Gossiping in Chordal Rings under the Line Model

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Abstract

This paper is devoted to the gossip (or all-to-all) problem in the chordal ring under the one-port *line model*. The line model assumes long distance calls between non neighboring processors. In this sense, the line model is strongly related to circuit-switched networks, wormhole routing, optical networks supporting wavelength division multiplexing, ATM switching, and networks supporting connected mode routing protocols.

Since the chordal rings are competitors of networks as meshes or tori because of theirs short diameter and bounded degree, it is of interest to ask whether they can support intensive communications (typically all-to-all) as efficiently as these networks. We propose polynomial algorithms to derive optimal or near optimal gossip protocols in the chordal ring.

1 Introduction

In the study of the properties of interconnection networks, the problem of dissemination of information is an important and a very active research area [12, 26]. Indeed, the ability of an interconnection network to effectively disseminate the information among its processors (e.g., accumulation, broadcast or gossip) is a "pertinent" measure to determine the best communication structures for parallel and/or distributed computers. Assume that every node of a network holds a piece of information. Broadcast is the information dissemination problem that consists, for one node of a network, of sending its piece of information to all the other nodes. The accumulation problem can be considered as the reverse of broadcast problem. In the accumulation problem, every vertex has to send its piece of information to one specific vertex of the network. Finally, gossiping is a simultaneous broadcast from every node of the network. Due to their complexity, these three communication primitives are often provided at the software level. Most of the communication libraries available on parallel systems (as MPI [24]) provide access to such communication procedures. More generally, these three communication patterns are fundamental primitives used in many algorithms for the programming, and for the control

*Additional support by the DRET of the DGA.
†Partial support by UPC-PR-9704.

of parallel and distributed systems. For example, they are used for barrier synchronization or cache coherence [28], for parallel search algorithms [7], and for linear algebra algorithms [8].
In [10], Farley introduced the model called *Line Model* which satisfies the following: **(i)** a *call* involves exactly two nodes (these two nodes can be at distance more than 1), **(ii)** any two paths corresponding to simultaneous calls must be edge-disjoint. Furthermore, Farley assumed that nodes satisfy the $1-$*port* hypothesis, that is: **(iii)** a node can take part in one call at a time. The *vertex-disjoint paths mode* can be defined analogously to the line model by replacing hypothesis (ii) by the following **(iv)** any two paths corresponding to simultaneous calls must be vertex-disjoint. The calls are subject to different possible constraints: when two nodes are involved in the same call, they can either exchange all the informations they are aware of (*full-duplex* or *2-way* mode) or alternatively, the information can only flow in one direction (*half-duplex* or *1-way* mode).
A *round* is the set of all calls carried out simultaneously. The complexity of our communication protocols will be measured by the number of communication rounds required to complete these protocols. For a given graph $G = (V, E)$, and for any arbitrary node u in G, we denote by $b(G, u)$ (resp. $a(G, u)$) the minimum number of rounds for broadcasting from the source node u (resp. for accumulation) in the graph G. Similarly, the gossip time of G, denoted by $g(G)$, is the minimum number of rounds necessary to perform a gossip in G.
In [10], Farley proved that, in the $1-$port model, broadcast from any node in any $n-$node network can be performed in $\lceil \log_2 n \rceil$ rounds. His proof makes use of routing along the edge of a spanning tree of the network. However, the gossip problem is still open for arbitrary networks, that is, the complexity of gossiping in the $1-$port line model in arbitrary networks is not known. Hromkovič et al. [17] gave a lower bound for the gossip problem, and some results have been derived for tree-networks [9] and for planar graphs [16].
Chordal ring networks were introduced in [6]. They form a family of generalized loop networks [3]. The *chordal ring* of N vertices and chord c, denoted by $\mathbf{C(N, c)}$, is the graph with vertices labeled in $\mathbb{Z}_N$, and adjacencies given by $i \sim i \pm 1$, $i \sim i + c$ for every even vertex i. The structure of these graphs has been extensively studied. For example, Arden and Lee [1] studied the problem of the maximization of the number of nodes for a given diameter, and Yebra et al. [29] found a relationship between a certain type of plane tessellation and the chordal ring. Moreover, due to their simple structure, and their short diameter, chordal ring graphs are attractive topologies for interconnection networks. Chordal rings can support compact [25] and fault-tolerance [2] routing functions. Finally, Comellas and Hell [5] presented an optimal solution for the broadcast problem in chordal rings under the telephone model.
As [5], this paper is devoted to the study of communication problems in chordal rings. In particular, our aim is to find an algorithm for the gossip problem in the full-duplex edge-disjoint paths mode since the model is appropriate to networks supporting long distance calls such as wormhole or circuit-switched routing. The next section describes the method to find the gossip time in the chordal ring. Section 3 deals with some properties of the chordal ring and, finally, in Section 4 gossiping algorithms are described in order to give the upper bound of the gossip time in chordal rings.

2 Basic concepts

An interconnection network is modeled by a connected undirected graph $G = (V, E)$, where the vertices in V correspond to the processors, and the edges in E represent the communication links of the network. Our gossip algorithms are based on the so-called 3-phase gossip method [15]. For that purpose, the Section 3 gives a decomposition of the chordal ring into disjoint cycles.

This decomposition is the base of all our gossip protocols of Section 4. In the full-duplex line model, Farley has shown the following:

Lemma 1 (Farley [10]) *Let G be a graph of n nodes. In the 2-way mode line model,*

$$b(G) = a(G) = \lceil \log_2 n \rceil.$$

Moreover, Hromkovič et al. [17] gave a lower bound for the gossip problem:

Lemma 2 (Hromkovič et al. [17]) *Let G be a graph of N nodes and of edge-bisection width B. In the 2-way mode line model*

$$g(G) \geq 2\lceil \log_2 N \rceil - \log_2 B - \log_2 \log_2 N - 2.$$

In order to get upper bounds, we use the "three–phase algorithm" method as in [15]. The three–phase algorithm is composed of an accumulation phase, a gossiping phase, and a broadcasting phase.

Algorithm 1 Three-phase gossip algorithm

Divide G into r connected components containing exactly one accumulation node each. These components are called *accumulation components*. $A(G)$ is the set of accumulation nodes.
/* *Accumulation phase* */
Each vertex $u \in A(G)$ accumulates the information from the nodes of its component.
/* *Gossip phase* */
Perform a gossip among the set $A(G)$ of accumulation nodes.
/* *Broadcast phase* */
Every node in $A(G)$ broadcasts information in its components.

To obtain an effective algorithm, we will look for a set of accumulation nodes such that the gossip phase can be performed as quickly as gossiping in a complete graph, and such that the size of the accumulation components is sufficiently small in order to minimize the time for the first and third phases. Moreover, these accumulation components should be connected so that the optimal $\lceil \log_2 N \rceil$-round accumulation and broadcast algorithms described in [10] (see Lemma 1) can be independently applied in all the components. For the gossip phase, our algorithms will be based on the two following algorithms 2 and 3. K_N stands for the complete graph of N nodes.

Algorithm 2 Gossiping in a K_N, N even

For j:=1 **to** $\lceil \log_2 N \rceil$ **do**
  For each vertex i, i even, **do in parallel**
    exchange information between node i and node $i + 2^j - 1 \pmod{N}$;

Algorithm 3 Gossiping in a K_N, N odd

m:= $\lfloor N/2 \rfloor$;
For each node i, $0 < i < N/2$, **do in parallel**
  exchange information between node i and node $i + m$;
if m is odd **then** $n' = m + 1$ **else** $n' = m + 2$;
Gossip in the complete graph of vertices $\{0, \ldots, n'\}$;

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For each node i, i < N/2, do in parallel
    exchange information between node i and node i + m;
```

In the three-phase gossip algorithm, a call between vertices i and j is replaced by a call between the accumulation node of the ith component and the accumulation node of the jth component. For a given call between the accumulation node x_i of the ith component and the accumulation node x_j of the jth component, that is for a given path $\mathcal{P}$ between x_i and x_j, the *length* of the call is defined as the number of components traversed by $\mathcal{P}$ plus one.
Let s be the maximum size of the components, then the accumulation and broadcast phases are done in $\lceil \log_2 s \rceil$ rounds each. Moreover, if r is the number of components, the gossiping phase needs $\lceil \log_2 r \rceil$ to be completed. Thus, we can conclude that the three-phase gossip algorithm needs $2\lceil \log_2 s \rceil + \lceil \log_2 r \rceil$ rounds to perform.
In order to apply this algorithm to the chordal ring graphs, we will present some properties of this family of graphs.

3 Definition of the chordal rings

Definition 1 *Let N be an even integer and c an odd integer between 1 and $N/2$. The* chordal ring *graph of order N and chord c, $\mathbf{C(N,c)}$, is the graph of order N, with vertices labeled in $\mathbb{Z}_N$, and adjacencies given by $i \sim i \pm 1$, $i \sim i + c$ for every even vertex i.*

Chordal ring graphs are connected and 3-regular. They are bipartite, with partition sets $V_0 = \{0, 2, 4, \dots N-2\}$ and $V_1 = \{1, 3, 5, \dots, N-1\}$.
For any two vertices x, y, we define $\alpha_{x,y} : \mathbb{Z}_N \to \mathbb{Z}_N$ as follows:
If $x - y \equiv 0 \pmod 2$ then $\alpha_{x,y}(i) = y - x + i$, otherwise $\alpha_{x,y}(i) = y + x - i$.
In both cases, $\alpha_{x,y}$ is an automorphism and it verifies $\alpha_{x,y}(x) = y$. So, $\mathbf{C(N,c)}$ is vertex-transitive. For more details on these graphs, see [2].
In this section we present upper bounds for the edge bisection width of a chordal ring $C(N,c)$. The *edge bisection width*, B, is the minimum number of edges which separate the graph into two sets of vertices of the same cardinality.

Lemma 3 *Let $\mathbf{C(N,c)}$ be the chordal ring of order N and chord c. Let B be the edge bisection width of $\mathbf{C(N,c)}$, then $B \le c + 2$.*

Proof.
First, we observe that the natural order for integer numbers gives a natural partition of the vertices, so we have an immediate bound for B :

$$\mathbb{Z}_N = [0, N/2-1] \cup [N/2, N-1] \text{ where } [a,b] = \{a, a+1, \dots, b\}.$$

Let $A = \{(i,j) \in E |\, i \in [0, N/2-1],\, j \in [N/2, N-1]\}$ be the set of edges between $[0, N/2-1]$ and $[N/2, N-1]$. Clearly, from the definition of the edge bisection width, we have that the number of edges of A is an upper bound for B.
For $N/2$ even we have
$A = \{(1, N-c+1), (3, N-c+3), \dots, (c-2, N-2)\} \cup$
$\cup \{(N/2-c+1, N/2+1), (N/2-c+3, N/2+3), \dots, (N/2-2, N/2+c-2)\} \cup$
$\cup \{(0, N-1), (N/2-1, N/2)\}$.
Thus $|A| = (c-1)/2 + (c-1)/2 + 2 = c + 1 \ge B$.
For $N/2$ odd, we apply the same argument as previously and we get $|A| = c + 2 \ge B$.

$\square$

From Lemmas 3 and 2, we can deduce a lower bound of the gossip time in chordal rings $\mathbf{C(N,c)}$.

Lemma 4 *Let $\mathbf{C(N,c)}$ be a chordal ring of order N and chord c. In the 2-way mode line model*

$$g(G) \geq 2\lceil \log_2 N \rceil - \log_2(c+2) - \log_2 \log_2 N - 2.$$

3.1 Decomposition of the chordal rings into cycles

Let us introduce a decomposition of the chordal ring $\mathbf{C(N,c)}$ of order N and chord c that will be particularly helpful for the design of our gossip algorithm. We can distinguish in $\mathbf{C(N,c)}$ a set of $a = \lfloor \frac{N}{c+1} \rfloor$ disjoint $(c+1)$-cycles, $C_0, \ldots, C_{a-1}$:

$$C_i = \{i(c+1), i(c+1)+1, \ldots, i(c+1)+c\}.$$

If $N = a(c+1)+b$ with $0 \leq b \leq c-1$ even, then the chordal ring $\mathbf{C(N,c)}$ consists of a cycles of length $c+1$, labeled from 0 to $a-1$, and a path of b vertices, with some additional edges joining these subgraphs.
Given a cycle C_i, a vertex is said to be the jth vertex of cycle C_i, $j = 0, \ldots, c$, if its label is equal to $i(c+1)+j$. Note that the vertex 0 of C_i is adjacent to vertex 1 and c of C_i. Moreover, for each i, $0 \leq i < \lfloor \frac{N}{c+1} \rfloor - 1$, cycles C_i and C_{i+1} share $(c+1)/2$ edges. Among these edges, $(c-1)/2$ edges are of type $[x, x+c]$ and one edge is of type $[x, x+1]$. In other words, for any even value j, the jth vertex in C_i is adjacent to the $j-1$st vertex in C_{i+1} by a chordal edge. See the decomposition of $\mathbf{C(64,7)}$ in Fig. 1.

4 Gossip Algorithm for the chordal ring $\mathbf{C(N,c)}$

In this section, we describe a polynomial time algorithm to compute an optimal communication scheme for gossiping in any chordal ring $\mathbf{C(N,c)}$. We consider two cases: $N = a(c+1)$ and $N = a(c+1)+b$ where $2 \leq b \leq c-1$. For these two cases, we split the graph into a certain number of cumulative components, and we choose the accumulation node in each component. Finally, we set the paths corresponding to calls such that the gossip phase between the accumulation nodes performs almost as quickly as gossiping in the complete graph.
By convention, the integers r and s represent respectively the number of components, and their maximum size.

4.1 Case $N = a(c+1)$

Since the graph is composed of a cycles of $c+1$ vertices (see Section 3.1), the components are naturally defined by groups of the cycles $C_0, \ldots, C_{a-1}$. Actually, there are two types of decomposition according to the parameter a:

- If $a < c+1$, then there are a components ($r = a$): each component is a cycle of $c+1$ vertices ($s = c+1$).

- If $a \geq c+1$, then there are $c-1$ components ($r = c-1$). More precisely, assuming that $a = \alpha(c-1)+\beta$ with $0 \leq \beta < c-1$, β of these components are the union of $\alpha+1$ consecutive cycles of $c+1$ vertices, and the $c-1-\beta$ remaining components are the union of α consecutive cycles of $c+1$ vertices ($s \leq (c+1)(\alpha+1)$. In fact $s = \alpha(c+1)$ if $\beta = 0$ and $s = (c+1)(\alpha+1)$ otherwise).

We label the components between 0 and $r-1$. In the kth component, the cycles are labeled between 1 and Γ, where Γ is the number of cycles in the component k. We denote by (k, i, j)

the vertex j of the ith cycle in the kth component. The kth component has $(k, 1, R = c-1)$ as accumulation node.
Since each component is connected, the accumulation phase and the broadcast phase can be performed using the algorithm in [10]. Now, we focus on the gossip phase, and we are interested in a path $\mathcal{P}_{i\to i+\ell \pmod r}$ corresponding to a call between the two accumulation nodes of component i and $i+\ell \pmod r$, $i = 0, \dots, r-1$ and $\ell = 1, \dots, R/2$.
For $\ell \leq \frac{c-1}{2}$ we define $\mathcal{P}_{i\to i+\ell}$ as a union of paths denoted by $P(i,0), P(i,1), \dots, P(i,\ell)$ such that $P(i,k)$ is as follows (let $R = c-1$):

1. if $0 \leq k \leq \ell - 1$, then we consider two cases:
   - if the component $i+k$ is composed of one cycle of $c+1$ vertices, then the path $P(i,k)$ is $\{(i+k, 1, R-2k), (i+k+1, 1, R-2k-1), (i+k+1, 1, R-2k-2)\}$
   - if the component $i+k$ is composed of Γ cycles of $c+1$ vertices, then the path $P(i,k)$ is $\{(i+k, \gamma, R-2k), (i+k, \gamma+1, R-2k-1), (i+k, \gamma+1, R-2k) | \gamma = 1, \dots, \Gamma-1\} \cup \{(i+k, \Gamma, R-2k), (i+k+1, 1, R-2k-1), (i+k+1, 1, R-2k-2)\}$.
2. if $k = \ell$, then the path $P(i,\ell)$ is a path of component $i+\ell$, from vertex $(i+\ell, 1, R-2\ell)$ to $(i+\ell, 1, R)$ passing through vertex $(i+\ell, 1, 0)$.

Note that the condition $2\ell \leq c-1$ assures that these paths are well defined.
At each round of the gossip phase, the exchange of information can be performed by using the paths defined above:

1. If the number of components is odd, $r = 2r'+1$, and we are in first or last round. Nodes i and $i+r'$, $1 \leq i \leq r'$, exchange their information by using paths $\mathcal{P}_{i\to i+r' \pmod r}$, with $r' \leq \frac{c-1}{2}$.
2. If the number of components is odd, $r = 2r'+1$ but we are neither in the first nor in the last round, we merge the components $n', \dots, r-1$ into a single n'-component, with $n' = r'$ if r' is odd and $n' = r'+1$ if r' is even. After this merging, the number of components is even, and we proceed as in the next case.
3. If the number of components is even, for some j the accumulation nodes of i and $i+2^j-1 \pmod r$ components exchange their information for all i even. If $2^j-1 \pmod r \leq \frac{c-1}{2}$ the paths corresponding to this round are $\mathcal{P}_{i\to i+\ell \pmod r}$ with $\ell = 2^j-1$, for all i even. When $2^j-1 \pmod r > \frac{c-1}{2}$ the paths corresponding to this round are $\mathcal{P}_{i\to i+\ell \pmod r}$ with $\ell = r - (2^j-1 \pmod r) \leq \frac{c-1}{2}$, for all i odd.

Let us prove that, at each round, the calls are pairwise edge-disjoint.

Lemma 5 *Let be $\ell \leq \frac{c-1}{2}$. For all $i < j$ and $i+\ell \pmod r \neq j$, the paths $\mathcal{P}_{i\to i+\ell}$ and $\mathcal{P}_{j\to j+\ell}$ are pairwise edge-disjoint.*

Proof. Assume w.l.g. that $i = 0$. If $j < k < \ell$, the paths $\mathcal{P}_{0\to\ell}$ and $\mathcal{P}_{j\to j+\ell}$ cross the k-th component. We need only to prove that vertices of the k-th component belonging to the path $\mathcal{P}_{0\to\ell}$ are different to vertices of the k-th component belonging to the path $\mathcal{P}_{j\to j+\ell}$.

The third component of the vertex coordinates in $P(0,k)$ is $R-2k$ or $R-2k-1$, while the third component of the vertex coordinates in $P(j,k-j)$ is $R-2k-2j$ or $R-2k-2j-1$. Thus, the assertion holds. $\square$

In Fig. 1 there is an example of the paths $\mathcal{P}_{i \to i+3}$ corresponding to the round $t=2$, in the chordal ring $\mathbf{C(64,7)}$. Since in the decomposition of this graph all the groups have only one cycle, the vertex j in cycle i is labeled by $(i,1,j)$.

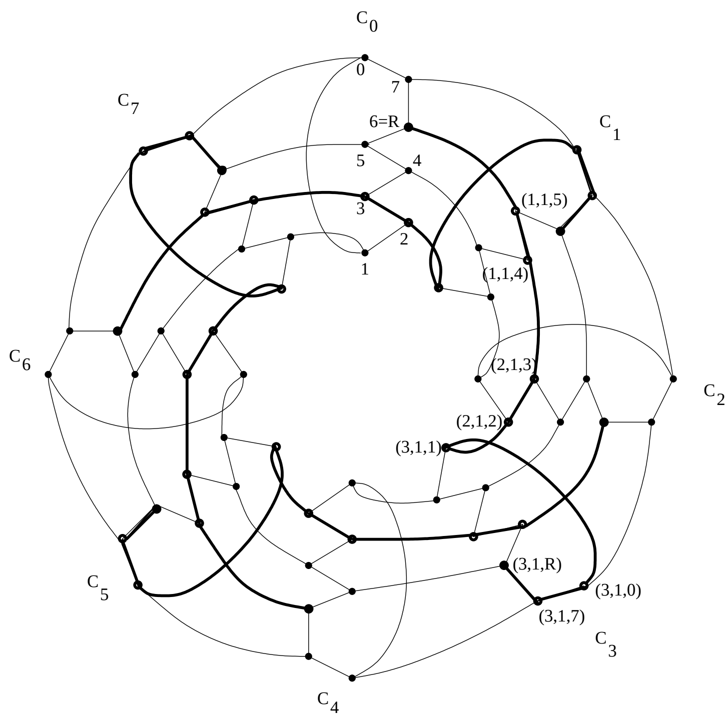

Figure 1: The chordal ring $\mathbf{C(64,7)}$.

Therefore, we conclude that the decomposition in cycles as defined in Section 3.1 allows a gossip among accumulation nodes that performs as quickly as gossiping in the complete graph.

4.2 The upper bound in case $N = a(c+1)$

Since the gossip algorithm performs in $2\lceil \log_2 s \rceil + \lceil \log_2 r \rceil = 2\lceil \log_2 N \rceil - \lceil \log_2 r \rceil + O(1)$, the calculation of the number of rounds gives:

- $a \geq c+1$, then $r = c-1$ and $s = (\alpha+1)(c+1)$.

$$g(\mathbf{C}(N,c)) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 c \rceil + O(1)$$

- $a < c+1$, then $r = a$ and $s = c+1$.

$$g(\mathbf{C}(N,c)) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 a \rceil + O(1)$$

4.3 The general case: $N = a(c+1) + b$, $0 < b < c$.

Note that, since N and $c+1$ are even, b is necessarily even. We consider two cases: $a \geq b/2$ and $a < b/2$.

4.3.1 Case 1. $a \geq b/2$

A graph of type $\mathcal{A}$ is a subgraph of the chordal ring that contains $c+3$ consecutive vertices on the ring, two chordal edges of type $[x, x+c]$, and $c+2$ edges of type $[x, x+1]$. If i is the smallest label of nodes of a subgraph H of type $\mathcal{A}$, then the vertex of label $i+j$ is called the jth vertex of H (the vertex 0 of H is incident to vertices 1 and c in H).
There are two types of decomposition, according to the parameter a. In both cases, we split the graph in $a-b/2$ cycles of $c+1$ vertices and in $b/2$ graphs of type $\mathcal{A}$.

- If $a < c+1$, then there are a components. If $0 \leq i \leq b/2-1$, then the component i is a subgraph of type $\mathcal{A}$, A_i, and if $b/2 \leq i \leq a-1$, then the component i is a cycle of length $c+1$, C_i. The graph is the union of consecutive subgraphs

$$A_0, A_1, \dots, A_{b/2-1}, C_{b/2}, \dots, C_{a-1}$$

  And for any i, $0 \leq i \leq a-1$, the subgraph of index i of this sequence is connected to the subgraphs of index $i-1$ and $i+1$, modulo a, by $(c+1)/2$ edges.

- If $a \geq c+1$, then we define $c-1$ components, each of them containing at most one subgraph of type $\mathcal{A}$ and possibly many cycles, as follows. Assuming that $a = \alpha(c-1)+\beta$, with $0 \leq \beta < c-1$, then the first β components are constituted by $\alpha+1$ consecutive subgraphs, and the $c-1-\beta$ remainig components are constituted by α consecutive subgraphs. Moreover, only the first subgraph in the first $b/2$ components is a subgraph of type $\mathcal{A}$.

  More precisely, the component i, $0 \leq i \leq b/2-1$, is the union of consecutive subgraphs $A_i, C_{i,1}, \dots, C_{i,\Gamma_i}$, and the component i, $b/2 \leq i < c-1$, is the union of consecutive cycles of length $c+1$, $C_{i,0}, C_{i,1}, \dots, C_{i,\Gamma_i}$, where A_i is a subgraph of type $\mathcal{A}$, $C_{i,j}$ is a cycle of length $c+1$, and the value of Γ_i is α if $i \leq \beta-1$, and $\alpha-1$ otherwise.

  The graph can be described by the following sequence:

$$\underbrace{A_0, C_{0,1}, \dots, C_{0,\Gamma_0}}_{\text{component } 0}, \underbrace{A_1, C_{1,1}, \dots, C_{1,\Gamma_1}}_{\text{component } 1}, \dots, \underbrace{A_{b/2-1}, C_{b/2-1,1}, \dots, C_{b/2-1,\Gamma_{b/2-1}}}_{\text{component } b/2-1},$$

$$\underbrace{C_{b/2,0}, C_{b/2,1}, \dots, C_{b/2,\Gamma_{b/2}}}_{\text{component } b/2}, \dots, \underbrace{C_{c-2,0}, C_{c-2,1}, \dots, C_{c-2,\Gamma_{c-2}}}_{\text{component } c-2},$$

  In each component, the subgraph of type $\mathcal{A}$ is labeled 1, and the remaining cycles are labeled $2, 3, \dots$

The component k has $(k, 1, c-1)$ as its accumulation node if component k does not contain a subgraph of type $\mathcal{A}$, and $(k, 1, c+1)$ otherwise. We set $R = c-1$.
The accumulation phase and the broadcast phase can be performed using the algorithm in [10] since each component is connected. Now, we focus on the gossip phase, and we are interested in the call $\mathcal{P}'_{i \to i+\ell}$ between the two accumulation nodes of the component i and the component $i+\ell$.
As in the previous case, for $\ell \leq \dfrac{c-1}{2}$ we define $\mathcal{P}'_{i \to i+\ell}$ as the union of paths denoted by $P'(i,0), P'(i,1), \dots, P'(i,\ell)$ such that $P'(i,k)$, $0 \leq k \leq \ell$, is as follows:

1. if the $i+k+1$ component does not contain a graph of type $\mathcal{A}$ or $k = \ell$, then $P'(i,k) = P(i,k)$ [1];

[1] defined in Section 4.1

2. if $k \neq \ell$ and the component $i+k+1$ contains a graph of type $\mathcal{A}$, then we consider four cases:

   - if the component $i+k$ contains only one graph of type $\mathcal{A}$, then the path $P'(i,k)$ is $\{(i+k,1,R-2k+2),(i+k+1,1,R-2k-1),(i+k+1,1,R-2k)\}$
   - if the component $i+k$ contains only a cycle of $c+1$ vertices, then the path $P'(i,k)$ is $\{(i+k,1,R-2k),(i+k+1,1,R-2k-1),(i+k+1,1,R-2k)\}$
   - if the component $i+k$ is composed of one graph of type $\mathcal{A}$ and $\Gamma-1$ cycles of $c+1$ vertices, then the path $P'(i,k)$ is $\{(i+k,1,R-2k+2),(i+k,2,R-2k-1),(i+k,2,R-2k)\} \cup \{(i+k,\gamma,R-2k),(i+k,\gamma+1,R-2k-1),(i+k,\gamma+1,R-2k)|\gamma = 2,\ldots,\Gamma-1\} \cup \{(i+k,\Gamma,R-2k),(i+k+1,1,R-2k-1),(i+k+1,1,R-2k-2)\}$.
   - if the component $i+k$ is composed of Γ cycles of $c+1$ vertices, then the path $P'(i,k)$ is $\{(i+k,\gamma,R-2k),(i+k,\gamma+1,R-2k-1),(i+k,\gamma+1,R-2k)|\gamma = 1,\ldots,\Gamma-1\} \cup \{(i+k,\Gamma,R-2k),(i+k+1,1,R-2k-1),(i+k+1,1,R-2k-2)\}$.

It is easy to see that at each round of the gossip phase, the exchange of information can be performed by using the paths defined above. And the next lemma holds.

Lemma 6 *Let be $\ell \leq \frac{c-1}{2}$. For all $i < j$ and $i+\ell \pmod r \neq j$, the paths $\mathcal{P}'_{i\to i+\ell}$ and $\mathcal{P}'_{j\to j+\ell}$ are pairwise edge-disjoint.*

The proof is analogous to the proof of Lemma 5.

4.3.2 Case 2. $a < b/2$

Since $b < c+1$, it implies that $a < (c+1)/2$. By using the cycle decomposition, we split the graph into $r = a+1$ components. The components $1\ldots a$ contain one $c+1$–cycle and the component 0 is a path of b vertices. The vertex j of the b–path in component 0 has label $(0,1,j)$ and the vertices in the other components are labeled as in Section 4.1. We take $(k,1,R=b-2)$, $k=0\ldots a$, as accumulation vertices (in Section 4.1, $R=c-1$). Let us notice that there are $b/2+1$ edges connecting component $r-1$ with component 0.
This decomposition allows us to define an algorithm as in Section 4.1.

4.4 The upper bound in case $N = a(c+1)+b$

Since the gossip algorithm performs in $2\lceil\log_2 s\rceil + \lceil\log_2 r\rceil$, the calculation of the number of rounds gives:

- $a \geq b/2$:

  $a \geq c+1$, then $r = c-1$ and $s = (\alpha+1)(c+3)$.

  $$g(\mathbf{C}(N,c)) \leq 2\lceil\log_2 N\rceil - \lceil\log_2 c\rceil + O(1)$$

  $a < c+1$, then $r = a$ and $s = c+3$.

  $$g(\mathbf{C}(N,c)) \leq 2\lceil\log_2 N\rceil - \lceil\log_2 a\rceil + O(1)$$

- $a < b/2 < (c+1)/2$, then $r = \lfloor a/2\rfloor + 1$ and $s = 2(c+1)$:

  $$g(\mathbf{C}(N,c)) \leq 2\lceil\log_2 N\rceil - \lceil\log_2 a\rceil + O(1)$$

5 Conclusion

We have decomposed the chordal ring $\mathbf{C(N, c)}$ into r components of size s in order to apply the three–phase algorithm. This enables us to give an upper bound for the gossiping time under the full-duplex line model:

$$g(\mathbf{C}(N, c)) \leq 2\lceil \log_2 s \rceil + \lceil \log_2 r \rceil$$

According to the different values of c and $N = a(c+1) + b$ we can reduce the results into the following two cases:

- $a \geq c+1$
$$g(\mathbf{C}(N, c)) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 c \rceil + O(1)$$
- $a < c+1$
$$g(\mathbf{C}(N, c)) \leq 2\lceil \log_2 N \rceil - \lceil \log_2 a \rceil + O(1)$$

From Lemma 4 we can conclude that in the first of the above cases our bound is optimal. In the second one, we expect that a better approximation of the edge–bisection width could prove the optimality of this algorithm.

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