}$ is an effective Gröbner structure, i.e. that $\mathfrak{R}$ allows the algorithmic computation of Gröbner bases, were derived. The motivation to start from $\mathfrak{R}$ was to maintain as much as possible generality. But it proved to be a disadvantage that the class of rings covered by the results remains widely hidden. Therefore, in this paper we use an opposite approach. We start with a well-ordered monoid Γ and a ring R obtained by adjunction of finitely many elements $X = \{X_1, \ldots, X_n\}$ to a ground ring Q. Then we associate a natural graded structure $\mathfrak{R}$ to these objects and investigate the constructivity in dependence on Q, Γ, and the defining relations of R. More precisely, our aim is to find classes of rings whose natural graded structures allow the reduction of large subproblems to the valuation monoid Γ and the ground ring Q in order to obtain uniform algorithmic solutions.

The constructive instances of graded structures corresponding to successful generalizations of Buchberger's method can be divided in two main directions. The first considers polynomial rings $R = Q[X]$ in finitely many variables $X = \{X_1, \ldots, X_n\}$ with more general ground rings Q than only fields. For instance, there were investigated situations with principal ideal domains Q (cf. [15, 24]) or, even more general, commutative rings Q in which linear equations are solvable (cf. [1, 8, 20, 28, 30, 31]). R is a graded ring with respect to the commutative monoid freely generated by X in all these cases. The second direction of generalizations keeps Q a (skew) field but relaxes the property that R is a graded ring. Examples are enveloping algebras of Lie algebra [7], algebras of solvable type [16], G-algebras [2, 3], and solvable polynomial rings [17]. The constructive instances of natural graded structures investigated here include all the above types of rings but, in addition, also combinations of the two main directions are subsumed. Of course, the extensions which do not fit in the frame of graded structures, e.g. group rings (cf. [18, 19, 27]) and reduction rings (cf. [10, 29]), are not covered here.

The paper is organized as follows. In Section 2 we present a short introduction to the theory of graded structures. Then we define the notion of natural Γ-graded structures $\mathfrak{R}$ of extension rings R of Q generated by a set X in Section 3. The fourth section considers necessary conditions for Q and Γ of an effective left, right, or two-sided Gröbner structure $\mathfrak{R}$. The presentation of R

by truncated Gröbner bases in the free extension ring of Q by X is subject of Section 5. Sections 6 and 7 provide algorithmic solutions for problems in the associated graded ring G of $\mathfrak{R}$ which are fundamental for the computation of Gröbner bases. Assumptions ensuring ascending chain conditions for one- or two-sided ideals of G are considered in Section 8. Section 9 shows that the conditions introduced so far allow the algorithmic computation of left syzygy modules of homogeneous left ideals of the associated graded ring. In particular, this finishes the proof of the first main result of the paper which concerns effective left Gröbner structures and is summarized in Theorem 6. Section 10 deals with the two-sided case. Some effective left Gröbner structures $\mathfrak{R}$ allow the application of a generalized Kandri-Rody/Weispfenning closure technique (see [16]) in order to compute Gröbner bases of two-sided ideals (see Theorem 7). Theorem 8 generalizes a result of Mora who was the first presenting algebras in which Gröbner bases of two-sided ideals can be computed in an algorithmic way while, in general, one-sided ideals are even not finitely generated in these algebras (see [23]). The aim of Section 11 is to give an impression when a graded structure can be an effective Gröbner structure though it does not satisfy the assumptions of Theorems 6-8. We close the paper by presenting examples of effective Gröbner structures in Section 12.

Finally, we remark that *ring* always stands for associative ring with unit element in this paper. In particular, also ring extensions are considered only in this class.

2 Graded structures

Let R be a ring with unit element and $(\Gamma, \prec)$ a well-ordered monoid. Let ϵ denote the unit element of Γ and note the well-known fact that ϵ is the minimal element of Γ with respect to $\prec$. Finally, let $\varphi : R \setminus \{0\} \to \Gamma$ be a Γ-pseudo valuation function, i.e. it satisfies

$$\begin{array}{rcl} \varphi(u) & = & \epsilon \\ a+b \neq 0 & \Longrightarrow & \varphi(a+b) \preceq \max(\varphi(a), \varphi(b)) \\ ab \neq 0 & \Longrightarrow & \varphi(ab) \preceq \varphi(a) \circ \varphi(b) \end{array}$$

for all invertible elements $u \in R$ and all non-zero elements $a, b \in R$. For each $\gamma \in \Gamma$ the set $\mathcal{F}_\gamma = \{a \mid \varphi(a) \preceq \gamma\} \cup \{0\}$ is an additive subgroup of R and it is easy to prove that the family $\mathfrak{F} = (\mathcal{F}_\gamma)_{\gamma \in \Gamma}$ is a filtration of R. For each $\gamma \in \Gamma$ we define the quotient $G_\gamma = \mathcal{F}_\gamma / \widehat{\mathcal{F}}_\gamma$ of $\mathcal{F}_\gamma$ by its subgroup $\widehat{\mathcal{F}}_\gamma = \{0\} \cup \bigcup_{\gamma' \prec \gamma} \mathcal{F}_{\gamma'}$. For $a \in \mathcal{F}_\gamma$ we introduce the denotation $[a]_{\widehat{\mathcal{F}}_\gamma}$ for the residue class $a + \widehat{\mathcal{F}}_\gamma \in G_\gamma$. The equation

$$\forall a, b \in R \setminus \{0\} \ : \ [a]_{\widehat{\mathcal{F}}_{\varphi(a)}} [b]_{\widehat{\mathcal{F}}_{\varphi(b)}} = [ab]_{\widehat{\mathcal{F}}_{\varphi(a) \circ \varphi(b)}}$$

determines a multiplication which makes the direct sum $G = \bigoplus_{\gamma \in \Gamma} G_\gamma$ a Γ-graded ring with unit element $[1]_{\widehat{\mathcal{F}}_\epsilon}$. G with this multiplication is called the associated graded ring of the filtered structure $(R, \mathfrak{F})$. The elements $u \in G_\gamma$ are homogeneous of degree γ (denotation $\deg(u) = \gamma$). R and G are connected via the function $\mathrm{in} : R \to G$ assigning each element $a \in R$ its initial form $\mathrm{in}(a) = [a]_{\widehat{\mathcal{F}}_{\varphi(a)}}$ (by definition $\mathrm{in}(0) = 0$). Let $\hat{G} = \bigcup_{\gamma \in \Gamma} G_\gamma$ denote the set of all homogeneous elements of G and $\mathrm{in}^* : \hat{G} \to R$ an arbitrary section of in, i.e. $\mathrm{in}(\mathrm{in}^*(u)) = u$ for all homogeneous elements $u \in G$.

Definition 1 *With the above notation we call* $\mathfrak{R} = (R, \Gamma, \varphi, G, \mathrm{in})$ *a* graded structure. *Furthermore, a set $F \subset R$ is called a Gröbner basis of the left (right,two-sided) ideal I generated by F if* $\mathrm{in}(F)$ *and* $\mathrm{in}(I)$ *generate the same left (right,two-sided) ideal of G.*

Definition 2 *A graded structure* $\mathfrak{R} = (R, \Gamma, \varphi, G, \mathrm{in})$ *is called an* effective left (right) Gröbner structure *if the following conditions are satisfied:*

i) the rings R and G and the ordered monoid Γ are effective algebraic structures,

ii) φ and in *are computable functions, and there exists a computable section* in^* *of the initial mapping,*

iii) the membership problem of homogeneous left (right) ideals of G given by an arbitrary finite homogeneous generating set is decidable,

iv) for any finite set $H \subset G$ of homogeneous elements there can be computed a finite homogeneous generating set of the left (right) syzygy module $\mathrm{LSyz}(H)$ *of H, and*

v) G is a left (right) noetherian ring.

Before, we consider the two-sided case we will briefly discuss the syzygy problem of two-sided ideals. Let E denote the subring of G which is generated by the unit element $[1]_{\widehat{\mathcal{F}}_e}$. G is left and right E-module, so the tensor product $G \otimes_E G$ is a well-defined E-bimodule. In the following we consider $G \otimes_E G$ with its natural G-bimodule structure. Let $H = \{h_1, \ldots, h_k\} \subset G$ be a finite subset of G and $S_H : (G \otimes_E G)^k \to G$ denote the G-bimodule homomorphism defined by $S_H\left(\sum_{j=1}^m a_j e_{i_j} b_j\right) = \sum_{j=1}^m a_j h_{i_j} b_j$, where $1 \le i_j \le k$ and $a_j e_{i_j} b_j$ denotes the tensor $a_j \otimes b_j$ belonging to the i_j-th copy of $G \otimes_E G$. For any H the kernel $\ker S_H$ forms a G-submodule of $(G \otimes_E G)^k$, the so-called *syzygy module* $\mathrm{Syz}(H)$ of H. Even for noetherian rings G the G-bimodule $(G \otimes_E G)^k$ needs not to be noetherian, too. Therefore, a straight forward generalization of condition *iv)* would be to strong. Mora solved the problem in [22] by asking for the computability of a finite non-trivial homogeneous generating set of $\mathrm{Syz}(H)$. A homogeneous syzygy $\sum_{j=1}^m a_j e_{i_j} b_j \in \mathrm{Syz}(H)$ is called *trivial* if the element $lift_F\left(\sum_{j=1}^m a_j e_{i_j} b_j\right) = \sum_{j=1}^m \mathrm{in}^*(a_j) f_{i_j} \mathrm{in}^*(b_j)$ can be reduced to zero modulo F for any set $F = \{f_1, \ldots, f_k\} \subset R$ such that $\mathrm{in}(f_i) = h_i$ $(i = 1, \ldots, k)$. If B together with the trivial syzygies of H generate the syzygy module $\mathrm{Syz}(H)$ then B is called a *non-trivial generating set* of $\mathrm{Syz}(H)$.

Definition 3 *A graded structure* $\mathfrak{R} = (R, \Gamma, \varphi, G, \mathrm{in})$ *is called an* effective two-sided Gröbner structure *if the following conditions hold:*

i) the rings R and G and the ordered monoid Γ are effective algebraic structures,

ii) φ and in *are computable functions, and there exists a computable section* in^* *of the initial mapping,*

iii) the membership problem of homogeneous two-sided ideals of G given by an arbitrary finite homogeneous generating set is decidable,

iv) for any finite set $H \subset G$ of homogeneous elements there can be computed a finite non-trivial homogeneous generating set of the syzygy module $\mathrm{Syz}(H)$*, and*

v) G satisfies the ascending chain condition for two-sided ideals.

Let $A \subseteq G$ be an arbitrary subring generated by the initial forms $\mathrm{in}(a)$ of elements a belonging to the center of R. Obviously, A is contained in the center of G. By $\mathrm{Syz}_A(H)$ we denote the image of the syzygy module of H under the natural G-bimodule homomorphism $\tau : (G \otimes_E G)^k \to (G \otimes_A G)^k$. Since all syzygies belonging to the intersection $\ker \tau \cap \mathrm{Syz}(H)$ are trivial the following criterion can be used for the verification of condition *iv)*: if $\mathrm{Syz}_A(H)$ is finitely generated then $\mathrm{Syz}(H)$ has a finite non-trivial generating set and for any generating set B of $\mathrm{Syz}_A(H)$ the set $\{b \mid \tau(b) \in B\}$ is non-trivial generating set of $\mathrm{Syz}(H)$.

Let $\mathfrak{R} = (R, \Gamma, \varphi, G, \mathrm{in})$ be an effective left (right, two-sided) Gröbner structure. Then for any finite subset $F \subset R$ there can be computed a left (right, two-sided) Gröbner basis of the left (right, two-sided) ideal of R generated by F in an algorithmic way [22]. Given $\mathfrak{R}$ it remains to check that the conditions *i)-v)* are satisfied. The large generality of the concept of graded structures is its power but as soon as effectiveness is concerned it becomes also its main difficulty. At the level of Definitions 2 and 3 no restrictions apply to the algorithms solving conditions *iii)* and particularly *iv)*. This is motivation to look for subclasses of effective graded structures which have uniform algorithms for deciding membership problems and computing syzygy modules of homogeneous ideals of the associated graded ring.

3 Natural graded structures of extension rings

We consider a ring R with a finite minimal generating set $X = \{X_1, \ldots, X_n\}$ over some ground ring Q. For an arbitrary well-ordered monoid $(\Gamma, \prec)$ with a minimal generating set $Y = \{Y_1, \ldots, Y_n\}$ the condition

$$a \in \mathcal{F}_\gamma \quad :\Longleftrightarrow \quad \begin{array}{l} a \text{ is a finite sum of terms } r_0 X_{i_1} r_1 \cdots X_{i_k} r_k, \\ \text{where } r_0, \ldots, r_k \in Q \text{ and } Y_{i_1} \circ \cdots \circ Y_{i_k} \preceq \gamma\,. \end{array}$$

defines a Γ-filtration $\mathfrak{F} = (\mathcal{F}_\gamma)_{\gamma \in \Gamma}$ of R.

Definition 4 *For R, $(\Gamma, \prec)$, and $\mathfrak{F}$ as above, the Γ-graded structure $\mathfrak{R} = (R, \Gamma, \varphi, G, \mathrm{in})$ induced by the function*

$$\varphi(a) := \min\{\gamma \in \Gamma \mid a \in \mathcal{F}_\gamma\},\ a \in R \setminus \{0\}$$

will be called the natural Γ*-graded structure of R.*

There is a natural isomorphism between the subring $Q \subseteq R$ and the subring $G_\epsilon \subseteq G$ formed by all homogeneous elements of degree ϵ, where ϵ denotes the unit element of Γ. In the following G_ϵ and Q will be identified. Then G is left and right Q-module.

We will restrict our investigations to such situations where each quotient $G_\gamma = \mathcal{F}_\gamma / \widehat{\mathcal{F}}_\gamma$, $\gamma \in \Gamma$, contains an element $\mathfrak{g}_\gamma$ which generates it as left and as right Q-module. In particular, all G_γ are cyclic and for each $\gamma \in \Gamma$ there exists a homomorphism $\sigma_\gamma : Q \to Q$ satisfying

$$\mathfrak{g}_\gamma a - \sigma_\gamma(a)\mathfrak{g}_\gamma = 0 \text{ for all } a \in Q\ . \tag{1}$$

For an arbitrary section in^* of the initial mapping the elements $\mathrm{in}^*(\mathfrak{g}_{Y_i})$, $i = 1, \ldots, n$, generate R over Q. Hence, without loss of generality we may assume that the elements of the generating set X allow a section having the property

$$\mathrm{in}^*(\mathfrak{g}_{Y_i}) = X_i \quad (i = 1, \ldots, n)\ .$$

A cyclic left Q-module M is determined by its *annihilating left ideal*

$$\mathrm{ann}_L M = \{a \in Q \mid am = 0 \text{ for all } m \in M\}$$

up to isomorphism. We have $M \simeq Q/\mathrm{ann}_L M$. An analogous statement holds for right Q-modules M and annihilating right ideals $\mathrm{ann}_R M$. Both left and right annihilator are even two-sided ideals. For Q-modules M containing an element $\mathfrak{g}$ which generates it as left and as right module we have the ring isomorphism

$$Q/\mathrm{ann}_L M \simeq Q/\mathrm{ann}_R M \tag{2}$$

and it holds

$$a \in \mathrm{ann}_L M \Longleftrightarrow a\mathfrak{g} = 0 \quad , \quad a \in \mathrm{ann}_R M \Longleftrightarrow \mathfrak{g}a = 0\ .$$

We remark that the restriction to cyclic modules G_γ is typical but not necessary for Gröbner basis investigations. For instance, the main theorem on abelian groups can be applied successfully in many situations where the G_γ are of higher dimension. Möller and Mora investigated such situations in [21]. Also Hironaka's standard bases in power series rings refer to a grading with non-cyclic homogeneous summands (see [14])[1]. Pesch introduced a Gröbner theory in iterated Ore extensions (see [25]). Though, there is a natural translation of Pesch's method in the language of graded structures the result is not one of the known constructive instances. The direct summands G_γ of the associated graded ring are only cyclic as left Q-modules but higher dimensional as right Q-modules.

[1] Note, Hironaka's grading is based on an order $\prec$ which is not well-founded. This leads to additional computability problems which were discussed in [5].

4 Conditions on Q and Γ

If the natural graded structure $\mathfrak{R}$ is an effective left Gröbner structure then Q must be a computable, noetherian ring with decidable left ideal membership problem. Moreover, for any finite subset $H \subset Q$ a finite generating set of the left syzygy module $\mathrm{LSyz}(H)$ can be computed. To sketch a proof consider the extension left ideal $G \cdot I$ of the left ideal $I \subset Q$. G needs not to be a flat extension of Q, for instance, the left syzygy module of $G \cdot I$ is not necessarily generated by homogeneous left syzygies of degree ϵ. But taking into account that G is a graded ring the computability conditions carry over from G to Q. Analogous arguments can be applied in the right and two-sided case.

Assume that the natural graded structure $\mathfrak{R}$ of the monoid ring $R = Q[\Gamma]$ is an effective left, right, and two-sided Gröbner structure. Then also Γ has to fulfill rather strong conditions. So, Γ must be a computable well-ordered monoid. Furthermore, it has to satisfy a generalization of Dickson's Lemma [12], i.e. for any infinite sequence $\gamma_1, \gamma_2, \ldots$ of elements of Γ there exist positive integers $i < j$ and $k < l$ such that γ_i is a left divisor of γ_j and γ_k is a right divisor of γ_l. In this case we call Γ a noetherian monoid which reflects the fact that ascending chains of left, right, or two-sided monoid ideals, respectively, will always stabilize[2]. Further necessary conditions on Γ are that left, right, and two-sided divisibility of elements of Γ is decidable and that minimal common left, right, and two-sided multiples of finite subsets of Γ can be computed algorithmically. We remark, that the decidability of left or right divisibility is equivalent to the seemingly much harder condition, that the set of all decompositions into irreducible factors is finite and computable in an algorithmic way for all $\gamma \in \Gamma$. This is an easy consequence of the following facts. Any noetherian well-ordered monoid Γ satisfies the left and right cancellation law and any element $\gamma \neq \epsilon$ of Γ has only a finite number of decompositions into irreducible factors. It follows that the minimal generating set X of Γ is uniquely determined, finite, and consists exactly of the irreducible elements of $\Gamma \setminus \{\epsilon\}$.

5 Presentation of R by truncated Gröbner bases

Let $Q_C \subseteq Q$ denote the subring formed by all elements of Q which commute with the elements of R, i.e. $Q_C = \{a \in Q \mid \forall b \in R \,:\, ab = ba\}$. Note, that at least the subring Q_U of Q generated by 1 is contained in Q_C.

For an arbitrary intermediate ring $Q_U \subseteq \hat{Q} \subseteq Q_C$ we introduce the notation $A_{\hat{Q}}$ for the ring $\langle Q, X\rangle_{\hat{Q}}$ which is freely generated by X in the class of extension rings of Q whose center contains $\hat{Q}$. For any such $\hat{Q}$ the ring R is a homomorphic image of $A_{\hat{Q}}$. Let $K_{\hat{Q}}$ denote the kernel $\ker \iota_{\hat{Q}}$ of the natural endomorphism $\iota_{\hat{Q}} : A_{\hat{Q}} \to R$ acting identically on X, and identify $R = A_{\hat{Q}}/K_{\hat{Q}}$.

[2] If the graded structure of $R = Q[\Gamma]$ is only required to be an effective two-sided Gröbner structure then a weaker generalization of Dickson's Lemma providing only the ascending chain condition for two-sided ideals would be sufficient. But for simplicity we consider only the strongest generalization which is suitable for all three types of ideals.

We fix an intermediate ring $Q_U \subseteq \hat{Q} \subseteq Q_C$ for which Q is a computable $\hat{Q}$-module and denote $A = A_{\hat{Q}}$, $K = K_{\hat{Q}}$, and $\iota = \iota_{\hat{Q}}$. A is a computable ring and, hence, the ring R is computable iff the membership problem of K is decidable.

Let $\langle Y \rangle$ denote the word monoid freely generated by Y and ordered by a well-founded order $\prec_A$ satisfying $u \prec_A v \Longrightarrow \nu(u) \preceq \nu(v)$ for all $u, v \in \langle Y \rangle$, where $\nu : \langle Y \rangle \to \Gamma$ denotes the natural homomorphism. Following the ideas from Section 3 we associate to A a graded structure $\mathfrak{A} = (A, \langle Y \rangle, \varphi_A, G_A, \mathrm{in}_A)$ which induces a notion of Gröbner bases for arbitrary ideals of A with respect to $\mathfrak{A}$. Though, in general, there is no algorithm for computing such Gröbner bases at least the existence of possibly infinite Gröbner bases with respect to $\mathfrak{A}$ is ensured for any one- or two-sided ideal.

For the rest of the paper we assume that $\prec_A$ has the property that for all $\gamma \in \Gamma$ the element $\mathfrak{g}'_\gamma = \mathrm{in}(X_{i_1} \ldots X_{i_k})$, where $Y_{i_1} \ldots Y_{i_k} = \min\{u \in \langle Y \rangle \mid \nu(u) = \gamma\}$, generates G_γ as left and as right Q-module[3]. Furthermore, without loss of generality, we assume the choice $\mathfrak{g}_\gamma = \mathfrak{g}'_\gamma$, $\gamma \in \Gamma$, for the generators distinguished in Section 3.

Let us investigate the structure of a (possibly infinite) Gröbner basis H of $K = \ker \iota$ with respect to $\mathfrak{A}$.

According to equations (1) the kernel $\ker \iota$ contains elements of the form

$$X_i \alpha - \sigma_{Y_i}(\alpha) X_i + p_{i,\alpha} \ , \tag{3}$$

where $\alpha \in Q$ and $p_{i,\alpha} = 0$ or $p_{i,\alpha} \in R$ with $\nu(\varphi_A(p_{i,\alpha})) \prec \nu(Y_i) = Y_i$.

Consider an arbitrary $t = Y_{i_1} \cdots Y_{i_k} \in \langle Y \rangle$ and let $Y_{j_1} \cdots Y_{j_l} = \min\{u \in \langle Y \rangle \mid \nu(u) = \nu(t)\}$. Then K contains an element

$$X_{i_1} \cdots X_{i_k} - \alpha_t X_{j_1} \cdots X_{j_l} + q_t \ , \tag{4}$$

where $\alpha_t \in Q$ and $q_t = 0$ or $\nu(\varphi_A(q_t)) \prec \nu(t)$. Furthermore, in the special case $l = k$ and $j_1 = i_1, \ldots, j_k = i_k$ the ideal K contains elements

$$\beta_{t,q} X_{j_1} \cdots X_{j_l} + r_{t,q} \ , \tag{5}$$

where $\beta_{t,q} \in \mathrm{ann}_L G_{\nu(t)}$ and $r_{t,q} = 0$ or $\nu(\varphi_A(r_{t,q})) \prec \nu(t)$ $(q = 1, 2, \ldots)$.

Since $Q \cap K = \{0\}$ there exists a Gröbner bases H of K which consists only of elements of types (3), (4), and (5). Recall, that for effective Gröbner structures $\mathfrak{R}$ the kernel K must have decidable membership problem. Instead we assume the stronger condition that K is given by a finite truncated Gröbner basis with respect to $\mathfrak{A}$, where

Definition 5 *$H_{\mathrm{trunc}} \subseteq H$ is called a* truncation *of the Gröbner basis H of K with respect to $\mathfrak{A}$ if it satisfies the following conditions:* i) *all elements of $H \setminus H_{\mathrm{trunc}}$ are of type (5),* ii) $\nu(\varphi_A(h)) \nmid \nu(\varphi_A(h'))$ *for all $h' \in H_{\mathrm{trunc}}$ and $h \in H \setminus H_{\mathrm{trunc}}$, and* iii) *for all $h \in H \setminus H_{\mathrm{trunc}}$ there exists a divisor $\gamma \in \Gamma$ of $\nu(\varphi_A(h))$ such that $\nu(\varphi_A(h')) \nmid \gamma$ for all $h' \in H \setminus H_{\mathrm{trunc}}$ and $G_{\nu(\varphi_A(h))} \cong G_{\gamma'}$ for all γ', where $\gamma \mid \gamma' \mid \nu(\varphi_A(h))$.*

[3] In particular, we assume the existence of such an order.

6 Computation of annihilating ideals of G_γ

Given a finite truncated Gröbner basis H_{trunc} of K with respect to $\mathfrak{A}$ it is possible to compute a finite generating set of the annihilating left ideal $\text{ann}_L G_\gamma$ for any given $\gamma \in \Gamma$ in an algorithmic way.

Let $t = Y_{i_1} \cdots Y_{i_k} \in \langle Y \rangle$ be the (w.r.t. $\prec_A$) minimal word such that $\nu(t) = Y_{i_1} \circ \cdots \circ Y_{i_k} = \gamma$. Then we have

$$\alpha \in \text{ann}_L G_\gamma \Longleftrightarrow \alpha X_{i_1} \cdots X_{i_k} \text{ is reducible modulo } H \ , \tag{6}$$

where H is an arbitrary Gröbner basis of ker ι with respect to $\mathfrak{A}$. For the rest of this paragraph we write shortly $\alpha X_{j_1} \cdots X_{j_l}$ for the monomial $\alpha X_{j_1} \cdots X_{j_l} + \widehat{\mathcal{F}}_{A,Y_{j_1} \cdots Y_{j_l}} \in G_A$ of the associated graded ring of $\mathfrak{A}$.

Consider $h \in H$ with initial form $\text{in}_A(h) = \beta_h X_{i_j} \cdots X_{i_m}$. Then the product $X_{i_1} \cdots X_{i_{j-1}} \text{in}_A(h) X_{i_{m+1}} \cdots X_{i_k}$ is congruent to the monomial $\beta_{h,t} X_{i_1} \cdots X_{i_k} \in G_A$, where $\beta_{h,t} = \sigma_{Y_{i_1}}(\cdots(\sigma_{Y_{i_{j-1}}}(\beta_h)))$, modulo the two-sided ideal generated by the initial forms of the elements of H which belong to type (3). Obviously, $\beta_{h,t} \in \text{ann}_L G_\gamma$ for all so-constructed elements $\beta_{h,t} \in Q$. Furthermore, the right hand side of condition (6) means that for all $\alpha \in \text{ann}_L G_\gamma$ the homogeneous element $\alpha X_{i_1} \cdots X_{i_k} \in G_A$ must be a linear combination of homogeneous elements $X_{i_1} \cdots X_{i_{j-1}} \text{in}_A(h) X_{i_{m+1}} \cdots X_{i_k}$, where $h \in H$, $\varphi_A(h) = Y_{i_j} \cdots Y_{i_m}$, and $1 \leq j \leq m \leq k$. Hence, the annihilating left ideal $\text{ann}_L G_\gamma$ is generated by the above elements $\beta_{h,t}$.

If $\varphi_A(h) \nmid t$ for all $h \in H \setminus H_{\text{trunc}}$ then this provides an algorithm for the computation of a finite generating set of $\text{ann}_L G_\gamma$. Otherwise, we first need to complete H_{trunc} to a truncation H'_{trunc} of a Gröbner basis H' of K such that $\varphi_A(h) \nmid t$ for all $h \in H' \setminus H'_{\text{trunc}}$. From Definition 5 it follows that the completion can be done by adding the (finitely many) elements to H_{trunc} which are remainder of a product $X_{j_1} \cdots X_{j_m} h X_{p_1} \cdots X_{p_l}$, where $h \in H_{\text{trunc}}$ and $Y_{j_1} \circ \cdots \circ Y_{j_m} \circ \nu(\varphi_A(h)) \circ Y_{p_1} \circ \cdots \circ Y_{p_l} = \gamma$, modulo H_{trunc}.

For constructive G there are also computable homomorphisms $\widehat{\sigma_\gamma} : Q \to Q$ satisfying $a\mathfrak{g}_\gamma = \mathfrak{g}_\gamma \widehat{\sigma_\gamma}(a)$. This allows the transformation of the truncated Gröbner basis H_{trunc} in an equivalent system with all coefficients right of the products $X_{i_1} \cdots X_{i_k}$. Therefore, finite generating sets of the right annihilating ideals $\text{ann}_R G_\gamma$ can be computed in a similar way.

7 Ideal membership in the associated graded ring

Let $u_1, \ldots, u_k$, and v be non-zero homogeneous elements of the associated graded ring G of the natural graded structure $\mathfrak{R}$. Can we decide $v \in J$, where J is the left, respectively two-sided, ideal generated by the elements $u_1, \ldots, u_k$? Our previous assumptions on Q, Γ, and K will turn out to be already sufficient to answer this question positively.

Let $\deg u_i = \gamma_i$ and $\deg v = \gamma$ denote the degrees of the homogeneous elements $u_1, \dots, u_k$, and v. Then the elements can be assumed to be presented in the form $u_i = \alpha_i \mathfrak{g}_{\gamma_i}$, and $v = \beta \mathfrak{g}_\gamma$, where $\alpha_1, \dots, \alpha_k, \beta \in Q$.

First consider left ideals J. The set $M = \{(\omega, i) \mid 1 \le i \le k \wedge \omega \circ \gamma_i = \gamma\}$ is finite and can be computed in an algorithmic way since divisibility in Γ is decidable. By constructivity of G there is an algorithm transforming each product $\mathfrak{g}_\omega \alpha_i \mathfrak{g}_{\gamma_i}$, $(\omega, i) \in M$, in the form $\mathfrak{g}_\omega \alpha_i \mathfrak{g}_{\gamma_i} = \alpha'_{\omega,i} \mathfrak{g}_\gamma$, where $\alpha'_{\omega,i} \in Q$. Obviously,

$$\begin{aligned} v \in J \quad &\Longleftrightarrow \quad \exists \beta_{\omega,i} \in Q \ : \ v = \sum_{(\omega,i) \in M} \beta_{\omega,i} \mathfrak{g}_\omega u_i \\ &\Longleftrightarrow \quad \beta \in Q \cdot (\alpha'_{\omega,i}) + \operatorname{ann}_L G_\gamma \ . \end{aligned} \tag{7}$$

Now, consider the two-sided ideal generated by $u_1, \dots, u_k$. We can compute the set $M = \{(\omega, i, \omega') \mid 1 \le i \le k \wedge \omega \circ \gamma_i \circ \omega' = \gamma\}$, which is finite according to our assumptions. Applying similar arguments as in the left ideal case and taking into account that $\operatorname{ann}_L G_\gamma$ is even two-sided it follows

$$\begin{aligned} v \in J \quad &\Longleftrightarrow \quad \exists \beta_{\omega,i,\omega';j}, \beta'_{\omega,i,\omega';j} : v = \sum_{(\omega,i,\omega') \in M} \sum_{j=1}^{m_{\omega,i,\omega'}} \beta_{\omega,i,\omega';j} \mathfrak{g}_\omega u_i \mathfrak{g}_{\omega'} \beta'_{\omega,i,\omega';j} \\ &\Longleftrightarrow \quad \beta \equiv \sum \beta_{\omega,i,\omega';j} \alpha'_{\omega,i,\omega'} \sigma_\gamma \left(\beta'_{\omega,i,\omega';j} \right) \mod \operatorname{ann}_L G_\gamma \\ &\Longleftrightarrow \quad \beta \in Q \cdot (\alpha'_{\omega,i,\omega'}) \cdot Q + \operatorname{ann}_L G_\gamma \ , \end{aligned} \tag{8}$$

where $\alpha'_{\omega,i,\omega'} \mathfrak{g}_\gamma = \mathfrak{g}_\omega u_i \mathfrak{g}_{\omega'}$.

In conclusion we proved that the membership problem of a (left) homogeneous ideal of G can be reduced to the membership problem of a (left) ideal of Q. It is well-known that the decidability of $v \in J$? ensures the existence of an algorithm computing a representation of v in terms of $u_1, \dots, u_k$ for any $v \in J$. However, due to its inefficiency, this general algorithm resulting from the theory is of no practical importance. Note, our above considerations prove not only decidability but provide also nice formulae transforming solutions of (7) and (8), respectively, in representations of v. Let $\delta_1, \dots, \delta_m$ generate $\operatorname{ann}_L G_\gamma$ as a left ideal. We have:

$$\begin{aligned} & \beta = \sum_{(\omega,i) \in M} \beta_{\omega,i} \alpha'_{\omega,i} + \sum_{j=1}^{m} \mu_j \delta_j \\ \Longrightarrow \quad & v = \sum_{i=1}^{k} \left(\sum_{(\omega,i) \in M} \beta_{\omega,i} \mathfrak{g}_\omega \right) u_i \end{aligned}$$

and

$$\beta = \sum_{(\omega,i,\omega') \in M} \sum_{j=1}^{m_{\omega,i,\omega'}} \beta_{\omega,i,\omega';j} \alpha'_{\omega,i,\omega'} \beta'_{\omega,i,\omega';j} + \sum_{j=1}^{m} \mu_j \delta_{i_j} \mu'_j$$

$$\Longrightarrow \quad v = \sum_{(\omega,i,\omega')\in M} \sum_{j=1}^{m_{\omega,i,\omega'}} (\beta_{\omega,i,\omega';j}\mathfrak{g}_\omega)\, u_i \left(\mathfrak{g}_{\omega'}\widehat{\sigma_\gamma}\left(\beta'_{\omega,i,\omega';j}\right)\right)$$

Hence, under some obvious conditions on the efficiency of calculations in Q, Γ, and G we obtain also efficient algorithms for the computation of representations of v in terms of $u_1, \ldots, u_k$.

8 Noetherianity of G

Until now our conditions on Q, Γ, and K influenced mainly the Q-module structure but there are still to many freedoms in the ring structure of R and G. In particular, we have not yet enough control about the zero divisors of G.

Consider, for instance, the following extremal case. Let Γ be the free commutative monoid generated by Y and assume that the elements of Q commute with the elements of Γ. Moreover, let $X_iX_j \in K$ for all $1 \leq j < i \leq n$ and K contain no element of type (5). Then G contains many zero-divisors, is not noetherian, and the syzygy modules of even many one-sided homogeneous principal ideals of G are not finitely generated. More generally, serious problems may arise if $\ker\,\iota$ contains elements of type (4) whose coefficient α_t is not invertible modulo $\mathrm{ann}_L G_{\nu(t)}$. Such kernel elements can, but need not, cause a non-noetherian associated graded ring G.

The condition

$$\forall \gamma, \omega \in \Gamma \ : \ G_\gamma G_\omega = G_{\gamma\circ\omega} \ . \tag{9}$$

is equivalent to the property that for any $t = Y_{i_1}\cdots Y_{i_k} \in \langle Y\rangle$ there exists $X_{i_1}\cdots X_{i_k} - \alpha_t X_{j_1}\cdots X_{j_l} + q_t \in \ker\,\iota$ of type (4) such that α_t is a unit modulo $\mathrm{ann}_L G_{\nu(t)}$. Note, in Section 5 we had to assume the existence of an order $\prec_A$ such that the initial form $\mathrm{in}(X_{i_1}\ldots X_{i_k})$, where $Y_{i_1}\ldots Y_{i_k} = \min\{u \in \langle Y\rangle \mid \nu(u) = \gamma\}$, generates G_γ as left and as right Q-module for all $\gamma \in \Gamma$. It is easy to observe that in case condition (9) holds any $\prec_A$ has this property.

If Q and Γ are noetherian and G satisfies condition (9) then the associated graded ring G is left and right noetherian. We show that any infinite sequence of non-zero homogeneous elements $u_1 = \mathfrak{g}_{\gamma_1}\alpha_1, u_2 = \mathfrak{g}_{\gamma_2}\alpha_2, \ldots$ of G contains $u_l \in G\cdot(u_1,\ldots,u_{l-1})$. Since Γ is noetherian there exists an infinite subsequence $u_{i_1}, u_{i_2}, \ldots$ such that the degree of u_{i_k} is a right multiple of the degree of u_{i_j} for all $j < k$. Moreover, by condition (9) for all $j < k$ it follows the existence of a homogeneous element $v_{j,k}$ such that $\mathfrak{g}_{\gamma_{i_k}} = v_{j,k}\mathfrak{g}_{\gamma_{i_j}}$. Furthermore, from the noetherianity of Q we deduce the existence of an index $l > 1$ such that α_{i_l} belongs to the left ideal of Q generated by the elements $\alpha_{i_1},\ldots,\alpha_{i_{l-1}}$. Consequently, there exist $\beta_1,\ldots,\beta_{l-1} \in Q$ such that $u_{i_l} = \mathfrak{g}_{\gamma_{i_l}}\alpha_{i_l} = \sum_{r=1}^{l-1} v_{r,l}\mathfrak{g}_{\gamma_{i_r}}\beta_r\alpha_{i_r} = \sum_{r=1}^{l-1} v_{r,l}\hat{\sigma}_{\gamma_{i_r}}(\beta_r)\mathfrak{g}_{\gamma_{i_r}}\alpha_{i_r} = \sum_{r=1}^{l-1}(v_{r,l}\hat{\sigma}_{\gamma_{i_r}}(\beta_r))u_{i_r}$. Hence, u_{i_l} belongs to the left ideal of G generated by $u_1,\ldots,u_{i_{l-1}}$ and it follows that G is a left noetherian ring. Starting with representations $u_i = c'_i\mathfrak{g}_{\gamma_i}$ we can prove in the same way that G is right noetherian and, hence, noetherian.

Next we change condition (9) in such a way that G still satisfies the ascending chain condition for two-sided but not longer necessarily for left or right ideals. Instead of (9) we assume now that the elements of Q commute with the elements of X and that for all $\omega \in \Gamma$ and divisors $\gamma \in \Gamma$ there exists a decomposition $\gamma' \circ \gamma \circ \gamma'' = \omega$ such that

$$G_{\rho'} G_{\rho} G_{\rho''} = G_{\rho' \circ \rho \circ \rho''} \tag{10}$$

for all divisor triples $\rho' \mid \gamma'$, $\rho \mid \gamma$, $\rho'' \mid \gamma''$. We will show that any infinite sequence $u_1 = \mathfrak{g}_{\gamma_1}\alpha_1, u_2 = \mathfrak{g}_{\gamma_2}\alpha_2, \ldots$ of homogeneous elements of G contains an element $u_k \in G(u_1, \ldots, u_{k-1})G$. Since Γ is noetherian it is sufficient to prove the assertion for sequences satisfying $\gamma_i \mid \gamma_j$ for all $i < j$. Since Q is noetherian there exists k such that $\alpha_k \in Q(\alpha_1, \ldots, \alpha_{k-1})Q$. For all $i < k$ there exist $\gamma_i', \gamma_i'' \in \Gamma$ and $\beta_i \in Q$ such that $\beta_i \mathfrak{g}_{\gamma_i'} u_i \mathfrak{g}_{\gamma_i''} = \mathfrak{g}_{\gamma_k}\alpha_i$ according to the above assumptions. Hence, $u_k \in G(u_1, \ldots, u_{k-1})G$ and we are done.

Given a truncated Gröbner basis of K condition (9) could be verified using a simple criterion checking whether the coefficients α_t appearing in the elements of type (4) are invertible modulo $\mathrm{ann}_L G_{\nu(t)}$. When Γ is commutative and $G_{Y_{i_1} \circ Y_{i_2} \circ \cdots \circ Y_{i_k}} = G_{Y_{i_1}} G_{Y_{i_2}} \cdots G_{Y_{i_k}}$ for all $1 \leq i_1 \leq \cdots \leq i_k \leq n$ [4] then a similar criterion allows the verification of condition (10). For each pair (i, j) such that $1 \leq j < i \leq n$ the ideal K contains an element $X_i X_j - \alpha_{i,j} X_j X_i + q_{i,j}$ of type (4) and it is obvious how to construct these elements from an arbitrary truncated Gröbner basis of K with respect to $\mathfrak{A}$. Condition (10) holds iff for each $1 \leq j \leq n$ we have at least one of the following two properties: *i)* $\alpha_{i,j}$ is invertible modulo $\mathrm{ann}_L G_{Y_j \circ Y_i}$ for all $j < i \leq n$ or *ii)* $\alpha_{j,i}$ is invertible modulo $\mathrm{ann}_L G_{Y_i \circ Y_j}$ for all $1 \leq i < j$. Let $\gamma \mid \omega$, an example of a suitable decomposition $\omega = \gamma' \circ \gamma \circ \gamma''$ can be obtained by gathering all variables of the quotient $\frac{\omega}{\gamma}$ whose index j satisfies condition *ii)* in γ' and the rest in γ''. Now, let us consider the opposit direction, i.e. for some $1 \leq j \leq n$ neither condition *i)* nor *ii)* holds. Then there exist $i < j$ and $i' > j$ such that $\alpha_{j,i}$ and $\alpha_{i',j}$ are not invertible modulo the corresponding annihilating left ideals and for $\omega = Y_i \circ Y_j \circ Y_{i'}$ and $\gamma = Y_i \circ Y_{i'}$ no decomposition fulfills condition (10).

9 Effective left or right Gröbner structures

As an immediate consequence of condition (9) we obtain that the product $\mathfrak{g}_\omega \mathfrak{g}_\gamma$ generates $G_{\omega \circ \gamma}$ as left and as right Q-module for all $\omega, \gamma \in \Gamma$. In particular, $\alpha \mathfrak{g}_\omega \mathfrak{g}_\gamma = 0$ iff $\alpha \in \mathrm{ann}_L G_{\omega \circ \gamma}$ and, hence, $\mathrm{ann}_L G_\omega \subseteq \mathrm{ann}_L G_{\omega \circ \gamma}$. Consequently, the quotient ring $Q/\mathrm{ann}_L G_{\omega \circ \gamma}$ is a homomorphic image of the quotient ring $Q/\mathrm{ann}_L G_\omega$. Applying similar arguments to right annihilating ideals it follows that $Q/\mathrm{ann}_R G_{\omega \circ \gamma}$ is homomorphic image of the quotient ring $Q/\mathrm{ann}_R G_\gamma$. Isomorphisms (2) imply the existence of ring epimorphisms $\rho_{\omega,\gamma} : Q/\mathrm{ann}_L G_\gamma \to$

[4] Note, the most important case covered by these conditions is when Γ is the commutative monoid freely generated by Y and the order $t \prec_A s \iff \nu(t) \prec \nu(s) \vee (\nu(t) = \nu(s) \wedge t <_l s)$, where $<_l$ denotes the lexicographical order extending $Y_1 <_l Y_2 <_l \ldots <_l Y_n$, can be used in the construction of $\mathfrak{A}$ (see Section 5).

$Q/\mathrm{ann}_L G_{\omega\circ\gamma}$ for all $\omega, \gamma \in \Gamma$. If the epimorphism $\rho_{\omega,\gamma}$ is not injective then for any $\omega' \in \Gamma$ also the composition $\rho_{\omega,\gamma} \circ \rho_{\omega',\omega\circ\gamma} : Q/\mathrm{ann}_L G_\gamma \to Q/\mathrm{ann}_L G_{\omega'\circ\omega\circ\gamma}$ is surjective but not injective. Since Q is noetherian the existence of a non-injective epimorphims implies that the rings are not isomorphic. In conclusion we proved that for all $\gamma \in \Gamma$ the set

$$\Gamma_\gamma = \{\omega \in \Gamma \mid G_\gamma \not\cong G_{\omega\circ\gamma}\} \tag{11}$$

is either empty or a left monoid ideal of Γ. The condition $\omega \notin \Gamma_\gamma$ is equivalent to $\mathrm{ann}_R G_\gamma = \mathrm{ann}_R G_{\omega\circ\gamma}$.

Note, given a finite truncated Gröbner basis H_{trunc} of K there is an obvious algorithm for the computation of a finite generating set[5] Δ_γ of Γ_γ for an arbitrary given $\gamma \in \Gamma$. Roughly, the idea behind is to extract a generating set Δ_γ from the set of all elements $\omega \in \Gamma$ for which $\omega \circ \gamma$ is a minimal common multiple of γ and the elements of some subset of $\{\nu(\varphi_A(h)) \mid h \in H_{\mathrm{trunc}}\}$. The existence of such a generating set follows immediately from Definition 5.

This can be applied to the algorithmic computation of the left syzygy module $\mathrm{LSyz}(U)$ for an arbitrary finite set U of homogeneous non-zero elements of G. We define recursively $\Omega(U)_{i+1} = \{\gamma' \circ \gamma \mid \gamma \in \Omega(U)_i \wedge \gamma' \in \Delta_\gamma\}$, where the initial value $\Omega(U)_0 \subseteq \Gamma$ is the set of all minimal common right multiples of the degrees of elements of U. Each set $\Omega(U)_i$ is finite and can be computed algorithmically. If $\Omega(U)_i = \emptyset$ then $\Omega(U)_j = \emptyset$ for all $j > i$. By the properties of Q there cannot exist an infinite sequence $Q/\mathrm{ann}_L G_{\gamma_0} \longrightarrow Q/\mathrm{ann}_L G_{\gamma_1} \longrightarrow \cdots$ of non-injective ring epimorphisms. Hence, there exists a natural number i_0 such that $\Omega(U)_{i_0} = \emptyset$ and, therefore, $\Omega(U) = \bigcup_{i=1}^{\infty} \Omega(U)_i = \bigcup_{i=1}^{i_0-1} \Omega(U)_i$ is finite and can be computed algorithmically. For arbitrary given $\gamma \in \Omega(U)$ there can be computed a finite generating set of the left syzygy module of $\{\delta + \mathrm{ann}_L G_\gamma \mid \exists u \in U\, \exists \omega \in \Gamma : \mathfrak{g}_\omega u = \delta \mathfrak{g}_\gamma\} \subset Q/\mathrm{ann}_L G_\gamma$ according to the properties of Q. These generating left syzygies can be lifted to homogeneous left syzygies of degree γ of U by multiplying each of their components from the right by the corresponding element $\mathfrak{g}_\omega$. Any homogeneous left syzygy of degree γ of U is contained in the left G-module generated by the set B_γ formed by the lifted left syzygies. Next, we show that any homogeneous left syzygy $s = \sum_{u\in U} h_u e_u$ of U, whose degree is a common right multiple of the degrees of all elements of U, belongs to the left G-module generated by the union $B(U) = \bigcup_{\gamma\in\Omega(U)} B_\gamma$. Let γ be a maximal right divisor of $\deg s$ which is contained in $\Omega(U)$ and $\omega \in \Gamma$ be such that $\omega \circ \gamma = \deg s$. According to condition (9) there exist homogeneous elements v_u such that $\mathfrak{g}_\omega v_u = h_u$ and, hence, s can be written in the form $s = \mathfrak{g}_\omega \sum_{u\in U} v_u e_u$. $\sum_{u\in U} v_u u$ is a homogeneous element of G of degree γ and, therefore, can be written in the form $\mathfrak{g}_\gamma d$, where $d \in Q$. Furthermore, $\mathfrak{g}_\omega \mathfrak{g}_\gamma d = 0$ since s is a left syzygy of U. Consequently, $d \in \mathrm{ann}_R(G_{\deg s}) \supseteq \mathrm{ann}_R(G_\gamma)$. By definition of $\Omega(U)$ the inclusion is even equality and, therefore, s is a multiple of a homogeneous left syzygy of U which has a degree contained in $\Omega(U)$.

In conclusion, the set $B(U) \cup \bigcup_{U'\subset U} \mathrm{LSyz}(U')$, where $B(U) = \bigcup_{\gamma\in\Omega(U)} B_\gamma$, generates $\mathrm{LSyz}(U)$ and induction on the number of elements of U yields that

[5]Formally, $\Delta_\gamma = \emptyset$ is considered as generating set of $\Gamma_\gamma = \emptyset$.

a finite homogeneous generating set of $\mathrm{LSyz}(U)$ can be constructed in an algorithmic way.

Theorem 6 *Let Q be a computable noetherian ring with decidable ideal membership and solvable syzygy problem for left, right, and two-sided ideals, and $\hat{Q}$ a subring of the center of Q such that Q is a computable $\hat{Q}$-module. Furthermore, let Γ be a computable well-ordered monoid which is noetherian and allows algorithmic computation of minimal common multiples and factorial decompositions. Finally, let $R = \langle Q, X\rangle_{\hat{Q}}/K$ be given by a finite truncated Gröbner basis H_{trunc} of the two-sided ideal K and let the associated graded ring G belonging to the natural graded structure $\mathfrak{R} = (R, \Gamma, \varphi, G, \mathrm{in})$ satisfy condition (9).*

Then $\mathfrak{R}$ is an effective left Gröbner structure.

Conditions *i)-v)* of Definition 2 have been verified already. $\square$

Analogous considerations prove that any graded structure $\mathfrak{R}$ fulfilling the assumptions of the above theorem is also an effective right Gröbner structure. However, the assumptions could be slightly relaxed by assuming only the conditions on Q and Γ which refer to left (right) ideals. Among these marginal cases there are graded structures $\mathfrak{R}$ which are only an effective left (right) but not an effective right (left) Gröbner structure.

10 Effective two-sided Gröbner structures

Under some additional assumptions the graded structures considered in Theorem 6 allow also the computation of Gröbner bases of two-sided ideals of R using a generalized Kandri-Rody/Weispfenning closure technique [16].

Theorem 7 *Let Q be a computable noetherian ring with decidable ideal membership and solvable syzygy problem for left, right, and two-sided ideals, and $\hat{Q}$ a subring of the center of Q such that Q is a computable $\hat{Q}$-module. Furthermore, let Γ be a computable well-ordered commutative monoid which is noetherian and allows algorithmic computation of minimal common multiples and factorial decompositions. In addition, let there exist computable functions $\kappa : Q \times Q \to Q$ and $\kappa_Y : Y \times Q \to Q$ satisfying $\alpha \cdot \beta = \kappa(\alpha, \beta) \cdot \alpha$ respectively $\alpha \cdot \mathfrak{g}_{Y_i} = \kappa_Y(Y_i, \alpha) \cdot \mathfrak{g}_{Y_i} \cdot \alpha$ for all $\alpha, \beta \in Q$ and $i = 1, \ldots, n$. Finally, let $R = \langle Q, X\rangle_{\hat{Q}}/K$ be given by a finite truncated Gröbner basis H_{trunc} of K and the associated graded ring G of the natural graded structure $\mathfrak{R} = (R, \Gamma, \varphi, G, \mathrm{in})$ satisfy condition (9).*

Then $\mathfrak{R}$ is an effective two-sided Gröbner structure and each two-sided Gröbner basis F of an arbitrary two-sided ideal $I \subseteq R$ is also a left and a right Gröbner basis of I.

It remains to consider the solution of the syzygy problem. Note, $\hat{Q}$ is a subring of the center of R and the initial mapping acts identically on $\hat{Q}$. Therefore, according to the criterion presented behind Definition 3 it suffices to show that for an arbitrary finite set U of non-zero homogeneous elements of G there can be computed a finite homogeneous generating set of $\mathrm{Syz}_{\hat{Q}}(U)$ in an algorithmic way.

For arbitrary $\alpha \in Q$ and $u = \beta \mathfrak{g}_\gamma \in U$, where $\beta \in Q$ and $\gamma \in \Gamma$, there can be computed the syzygy $s_{\alpha,u} = e_u\alpha - \delta_{\alpha,u}e_u \in \mathrm{Syz}_{\hat{Q}}(U)$, where $\delta_{\alpha,u} = \kappa(\beta, \sigma_\gamma(\alpha))$. In a similar way there can be computed a syzygy $s_{Y_i,u} = e_u\mathfrak{g}_{Y_i} - \delta_{Y_i,u}\mathfrak{g}_{Y_i}e_u \in \mathrm{Syz}_{\hat{Q}}(U)$ for given $i = 1,\dots,n$ and $u \in U$. $s_{Y_i,u}$ is uniquely determined up to a trivial summand $\beta\mathfrak{g}_{Y_i}e_u$, where $\beta \in \mathrm{ann}_L G_{Y_i\circ\gamma}$. Since H_{trunc} is finite the set $Z = \{\zeta \in Q \mid X_i\zeta - \sigma_{Y_i}(\zeta)X_i + p_{i,\zeta} \in H_{\mathrm{trunc}}\}$ of all highest coefficients of the elements of type (3) contained in the truncated Gröbner basis H_{trunc} is finite, too. Moreover, Z generates Q as a ring over $\hat{Q}$. So, we can compute finite sets $B_Z = \{s_{\zeta,u} \mid (\zeta,u) \in Z \times U\}$ and $B_Y = \{s_{Y_i,u} \mid (Y_i,u) \in Y \times U\}$. Next we will show that $B_Z \cup B_Y \cup \mathrm{LSyz}(U) \otimes_{\hat{Q}} 1$ generates $\mathrm{Syz}_{\hat{Q}}(U)$.

We have $e_u\zeta_1\cdots\zeta_k = (s_{\zeta_1,u} + \delta_{\zeta_1,u}e_u)\,\zeta_2\cdots\zeta_k$ and by induction on k it follows $s_{\zeta_1\cdots\zeta_k,u} \in GB_ZG$ for all $u \in U$ and all products $\zeta_1\cdots\zeta_k$, where $\zeta_1,\dots,\zeta_k \in Z$. Hence, $s_{\alpha,u} \in GB_ZG$ for all $\alpha \in Q$ and $u \in U$. Next, we will prove the existence of a syzygy $s_{\gamma,u} = e_u\mathfrak{g}_\gamma - \delta_{\gamma,u}\mathfrak{g}_\gamma e_u \in G(B_Z \cup B_Y)G$ for all $\gamma \in \Gamma$ and $u \in U$ by induction on the length k of an arbitrary representation $\gamma = Y_{i_1} \circ \cdots \circ Y_{i_k}$. The initial step $k = 1$ is obvious. Consider $k > 1$ and set $\gamma' = Y_{i_1} \circ \cdots \circ Y_{i_{k-1}}$. We have $e_u\mathfrak{g}_{\gamma'\circ Y_{i_k}} = e_u\mathfrak{g}_{Y_{i_k}}\mathfrak{g}_{\gamma'}\alpha = s_{Y_{i_k},u}\mathfrak{g}_{\gamma'}\alpha + \delta_{Y_{i_k},u}\mathfrak{g}_{Y_{i_k}}e_u\mathfrak{g}_{\gamma'}\alpha$ for some $\alpha \in Q$ and by induction hypothesis there exists $s_{\gamma',u} = e_u\mathfrak{g}_{\gamma'} - \delta_{\gamma',u}\mathfrak{g}_{\gamma'}e_u \in G(B_Z \cup B_Y)G$. Hence, $e_u\mathfrak{g}_{\gamma'\circ Y_{i_k}} = s_{Y_{i_k},u}\mathfrak{g}_{\gamma'}\alpha + \delta_{Y_{i_k},u}\mathfrak{g}_{Y_{i_k}}s_{\gamma',u}\alpha + \delta_{Y_{i_k},u}\mathfrak{g}_{Y_{i_k}}\delta_{\gamma',u}\mathfrak{g}_{\gamma'}e_u\alpha = s_{Y_{i_k},u}\mathfrak{g}_{\gamma'}\alpha + \delta_{Y_{i_k},u}\mathfrak{g}_{Y_{i_k}}s_{\gamma',u}\alpha + \delta_{Y_{i_k},u}\mathfrak{g}_{Y_{i_k}}\delta_{\gamma',u}\mathfrak{g}_{\gamma'}s_{a,u} + \delta_{Y_{i_k},u}\mathfrak{g}_{Y_{i_k}}\delta_{\gamma',u}\mathfrak{g}_{\gamma'}\delta_{\alpha,u}e_u$. This finishes the induction proof. As an immediate consequence we obtain that for any homogeneous syzygy $s \in \mathrm{Syz}_{\hat{Q}}(U)$ there exists a homogeneous left syzygy $s' \in \mathrm{LSyz}(U)$ such that $s - s' \otimes 1 \in G(B_Z \cup B_Y)G$. Therefore, $B_Z \cup B_Y \cup \mathrm{LSyz}(U) \otimes_{\hat{Q}} 1$ generates $\mathrm{Syz}_{\hat{Q}}(U)$. Application of Theorem 6 yields that $\mathfrak{R}$ is an effective two-sided Gröbner structure.

From the above investigations it follows that for any homogeneous elements $u, v \in G$ there exists a homogeneous element $w \in G$ of the same degree as v such that $uv = wu$. Hence, any homogeneous left ideal of G is even two-sided. Therefore, left and two-sided initial ideal coincide for any two-sided ideal $I \subseteq R$. Moreover, the left and the two-sided ideal generated by the initial parts of a subset of I are equal. Consequently, any Gröbner basis of the two-sided ideal I is also a Gröbner basis of I considered as left ideal according to Definition 1. Analogous arguments apply to I considered as a right ideal. □

The requirement of the existence of the functions κ and κ_Y might seem rather technical. It could be replaced by one of the stronger conditions that Q is a skew field or $Q = \hat{Q}$. In fact these both situations are the most interesting applications.

At the end of the previous section we mentioned marginal cases of Gröbner structures which are effective only with respect to one side. An interesting open question is whether relaxing the conditions on Q there can be obtained graded structures which are an effective Gröbner structure with respect to two-sided and left (or right) ideals but not with respect to right (or left) ideals. Outside the theory of graded structures such a behavior is known from the investigations of Madlener and Reinert in group rings (see [19]).

Roughly, the idea behind the Kandri-Rody/Weispfenning closure technique consists in computing left Gröbner bases and checking whether the generated left ideal is closed under multiplication with variables from the right. If this is not the case then the non-zero remainders are added to the basis and the cycle of left Gröbner basis computation and saturation with right multiples is repeated. In our situation the generating set $B_Z \cup B_Y \cup \mathrm{LSyz}(U) \otimes_{\hat{Q}} 1$ of the syzygy module allows a similar procedure. The syzygies contained in B_Z and B_Y represent the multiples considered in the saturation step of the left Gröbner basis.

Mora considered a class of non-commutative algebras which allow the computation of Gröbner bases for two-sided but not necessarily for one-sided ideals (see [23]). The reason is that the associated graded ring satisfies the ascending chain condition for two-sided but not for one-sided ideals. The following theorem based on condition (10) generalizes Mora's result.

Let $\gamma' \circ \gamma \circ \gamma'' = \omega$ and γ', γ'' satisfy the assumptions of condition (10). In particular, we have $G_{\gamma'} G_\gamma = G_{\gamma' \circ \gamma}$ and $G_{\gamma' \circ \gamma} G_{\gamma''} = G_\omega$. Applying similar arguments as in the previous section to arbitrary $\omega', \omega'' \in \Gamma$ we obtain an epimorphism sequence $G_\gamma \to G_{\gamma' \circ \gamma} \to G_\omega \to G_{\omega' \circ \omega \circ \omega''}$. Hence, for all $\gamma \in \Gamma$ the set

$$\widehat{\Gamma}_\gamma = \{\omega \in \Gamma \ : \ \gamma \mid \omega \wedge G_\gamma \ncong G_\omega\} \tag{12}$$

is either empty or a monoid ideal of Γ. A finite generating set Δ_γ of $\widehat{\Gamma}_\gamma$ can be computed using a truncated Gröbner basis of K.

Theorem 8 *Let Q be a computable noetherian commutative ring with decidable ideal membership and solvable syzygy problem. Furthermore, let Γ be a computable well-ordered commutative monoid which is noetherian and allows algorithmic computation of minimal common multiples and factorial decompositions. Finally, let $R = \langle Q, X \rangle_Q / K$ be given by a finite truncated Gröbner basis H_{trunc} of the two-sided ideal K and let the associated graded ring G of the natural graded structure $\mathfrak{R} = (R, \Gamma, \varphi, G, \mathrm{in})$ satisfy condition (10).*

Then $\mathfrak{R}$ is an effective two-sided Gröbner structure.

It remains the verification of conditions *iv)* and *v)* of Definition 3.

First, we will show that any infinite sequence $u_1 = \mathfrak{g}_{\gamma_1}\alpha_1, u_2 = \mathfrak{g}_{\gamma_2}\alpha_2, \ldots$ of homogeneous elements of G contains an element $u_k \in G(u_1, \ldots, u_{k-1})G$. Since Γ is noetherian it is sufficient to prove the assertion for sequences satisfying $\gamma_i \mid \gamma_j$ for all $i < j$. By noetherianity of Q there exists k such that $\alpha_k \in (\alpha_1, \ldots, \alpha_{k-1})Q$. By condition (10) it follows the existence of $\gamma_i', \gamma_i'' \in \Gamma$ and $\beta_i \in Q$ such that $\beta_i \mathfrak{g}_{\gamma_i'} \mathfrak{g}_{\gamma_i} \mathfrak{g}_{\gamma_i''} = \mathfrak{g}_{\gamma_k}$ for all $i < k$. Hence, $u_k \in G(u_1, \ldots, u_{k-1})G$ and, consequently, G satisfies the ascending chain condition for two-sided ideals.

For the rather technical and lengthy proof of condition *iv)* we refer to [5, Theorem 5.23]. Here, we will sketch only the main ideas. For any $Y_i \in Y$ and $u \in U$ there exists a homogeneous syzygy $s_{Y_i,u} = \alpha_{Y_i,u} e_u \mathfrak{g}_{Y_i} - \beta_{Y_i,u} \mathfrak{g}_{Y_i} e_u \in \mathrm{Syz}_Q(U)$, where at least one of the elements $\alpha_{Y_i,u}, \beta_{Y_i,u} \in Q$ is a unit. Let $B_Y =$

$\{s_{Y_i,u} \mid (Y_i,u) \in Y \times U\}$. For any homogeneous syzygy $s = \sum_{i=1}^{k} v_i e_{u_i} w_i \in \mathrm{Syz}_{\hat{Q}}(H)$ whose degree is a multiple of the degrees of all $u \in U$ there exists a homogeneous syzygy $s' = \mathfrak{g}_\delta \left(\sum_{i=1}^{k} v'_i e_{u_i} w'_i\right) \mathfrak{g}_{\delta'}$ such that $s - s' \in GB_Y G$ and $\deg(v'_i) \circ \deg(u_i) \circ \deg(w'_i)$ is a minimal common multiple of the degrees of the elements of U. Let $\Omega(U)_0$ be the set of all minimal common multiples of the degrees of $u \in U$ and define recursively $\Omega(U)_{i+1} = \bigcup_{\gamma \in \Omega(U)_i} \Delta_\gamma$. Then the set $\Omega(U) = \bigcup_{i=0}^{\infty} \Omega(U)_i$ is finite and can be constructed algorithmically. Finally, the set $B_Y \cup \bigcup_{\gamma \in \Omega(U)} C_\gamma \cup \bigcup_{U' \subset U} \mathrm{Syz}_Q(U')$, where the C_γ are finite generating sets of the Q-modules of all homogeneous syzygies of U of degree γ, generates $\mathrm{Syz}_Q(U)$. □

11 Open problems

Before we could prove that a natural graded structure $\mathfrak{R} = (R, \Gamma, \varphi, G, \mathrm{in})$ is an effective Gröbner structure we had to introduce a series of conditions on the objects Q, Γ, and K. In this section we deal with the question which conditions could be relaxed without loosing the effective Gröbner structure property. For natural graded structures $\mathfrak{R}$ which are left, right, and two-sided Gröbner structure our conditions on Q are necessary and cannot not be relaxed in any way. If $\mathfrak{R}$ is required to be an effective Gröbner structure with respect to only one side, left, right, or two-sided, then the necessity of the conditions follows only for ideals of Q belonging to the same side. Under the condition that the natural graded structure of the monoid ring $Q\langle\Gamma\rangle$ has to be an effective Gröbner structure similar statements apply to the assumptions on Γ. In marginal cases with many homogeneous summands of G being the zero module, e.g. if $G_\gamma = 0$ for all $\gamma \in \Gamma \setminus \{\epsilon\}$, the conditions on Γ could be relaxed. But in such situations the linkage between the ring R and the monoid Γ is so weak that often a graded structure of R with respect to a suitable submonoid of Γ satisfying our assumptions can be used. Open questions are when such a submonoid exists and how it can be constructed. Moreover, special situations with ground rings Q and valuation monoids Γ satisfying only the conditions corresponding to ideals of a fixed side remain open for future investigations.

In Section 3 we gave examples showing that the restriction to graded structures whose associated graded ring has cyclic homogeneous summands is serious. But, in spite the described examples, this condition is very typical for Gröbner basis investigations. Even Pesch makes use of it by mainly working with the left module structure. Nevertheless, there remains an open research direction.

The condition that R has to be given by a finite truncated Gröbner basis of the kernel K of a homomorphism $\iota : \langle Q, X\rangle_{\hat{Q}} \to R$ and conditions (9) and (10) are the most interesting restrictions and will be discussed now.

Assume, there exists an infinite sequence $\gamma_1, \gamma_2, \ldots \in \Gamma$ of right multiples such that $G_{\omega_i} G_{\gamma_i} \subsetneq G_{\gamma_{i+1}}$, where $\gamma_{i+1} = \omega_i \circ \gamma_i$, for all $i = 1, 2, \ldots$. Then the left ideal $G \cdot (\mathfrak{g}_{\gamma_1}, \mathfrak{g}_{\gamma_2}, \ldots)$ is not finitely generated. Hence, such a sequence can not exist in the associated graded ring of an effective left Gröbner structure.

Though, an effective left Gröbner structure need not necessarily to satisfy condition (9) the above observation shows that the cases lying outside are rather marginal.

In the following, we will consider the condition that K can be presented by a finite truncated Gröbner basis. If the ring Q is finitely generated over $\hat{Q}$ and there exists a finite confluent system of rewriting rules for Γ then there are Gröbner bases of K which contain only finitely many elements of types (3) and (4) and it remains to consider the number of elements of type (5). For effective left Gröbner structures there can be computed a finite homogeneous generating set of the left syzygy module of the principal left ideal generated by $\mathfrak{g}_\gamma \in G$ for any given $\gamma \in \Gamma$. The coefficients of the left syzygies of degree γ generate the annihilating left ideal of the homogeneous summand G_γ of the associated graded ring and, hence, the annihilating left ideals $\mathrm{ann}_L G_\gamma$ are computable for any effective left Gröbner structure $\mathfrak{R}$ satisfying the above assumptions. Moreover, all $\omega \in \Gamma$ which are minimal right multiples of γ with the property that there exists a non-injective epimorphism from G_γ onto G_ω appear among the degrees of the left syzygies in an arbitrary homogeneous generating set of $\mathrm{LSyz}\,(G \cdot \mathfrak{g}_\gamma)$. Hence, if (9) holds then a finite generating set Δ_γ of the left monoid ideal (or empty set) Γ_γ defined in (11) can be computed in an algorithmic way for any given $\gamma \in \Gamma$. If $G_\gamma \cong G_\omega$ for some proper divisor ω of $\gamma \in \Gamma$ then no elements of type (5) with highest degree γ need to be contained in a truncated Gröbner basis. Hence, we have to compute the set of all $\gamma \in \Gamma$ such that $G_\gamma \ncong G_\omega$ for all proper divisors ω. Let $\Omega(\{1\})$ be defined as in Section 9 before Theorem 6. $\Omega(\{1\})$ can be computed in an algorithmic way since it requires only computations of generating sets Δ_γ. Moreover, $\Omega(\{1\})$ is just the set of degrees where "essentially new" elements of type (5) can appear in a Gröbner basis of K. For each of the finitely many elements $\gamma \in \Omega(\{1\})$ there can be computed a finite generating set of $\mathrm{ann}_L G_\gamma$. A possible set of highest coefficients of Gröbner basis elements of type (5) with highest term $X_{i_1} \cdots X_{i_k}$, where $Y_{i_1} \cdots Y_{i_k} \in \langle Y \rangle$ is the minimal representant of γ, can be found among the generators of $\mathrm{ann}_L G_\gamma$. Note, we proved not only the existence of a finite truncated Gröbner basis of K but showed also how its initial forms can be constructed.

Similar considerations can be done in the two-sided case. In fact, the remaining gaps are larger than here, but, the most interesting cases are again covered by our Theorems.

12 Examples

Classical examples such as polynomial rings, enveloping algebras of Lie algebras, or algebras of solvable type satisfy the assumptions of Theorem 7 if we consider only computable coefficient fields. Moreover, Theorem 7 applies also to polynomial rings over computable Euclidean rings or, more general, computable commutative rings in which linear equations are solvable. But generalizing both types of examples we can also consider rings $R = \langle Q, X \rangle_Q / K$, where Q is a computable commutative ring in which linear equations are solvable and K has a

Gröbner basis consisting of an element $X_jX_i + e_{i,j}X_iX_j + p_{i,j}$, where $e_{i,j}$ is a unit of Q and $p_{i,j}$ has leading term smaller than X_iX_j, for each pair $1 \leq i < j \leq n$. The natural graded structure of R with respect to the free commutative monoid Γ generated by Y is similar to that used in algebras of solvable type but with a more general coefficient domain.

Note, the assumption that K can be represented by a Gröbner basis of the form $\{X_jX_i + e_{i,j}X_iX_j + p_{i,j} \mid 1 \leq i < j \leq n\}$ is (implicitly) used in almost all investigations of Gröbner bases in non-commutative rings to which a $\mathbb{N}^n$-graded structure can be associated (cf. [7, 13, 16, 17]). Additional Gröbner basis elements first appeared in [2]. Here, we are faced with new problems in comparison to [2] which arise from dropping the assumption that the coefficient ring has to be a subfield of the center of R. The following examples will show some typical new situations where our results are applicable. Theorems 6-8 assumed that a finite truncated Gröbner basis of K with respect to $\mathfrak{A}$ is given a priori. However, also if $R = A/K$ is given by an arbitrary finite generating set of K there is a good chance to compute a truncated Gröbner basis of K. There has to be calculated a (truncated) Gröbner basis in a free extension ring $A = \langle Q, X \rangle_{\hat{Q}}$. The decision of ideal membership and the computation of syzygy modules of finitely generated $\langle X \rangle$-homogeneous ideals requires only the application of simple well-known algorithms for $\langle X \rangle$-graded rings. Hence, the general method for computing Gröbner bases in graded structures becomes semi-algorithmic for free extension rings A, i.e. if there exists a finite Gröbner basis of K then it will be computed in finite time. If K has no finite Gröbner basis with respect to $\mathfrak{A}$ but a finite truncation then eventually the Gröbner method will have computed such a truncation. However, it is a (probably undecidable) problem to realize that the algorithm can be stopped. The examples were calculated using the special computer algebra system FELIX (see [6]).

Example 1: Consider the ring $A = \mathbb{Z}\langle x, y, z \rangle = \langle \mathbb{Z}, \{x, y, z\} \rangle_{\mathbb{Z}}$ which is freely generated by $\{x, y, z\}$ in the class of all extension rings of the integers $\mathbb{Z}$.[6] Let $\langle x, y, z \rangle$ denote the word monoid and Γ the monoid of commutative terms in the variables $\{x, y, z\}$. We order Γ by the total degree order $\prec$ extending $z \prec y \prec x$ and $\langle x, y, z \rangle$ by the well-founded order $\prec_A$ which compares the words first (forgetting non-commutativity) according to $\prec$ and second applies the lexicographical order $<_l$ extending $x <_l y <_l z$ for breaking ties. Let $\mathfrak{A}$ denote the natural $\langle x, y, z \rangle$-graded structure of A and consider the two-sided ideal $K \subseteq A$ generated by $\{yx - 3xy - 3z, zx - 2xz + y, zy - yz - x\}$. During the computation of a Gröbner basis of K with respect to $\mathfrak{A}$ the following elements are constructed:

$$yx - 3xy - 3z, zx - 2xz + y, zy - yz - x,$$

$$6yz + 3x, 9xz - 3y, 12xy + 9z, 12y^2 - 27z^2, x^2 + 2y^2 - 6z^2$$

[6] Note, the condition that $\mathbb{Z}$ is contained in the center of A is trivially satisfied since only rings with unit element are considered.

$$9z^3-30xy-21z, 4y^3+9yz^2+3y, 4xy^2+3yz+3x, 3xyz-3y^2+9z^2,$$
$$3yz^3-90xy^2-3xz^2-3yz-36x, 2y^3z-3xy^2+3yz, xy^2z-3y^3-3xz,$$
$$y^3z^3-2xy^4-3y^3z-3yz^3+xy^2-3yz,$$
$$xy^3z+3y^4-6y^2z^2, xy^4z+y^5+y^3z^2+2y^3-3yz^2, xy^5z-y^6+3y^4z^2, \ldots$$

Reducing $(xy^{j-1}z+p_{j-1})y$ modulo this intermediate basis we observe by induction that K contains an element of the form xy^jz+p_j, where $\varphi_A(p_j) \prec_A xy^jz$, for any positive integer $j>1$. In fact, only such elements are necessary in order to complete the above intermediate basis to an infinite Gröbner basis of K with respect to $\mathfrak{A}$ and a finite Gröbner basis does not exist. But according to Definition 5 the above set is already a truncated Gröbner basis of K with respect to $\mathfrak{A}$ and even the elements of the last row can be removed. The ring $R=A/K$ satisfies the assumptions of Theorem 8 and therefore, the natural Γ-graded structure of R is an effective two-sided Gröbner structure. The assumptions of Theorem 6 and 7 are violated since the coefficient of xy in $yx-3xy-3z$ is not invertible modulo the annihilating ideal $\mathrm{ann}_L G_{xy}=12\mathbb{Z}$.

Example 2: Consider the graded structure $\mathfrak{A}$ from the previous example and let K be the two-sided ideal generated by the elements $yx-3xy-z, zx-xz+y$, and $zy-yz-x$. We are interested in the natural Γ-graded structure of $R=A/K$. The generators look similar to the defining relations of an algebra of solvable type. But even if we allow rational coefficients the behavior of our ring is much different since the terms $x^iy^jz^k$ $(i,j,k=0,1,2,\ldots)$ are linearly dependent. The elements

$$yx-3xy-z, zx-xz+y, zy-yz-x,$$
$$8xy+2z, 4xz-2y, 4yz+2x,$$
$$2x^2-2y^2, 4y^2-2z^2, 2z^3-2xy$$

form a finite Gröbner basis of the two-sided ideal $K \subset \mathbb{Z}\langle x,y,z\rangle$ with respect to $\mathfrak{A}$ from example 1.

Since $\mathrm{ann}_L G_{xy}=8\mathbb{Z}$, we have $\mathfrak{g}_{xy}=\mathfrak{g}_x\mathfrak{g}_y=3\mathfrak{g}_y\mathfrak{g}_x$ in the associated graded ring of the natural Γ-graded structure $\mathfrak{R}$ of R. Hence, condition (9) holds. The other assumptions of Theorem 7 are obvious. Consequently, finite Gröbner bases with respect to $\mathfrak{R}$ can be computed using the algorithms presented in this paper for arbitrary ideals of R.

Example 3: Let $\mathcal{W}=\langle\mathbb{Q},\{p,q\}\rangle_{\mathbb{Q}}/(qp-pq-1)$ and consider the ring $R=\langle\mathcal{W},\{x,y\}\rangle_{\mathbb{Q}}/K$, where K ist the two-sided ideal of $\langle\mathcal{W},\{x,y\}\rangle_{\mathbb{Q}}$ given by the Gröbner basis

$$xp-qx, xq+px, yp-qy, yq+py, yx-xy+y^2$$

with respect to the natural graded structure induced by the well-ordered monoid $(\langle x,y\rangle, \prec_A)$, where $\prec_A$ compares words by first forgetting non-commutativity and applying the lexicographical order $\prec$ of the free commutative monoid which extends $y \prec x$ and second breaking ties by comparing the non-commutative words with respect to the lexicographical order extending $x <_l y$. Note, not all coefficients but only the rational numbers commute with the variables of the ring R. Functions κ and κ_Y as required in Theorem 7 do not exist but at least the assumptions of Theorem 6 are fulfilled in this situation. For this reason finite Gröbner bases of left ideals $I \subseteq R$ can be computed using the algorithms presented in this paper. Consider the homogeneous element $u = p\mathfrak{g}_{x^2y}$ of the associated graded ring G of R. Since $up - au = (-pq - ap)\mathfrak{g}_{x^2y} \neq 0$ for all $a \in \mathcal{W}$ the two-sided ideal generated by u is strictly larger than the left ideal generated by u. Hence, homogeneous left ideals of G need not to be two-sided and, therefore, two-sided Gröbner bases need not to be left Gröbner bases. However, though neither Theorem 7 nor Theorem 8 is applicable it remains an open question whether the natural graded structure of R is an effective two-sided Gröbner structure.

Example 4: Once again, let us consider the graded structure $\mathfrak{A}$ from example 1 and let K be generated by $yx - 3xy, zx + y^2, zy - yz + z^2$. Since $R = A/K$ is a $\mathbb{N}$-graded ring it is easy to observe that $\mathrm{ann}_L G_{xz}$ and $\mathrm{ann}_L G_{xy}$ are zero ideals. Therefore, the coefficient 3 of xy in the first generator and the coefficient 0 of xz in the second generator are both not invertible modulo the corresponding annihilating left ideal and, hence, neither Theorem 6 nor Theorem 7 can be applied to R. The elements

$$\begin{gathered} yx - 3xy, zx + y^2, zy - yz + z^2, \\ 2y^3 + y^2z - 2yz^2 + 2z^3, \\ 14yz^3 - 28z^4, y^2z^2 - 4yz^3 + 6z^4, 27xy^2z - 54xyz^2 + 54xz^3 + y^4, \\ 14z^5, 2yz^4 - 6z^5, y^4z, y^5, 2xyz^3 - 4xz^4, 27xy^3z, \\ 2z^6, 2xz^5 \end{gathered}$$

form a Gröbner basis of K with respect to $\mathfrak{A}$. Consider arbitrary monoid elements $\omega = x^iy^jz^k$ and $\gamma = x^{i'}y^{j'}z^{k'}$ such that $\gamma \mid \omega$. Then condition (10) holds for the decomposition $\gamma' \circ \gamma \circ \gamma''$, where $\gamma' = x^{i-i'}$ and $\gamma'' = y^{j-j'}z^{k-k'}$. Hence, the assumptions of Theorem 8 are satisfied and the natural Γ-graded structure $\mathfrak{R}$ of R is an effective two-sided Gröbner structure. A finite Gröbner basis can be computed for any two-sided ideal of R using the algorithms presented in this paper.

Note, R does not satisfy the ascending chain condition for left ideals, e.g. the left ideal $R \cdot (xz, xz^2, xz^3, \ldots)$ has no finite generating set. Hence, it is proved that $\mathfrak{R}$ is not an effective left Gröbner structure.

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